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SYMMETRIC IDEALS OF GENERALIZED SUMMING MULTILINEAR OPERATORS

ARIEL S. SANTIAGO[1,†] & GERALDO BOTELHO[2,‡]

[1]Instituto de Ciências Exatas, UFMG, MG, Brasil, [2]Faculdade de Matemática, UFU, MG, Brasil,

[†]arielsantossantiago@hotmail.com [‡]botelho@ufu.br

Abstract

Let $X_1, \dots, X_n, Y$ be classes of Banach spaces-valued sequences. An n-linear operator A between Banach spaces belongs to the ideal of $(X_1, \dots, X_n; Y)$-summing multilinear operators if $(A(x_j^1, \dots, x_j^n))_{j=1}^\infty$ belongs to Y whenever $(x_j^k)_{j=1}^\infty$ belongs to $X_k, k = 1, \dots, n$. In this paper we develop techniques to generate non trivial symmetric ideals of this type. Illustrative examples are provided.

1 Introduction

For Banach spaces $E, E_1, \dots, E_n, F$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\mathcal{L}(E_1, \dots, E_n; F)$ denotes the space of continuous n-linear operators from $E_1 \times \cdots \times E_n$ to F. If $E = E_1 = \cdots = E_n$, we write $\mathcal{L}(^nE; F)$. An ideal of n-linear operators is a subclass $\mathcal{M}_n$ of the class of continuous n-linear operators between Banach spaces such that, for all $E_1, \dots, E_n, F$,

$$\mathcal{M}_n(E_1, \dots, E_n; F) := \mathcal{M}_n \cap \mathcal{L}(E_1, \dots, E_n; F)$$

is a linear subspace of $\mathcal{L}(E_1, \dots, E_n; F)$ containing the n-linear operators of finite type and satisfying the following ideal property: if $A \in \mathcal{M}_n(E_1, \dots, E_n; F)$, $u_j \colon G_j \longrightarrow E_j$, $j = 1, \dots, n$, and $t \colon F \longrightarrow H$ are bounded linear operators, then the composition $t \circ A \circ (u_1, \dots, u_n)$ belongs to $\mathcal{M}_n(G_1, \dots, G_n; H)$. Details can be found in [1, 2].

In this work we are interested in ideals defined, or characterized, by the transformation of vector-valued sequences. We follow the approach from [3], which we describe next.

The symbol $E \overset{1}{\hookrightarrow} F$ means that E is a linear subspace of F and $\|\cdot\|_F \leq \|\cdot\|_E$ on E. By $c_{00}(E)$ and $\ell_\infty(E)$ we denote the spaces of eventually null and bounded E-valued sequences. By $(e_j)_{j=1}^\infty$ we denote the canonical sequences. A *sequence class* is a rule that assigns, to each Banach space E, a Banach space $X(E)$ of E-valued sequences such that $c_{00}(E) \subseteq X(E) \overset{1}{\hookrightarrow} \ell_\infty(E)$ and $\|e_j\|_{X(\mathbb{K})} = 1$ for every $j \in \mathbb{N}$, in symbols $E \mapsto X(E)$.

Let $1 \leq p < \infty$. The following correspondences are examples of sequence classes: $E \mapsto \ell_\infty(E)$, $E \mapsto c_0(E)$, $E \mapsto c(E)$, $E \mapsto c_0^w(E)$, $E \mapsto \ell_p(E)$, $E \mapsto \ell_p^w(E)$, $E \mapsto \ell_p^u(E)$, $E \mapsto Rad(E)$, $E \mapsto RAD(E)$, $E \mapsto \ell_p\langle E\rangle$, $E \mapsto \ell_p^{mid}(E)$. All sequences spaces aforementioned are endowed with their natural norms.

An n-linear operator $A \in \mathcal{L}(E_1, \dots, E_n; F)$ is said to be $(X_1, \dots, X_n; Y)$-summing if $(A(x_j^1, \dots, x_j^n))_{j=1}^\infty \in Y(F)$ whenever $(x_j^k)_{j=1}^\infty \in X_k(E_k), k = 1, \dots, n$. The ideal of $(X_1, \dots, X_n; Y)$-summing n-linear operators is denoted by $\Pi_{X_1, \dots, X_n; Y}$.

The symmetrization of the n-linear operator $A \in \mathcal{L}(^nE; F)$ is denoted by A_s. According to [1], an ideal $\mathcal{M}_n$ of n-linear operators is said to be *symmetric* if A_s belongs to $\mathcal{M}_n$ whenever A belongs to $\mathcal{M}_n$. The purpose of this work is to develop techniques to generate symmetric ideals of the kind $\Pi_{X_1, \dots, X_n; Y}$ that are non trivial in the sense that $X_i \neq X_j$ for some i, j and $Y \neq \ell_\infty(\cdot)$.

2 Main Results

Definition 2.1. *The sequence classes $X_1, \dots, X_n$ are said to be jointly dominated if there exists a finitely determined sequence class X such that $X_i(E) \overset{1}{\hookrightarrow} X(E)$ and $\|\cdot\|_{X_i(E)} \leq \|\cdot\|_{X(E)}$ on $c_{00}(E)$ for every Banach space E and all i.*

Theorem 2.1. *If the sequence classes $X_1, \ldots, X_n$ are jointly dominated, then the ideal $\Pi_{X_1,\ldots,X_n;Y}$ is symmetric for every finitely determined sequence class Y.*

Definition 2.2. *Let X be a sequence class.*
(a) For a Banach space E, we define $\overline{c_{00}}^X(E)$ as the closure of $c_{00}(E)$ in $X(E)$ endowed with the norm $\|\cdot\|_{X(E)}$.
(b) X is finitely shrinking if, regardless of the Banach space E, the sequence $(x_j)_{j=1}^\infty \in X(E)$ and $k \in \mathbb{N}$, it holds

$$(x_j)_{j\neq k} := (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots) \in X(E) \text{ and } \|(x_j)_{j\neq k}\|_{X(E)} \leq \|(x_j)_{j=1}^\infty\|_{X(E)}.$$

(c) Suppose that X is finitely shrinking. If $(x_j)_{j=1}^\infty \in X(E)$, then $(x_n, x_{n+1}, \ldots) \in X(E)$ for every $n \in \mathbb{N}$. For a Banach space E, we define

$$X^u(E) := \left\{(x_j)_{j=1}^\infty \in X(E) : \lim_n \|(x_n, x_{n+1}, \ldots)\|_{X(E)} = 0\right\}, \quad \text{endowed with the norm } \|\cdot\|_{X(E)}.$$

Corollary 2.1. *Let X and Y be finitely determined sequence classes with X finitely shrinking. The ideal $\Pi_{X_1,\ldots,X_n;Y}$ is symmetric for every $n \geq 2$ and all $X_i \in \{\overline{c_{00}}^X, X^u, X\}$, $i = 1, \ldots, n$.*

Example 2.1. *Let $1 \leq p < \infty$. $\Pi_{\ell_p^{\theta_1},\ldots,\ell_p^{\theta_n};Y}$ and $\Pi_{X_1,\ldots,X_n;Y}$ are non trivial sequential symmetric ideals for every $n \geq 2$, all $\theta_1, \ldots, \theta_n \in \{u, w\}$ with $\theta_i = u$ and $\theta_j = w$ for some i and j, all $X_1, \ldots, X_n \in \{\ell_p^{mid}, (\ell_p^{mid})^u\}$ (respectively, $\{RAD, Rad\}$) with $X_i = \ell_p^{mid}$ (respectively, $X_i = RAD$) and $X_j = (\ell_p^{mid})^u$ (respectively, $X_j = Rad$) for some i and j, and every finitely determined sequence class $Y \neq \ell_\infty(\cdot)$.*

Several usual sequence classes are finitely determined, such as $\ell_p(\cdot), \ell_p^w, \ell_\infty$, RAD, $\ell_p\langle\cdot\rangle$ and ℓ_p^{mid}; and the bad news is that some are not, such as $c_0(\cdot), c_0^w$, $c(\cdot), \ell_p^u$, Rad and $(\ell_p^{mid})^u$. In this case, we have the following theorem.

Definition 2.3. *For a sequence class X and regardless of the Banach space E we define*

$$X^{fd}(E) := \left\{(x_j)_{j=1}^\infty \in E^{\mathbb{N}} : \|(x_j)_{j=1}^\infty\|_{X^{fd}(E)} := \sup_k \|(x_j)_{j=1}^k\|_{X(E)} < \infty\right\}.$$

We say that X is finitely zero invariant if the sequence $(x_j)_{j=1}^\infty \subseteq E$ and $k \in \mathbb{N}$ for which $x_k = 0$, it holds

$$(x_j)_{j=1}^\infty \in X(E) \Leftrightarrow (x_j)_{j\neq k} \in X(E) \text{ and } \|(x_j)_{j=1}^\infty\|_{X(E)} = \|(x_j)_{j\neq k}\|_{X(E)}.$$

Theorem 2.2. *Let X and Y be finitely shrinking and finitely zero invariant sequence classes. For every $n \in \mathbb{N}$, the ideal $\Pi_{X_1,\ldots,X_n;Y}$ is symmetric whenever $X_i \in \{X^u, X, X^{fd}\}$ for every $i = 1, \ldots, n$, and $X_k = X^u$ for some $k \in \{1, \ldots, n\}$. Moreover, in this case $\Pi_{X_1,\ldots,X_n;Y} = \Pi_{X_1,\ldots,X_n;Y^u}$.*

Example 2.2. *The ideals $\Pi_{\ell_p^{\theta_1},\ldots,\ell_p^{\theta_n};Y}$ and $\Pi_{X_1,\ldots,X_n;Y}$ from Example 2.5 are non trivial sequential symmetric ideals for every finitely shrinking and finitely zero invariant sequence class $Y \neq \ell_\infty(\cdot)$.*

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GENERAL CRITERIA FOR A STRONGER NOTION OF LINEABILITY

G. RIBEIRO[1], VINÓCIUS V. FÁVARO[2], D. PELLEGRINO[1] & ANSELMO RAPOSO JR.[3]

[1]Departamento de Matemática, UFPB, PB, Brasil, [2] Faculdade de Matemática, UFU, MG, Brasil, [3] Departamento de Matemática, UFMA, MA, Brasil.

Abstract

In this work, among other results, we prove some general criteria for the notion of (α, β)-spaceability and, as applications, we extend recent results of different authors.

1 Introduction

The notions of lineability and spaceability were introduced in the seminal paper [2] by Aron, Gurariy and Seoane-Sepúlveda and its essence is to investigate linear structures within exotic settings. More precisely, $A \subset V$ is α-lineable in a vector space V, if $A \cup \{0\}$ contains an α-dimensional subspace of V. In addition, if V is endowed with a topology, we say that A is α-spaceable in V if there is an α-dimensional closed subspace of V contained in $A \cup \{0\}$.

It turns out that the vast literature related to this subject has shown that positive results of lineability and spaceability are rather common; we refer the reader to [1]. Recently, in [7] the notions of (α, β)-lineability and (α, β) -spaceability were introduced as an attempt to investigate how far positive results of lineability and spaceability remain valid under stricter assumptions.

From now on $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{K}$ represents the real scalar field $\mathbb{R}$ or the complex scalar field $\mathbb{C}$. All vector spaces are considered over $\mathbb{K}$.

Let V be a vector space and let A be a non-void subset of V.

- If α, β are cardinal numbers, $\alpha \leq \beta$, and then A is called (α, β)-lineable if it is α-lineable and for each α-dimensional subspace $W_\alpha \subset A \cup \{0\}$ there is a β-dimensional subspace W_β such that

$$W_\alpha \subset W_\beta \subset A \cup \{0\}. \tag{1}$$

- When V is endowed with a topology and the subspace W_β satisfying (1) can always be chosen closed, we say that A is (α, β)-spaceable. Moreover, we shall also say that A is (α, β)-dense lineable if it is α-lineable and for each α-dimensional subspace $W_\alpha \subset A \cup \{0\}$ there is a β-dimensional dense subspace W_β such that

$$W_\alpha \subset W_\beta \subset A \cup \{0\}.$$

The letters α, β will always represent cardinal numbers, $card(A)$ denotes the cardinality of the set A, $\aleph_0 := card(\mathbb{N})$ and $\mathfrak{c} := card(\mathbb{R})$.

2 Main Results

Theorem 2.1. *Let $\alpha \geq \aleph_0$ and V be an F-space. Let A, B be subsets of V such that A is α-lineable and B is 1-lineable. If $A \cap B = \varnothing$ and A is stronger than B, then A is not (α, β)-spaceable, regardless of the cardinal number β.*

Corollary 2.1. *Let $\alpha \geq \aleph_0$ and β be a cardinal number. The set $\mathcal{ND}\,[0,1]$ is not (α,β)-spaceable.*

Corollary 2.2. *Let $\alpha \geq \aleph_0$ and β be a cardinal number. For $p > 0$, the set $L_p[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q[0,1]$ is not (α,β)-spaceable.*

Corollary 2.3. *If $F = c_0$ or $F = c$, then $\ell_\infty \setminus F$ is not (α,β)-spaceable if $\alpha \geq \aleph_0$.*

Theorem 2.2. *Let V be an infinite dimensional Banach space and W be a closed vector subspace of V. If W has infinite codimension, then $V \setminus W$ is $(n,\mathfrak{c})$-spaceable, for every $n \in \mathbb{N}$.*

Corollary 2.4. *Let $F = c$ or c_0. Then $\ell_\infty \setminus F$ is $(\alpha,\mathfrak{c})$-spaceable if, and only if, $\alpha < \aleph_0$.*

Given a topological vector space V, let $\mathcal{B}_V$ be the set of all basis for the topology of V. Since cardinal numbers are well-ordered, we can consider $B_0 \in \mathcal{B}_V$ of minimal cardinality. The cardinality of B_0 is called the *weight* of V and denoted by $w(V)$.

Theorem 2.3. *Let $V \neq \{0\}$ be a topological vector space and $W \subset V$ be a linear subspace such that $w(V) \leq \dim(V/W)$. Then $V \setminus W$ is (α,β)-dense lineable for each $\alpha < \dim(V/W)$ and*

$$\max\{\alpha, w(V)\} \leq \beta \leq \dim(V/W).$$

Corollary 2.5. *Let $0 < p < \infty$. The set $L_p[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q[0,1]$ is (α,β)-dense lineable, for each $\alpha < \mathfrak{c}$ and*

$$\max\{\alpha, \aleph_0\} \leq \beta \leq \mathfrak{c}.$$

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MAXIMAL IDEALS OF GENERALIZED SUMMING LINEAR OPERATORS

JAMILSON R. CAMPOS[1], GERALDO BOTELHO[2] & LUCAS NASCIMENTO[3]

[1]Departamento de Ciências Exatas, UFPB, PB, Brasil, Faculdade de Matemática, UFU, MG, Brasil, [3]IFCE, CE, Brasil

Abstract

We prove when a Banach ideal of linear operators defined, or characterized, by the transformation of vector-valued sequences is maximal. Known results are recovered as particular cases and new information is obtained.

1 Introduction

The theory of operator ideals is central in modern mathematical analysis and, in this context, maximal ideals play a key role. A number of important operator ideals are defined, or characterized, by the transformation of vector-valued sequences; and some of these ideals are known to be maximal. A unifying approach to this kind of operators ideals was proposed in [1] using the concept of *sequence classes*.

The purpose of this talk is to present results on the maximality of the Banach operator ideals of the type $\mathcal{L}_{X;Y^{dual}}$ obtained in the work [3]. The main result establishes conditions on the sequences classes X and Y under which the Banach operator ideal $\mathcal{L}_{X;Y^{dual}}$ is maximal. We prove our main results defining, developing and applying a tensor quasi-norm $\alpha_{X;Y}$ determined by the sequence classes X and Y. The tensor quasi-norms $\alpha_{X;Y}$ are based on – and can be regarded as generalizations of – the classical Chevet-Saphar tensor norms (see [6]).

Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ are denoted by E and F. The symbol $E \overset{1}{\hookrightarrow} F$ means that E is a linear subspace of F and $\|x\|_F \leq \|x\|_E$ for every $x \in E$. The symbol $(x_j)_{j=1}^n$, where $x_1, \ldots, x_n \in E$, stands for the sequence $(x_1, x_2, \ldots, x_n, 0, 0, \ldots) \in E^{\mathbb{N}}$. By ε we denote the injective tensor norm on the tensor product of Banach spaces.

The theory and symbology of the sequence class environment and about the dual of sequence classes are present in [1] and [2]. The other notations and symbols used here are either usual in functional analysis or are present in the references already cited.

2 Main Results

We need $\mathcal{L}_{X;Y^{dual}}$ to be a Banach operator ideal. So, whenever we refer to $\mathcal{L}_{X;Y^{dual}}$ we assume that the sequence classes X and Y are linearly stable, Y is spherically complete and $X(\mathbb{K}) \overset{1}{\hookrightarrow} Y^{dual}(\mathbb{K})$. For the other properties and definitions concerning sequence classes used here we refer to [1, 2] and [3].

Let X and Y be sequence classes. For Banach spaces E and F, consider the map $\alpha_{X,Y} \colon E \otimes F \longrightarrow \mathbb{R}$ given by $\alpha_{X,Y}(u) = \inf \left\{ \left\|(x_j)_{j=1}^n\right\|_{X(E)} \cdot \left\|(y_j)_{j=1}^n\right\|_{Y(F)} : u = \sum_{j=1}^n x_j \otimes y_j \right\}$.

The idea is to identify conditions on the sequences classes X and Y, as weak as possible, so that $\alpha_{X,Y}$ is a quasi-norm on $E \otimes F$. So, we have the following

Proposition 2.1. *If X and Y are monotone sequence classes and $\varepsilon \leq \alpha_{X,Y}$, then, for all Banach spaces E and F, $\alpha_{X,Y}$ is a quasi-norm on $E \otimes F$ such that $\alpha(x \otimes y) \leq \|x\| \cdot \|y\|$ for all $x \in E$ and $y \in F$.*

Proposition 2.2. *If X and Y are monotone, linearly stable and finitely injective sequences classes and $\varepsilon \leq \alpha_{X,Y}$, then $\alpha_{X,Y}$ is a tensor quasi-norm.*

For $M \in \mathcal{F}(E)$ (finite dimensional subspaces of E) we denote by $I_M \colon M \longrightarrow E$ the inclusion operator and for $L \in \mathcal{CF}(F)$ (finite codimensional subspaces of E) we denote by $Q_L \colon F \longrightarrow F/L$ the quotient operator.

Here is our main result:

Theorem 2.1. *Suppose that $\alpha_{X,Y}$ is a tensor quasi-norm and that X and Y are finitely determined or finitely dominated. For an operator $T \in \mathcal{L}(E;F)$, $T \in \mathcal{L}_{X;Y^{dual}}(E;F)$ if and only if*

$$s := \sup\left\{\|Q_L \circ T \circ I_M\|_{X;Y^{dual}} : (M,L) \in \mathcal{F}(E) \times \mathcal{CF}(F)\right\} < \infty,$$

and, in this case, $\|T\|_{X;Y^{dual}} = s$. In particular, the Banach operator ideal $\mathcal{L}_{X;Y^{dual}}$ is maximal.

The next corollary is combination of the theorem above with a well-known characterization of maximal ideals.

Corollary 2.1. *Let $u \in \mathcal{L}(E;F)$ be given. Under the assumptions of Theorem 2.1 we have $u \in \mathcal{L}_{X;Y^{dual}}(E;F)$ if and only if $u^{**} \in \mathcal{L}_{X;Y^{dual}}(E^{**};F^{**})$ and $\|u\|_{X,Y^{dual}} = \|u^{**}\|_{X,Y^{dual}}$.*

Some examples of applications of our results:
(a) Theorem 2.1 recovers the following well known facts.
• The Banach ideal of absolutely (q,p)-summing operators: $\Pi_{q,p} := \mathcal{L}_{\ell_p^w(\cdot);\ell_q(\cdot)} = \mathcal{L}_{\ell_p^w(\cdot);[\ell_{q^*}(\cdot)]^{dual}}$, with $1 \leq p \leq q < \infty$, is maximal. In particular, the ideal Π_p of absolutely p-summing operators is maximal.
• The Banach ideal of Cohen strongly (q,p)-summing operators: $\mathcal{D}_{q,p} := \mathcal{L}_{\ell_p(\cdot);\ell_q\langle\cdot\rangle} = \mathcal{L}_{\ell_p(\cdot);[\ell_{q*}^w(\cdot)]^{dual}}$, $1 \leq p \leq q < \infty$, is maximal. Although we found no reference to quote, we believe this is a well known fact.
• The Banach ideal of cotype q operators: $\mathfrak{C}_q := \mathcal{L}_{RAD(\cdot);\ell_q(\cdot)} = \mathcal{L}_{RAD(\cdot);[\ell_{q^*}(\cdot)]^{dual}}$, $2 \leq q < \infty$, is maximal [6, 17.4].
(b) Just to illustrate the new information that can be obtained from Theorem 2.1 we mention that the Banach ideals $\mathcal{L}_{\ell_p^{mid}(\cdot);\ell_q\langle\cdot\rangle} = \mathcal{L}_{\ell_p^{mid}(\cdot);[\ell_{q*}^w(\cdot)]^{dual}}$ *and* $\mathcal{L}_{\ell_p^{mid}(\cdot);\ell_q(\cdot)} = \mathcal{L}_{\ell_p^{mid}(\cdot);[\ell_{q*}(\cdot)]^{dual}}$, which were studied in [4, 5], are maximal.

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GROTHENDIECK'S COMPACTNESS PRINCIPLE ON BANACH LATTICES

JOSÉ LUCAS P. LUIZ[1], GERALDO BOTELHO[2] & VINÍCIUS C. MIRANDA[2]

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IFNMG - Campus Araçuaí, MG, Brasil[1], Faculdade de Matemática, UFU, MG, Brasil[2]

lucasvt09@hotmail.com[†] botelho@ufu.br[‡] colferaiv@gmail.com[§]

Abstract

In this talk we present recent results obtained by the authors about the Grothendieck compactness principle for the absolute weak topology in the context of Banach lattices.

1 Introduction

Grothendieck's compactness principle was proved by Grothendieck in 1955 and claims the following: in a Banach space E, every compact subset is contained in the closed convex hull of a null sequence (see [6]).

In 2012, Dowling, Freeman, Lennard, Odell, Randrianantoanina and Turett studied the Grothendieck compactness principle for the weak topology and proved the following: every weakly compact subset of a Banach space E is contained in the closed convex hull of a weakly null sequence if and only if E has the Schur property (that is, every weakly null sequence in E is norm null) (see [3]). Later, another proof appeared in [7] using an operator theoretic approach.

Other works about this compactness-type principle have appeared in the literature in the last years from different approaches (see [2], [4] and [5]). In this talk, we present recent results about this principle for the absolute weak topology on Banach lattices.

2 Main Results

Before we present our main result (Theorem 2.3) we need to prove some results about the absolute weak topology on Banach lattice whose analogues are well known for the weak topology on Banach spaces. Below we list some of these results. By B_E we denote the closed unit ball of the Banach space E.

The first result is a Smulyan-type theorem for the absolute weak topology:

Theorem 2.1. *Absolutely weakly compact subsets of Banach lattices are absolutely weakly sequentially compact.*

The next result shows that there is no Alaoglu's theorem for the absolute weak topology:

Proposition 2.1. *Let E be a Banach lattice. If B_{E^*} is absolutely weak* compact, then E has order continuous norm.*

The following is a partial Eberlein-type theorem for the absolute weak topology.

Theorem 2.2. *Let K be an absolutely weakly sequentially compact subset of a Banach lattice E. If E is separable or $B_{E^{**}}$ is absolutely weak* compact, then K is absolutely weakly compact.*

Proposition 2.2. *Let E be Banach lattice such that E^* and E^{**} have order continuous norms and E^{**} is atomic. Then $B_{E^{**}}$ is absolutely weak* compact. In particular, $B_{E^{**}} = B_E$ is absolutely weak* compact for every reflexive atomic Banach lattice.*

We also present the following characterizations of the positive Schur property (meaning that every positive weakly null sequence is norm null) by means of the absolute weak topology.

Proposition 2.3. *The following are equivalent for a Banach lattice E:*
(a) E has the positive Schur property.
(b) Absolutely weakly null sequences in E are norm null.
(c) Positive absolutely weakly null sequences in E are norm null.
(d) Positive disjoint absolutely weakly null sequences in E are norm null.
(e) Disjoint absolutely weakly null sequences in E are norm null.

These preliminary results allow us to present and prove our main result:

Theorem 2.3. *A Banach lattice E has the positive Schur property if and only if every absolutely weakly compact subset of E is contained in the closed convex hull of an absolutely weakly null sequence.*

It is natural to wonder how the dual positive Schur property (meaning that positive weak*-null sequences in the dual space are norm null) can be connected to the absolute weak* topology. In this direction we prove the following corollary.

Corollary 2.1. *The following are equivalent for a Banach lattice E:*
(a) E has the dual positive Schur property.
(b) Every absolutely weak* null sequence in E^* is norm null.
(c) E has the positive Grothendieck property and every sequentially absolutely weak*-compact subset of E^* is contained in the closed convex hull of an absolutely weak* null sequence.

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A MULTIPOLYNOMIAL EXTENSION OF THE KAHANE–SALEM–ZYGMUND INEQUALITY

LINDINÊS COLETA[1], NACIB GURGEL ALBUQUERQUE[1] & THIAGO VELANGA[2]

Departamento de Matemática, UFPB, PB, Brasil[1], Departamento de Matemática, UNIR, RO, Brasil[2]

lindinescoleta@gmail.com[†] nacib.albuquerque@academico.ufpb.br[‡] thiagovelanga@unir.br[§]

Abstract

The classical Kahane-Salem-Zygmund inequality is investigated within the multipolynomials framework, a scenario encompassing multilinear and polynomial topics. The study compares the results obtained concerning the constants involved and the multilinear forms and polynomials derived from them. We provide a multipolynomial version of the inequality with optimal constants, among other results obtained.

1 Introduction

Here we present a multipolynomial version of the Kahane-Salem-Zygmund (KSZ) inequality, which one depends on the multilinear result used as tool. Before further going, let us recall some basic facts about polynomials and multipolynomials. We recall that if E and F are Banach spaces, a mapping $P : E \to F$ is said to be an m-homogeneous polynomial when there exists an m-linear mapping $A : E^m \to F$ which is equal to P on the diagonal (we refer [4] or [3] to the basics of the theory).

The concept of homogeneous polynomial mentioned above has a natural generalization, due to I. Chernega and A. Zagorodnyuk in [2, Definition 3.1]. Namely, given positive integers m and $n_1, \ldots, n_m$, a mapping

$$P : E_1 \times \cdots \times E_m \to F$$

is said to be an $(n_1, \ldots, n_m)$-*homogeneous polynomial* if, for each i with $1 \leq i \leq m$, the mapping

$$P\left(z_1, \ldots, z_{i-1}, \cdot, z_{i+1}, \ldots, z_m\right) : E_i \to F$$

is an n_i-homogeneous polynomial for all fixed $z_j \in E_j$ with $j \neq i$. It reduces to an m-linear mapping when $m > 1$ and $n_1 = \ldots = n_m = 1$ and to an n_1-homogeneous polynomial when $m = 1$. Continuous multipolynomials are all those bounded on the product of unit balls B_{E_i} of E_i $(i = 1, \ldots, m)$. In that case,

$$\|P\| := \sup\left\{\|P(z_1, \ldots, z_m)\| : z_i \in B_{E_i}, i = 1, \ldots, m\right\}$$

defines a norm on the vector space $\mathcal{P}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$ of all continuous $(n_1, \ldots, n_m)$-homogeneous polynomials from $E_1 \times \cdots \times E_m$ into F (see [2] and [6] for the basics of theory).

Let us fix some useful and standard notation. For $1 \leq p < \infty$, we consider the Banach space ℓ_p^n, which is just $\mathbb{K}^n$ with the norm $\|z\|_p = (\sum_{j=1}^n |z_j|^p)^{1/p}$, whereas ℓ_∞^n is just $\mathbb{K}^n$ with the norm $\|z\|_\infty = \max_{j=1,\ldots,n} |z_j|$. For $1 < p < \infty$, the conjugate number is denoted as p^*, that is $1/p + 1/p^* = 1$. We use the convention that 1 and ∞ are conjugated to each other. For $z = (z_1, \ldots, z_n) \in \ell_p^n$, we shall write z^α to describe the product $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, where α denotes an n-tuple $(\alpha_1, \ldots, \alpha_n)$ of non-negative integers; we will let $|\alpha|$ and $\alpha!$ represent the sum $\alpha_1 + \cdots + \alpha_n$ and the product $\alpha_1! \cdots \alpha_n!$, respectively. Finally, for fixed m and $n_1, \ldots, n_m$ positive integers, we shall write $M := \sum_{i=1}^m n_i$.

2 Main Results

We apply the KSZ multilinear version due to Pellegrino et al. [5, Theorem 1.1], and it is a refined version of the KSZ multilinear inequality with optimal exponents.

Theorem 2.1. *[5, Theorem 1.1] Let m and n be positive integers, and let $p_1, \ldots, p_m \in [1, \infty]$. Then, there exist a universal constant C_m, a choice of signs $+1$ and -1, and an m-linear form $A_{m,n} : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ of the type*

$$A_{m,n}(z_1, \ldots, z_m) = \sum_{j_1,\ldots,j_m=1}^{n} \pm z_{1j_1} \cdots z_{mj_m},$$

such that

$$\|A_{m,n}\| \leq C_m \cdot n^{\frac{1}{\min_k\left\{\max\left\{2,p_k^*\right\}\right\}} + \sum_{k=1}^{m} \max\left\{\frac{1}{2} - \frac{1}{p_k}, 0\right\}}. \tag{1}$$

Moreover, the exponent in the right-hand side of inequality (1) is optimal.

We get the following multipolynomial extension of the Theorem 2.1.

Theorem 2.2. *Let $m, n,$ and $n_1, \ldots, n_m$ be positive integers, and let $p_1, \ldots, p_m \in [1, \infty]$. Then, there exist a universal constant C_M, a choice of signs $+1$ and -1, and an $(n_1, \ldots, n_m)$-homogeneous polynomial $P_{M,n} : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \longrightarrow \mathbb{K}$ of the type*

$$P_{M,n}(z_1, \ldots, z_m) = \sum_{|\alpha_1|=n_1,\ldots,|\alpha_m|=n_m} \pm\, c_\alpha z_1^{\alpha_1} \cdots z_m^{\alpha_m}, \tag{2}$$

with $c_\alpha = \prod_{i=1}^{m} \binom{n_i}{\alpha_i} - 2k$ for some non-negative integer k with $0 \leq k \leq [\prod_i \binom{n_i}{\alpha_i}]/2$ e at least n^m coefficients $c_\alpha = 1$, such that

$$\|P_{M,n}\| \leq C_M \cdot n^{\frac{1}{\min_{1\leq i\leq m}\{\max\{2,p_i^*\}\}} + \sum_{i=1}^{m} \max\left\{n_i\left(\frac{1}{2} - \frac{1}{p_i}\right), 0\right\}}. \tag{3}$$

Theorem 2.2 reduces to [5, Theorem 1.1] when $m > 1$ and $n_1 = \ldots = n_m = 1$. Another application by assuming $m = 1$, $n_1 = d$, and $p_1 = p$, on the other hand, provides a particular version for homogeneous polynomials; we have the following corollary of the theorem.

Corollary 2.1. *Let n and d be positive integers, and let $p \in [1, \infty]$. Then, there exist a universal constant C_d, a choice of signs $+1$ and -1, and a d-homogeneous polynomial $P_{d,n} : \ell_p^n \longrightarrow \mathbb{K}$ of the type $P_{d,n}(z) = \sum_{|\alpha|=d} \pm c_\alpha z^\alpha$, with at least n coefficients $c_\alpha = 1$, such that $\|P_{d,n}\| \leq C_d \cdot n^{\max\left\{d\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{1}{2}, 1-\frac{1}{p}\right\}}$.*

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ADJOINTS AND SECOND ADJOINTS OF ALMOST DUNFORD-PETTISOPERATORS

LUIS ALBERTO GARCIA[1,†] & GERALDO BOTELHO[2,‡]

IME, USP, SP, Brasil[1], Faculdade de Matemática, UFU, MG, Brasil[2]

luisgarcia@ime.usp.br[†]., lcff@ime.unicamp.br [‡].

Abstract

First we prove that the second adjoint of every bounded linear operator taking values in a Dedekind σ-complete Banach lattice containing a copy of ℓ_∞ is almost Dunford-Pettis. Next we generalize some known results concerning conditions on the Banach lattices E and F under which the adjoint T^* and the second adjoint T^{**} of any positive almost Dunford-Pettis operator $T\colon E \longrightarrow F$ are almost Dunford-Pettis. Finally we prove when T^* and T^{**} are almost Dunford-Pettis whenever T is a (not necessarily almost Dunford-Pettis) order weakly compact operator.

1 Introduction

A linear operator between Banach spaces is *completely continuous*, or a *Dunford-Pettis operator*, if it sends weakly null sequences to norm null sequences. The lattice counterpart of this important operator ideal is the class of almost Dunford-Pettis operators: an operator from a Banach lattice to a Banach space is *almost Dunford-Pettis* if it sends disjoint weakly null sequences to norm null sequences. Almost Dunford-Pettis operators have attracted the attention of many experts, for example see [2, 4, 5, 6].

In this work we investigate when the adjoint and the second adjoint of a linear operator between Banach lattices are almost Dunford-Pettis. These and related problems were treated in, e.g., [2, 4, 5]. The results we prove in this work improve upon all known results on the topic we are aware of.

To see that second adjoints are not always almost Dunford-Pettis, recall that a Banach lattice E has the positive Schur property if positive (or disjoint, or positive disjoint) weakly null sequences in E are norm null. The literature on this property is extensive. By id_E we denote the identity operator on E. If the Banach lattice E fails the positive Schur property, which is a quite usual occurrence, then $id_E^{**} = id_{E^{**}}$ is not almost Dunford-Pettis.

2 Main Results

By T^* we denote the adjoint of a linear operator T and by T^{**} its second adjoint. For basic notions, notation and results on Banach lattices we refer to [1, 7]. Operators are always supposed to be linear.

Theorem 2.1. *Let E be a Banach lattice and F be a Dedekind σ-complete Banach lattice containing a copy of ℓ_∞. The following are equivalent.*
(a) For every bounded operator $T\colon E \longrightarrow F$, T^{**} is almost Dunford-Pettis.
(b) For every regular operator $T\colon E \longrightarrow F$, T^{**} is almost Dunford-Pettis.
(c) For every positive operator $T\colon E \longrightarrow F$, T^{**} is almost Dunford-Pettis.
(d) E^{**} has the positive Schur property.

Note that adjoint and second adjoint of almost Dunford-Pettis operators not always are almost Dunford-Pettis.

Example 2.1. *(a) The identity id_{ℓ_1} is positive and almost Dunford-Pettis because ℓ_1 is a Schur space and $id_{\ell_1}^* = id_{\ell_\infty}$ is not almost Dunford-Pettis because ℓ_∞ fails the positive Schur property [8, p. 82].*

(b) Put $E = \left(\bigoplus_{n\in\mathbb{N}} \ell_\infty^n\right)_1$. *(i) The identity on* E *is positive and almost Dunford-Pettis because* E *has the positive Schur property [8, p. 17], whereas* $id_{E^{**}} = id_E^{**}$ *is not almost Dunford-Pettis because* E^{**} *fails the positive Schur property [3, Example 2.8]. (ii) Calling on [3, Example 2.8] again,* E^* *contains a lattice copy of* ℓ_1*, so the norm of* E^{**} *is not order continuous [7, Theorem 2.4.14]. Since dual Banach lattices are Dedekind complete and, as we have already mentioned,* E^{**} *fails the positive Schur property, Theorem 2.1 gives a positive operator* $T\colon E \longrightarrow E^{**}$ *such that* T^{**} *fails to be almost Dunford-Pettis. The positive Schur property of* E *assures that* T *is almost Dunford-Pettis.*

Theorem 2.2. *The following are equivalent for the Banach lattices* E *and* F*.*
(a) If $T\colon E \longrightarrow F$ *is positive and almost Dunford-Pettis, then* T *is positively limited, hence* T^* *is almost Dunford-Pettis.*
(b) E^* *has order continuous norm or* F *has the dual positive Schur property.*

Remark 2.1. *(1) Theorem 2.2 gets the same conclusion of the combination of [4, Theorem 3] and [5, Theorem 4.3] without asking* F *to have property (d).*
(2) Theorem 2.2 gets a stronger conclusion on one of the implications in [2, Theorem 5.1] with the same assumption.

Theorem 2.3. *Let* E, F *be Banach lattices such that* E^* *and* F *haver order continuous norms. If* $T\colon E \longrightarrow F$ *is positive and almost Dunford-Pettis, then* T^* *is positively limited, that is,* T^* *is almost limited, hence* T^{**} *is almost Dunford-Pettis.*

Theorem 2.4. *Let* E *be a Banach lattice in which every positive bounded disjoint sequence is order bounded. Then,*
(a) no matter the Banach lattice F*, every order weakly compact regular operator* $T\colon E \longrightarrow F$ *is positively limited, hence* T^* *is almost Dunford-Pettis.*
(b) no matter the Banach space G*, the adjoint* T^* *of every bounded order weakly compact operator* $T\colon E \longrightarrow G$ *is almost limited, hence* T^{**} *is almost Dunford-Pettis.*

The Theorem 2.4 applies to the Banach lattice $E = C(K)$, where K is a compact Hausdorff space.

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ORLICZ SEQUENCES AND LIMITED SETS

JAMILSON R. CAMPOS[1,†] & LUIZ F. P. SOUSA[2,‡]

Departamento de Matemática, UFPB, PB, Brasil[1], Departamento de Ciências Exatas, UFPB, PB, Brasil[2]

jamilson@dcx.ufpb.br[†] louiz1998@gmail.com[‡]

Abstract

We introduce the space of weak Orlicz sequences and the concept of Orlicz-limited sets, which generalizes the notion of p-limited sets. Some features and results concerning these notions are established and, as a byproduct, another "mid-type" sequence space is obtained.

1 Introduction

In [1], A. Karn and D. Sinha introduce and study the concept p-limited sets generalizing the concept of limited sets. Here a subset S of a Banach space X is limited if for every sequence $(f_n)_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty} f_n(x) = 0$, for all $x \in X$, we have $f_n \to 0$ uniformly on S. Not every set that enjoys this property is compact, an error by I. M. Gelfand, and a counterexample of this is presented by R. Phillips in [3], which motivated the concept of a limited set.

In this work we introduce this "limited theory", mentioned above, in the context of the Orlicz spaces.

Given an Orlicz function M and a Banach space E, one can define the Orlicz sequence space $\ell_M(E)$ as the set of all sequences $(x_j)_{j=1}^{\infty}$ in E that satisfy $\sum_{j=1}^{\infty} M\left(\frac{\|x_j\|}{\eta}\right) < \infty$, for some $\eta > 0$. The subspace of $\ell_M(E)$, formed by sequences $(x_j)_{j=1}^{\infty}$ satisfying the same property for all $\eta > 0$ is denoted by $h_M(E)$.

The space $\ell_M(E)$ is complete with the norm $\|\cdot\|_M$, given by $\|(x_j)_{j=1}^{\infty}\|_M = \inf\left\{\eta > 0 : \sum_{j=1}^{\infty} M\left(\frac{\|x_j\|}{\eta}\right) \leq 1\right\}$. We refer to book [2] for further study of Orlicz sequence space.

For an Orlicz function M, we will denote its complementary Orlicz function by M^*. Also, the letters E and F will denote Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

2 Main Results

We define the set of all weak Orlicz sequences in E by $\ell_M^w(E) := \{(x_j)_{j=1}^{\infty} : (\varphi(x_j))_{j=1}^{\infty} \in \ell_M, \forall \varphi \in E'\}$. This set is actually a linear space and the expression $\|(x_j)_{j=1}^{\infty}\|_{w,M} := \sup_{\varphi \in B_{E'}} \|(\varphi(x_j))_{j=1}^{\infty}\|_M$ defines a complete norm in $\ell_M^w(E)$.

Every sequence $(x_j)_{j=1}^{\infty} \in \ell_M^w(E)$ can be identified with an continuous operator $\xi_x \colon h_{M^*} \longrightarrow E$ given by $\xi_x(\alpha) = \sum_{j=1}^{\infty} \alpha_j x_j$, $\alpha = (\alpha_j)_{j=1}^{\infty} \in h_{M^*}$. The map $x \mapsto \xi_x$ is an isometric isomorphism between $\ell_M^w(E)$ and $\mathcal{L}(h_{M^*}; E)$.

We also have $\ell_M(E) \subseteq \ell_M^w(E)$ with $\|\cdot\|_{w,M} \leq \|\cdot\|_M$ and the equality $\ell_M(E) = \ell_M^w(E)$ does not occur in general.

Theorem 2.1 (Dvoretzky-Rogers type)**.** *Let M be an Orlicz function such that ℓ_M has no copy of c_0. If E is an infinite-dimensional Banach space, then $\ell_M(E) \subsetneq \ell_M^w(E)$.*

It is not difficult to show that the space $\ell_M^{w^*}(E') := \left\{(\varphi_j)_{j=1}^{\infty} \text{ in } E' : (\varphi_j(x))_{j=1}^{\infty} \in \ell_M, \forall x \in E\right\}$ is a Banach space equipped with the norm given by $\|(\varphi_j)_{j=1}^{\infty}\|_{w^*,M} = \sup_{x\in E} \|(\varphi_j(x))_{j=1}^{\infty}\|_M$. We also prove an isometric isomorphism between $\ell_M^{w^*}(E')$ and $\mathcal{L}(E; \ell_M)$.

Here is one of our main definitions.

Definition 2.1. *A non-empty subset S of a Banach space E is said to be ℓ_M-limited if for every sequence $(\varphi_n)_{n=1}^{\infty}$ in $\ell_M^{w^*}(E')$, there is $(\alpha_n)_{n=1}^{\infty} \in \ell_M$ such that $|\varphi_n(x)| \leq \alpha_n$, for every $x \in S$ and all $n \in \mathbb{N}$.*

Lemma 2.1. *Let M be an Orlicz function and $x = (x_j)_{j=1}^{\infty} \in \ell_M^w(E)$. The set $\xi_x(B_{h_{M^*}})$ is ℓ_M-limited if and only if we have $((\varphi_n(x_j))_{j=1}^{\infty})_{n=1}^{\infty} \in \ell_M(\ell_M)$ for all $(\varphi_n)_{n=1}^{\infty} \in \ell_M^{w^*}(E')$.*

Definition 2.2. *Let $x = (x_j)_{j=1}^{\infty}$ a sequence in E. We say that x is mid Orlicz if it satisfies the conditions of Lemma 2.1. We will denote the space of all mid Orlicz sequences by $\ell_M^{\mathrm{mid}}(E)$.*

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SEQUENCE SPACES RELATED TO BILINEAR APPLICATIONS

JAMILSON R. CAMPOS[1,†], MARCOS MONTEIRO[2,‡] & JOEDSON SANTOS[3,§]

Departamento de Ciências Exatas, UFPB, PB, Brasil[1], Departamento de Matemática, UFS, SE, Brasil[2]
Departamento de Matemática, UFPB, PB, Brasil[3]

jamilson@dcx.ufpb.br[†] aurelio@mat.ufs.br[‡] joedson.santos@academico.ufpb.br[§]

Abstract

In this talk we present some contributions to the theory of sequence spaces associated to a bilinear application. In addition, we present and study some classes of linear operators associated with this theory.

1 Introduction

A new kind of summability in Banach spaces was introduced by O. Blasco, T. Signes and J. M. Calabuig in [1] from ideas about bilinear convolution. Given Banach spaces X, Y, Z, a real number $1 \leq p < \infty$ and an admissible bilinear application $B : X \times Y \longrightarrow Z$, they define a (B,p)-summable sequence in X as those sequences $(x_j)_{j=1}^{\infty}$ such that $\sum_{j=1}^{\infty} \|B(x_j, y)\|_Z^p < \infty,\ \forall y \in Y$. The space of all such sequences is denoted by $\ell_{B,p}(X)$.

In this talk, we study this kind of summability from the point of view of operator theory, presenting some contributions to the environment of (B,p)-summable sequences and introducing some classes of operators associated with this theory. These results come from our work in [4].

Here the letters X, Y, Z, H and F denote Banach spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The symbol $X \overset{1}{\hookrightarrow} Y$ means that X is a linear subspace of Y and $\|x\|_Y \leq \|x\|_X$ for every $x \in X$, and $X \overset{1}{=} Y$ means that X and Y are isometrically isomorphic. The symbol $(x_j)_{j=1}^{n}$, where $x_1, \ldots, x_n \in X$, stands for the sequence $(x_1, x_2, \ldots, x_n, 0, 0, \ldots) \in X^{\mathbb{N}}$. Other notations and symbols used here are well known or can be found in [2, 3].

2 Main Results

We say that a continuous bilinear application $B : X \times Y \longrightarrow Z$ is admissible if, $\forall y \in Y,\ B(x,y) = 0 \Rightarrow x = 0$. The expression $\|\cdot\|_{B,p}\colon \ell_{B,p}(X) \longrightarrow [0,\infty),\ (x_j)_{j=1}^{\infty} \longmapsto \sup\limits_{y \in B_Y} \|(B(x_j,y))_{j=1}^{\infty}\|_p$ defines a norm on $\ell_{B,p}(X)$. Another useful equivalent assertion is: $(x_j)_{j=1}^{\infty} \in \ell_{B,p}(X)$ if $(B_y(x_j))_{j=1}^{\infty} \in \ell_p(Z)$, for all $y \in Y$, where $B_y := B(\cdot, y) : X \to Z$.

In addition to the examples of spaces $\ell_{B,p}(X)$ given in [3], we present a new example: For $B : X \times \ell_p^w(X') \longrightarrow \ell_p$ given by $B(x, (x_k^*)_{k=1}^{\infty}) = (x_k^*(x))_{k=1}^{\infty}$, we have $\ell_{B,p}(X) \overset{1}{=} \ell_p^{mid}(X)$, space studied in [2].

If X is a B-normed space (that satisfies: there is a constant $C > 0$ such that $\|x\| \leq C \sup_{y \in B_Y} \|(B(x,y))\|$, for all $x \in X$), then $\ell_{B,p}(X)$ is a Banach space and $\ell_{B,p}(X) \overset{1}{\hookrightarrow} \ell_p^w(X)$.

Let us introduce the weakly (B,p)-summability for sequences in X as a generalization of the concept of B-unconditionally summability introduced in [1].

Definition 2.1. *A sequence $(x_j)_{j=1}^{\infty}$ in X is weakly (B,p)-summable if $(B(x_j,y))_{j=1}^{\infty} \in \ell_p^w(Z)$, for all $y \in Y$ (or equivalently, $(z^*(B(x_j,y)))_{j=1}^{\infty} \in \ell_p$, whenever $y \in Y$ and $z^* \in Z'$).*

The notation $\ell_{B,p}^w(X)$ will be used to the linear space of all weakly (B,p)-summable sequences and the expression $\|\cdot\|_{B,p,w} : \ell_{B,p}^w(X) \longrightarrow [0,\infty),\ (x_j)_{j=1}^{\infty} \longmapsto \sup\limits_{y \in B_Y} \|(B(x_j,y))_{j=1}^{\infty}\|_{w,p}$ defines a norm on $\ell_{B,p}^w(X)$.

Definition 2.2. *Let X be a B-normed space. We say that*

a) *$T \in \mathcal{L}(X;F)$ is $(p;B,q)$-summing, if $(T(x_j))_{j=1}^{\infty} \in \ell_p(F)$ whenever $(x_j)_{j=1}^{\infty} \in \ell_{B,q}(X)$.*

b) *$T \in \mathcal{L}(H;X)$ is weakly $(B,p;q)$-summing, if $(T(u_j))_{j=1}^{\infty} \in \ell_{B,p}(X)$ whenever $(u_j)_{j=1}^{\infty} \in \ell_q^w(H)$.*

Keeping in mind the definition of space $\ell_{B,p}(X)$, note that an operator T is weakly $(B,p;q)$-summing if and only if $(B_y \circ T(u_j))_{j=1}^{\infty} \in \ell_p(Z)$, for all $y \in Y$, and whenever $(u_j)_{j=1}^{\infty} \in \ell_q^w(H)$. We denote the space of all $(p;B,q)$-summing and weakly $(B,p;q)$-summing operators by $\Pi_{p,q}^B(X;F)$ and $W_{p,q}^B(H;X)$, respectively. If $p=q$, we write only $\Pi_p^B(X;F)$ and $W_p^B(H;X)$. These are normed spaces (as proved in [4]) with norms denoted by $\pi_{p,q}^B(\cdot)$ and $w_{p,q}^B(\cdot)$, respectively.

Here are our main results.

Theorem 2.1 (Pietsch domination-type I)**.** *An operator $T \in \mathcal{L}(H;X)$ is weakly (B,p)-summing if and only if for all $y \in Y$, there exist a constant $C_y > 0$ and a regular probability measure μ on the borelians of $B_{H'}$, with the weak star topology, such that $\|B_y \circ T(u)\|_Z \leq C_y \left(\int_{B_{U'}} |u^*(u)|^p \, d\mu(u^*)\right)^{\frac{1}{p}}$, for all $u \in H$.*

Theorem 2.2 (Pietsch domination-type II)**.** *Let $B : X \times Y' \longrightarrow Z$ be a bilinear application with X B-normed. If for all $x \in X$ we have $B_x := B(x,\cdot) \in \mathcal{L}_{w^*,\|\cdot\|}(Y';Z)$, then $T \in \Pi_{p,q}^B(X;F)$ if and only if there exist a constant $C > 0$ and a probability measure μ on the borelians of $B_{Y'}$, with the weak star topology, such that $\|T(x)\|_F \leq C \left(\int_{B_{Y'}} \|B(x,y^*)\|_Z^p \, d\mu(y^*)\right)^{\frac{1}{p}}$, for all $x \in X$.*

Theorem 2.3 (Splitting property)**.** *Let $1 \leq q \leq p, r < \infty$ be such that $1/q = 1/p + 1/r$. Let $B : X \times Y' \longrightarrow Z$ be a bilinear application with X B-normed and such that $B_x \in \mathcal{L}_{w^*,\|\cdot\|}(Y';Z)$, for all $x \in X$. Then, every $T \in \Pi_r^B(X;F)$ is $(q;p)$-mixing, that is, for every $n \in \mathbb{N}$ and $(x_j)_{j=1}^n \in X^n$, there exist $(\lambda_j)_{j=1}^n \in \mathbb{K}^n$ and $(y_j)_{j=1}^n \in F^n$, such that $T(x_j) = \lambda_j y_j, \ 1 \leq j \leq n$. Besides that, $\|(\lambda_j)_{j=1}^n\|_r \|(y_j)_{j=1}^n\|_{w,p} \leq \pi_r^B(T)\|(x_j)_{j=1}^n\|_{B,q}$.*

Inclusion and coincidence results relating the classes studied and other classes of operators, consequences of the above theorems, will be presented in this talk.

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A COPIES OF ℓ_∞ IN THE BANACH SPACE $\mathcal{P}_{W^*K}(^N X^*, Y)$

MICHAEL A. RINCÓN-VILLAMIZAR[1], SERGIO A. PÉREZ[1] & DIEGO A. REYES[1]

Universidad Industrial de Santander[1]

marinvil@uis.edu.co† saperezl@uis.edu.co‡ dijorero@correo.uis.edu.co§

Abstract

Let $\mathcal{P}_{w^*K}(^n X^*, Y)$ denote the Banach space of all compact n-homogeneous polynomials from X^* to Y which are w^*-w-continuous, endowed with the supremum norm. We show that $\mathcal{P}_{w^*K}(^n X^*, Y)$ contains ℓ_∞ iff either X or Y contains ℓ_∞.

1 Introduction

If X and Y are Banach spaces, we denote by $\mathcal{P}(\,^n X^*, Y)$ the Banach space of all n-homogeneous polynomials from X^* to Y, endowed with the norm:

$$\|P\| = \sup_{x^* \in B_{X^*}} \|P(x^*)\|, \quad P \in \mathcal{P}(\,^n X^*, Y).$$

Also let $\mathcal{P}_{w^*}(^n X^*, Y)$ be the closed subspace of $\mathcal{P}(\,^n X^*, Y)$ consisting of all those w^*-w-continuous members.

Definition 1.1. *A polynomial $P \in \mathcal{P}(\,^n X, Y)$ is called compact if $P(B_X)$ is a relative compact subset of Y.*

To go on, let $\mathcal{P}_{w^*K}(\,^n X^*, Y)$ be the closed subspace of all compact elements of $\mathcal{P}_{w^*}(\,^n X^*, Y)$. So, we have:

1. $\mathcal{P}(\,^1 X, Y) = L(X, Y)$, that is, the Banach space of all linear operators from X to Y;
2. $\mathcal{P}_{w^*}(\,^1 X^*, Y) = L_{w^*}(X^*, Y)$, the closed subspace of $L(X^*, Y)$ which consists of all w^*-w-continuous operators.
3. $\mathcal{P}_{w^*K}(\,^1 X^*, Y) = K_{w^*}(X^*, Y)$, that is, the Banach space of all compact linear operators from X to Y which are w^*-w-continuous.

In [1], L. Drewnowski proved:

Theorem 1.1. *$\mathcal{K}_{w^*}(X^*, Y)$ contains a subspace isomorphic to ℓ_∞ if and only if either X or Y contains a subspace isomorphic to ℓ_∞.*

The goal of the talk is to extend the aforementioned result to $\mathcal{P}_{w^*K}(\,^n X^*, Y)$. This result is part of the submitted paper [2].

2 Main Results

We will prove:

Theorem 2.1. *Let X and Y be Banach spaces and $n \in \mathbb{N}$. $\mathcal{P}_{w^*K}(\,^n X^*, Y)$ contains a subspace isomorphic to ℓ_∞ if and only if either X or Y contains a subspace isomorphic to ℓ_∞.*

To prove Theorem 2.1, we need the following tool: let X and Y be Banach spaces, and $n \geq 2$. If $P \in \mathcal{P}(^n X^*, Y)$, then we define $d(P)\colon X^* \to \mathcal{P}(^{n-1} X^*, Y)$ by

$$d(P)(x^*)(y^*) = \overset{\vee}{P}(x^*, y^*, y^*, \ldots, y^*), \quad x^*, y^* \in X^*.$$

The upcoming lemma establishes a fundamental property of the operator $d(P)$.

Proposition 2.1. *Let X and Y be Banach spaces, $n \geq 2$ and $P \in \mathcal{P}(^n X^*, Y)$. Then, $P \in \mathcal{P}_{w^*K}(^n X^*, Y)$ if and only if $d(P) \in \mathcal{K}_{w^*}(X^*, \mathcal{P}_{w^*K}(^{n-1} X^*, Y))$.*

The proof of Theorem 2.1 is based on the following result: for all Banach spaces X, Y and $n \geq 2$, the map

$$P \in \mathcal{P}_{w^*K}(^n X^*, Y) \mapsto d(P) \in \mathcal{K}_{w^*}(X^*, \mathcal{P}_{w^*K}(^{n-1} X^*, Y))$$

is an embedding.

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LINEABILITY OF SETS OF NON-INJECTIVE MAPS

MIKAELA A. OLIVEIRA[1,†] & GERALDO BOTELHO[2,‡]

IME, USP, SP, Brasil[1], Faculdade de Matemática, UFU, MG, Brasil[2]

mikaela_aires@ime.usp.br[†] botelho@ufu.br[‡]

Abstract

In the context of the recently introduced notions of (α, β)-lineability, where α and β are cardinal numbers, we generalize known results on $(1, \mathfrak{c})$-lineability of sets of non-injective linear operators between sequence spaces. In particular, our main result generalizes a result proved in [6] in several directions. Illustrative examples are provided.

1 Introduction

The study of lineability consists in the search for linear structures in subsets of vector spaces that are not vector subspaces. The study of lineability goes back to 2005 with Aron, Gurariy and Seoane-Sepúlveda [2]. However, well before that, although not yet formalized, results concerning the existence of linear substructures of certain sets were already being investigated, see, e.g. Gurariy [5] (1966).

In the last two decades, the search for linear structures in certain sets has been widely explored, and a huge number of papers in the area have been published every year in good journals. The references [1, 3] contain long lists of references in the area up to the time of their publication.

Several concepts related to lineability have appeared, such as, spaceability, α-lineability, where α is a cardinal number, dense-lineability, algebrability, latticeability, etc, each of them attracting the attention of many experts. In this work we are interested in the refinement recently introduced by V.V. Fávaro, D. Pellegrino and D. Tomaz in [6]. This refinement searches linear structures with more robust geometric properties of the investigated sets. Our focus is the following particular case of the more general concept introduced in [6]: A subset A is a topological linear space E is said to be $(1, \mathfrak{c})$-*lineable* if for every $x \in A$ there exists a $\mathfrak{c}$-dimensional subspace W of E such that $x \in W \subseteq A \cup \{0\}$. Of course, $\mathfrak{c}$ denotes the cardinality of $\mathbb{R}$.

Note that this concept is stronger than the earlier notions of lineability and $\mathfrak{c}$-lineability.

$(1, \mathfrak{c})$-lineability of several different sets have already been investigated. For instance, $(1, \mathfrak{c})$-lineability of sets of injective maps and of non injective maps were studied in [4, 6]. In this work we generalize the result on $(1, \mathfrak{c})$-lineability of sets of non injective maps obtained in [6].

2 Main Results

The main purpose of this work is to generalize the following result proved in [6] in several directions.

Theorem 2.1. *Let $p, q \geq 1$. Then the set of non injective bounded linear operators from ℓ_p to ℓ_q is $(1, \mathfrak{c})$-lineable.*

The main result of this work generalizes the theorem above by replacing the domain space with an arbitrary set (not necessarily a linear space), the target space with a q-normed sequence space satisfying some special properties, and the space of bounded linear operators with much more general sets. For example, our result covers the case of non injective operators from ℓ_p to ℓ_q for $0 < p, q \leq 1$. To state this result, we need the following definitions:

Definition 2.1. *(a) Let* $W \neq \{0\}$ *be a linear space and* $W^{\mathbb{N}} = \{(a_j)_{j=1}^{\infty} : a_j \in W \text{ for every } j \in \mathbb{N}\}$ *be the linear space of* W*-valued sequences endowed with the usual operations. By a spreading sequence space we mean a topological linear subspace* V *of* $W^{\mathbb{N}}$ *satisfying the following condition:*

If $(a_j)_{j=1}^{\infty} \in V$ *e* $\mathbb{N}_0 = \{j_1 < j_2 < j_3 < \cdots\}$ *is an infinite subset of* $\mathbb{N}$*, then the sequence* $(b_j)_{j=1}^{\infty}$ *defined as*

$$b_k = \begin{cases} a_i & \text{if } k = j_i, \\ 0 & \text{otherwise,} \end{cases}$$

belongs to V*. In this case, we the sequence* $(b_j)_{j=1}^{\infty}$ *is called the spreading of* $(a_j)_{j=1}^{\infty}$ *by* $\mathbb{N}_0$ *and write* $(b_j)_{j=1}^{\infty} = E((a_j)_{j=1}^{\infty}; \mathbb{N}_0)$.

(b) Let Ω *be a set,* $V^{\Omega} = \{f : \Omega \to V : f \text{ is function}\}$ *and* $0 < q \leq 1$ *A space of* V*-valued spreading functions is a linear subspace* A *of* V^{Ω} *with the pointwise operations endowed with a complete* q*-norm satisfying the following conditions:*

(i) If $f, g \in A$ *and* $\mathbb{N}_0 \subseteq \mathbb{N}$ *are such that* $f(x) = E(g(x); \mathbb{N}_0)$ *for every* $x \in \Omega$*, then* $\|f\|_A \leq \|g\|_A$.
(ii) If $f \in A$ *and* $u \colon V \to V$ *is a continuous linear operator, then* $u \circ f \in A$.

The main result of this work reads as:

Theorem 2.2. *Let* V *be a spreading sequence space and* A *be a space of* V*-valued spreading functions. Then the set* $\mathcal{N} = \{f \in A : f \text{ is non injective}\}$ *is either* $(1, \mathfrak{c})$*-lineable or* $\{0\}$.

Example 2.1. *The following generalizations of Theorem 2.1 follow from our main result.*
- *For* $p, q \geq 1$*, the set of non injective approximable linear operators from* ℓ_p *to* ℓ_q *is* $(1, \mathfrak{c})$*-lineable. Therefore, the corresponding sets of compact, completely continuous and weakly compact operators are* $(1, \mathfrak{c})$*-lineable as well.*
- *For* $0 < p, q < +\infty$*, the set of non injective bounded linear operators from* ℓ_p *to* ℓ_q *is* $(1, \mathfrak{c})$*-lineable.*

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GENERALIZED GEOMETRY AND TRANSITION PROBABILITIES

JURIAANS, S. O.[1,†] & QUEIROZ, P.C.[2,‡]

IME-USP[1], UFMA, Brazil[2]

ostanley@usp.br[†] pqueiroz@ime.usp.br[‡]

Abstract

Theories of Generalized Functions have been used to obtain solutions of differential equations containing singularities and nonlinearities and also to explain phenomena in physical reality such as in General Relativity and Fluid Dynamics. However the milieus in which this has always been done are the classical environments although results and developments in the generalized environments were used. Here we change the underlying structure $\mathbb{R}$ to $\widetilde{\mathbb{R}}$ and, relying on a Generalized Differential Calculus already developed, propose a Generalized Differential Geometry together with several other tools useful to fall back to classical milieus or classical solutions. In particular, given a classical oriented Riemannian manifold M, we embed M discretely into a generalized manifold M^* in such a way that M and its differential structure are the shadow of the differential structure of M^*. Among the tools we propose are a Fixed Point Theorem, based on the notion of hypersequences, the notion of support and ways to calculate generalized probabilities and transition probabilities. We extend an existing embedding theorem by proving that there exists an algebra embedding $\kappa : \hat{\mathcal{G}}(M) \longrightarrow \mathcal{C}^\infty(M^*, \widetilde{\mathbb{R}}_f)$, thus relating the generalized construction on classical manifolds to the classical construction on generalized manifolds.

1 Introduction

The theory of generalized functions goes back to Schwartz. More recently, J.F. Colombeau and E. Rosinger has undertaken the challenge of developing a nonlinear theory of generalized functions, thus extending Schwartz premier work. Colombeau's proposal has been extensively used. Several mayor contributions were given by prominent researchers in the field. In spite of these important contributions and development, somehow the underlying algebraic structure remained $\mathbb{R}$ or $\mathbb{C}$. It might be that one of the reasons to sticks to the classical underlying structure is the concern that introducing another underlying structure might lead to controversies either about the existence and rigor or about how much of nonstandard tools one needs to know to understand these structures. However, this should not really be concern since, for example, in [1], the Fermat reals ${}^\bullet\mathbb{R}$ were used as the algebraic underlying structure of a generalized differential calculus. The totally ordered topological ring ${}^\bullet\mathbb{R}$ is basically the union of halows of real numbers, each halow consisting of unique real number ${}^\circ x$ and elements $y = {}^\circ x + dt_a$, with $dt_a, a \in [1, \infty[$ being nilpotent elements. In particular, the group of invertible elements $Inv({}^\bullet\mathbb{R})$ is open but not dense and zero and nonzero infinitesimals are precisely the noninvertible elements.

It happens though that in applications and certain areas one must deal with infinitesimals and infinities which, in certain situations, are cancelled out by each other and thus suggesting that they are invertible elements. Can an environment be constructed in which infinitesimals and infinities coexists and some of which are invertible elements or at least invertible in some sense? There are several of such milieus and most of which are non-Archimedean rings. Recall that such non-Archimedean rings somehow originate with J. Tate. Here we focus on $\widetilde{\mathbb{R}}$ which was constructed in Colombeau's approach to generalized functions. Originally, it was just a ring where generalized functions took values, but, over time, it turned out to have a very rich topological and algebraic structure making it suitable to be the underlying algebraic milieu of a new Differential Calculus, a Generalized Differential Calculus. Let's sum up some of its features. Infinitesimals and infinities live like ebony and ivory in $\widetilde{\mathbb{R}}$ and when rendezvous occurs an interleaving of real numbers may be the result. Moreover, $Inv(\widetilde{\mathbb{R}})$ is open and dense, $\mathcal{B}(\widetilde{\mathbb{R}})$, its Boolean algebra of idempotent elements, consists of the characteristic functions of subsets of the real interval $I=]0,1]$ and if $x \notin Inv(\widetilde{\mathbb{R}})$ then there exist $e, f \in \mathcal{B}(\widetilde{\mathbb{R}})$ such that $e \cdot x = 0$ and $f \cdot x \in Inv(f \cdot \widetilde{\mathbb{R}})$.

Our purpose is to piece the puzzle using as pieces all the important concepts resulting in an ultra-metric milieu $\widetilde{\mathbb{R}}^n$, for each $n \in \mathbb{N}$, in which $\mathbb{R}^n$ is the shadow, or support, of points of $\widetilde{\mathbb{R}}^n$. In $\widetilde{\mathbb{R}}^n$, $\mathbb{R}^n - \{\vec{0}\}$ is a grid of equidistant points sitting between infinitesimals, the elements of $B_1(\vec{0}) - \{\vec{0}\}$, and infinities and hence, algebraically, it is the result of the rendezvous of such elements which go undetected in physical reality. The notion of the support of elements can be defined in each milieu and similar discrete embedding results hold. If $\Omega \subset \mathbb{R}^n$ is open, then there exists a discrete embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{C}^\infty(\widetilde{\Omega}_c, \widetilde{\mathbb{R}})$, where $\widetilde{\Omega}_c$ is a subset of $\widetilde{\mathbb{R}}^n$ consisting of those elements of $\overline{B}_1(\vec{0})$ whose support is contained in Ω and their norm is less than some real number. In particular, Dirac's infinity δ, becomes a $\mathcal{C}^\infty-$function on $\widetilde{\mathbb{R}}_c$ and $x\delta$ becomes nonzero and, when evaluated at certain infinitesimals, produces real values. Generalized Space-Time is constructed and applications to physical reality are given.

2 Main Results

Theorem 2.1 (Fixed Point Theorem). *Let $\Omega \subset \mathbb{R}^n$, $A = [(A_\varphi)_\varphi] \subset B_r(0) \cap \mathcal{G}_f(\Omega)$, $r < 2$, be an internal set, and $T : A \to A$ be a mapping with representative $(T_\varphi : A_\varphi \to A_\varphi)_{\varphi \in A_0(n)}$. If there exists $k = [(k_\varphi)_\varphi] \in \tilde{\mathbb{N}}$ such that each $T_\varphi^{k_\varphi}$ is a λ-contraction, then T^k is well-defined, continuous, and has a unique fixed point $f_0 \in A$.*

Theorem 2.2 (Down Sequencing Argument). *Let $f \in \mathcal{G}_f(\Omega)$ with $\Omega \subset \mathbb{R}^n$. If $f \in W^0_{m,r}[0]$ with $r > 0$ and $p_0 \in \mathbb{N}^n$, then $f \in W^{\|p_0\|}_{m,s}[0]$ where $s = 4^{-n\|p_0\|}r$, i.e., $W^0_{m,r}[0] \subset W^{\|p_0\|}_{m,s}[0]$.*

Theorem 2.3 (Embedding Theorem). *Let M be an $n-$dimensional orientable Riemannian manifold. There exists an $n-$dimensional $\mathcal{G}_f-$manifold M^*, in which M is discretely embedded, and an algebra monomorphism $\kappa : \hat{\mathcal{G}}(M) \longrightarrow \mathcal{C}^\infty(M^*, \widetilde{\mathbb{R}}_f)$ which commutes with derivation. Moreover, equations whose data have singularities or nonlinearities defined on M naturally extend to equations on M^* and, on M^*, these data become $\mathcal{C}^\infty-$functions.*

Theorem 2.4. *Let $\Omega \subset \mathbb{C}$ and let $f \in \mathcal{H}(\Omega)^*$ be holomorphic, $\mathcal{Z}_f = \{z \in \widetilde{\Omega}_c \,:\, f(z) = 0_{\widetilde{\mathbb{C}}}\}$ and $Z_f = \Omega \bigcap \mathcal{Z}_f$, the generalized and classical zero set of f. Then $\mathcal{Z}_f = Interl(Z_f)$. In particular, Z_f is the support of points of $\mathcal{Z}_f$. If given $f \in \mathcal{HG}(\Omega))$ a holomorphic net, $E \subset \Omega$ a set of uniqueness, such that $\widetilde{E} \subset \mathcal{Z}_f$, then $f = 0$. Consequently, $f = 0$ if and only if $supp(\mathcal{Z}_f)$ is a set of uniqueness.*

Theorem 2.5. *Let $T = [(T_\epsilon)] \in \mathcal{B}(\mathcal{G}_H)$ be a selfadjoint operator such that each T_ϵ is selfadjoint. Then $supp(\nu(T)) = \{\nu(T_0) : T_0 \in supp(T)\}$. In particular, the support of the generalized transition probabilities of T equals the transition probabilities of the elements of the support of T.*

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NONHOMOGENEOUS MHD EQUATIONS: CONVERGENCE OF THE SEMI-GALERKIN APPROXIMATIONS

FELIPE WERGETE CRUZ[1,†], JAUBER C. OLIVEIRA[2,‡] & MARKO A. ROJAS-MEDAR[3,§]

Departamento de Matemática, Universidade Federal de Pernambuco, Recife, Brazil[1],
Departamento de Matemática, Universidade Federal de Santa Catarina, Florianópolis, Brazil[2],
Departamento de Matemática, Universidad de Tarapacá, Arica, Chile[3]. This author was partially supported by: MATH-AMSUD project 21-MATH-03 (CTMicrAAPDEs), CAPES-PRINT 88887.311962/2018-00 (Brazil) [3]

felipe.wcruz@ufpe.br[†] j.c.oliveira@ufsc.br[‡] mmedar@academicos.uta.cl[§]

Abstract

The motion of incompressible electrical conducting fluids can be modeled by magnetohydrodynamics equations, which consider the Navier-Stokes equations coupled with Maxwell's equations. For the classical Navier-Stokes system, there exists an extensive study of the convergence rate for the Galerkin approximations. These results were extended to the equations of magnetohydrodynamics in [2]. In this work we extend these results to the equations of magnetohydrodynamics with variable density. We reach basically the same level of knowledge as in the case of the Navier-Stokes with variable density.

1 Introduction

This paper is concerned with the nonhomogeneous incompressible MHD system in 3D bounded domains. The governing equations are the following (see for instance the book [1]):

$$\begin{cases} \rho\mathbf{u}_t + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\Delta\mathbf{u} + \nabla\left(P + \frac{1}{2}|\mathbf{h}|^2\right) = \rho\mathbf{f} + (\mathbf{h}\cdot\nabla)\mathbf{h}, \\ \mathbf{h}_t + (\mathbf{u}\cdot\nabla)\mathbf{h} - \eta\Delta\mathbf{h} = (\mathbf{h}\cdot\nabla)\mathbf{u}, \\ \rho_t + \mathbf{u}\cdot\nabla\rho = 0, \\ \mathrm{div}\mathbf{u} = \mathrm{div}\mathbf{h} = 0. \end{cases} \tag{1}$$

Here, $t \geq 0$ is time and $x \in \Omega \subsetneq \mathbb{R}^3$ is the spatial coordinate. Moreover, the unknowns are $\rho(x,t) \in \mathbb{R}^+$, $\mathbf{u}(x,t) \in \mathbb{R}^3$, $P(x,t) \in \mathbb{R}$ and $\mathbf{h}(x,t) \in \mathbb{R}^3$, representing, respectively, the density, the incompressible velocity field, the hydrostatic pressure and the magnetic field of the fluid as functions of position x and time t. The function $|\mathbf{h}|^2/2$ is the magnetic pressure. Thus, we denote by $p \stackrel{\text{def}}{=} P + \frac{1}{2}|\mathbf{h}|^2$ the total pressure of the fluid. The positive constants μ and η represent, respectively, the viscosity and the resistivity coefficients. The later is inversely proportional to the electrical conductivity constant and acts as the magnetic diffusivity coefficient.

We complement the system (1) with the following initial and boundary conditions:

$$\begin{cases} (\rho, \mathbf{u}, \mathbf{h})\big|_{t=0} = (\rho_0, \mathbf{u}_0, \mathbf{h}_0) \quad \text{in } \Omega, \\ (\mathbf{u}(x,t), \mathbf{h}(x,t)) = (0,0) \quad \text{for all } (x,t) \in \Gamma \times (0,\infty), \end{cases} \tag{2}$$

where Γ is the boundary of Ω.

2 Main Result

We use the usual function spaces for the Navier-Stokes equations.

Concerning the existence of solutions for equations (1)-(2), they can be obtained by using a semi-Galerkin approximation, that is, we consider a Galerkin approximations

$$\mathbf{u}^k(x,t)=\sum_{i=1}^{k} c_{ik}(t)\mathbf{w}^i(x), \qquad \mathbf{h}^k(x,t)=\sum_{i=1}^{k} d_{ik}(t)\mathbf{w}^i(x)$$

where $\mathbf{w}^i$, $i = 1,\cdots$, are the eigenfunctions of the Stokes operator for the velocity and magnetic fields, respectively and an infinite dimensional approximation $\rho^k(x,t)$ for the density satisfying the following equations $(\mathbf{u}^k,\mathbf{h}^k,\rho^k)\in C^2([0,T];V_k)\times C^2([0,T];V_k)\times C^1(\overline{Q_T})$ of

$$\begin{cases} \rho^k_t+\mathbf{u}^k\cdot\nabla\rho^k=0 \ \text{ for } (x,t)\in Q_T, \\ \\ (\rho^k\mathbf{u}^k_t,\mathbf{v})+(\rho^k(\mathbf{u}^k\cdot)\nabla\mathbf{u}^k,\mathbf{v})+\mu(\nabla\mathbf{u}^k,\nabla\mathbf{v}) \\ \qquad =(\rho^k\mathbf{f},\mathbf{v})+((\mathbf{h}^k\cdot\nabla)\mathbf{h}^k),\mathbf{v}) \ \text{ for } t\in]0,T[,\forall\,\mathbf{v}\in V_k, \\ \\ (\mathbf{h}^k_t,\mathbf{w})+((\mathbf{u}^k\cdot\nabla)\mathbf{h}^k,\mathbf{w})+\eta(\nabla\mathbf{h}^k,\nabla\mathbf{w}) \\ \qquad =((\mathbf{h}^k\cdot\nabla)\mathbf{u}^k),\mathbf{w} \ \text{ for } t\in]0,T[,\forall\,\mathbf{w}\in V_k, \\ \\ \mathbf{u}^k(x,0)=P_k\mathbf{u}_0(x),\ \mathbf{h}^k(x,0)=P_k\mathbf{h}_0(x),\ \rho^k(x,0)=\rho_0(x) \ \text{ for } x\in\Omega. \end{cases} \tag{3}$$

Here V_k is the subspace generated by the first k eigenfunctions of the Stokes operator, we denote by λ_k the eigenvalue associated with the eigenfunction $\mathbf{w}_k$.

Theorem 2.1. *If $\mathbf{u}_0\ \mathbf{h}_0\in\ V\cap H^2(\Omega)$, $\rho_0\ \in\ W^{1,\infty}(\Omega)$ and $\mathbf{f}\in L^2(0;\overline{T};H^1(\Omega))$, $\mathbf{f}_t\in L^2(0,\overline{T};L^2(\Omega))$ then there exists $T>0$ with $T\le\overline{T}$ such that the system (1)-(2) has a unique strong solution in the class*

$$\mathbf{u},\ \mathbf{h}\in C\left([0,T];V\cap H^2(\Omega)\right), \tag{1}$$

$$\mathbf{u}_t,\ \mathbf{h}_t\in C\left([0,T];H\right)\cap L^2(0,T;V), \tag{2}$$

$$\rho\in W^{1,\infty}(\Omega\times[0,T]). \tag{3}$$

Moreover, the following bounds are true

(a)

$$\|\rho(t)-\rho^k(t)\|^2_{L^\infty(\Omega)}\le\frac{G_1(t)}{\lambda_{k+1}},$$

(b)

$$\|\mathbf{u}(t)-\mathbf{u}^k(t)\|^2+\|\mathbf{h}(t)-\mathbf{h}^k(t)\|^2\le G_2(t)\left(\frac{1}{\lambda^2_{k+1}}+\frac{1}{\lambda^{3/2}_{k+1}}\right)$$

for any $t\in[0,T]$. The continuous functions $G_i(t)$ depend on t, $i=1,2$.

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AN ANISOTROPIC SUMMABILITY AND MIXED SEQUENCES

RENATO MACEDO.[1], JAMILSON R. CAMPOS[1] & JOEDSON SANTOS[1]

Departamento de Matemática, UFPB, PB, Brasil[1]

renato.burity@academico.ufpb.br[†] jamilsonrc@gmail.com[‡] joedson.santos@academico.ufpb.br[§]

Abstract

In this work we define and study a new vector-valued sequence space, called the space of anisotropic (s, q, r)-summable sequences, that generalizes the space of $(s; q)$-mixed sequences. Furthermore, we define two classes of linear operators involving this new space and also show that they satisfy a Pietsch domination-type theorem.

1 Introduction

Throughout this work, X is a Banach space and we will consider: $\ell_p(X)$ (space of absolutely p-summable X-valued sequences with the usual norm $\|\cdot\|_p$); $\ell_p^w(X)$ (space of weakly p-summable X-valued sequences with the usual norm $\|\cdot\|_{w,p}$); $\ell_\infty(X)$ (space of limited X-valued sequences with the sup norm); $c_{00}(X)$ (space of eventually null X-valued sequences with the sup norm) and the symbol $X \overset{1}{\hookrightarrow} Y$ means that X is a subspace of Y and $\|x\|_X \leq \|x\|_Y$, for every $x \in X$. Now, we will define a more general family of sequence spaces, which we call an anisotropic sequence (s, q, r)-sumable, denoted by $\ell_{(s,q,r)}^A(X)$, with $1 \leq s, r, q < \infty$.

Definition 1.1. *Let $q < s$ and $r \leq s$ be real numbers. A sequence $(x_j)_{j=1}^\infty \in X^{\mathbb{N}}$ is said to be anisotropic (s, q, r)-summable if*

$$\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}|x_k^*(x_j)|^s\right)^{\frac{q}{s}} < \infty, \quad \text{whenever} \quad (x_k^*)_{k=1}^\infty \in \ell_r(X^*). \tag{1}$$

Remark 1.1. *Straightforward calculations show that if $s < r$ then $\ell_{(s,q,r)}^A(X) = \{0\}$ and if $s \leq q$ we obtain $\ell_{(s,q,r)}^A(X) = \ell_q^w(X)$.*

The expression

$$\|(x_j)_{j=1}^\infty\|_{A(s,q,r)} := \sup_{(x_k^*)_{k=1}^\infty \in B_{\ell_r(X^*)}} \left(\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}|x_k^*(x_j)|^q\right)^{s/q}\right)^{1/s} \tag{2}$$

defines a norm on the space $\ell_{(s,q,r)}^{\mathrm{A}}(X)$ and, moreover, $\left(\ell_{(s,q,r)}^{\mathrm{A}}(X); \|\cdot\|_{A(s,q,r)}\right)$ is Banach space.

2 Main Results

The space of anisotropic (s, q, r)-summable X-valued sequences satisfies the following statements:

(a) $\ell_q(X) \overset{1}{\hookrightarrow} \ell_{(s,q,r)}^{\mathrm{A}}(X) \overset{1}{\hookrightarrow} \ell_q^w(X)$.

(b) $c_{00}(X) \subseteq \ell_{(s,q,r)}^A(X) \overset{1}{\hookrightarrow} \ell_\infty(X)$.

(c) $\ell_{(s_1,q_1,r_1)}^{\mathrm{A}}(X) \overset{1}{\hookrightarrow} \ell_{(s_2,q_2,r_2)}^{\mathrm{A}}(X)$, whenever $r_2 \leq r_1 \leq q_1 \leq q_2$, $s_1 \leq s_2 < q_2$ and $s_1 < q_1$.

The next theorem will assure us that the space of $(s; q)$-mixed sequences with the norm $\|\cdot\|_{m(s;q)}$ (see [2, 3]), denoted by $(\ell_{m(s;q)}(X), \|\cdot\|_{m(s;q)})$, is a particular case of $\ell_{(s,q,r)}^A(X)$. Furthermore, it will provide us with another interpretation of $\|\cdot\|_{m(s;q)}$.

Theorem 2.1. *Let X be a Banach space. If $1 \leq q < s < \infty$, then*

$$\ell^m_{(s;q)}(X) = \ell^A_{(s,q,s)}(X) \quad \text{and} \quad \left\|(x_j)_{j=1}^\infty\right\|_{m(s;q)} = \left\|(x_j)_{j=1}^\infty\right\|_{A(s,q,s)}.$$

Now, we will study some classes of operators that are characterized by transformations of vector-valued sequences and deal with anisotropic summable sequences. In what follows, $1 \leq s, r, q, p < \infty$ are real numbers and $T : U \to X$ is a continuous linear operator between Banach spaces.

Definition 2.1. *Let $1 \leq p \leq q < s$ and $1 \leq r \leq s$ be real numbers. We say that T is weakly anisotropic $(s, q, r; p)$-summing ($T \in \mathcal{W}^A_{(s,q,r;p)}(U; X)$) if there is a constant $D > 0$ such that $\| (T(u_j))_{j=1}^\infty \|_{A(s,q,r)} \leq D\|(u_j)_{j=1}^\infty\|_{w,p}$, for every $(u_j)_{j=1}^\infty \in \ell^w_p(U)$.*

Definition 2.2. *Let $1 \leq q < s$, $q \leq p$ and $1 \leq r \leq s$ be real numbers. We say that T is anisotropic $(p; s, q, r)$-summing ($T \in \Pi^A_{(p;s,q,r)}(U; X)$) if there is a constant $C > 0$ such that $\|(T(u_j))_{j=1}^\infty\|_p \leq C \cdot \|(u_j)_{j=1}^\infty\|_{A(s,q,r)}$, whenever $(u_j)_{j=1}^\infty \in \ell^A_{(s,q,r)}(U)$.*

Furthermore, the infimum of all the constants D and C satisfying the inequalities above defines a norm for the classes $\mathcal{W}^A_{(s,q,r;p)}(U; X)$ and $\Pi^A_{(p;s,q,r)}(U; X)$, denoted by $w^A_{(s,q,r;p)}(\cdot)$ and $\pi^A_{(p;s,q,r)}(\cdot)$, respectively.

Proposition 2.1. *The pairs $\left(\mathcal{W}^A_{(s,q,r;p)}; w^A_{(s,q,r;p)}(\cdot)\right)$ and $\left(\Pi^A_{(p;s,q,r)}; \pi^A_{(p;s,q,r)}(\cdot)\right)$ are injective Banach ideals of operators.*

The next results show that these classes of operators satisfy a Pietsch domination-type theorem.

Theorem 2.2. *The operator $T \in \mathcal{W}^A_{(s,p,r;p)}(U; X)$ if and only if there are a positive constant C and a regular probability measure μ on the Borel subsets of B_{U^*}, with the weak star topology, such that*

$$\|\Psi_{x^*}(T(u))\|_s \leq C \left(\int_{B_{U^*}} |\varphi(u)|^p \, d\mu(\varphi) \right)^{\frac{1}{p}}, \quad \text{for all } u \in U \text{ and any } x^* = (x^*_k)_{k=1}^\infty \in B_{\ell_r(X^*)}. \tag{3}$$

Theorem 2.3. *The operator $T \in \Pi^A_{(p;s,p,r)}(U; X)$ if and only if there are a positive constant C and a regular probability measure μ on the Borel subsets of $B_{(\ell_{r^*}(U))^*}$, with the weak star topology, such that*

$$\|T(u)\| \leq C \left(\int_{B_{(\ell_{r^*}(U))^*}} \left(\sum_{k=1}^\infty |\varphi_{x^*}(u \cdot e_k)|^s \right)^{\frac{p}{s}} d\mu(\varphi_{x^*}) \right)^{\frac{1}{p}}, \quad \text{for every } \; u \in U. \tag{4}$$

This work is a part of the paper [1].

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MEASURE-THEORETIC UNIFORMLY POSITIVE ENTROPY ON THE SPACE OF PROBABILITY MEASURES.

RÔMULO M. VERMERSCH[1,†]

[1]Departamento de Matemática, UFSC, SC, Brasil

romulo.vermersch@ufsc.br[†]

Abstract

For a homeomorphism T on a compact metric space X, a T-invariant Borel probability measure μ on X and a measure-theoretic quasifactor $\widetilde{\mu}$ of μ, we study the relationship between the local entropy of the system (X, μ, T) and of its induced system $(\mathcal{M}(X), \widetilde{\mu}, \widetilde{T})$, where $\widetilde{T}$ is the homeomorphism induced by T on the space $\mathcal{M}(X)$ of all Borel probability measures defined on X.

1 Introduction

By a *m*easure-theoretic dynamical system (MDS) we mean a triple (X, μ, T), where (X, T) is a topological dynamical system and μ is a T-invariant Borel probability measure on X. The induced homeomorphism $\widetilde{T}$ is defined on the space of Borel probability measures $\mathcal{M}(X)$ by the formula $(\widetilde{T}(\mu))(A) := \mu(T^{-1}(A))$ $(\mu \in \mathcal{M}(X), A \subset X$ Borel set). A measure-theoretic *quasifactor* of $\mathfrak{X}$ [2] is a $\widetilde{T}$-invariant Borel probability measure $\widetilde{\mu}$ on $\mathcal{M}(X)$ which satisfies the so-called *barycenter equation*:

$$\int_X f(x) d\mu(x) = \int_{\mathcal{M}(X)} \int_X f(x) d\theta(x) d\widetilde{\mu}(\theta)$$

for every continuous function $f : X \to \mathbb{R}$. We denote by $Q(\mu)$ the set of all measure-theoretic quasifactors of μ. We say that a two-set Borel partition $\mathcal{P} = \{P_0, P_1\}$ of X is a *replete partition* if $\text{int}P_0 \neq \varnothing$ and $\text{int}P_1 \neq \varnothing$ [1]. We say that $\mathfrak{X}$ has μ-UPE if $h_\mu(T, \mathcal{P}) > 0$ for every replete partition $\mathcal{P}$. In other words, a μ-UPE system is a MDS that has *measure-theoretic uniformly positive entropy*.

In this work we are concerned with the relationship between the local entropy of the measure-theoretic dynamical systems (X, μ, T) and $(\mathcal{M}(X), \widetilde{\mu}, \widetilde{T})$, where $\widetilde{\mu}$ is a quasifactor of μ.

Let us introduce some elements of Kerr-Li machinery [6] that we shall use in the sequel. Let $\mathfrak{X} = (X, \mu, T)$ be a MDS and let $\boldsymbol{A} = (A_1, \ldots, A_k)$ be a tuple of subsets of X. For a subset D of X, we say that $J \subset \mathbb{N}$ is an *independence set for* $\boldsymbol{A}$ *relative to* D if for every nonempty finite subset $I \subset J$ and every map $\sigma : I \to \{1, \ldots, k\}$, we have

$$D \cap \bigcap_{j \in I} T^{-j} A_{\sigma(j)} \neq \emptyset.$$

For each $\delta > 0$, we denote by $\mathcal{B}(\mu, \delta)$ the collection of all Borel subsets D of X such that $\mu(D) \geq 1 - \delta$. For each $m \geqslant 1$ and $\delta > 0$, we define

$$\varphi(\boldsymbol{A}, \delta, m) = \min_{D \in \mathcal{B}(\mu, \delta)} \max \big\{ |\{1, \ldots, m\} \cap J| : J \text{ is an independence set for } \boldsymbol{A} \text{ relative to } D \big\}.$$

Now, put

$$\overline{I}_\mu(\boldsymbol{A}, \delta) := \limsup_{m \to \infty} \frac{\varphi(\boldsymbol{A}, \delta, m)}{m}.$$

Finally, let us define the *upper μ-independence density* of $\boldsymbol{A}$ as $\overline{I}_\mu(\boldsymbol{A}) := \sup_{\delta > 0} \overline{I}_\mu(\boldsymbol{A}, \delta)$.

The following useful characterization of μ-UPE is due to Kerr and Li [6].

Theorem 1.1. *Let (X, μ, T) be a MDS. Then, (X, μ, T) has μ-UPE if and only if for every pair $\boldsymbol{U} = (U_0, U_1)$ of nonempty disjoint open sets in X, one has $\overline{I}_\mu(\boldsymbol{U}) > 0$.*

Now, let us introduce some notations. For each $n \in \mathbb{N}$, let

$$\mathcal{M}_n(X) := \Big\{\frac{1}{n}\sum_{i=1}^{n}\delta_{x_i} \in \mathcal{M}(X) : x_1, \ldots, x_n \in X \text{ not necessarily distinct}\Big\},$$

where δ_x denotes the unit mass concentrated at the point x of X. It is classical that $\bigcup_{n\in\mathbb{N}} \mathcal{M}_n(X)$ is dense in $\mathcal{M}(X)$. Since $\mathcal{M}_n(X)$ is $\widetilde{T}$-invariant, we can consider the TDS $(\mathcal{M}_n(X), \widetilde{T})$, where we are also denoting by $\widetilde{T}$ the corresponding restricted map. For each $n \in \mathbb{N}$, let us consider $(X^{(n)}, \mu^{(n)})$ the canonical symmetric n-fold joining of (X, μ) [3], where $\mu^{(n)} := \mu \times \cdots \times \mu$ is the product measure on $X^{(n)} := X \times \cdots \times X$. We also consider $T_n := T \times \cdots \times T$ and, given any $\widetilde{\mu} \in Q(\mu)$, we consider the MDS $(\mathcal{M}_n(X), \widetilde{\mu}, \widetilde{T})$, where we are also denoting by $\widetilde{\mu}$ the corresponding normalized induced measure. Denote by S_n the group of all permutations of n elements and let us consider $\tau : X^{(n)} \to \hat{X}^{(n)} := X^{(n)}/S_n$ the quotient map. A typical element of $\hat{X}^{(n)}$ will be denoted by $\langle x_1, \ldots, x_n\rangle$. Moreover, we can consider the quotient measure $\tau_*(\mu^{(n)}) := \mu^{(n)} \circ \tau^{-1}$ on $\hat{X}^{(n)}$. Now let us consider the maps $\psi : (x_1, \ldots, x_n) \in X^{(n)} \mapsto (1/n)\sum_{l=1}^{n}\delta_{x_l} \in \mathcal{M}_n(X)$ and $\hat{\psi} : \langle x_1, \ldots, x_n\rangle \in \hat{X}^{(n)} \mapsto (1/n)\sum_{l=1}^{n}\delta_{x_l} \in \mathcal{M}_n(X)$. Clearly, $\hat{\psi}$ is a Borel isomorphism and we can consider the measure $(\hat{\psi} \circ \tau)_*(\mu^{(n)})$ on $\mathcal{M}_n(X)$.

2 Main Results

Theorem 2.1. *For every ergodic MDS (X, μ, T), the following assertions are equivalent:*

(i) (X, μ, T) *has* μ*-UPE;*

(ii) $(\mathcal{M}_n(X), (\hat{\psi} \circ \tau)_*(\mu^{(n)}), \widetilde{T})$ *has* $(\hat{\psi} \circ \tau)_*(\mu^{(n)})$*-UPE for some* $1 \le n < \infty$*;*

(iii) $(\mathcal{M}_n(X), (\hat{\psi} \circ \tau)_*(\mu^{(n)}), \widetilde{T})$ *has* $(\hat{\psi} \circ \tau)_*(\mu^{(n)})$*-UPE for every* $1 \le n < \infty$*.*

Theorem 2.2. *For every ergodic MDS (X, μ, T) and every $\widetilde{\mu} \in Q(\mu)$, if $(\mathcal{M}(X), \widetilde{\mu}, \widetilde{T})$ has $\widetilde{\mu}$-UPE, then (X, μ, T) has μ-UPE.*

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APPROXIMATION NUMBERS OF KERNELS SATISFYING AN ABSTRACT HÖLDER CONDITION.

THAÍS JORDÃO[1,†]

[1]ICMC, Universidade de São Paulo (USP), SP, Brasil

tjordao@icmc.usp.br[†]

Abstract

Kernels admitting Fourier series expansion with positive coefficients will be considered satisfying an extended Hölder condition defined in terms of generalized smoothness on the d-dimensional unit sphere of the Euclidean space. We provide sharp upper bounds for the Kolmogorov widths of the unit ball of reproducing kernel Hilbert spaces generated by kernel satisfiying an abstract Hölder condition. We estimate the widths from the decay rates for the sequence of eigenvalues of the integral operator.

1 Introduction

This work provides sharp upper bounds for the Kolmogorov widths in the case in which the kernel satisfies an abstract Hölder condition and is based on [1]. The *Kolmogorov n-width* of a subset A of a Hilbert space H ([2]) is the quantity $\delta_n(A;H)$ that measures how n-dimensional subspaces of H can approximate A. In other words, it is defined as

$$\delta_n(A;H) := \inf_{V_n\subset H} \sup_{f\in A} \inf_{f_n\in V_n} \|f-f_n\|_H,$$

where the first infimum is taken over all n-dimensional subspaces V_n of H. If the infimum is attained, for some n-dimensional subspace V of H, then V is called an *optimal subspace*. The estimation of the Kolmogorov n-th width of the unit ball of the reproducing kernel Hilbert space in $L^2(S^m)$ and the identification of the so-called optimal subspace usually suffice. These Kolmogorov widths can be computed through the eigenvalues of the integral operator associated to the kernel.

We consider the unit sphere S^m, $m \geq 2$, of $\mathbb{R}^{m+1}$ with its usual geodesic distance and write σ_m to denote the induced Lebesgue measure on S^m. If $K : S^m \times S^m \to \mathbb{R}$ is a symmetric and positive definite kernel on S^m, write $(\mathcal{H}(K), \|\cdot\|_{\mathcal{H}})$ to denote the unique separable Hilbert space of functions $f : S^m \to \mathbb{R}$ where K is a reproducing kernel. If K is continuous, the space $\mathcal{H}(K)$ is embeddable in the usual space $L^2(S^m) := L^2(S^m, \sigma_m)$. The integral operator $\mathcal{K} : L^2(S^m) \to L^2(S^m)$ given by

$$\mathcal{K}(f) = \int_{S^m} K(x,y)f(y)d\sigma_m(y), \quad f \in L^2(S^m), \tag{1}$$

is well-defined, compact, and self-adjoint. Its range is a dense subset of $\mathcal{H}(K)$ and, in addition,

$$\langle f,g\rangle_{L^2(S^m)} = \langle f,\mathcal{K}(g)\rangle_{\mathcal{H}(K)}, \quad f\in\mathcal{H}(K), \quad g\in L^2(S^m).$$

Since a version of the classical Mercer's Theorem hold, the integral operator $\mathcal{K}$ is positive and has countably many nonnegative eigenvalues, say, $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, with respective eigenfunctions $\varphi_1, \varphi_2, \ldots$. The set $\{\varphi_i : i = 1, 2, \ldots\}$ is orthonormal in $L^2(S^m)$ and orthogonal in $\mathcal{H}(K)$. Further,

$$K(x,y) = \sum_{i=1}^{\infty} \lambda_i \varphi_i(x)\varphi_i(y), \quad x, y \in S^m,$$

where the sum is absolutely and uniformly convergent. It follows that the set $\{\sqrt{\lambda_i}\,\varphi_i : i = 1, 2, \ldots\}$ is an orthonormal basis of $\mathcal{H}(K)$.

We will write $\Omega_n := \text{span}\,\{\sqrt{\lambda_i}\,\varphi_i : i = 1, \ldots, n\}$, for $n = 1, 2, \ldots$. The optimality of Ω_n ([2], Chapter 6) remains for this alternative definition of the Kolmogorov n-with for S the unit sphere in $\mathcal{H}(K)$, that is,

$$\delta_n\left(S; L^2(S^m)\right) = \sup_{f\in S}\left\| f - \sum_{i=1}^{n}\langle f, g_i\rangle_{\mathcal{H}(K)} g_i\right\|_2 = \sqrt{\lambda_n},$$

where $\{g_1, g_2, \ldots, g_n\}$ is an $\mathcal{H}(K)$-orthonormal basis of Ω_n.

2 Main Results

The Hölder condition used in the paper depends upon a fixed sequence of measures zonal Borel measures $\{\mu_t : t \in (0, \pi)\}$ generating convolution operators $T_t(f) = f * \mu_t$, $f \in L^2(S^m)$. Consider $\rho \in (0, 2]$ and $B : S^m \to [0, \infty)$ a function belonging to $L^\infty(S^m)$. The kernel $K : S^m \times S^m \longrightarrow \mathbb{R}$ is (μ_t, B, ρ)-*Hölder* if

$$|(K(x,\cdot) * \mu_t)(y) - K(x, y)| \le B(x)t^\rho, \quad t \in (0, \pi), \quad x, y \in S^m, \tag{2}$$

In the next result we write $v_m(t) = \sigma_m(supp(K^t(x,\cdot))$, for the volume of support of the function $y \in S^m \to K^t(x, y)$. This quantity does not depend upon x due to the invariance of σ_m with respect to orthogonal transformations on $\mathbb{R}^{m+1}$. Also, P_k^β represents the usual Gegenbauer polynomial of degree k associated to order dimension β and normalized as $P_k^\beta(1) = 1$.

Theorem 2.1. *Let $\{\mu_t : t \in (0, \pi)\}$ be a uniformly bounded family of measures such that, for each t, the multiplier of μ_t is*

$$\left\{c_{k,m}(v_m(t))^{-1}P^{\beta}_{\alpha(k)}(\cos t)(\sin t)^\gamma\right\}_{k=0}^{\infty},$$

where $\{c_{k,m}\}_{k=0}^\infty$ is a sequence of nonzero real numbers, $\alpha : \mathbb{Z}_+ \to \mathbb{Z}_+$ is strictly increasing, $\gamma > 0$ and 2β is an integer at least γ. Assume that for every positive integer l, there exists $q = q(l, m) > 0$ so that $ln \le \alpha(qn)$, $n = 1, 2, \ldots$. If K is (μ_t, B, ρ)-Hölder, then the sequence of eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of the integral operator $\mathcal{K}$ satisfies

$$\lambda_n = O(n^{-1-\rho/m}), \quad n \to \infty.$$

Theorem above is the key to obtain the following estimates for the Kolmogorov widths.

Theorem 2.2. *Under the assumptions in Proposition 2.1, if K is (μ_t, B, ρ)-Hölder, then*

$$\delta_n(S; L^2(S^m)) = O\left((n + 1)^{-1/2-\rho/2m}\right), \quad n \to \infty.$$

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THE BISHOP-PHELPS-BOLLOBÁS THEOREM FOR LIPSCHITZ MAPS.

THIAGO GRANDO[1,†]

Department of Mathematics, UNICENTRO, PR, Brazil[1].

tgrando@unicentro.br[†]

Abstract

In this talk we study a version of the Bishop-Phelps-Bollobás theorem called Lipschitz-Bishop-Phelps-Bollobás property (Lip-BPBp for short). Under appropriate conditions for a pointed metric space M and a function module space X, it was possible to prove that the pair (M, X) satisfies the Lip-BPBp.

1 Introduction

Let (M, d_1), (N, d_2) be metric spaces. A mapping $f : M \to N$ is called *Lipschitz* if there is a constant $C \geq 0$ such that $d_2(f(x_1), f(x_2)) \leq C.d_1(x_1, x_2)$, for every pair of points $x_1, x_2 \in M$. If, in addition, (M, d) is a *pointed* metric space (that is, there is a distinguished point in M denoted by 0) and Y is a Banach space then the space $\text{Lip}_0(M, Y)$ of all Lipschitz maps from M to Y which vanish at 0 is a Banach space when endowed with the norm $||F||_L = \sup\left\{\frac{||F(p)-F(q)||}{d(p,q)} : p, q \in M, \, p \neq q\right\}$. A map $F \in \text{Lip}_0(M, Y)$ *attain its norm in the strong sence* or *strongly attains its norm* if there exist $p, q \in M$, $p \neq q$ such that $\frac{||F(p)-F(q)||}{d(p,q)} = ||F||_L$. The set of those Lipschitz maps $F : M \to Y$ which strongly attain their norms is denoted by $\text{LipSNA}(M, Y)$. In this sense, a natural question is to decide for which metric spaces M the set $\overline{\text{LipSNA}(M, Y)}^{||.||} = \text{Lip}_0(M, Y)$, for all Banach space Y. In [2], the autors proved that $\text{LipSNA}([0, 1], \mathbb{R})$ is not dense in $\text{Lip}_0([0, 1], \mathbb{R})$. Considering M a *pointed* metric space, the function $\delta : M \to \text{Lip}_0(M, Y)^*$ defined by $\delta(f) = f(p)$, for every $p \in M$ and $f \in \text{Lip}_0(M, Y)$, is called the canonical isometric embedding. The norm-closed linear span of $\delta(M)$ in the Banach space $\text{Lip}_0(M, Y)^*$ is denoted by $\mathcal{F}(M)$, which is usually called *Lipschitz-free* space over M. It was proved in [1] that if $\mathcal{F}(M)$ has the Radon-Nykodym property (RNP), then $\overline{\text{LipSNA}(M, Y)}^{||.||} = \text{Lip}_0(M, Y)$, for all Banach space Y. Denoting by $\text{NA}(\mathcal{F}(M), Y)$ the set of norm-attaining operators from $\mathcal{F}(M)$ to Y, it is known that the density of $\text{LipSNA}(M, Y)$ in $\text{Lip}_0(M, Y)$ implies that $\overline{\text{NA}(\mathcal{F}(M), Y)} = L(\mathcal{F}(M), Y))$, but the reciprocal is not true, since $\text{LipSNA}([0, 1], \mathbb{R})$ is not dense in $\text{Lip}_0([0, 1], \mathbb{R})$, while $\text{NA}(\mathcal{F}([0, 1], \mathbb{R})$ is a dense set in $L(\mathcal{F}([0, 1], \mathbb{R})$, by the Bishop-Phelps Theorem. In [5], the authors presented an extension of the Biship-Phelps-Bollobás property to the Lipschitz context, which they called *Lipschitz-Bishop-Phelps-Bollobás property.*

Definition 1.1. *Let M be a pointed metric space and Y be a Banach space. We say that the pair (M, Y) has the Lipschitz-Bishop-Phelps-Bollobás property (Lip-BPBp for short), if given $\epsilon > 0$ there is $\eta(\epsilon) > 0$ such that for every norm-one $F \in Lip_0(M, Y)$ and every $p, q \in M$, $p \neq q$ such that $||F(p) - F(q)|| > (1 - \eta(\epsilon))d(p, q)$, there exist $G \in Lip_0(M, Y)$ and $r, s \in M$, such that*

$$\frac{||G(r) - G(s)||}{d(r, s)} = ||G||_L = 1, \quad ||G - F|| < \epsilon, \quad \frac{d(p, r) + d(q, s)}{d(p, q)} < \epsilon.$$

In the same article, they presented some pairs (M, Y) satisfying the *Lip-BPBp*. In this way, in [6], the authors study the stability behavior of the Bishop-Phelps-Bollobás property for Lipschitz maps. Among others, they proved that if M a pointed metric space such that $(M, \mathbb{R})$ has the *Lip-BPBp* then for every compact Hausdorff topological space K, the pair $(M, C(K))$ has the *Lip-BPBp* for Γ-flat operators, where Γ is the 1-norming set.

2 Main Result

In this note we study the Lipschitz-Bishop-Phelps-Bollobás property when the range space is a function module space. We obtained analogous result that presented in [6].

Definition 2.1. *Function Module is (the third coordinate of) a triple* $(K, (X_t)_{t\in K}, X)$*, where* K *is a nonempty compact Hausdorff topological space,* $(X_t)_{t\in K}$ *a family of Banach spaces, and* X *a closed* $C(K)$*-submodule of the* $C(K)$*-module* $\prod_{t\in K}^{\infty} X_t$ *(the* ℓ_∞*-sum of the spaces* X_t*) such that the following conditions are satisfied:*

1. *For every* $x \in X$*, the function* $t \to \|x(t)\|$ *from* K *to* $\mathbb{R}$ *is upper semi-continuous.*
2. *For every* $t \in K$*, we have* $X_t = \{x(t) : x \in X\}$*.*
3. *The set* $\{t \in K : X_t \neq 0\}$ *is dense in* K*.*

Theorem 2.1. *Let* $(K, (X_t)_{t\in K}, X)$ *be a function module space and* M *a pointed metric space. If* $(M, \mathbb{R})$ *has the Lip-BPBp then* (M, X) *has the Lip-BPBp for* Γ*-flat operators, where* Γ *is the 1-norming set.*

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SURJECTIVE ISOMETRIES IN THE LORENTZ SEQUENCE SPACES.

VICTOR S. RONCHIM[1,†]

Instituto de Matemática, IME-USP, SP, Brasil[1].

vronchim@ime.usp.br[†]

Abstract

It is well known that the surjective isometries in ℓ_p (for $1 \leq p \leq \infty$ and $p \neq 2$) are given by permutations with signs. In this work we prove that the isometries on the classical Lorentz sequence space, $d(w,1)$, and on its predual, $d_*(w,1)$, behave in the same way. This is a joint work with Christina Brech.

1 Introduction

We say that a Banach sequence space X has *standard group of isometries* if every $T \in \text{Isom}(X)$ (where $\text{Isom}(X)$ denotes the set of all isometries from X to X) is induced by a sequence of signs $\varepsilon \in \{-1,1\}^{\mathbb{N}}$ and a permutation $\pi \in S_\infty$, i.e.:

$$T(x_n) = (\varepsilon_n x_{\pi(n)})_{n\geq 1}$$

For example, the classical sequence spaces c, c_0 and ℓ_p (for $p \neq 2$) all have standard group of isometries (see [1]). Given a decreasing sequence of positive real numbers sequence $w = (w_n)_{n\geq 1}$ such that $w \in c_0 \setminus \ell_1$, we define the Lorentz sequence, denoted by $d(w,1)$ as follows:

$$d(w,1) = \left\{x = (x_n)_{n\geq 1} \in \mathbb{R}^{\mathbb{N}} : \|x\|_{w,1} < \infty\right\},$$

where $\|x\|_{w,1} = \sup\limits_{\pi \in S_\infty} \sum\limits_{i=1}^{\infty} |x_{\pi(n)}| w_n$ and S_∞ denotes the set of all permutations of natural numbers.

The space $d(w,1)$ endowed with $\|\cdot\|_{w,1}$ is a Banach sequence space and the canonical vectors $(e_n)_{n\geq 1}$ form a symmetric Schauder basis (as well as the classical ℓ_p spaces) and it is well known that $\|x\|_{w,1}$ is attained by its decreasing rearrangement, i.e., the sequence $x^* = (x_n^*)_{n\geq 1}$ that is obtained from reordering $(|x_n|)_{n\geq 1}$ as a decreasing sequence via a permutation.

In what follows, we will use the following notation: $W_n = W(n) = \sum\limits_{i=1}^{n} w_i$, $\text{supp}(x) = \{n \in \mathbb{N} : x(n) \neq 0\}$, $\text{ext}B_X$ denotes the set of extreme points of the closed unit ball of the Banach space X and all isometries here are surjective.

It is easy to check that the standard unitary vectors (e_n) form a boundedly complete Schauder basis for $d(w,1)$, thus it admits a predual. It is well known that the predual of the Lorentz sequence spaces is:

$$d_*(w,1) = \{x = (x_n) \in \mathbb{R}^{\mathbb{N}} : \lim_n \frac{\sum_{i=1}^n x^*(i)}{W(n)} = 0\},$$

endowed with the norm $\|(x_n)\|_W = \sup\limits_n \dfrac{\sum_{i=1}^n x^*(i)}{W(n)}$.

Unfortunately, to our knowledge, no isometry is explicitly given in the classical literature to describe the above identification. For instance, the Lorentz sequence space is a particular case of the Köthe space μ_w, and the above description is obtained in [3] passing through the α-dual space $\mu_w^\times$ and no isometry is given. Thus, we add one here for the sake of completeness, that is based on arguments from [2]:

Proposition 1.1. *The operator $\Phi : d_*(w,1)^* \to d(w,1)$ given by $\Phi(f) = (f(e_n))_{n\geq 1}$ is an isometry.*

2 Main Results

To show that the spaces $d(w,1)$ and $d_*(w,1)$ have standard group of isometries we exploit the fact that (surjective) isometries preserve extremal points of the unit ball, as well as the following characterization:

Theorem 2.1 ([4, Theorem 2.6]). *An element $x \in S_{d(w,1)}$ is an extreme point of $B_{d(w,1)}$ if, and only if, there exists $n_0 \in \mathbb{N}$ such that:*

$$x_n^* = \begin{cases} \dfrac{1}{W(n_0)} & , \text{ if } n \leq n_0 \\ 0 & , \text{ if } n > n_0 \end{cases}$$

and $w_1 > w_{n_0}$, when $n_0 > 1$.

Without passing through the rearrangement of x, the previous result states that $x \in \text{ext}B_{d(w,1)}$ if, and only if, $x = \frac{1}{W(n_0)}\varepsilon\chi_A$, where $|A| = n_0$, $\varepsilon \in \{-1,1\}^{\mathbb{N}}$ and, when A is not a singleton, $w_{n_0} < w_1$.

For $d(w,1)$ we will denote $k_0 = \max\{n \in \mathbb{N} : w_1 = w_{k_0}\}$. Notice that $k_0 \geq 1$ is well defined because $w \in c_0$ and:

$$w_1 = \ldots = w_{k_0} > w_{k_0+1} \geq \ldots$$

It follows from Theorem 2.1 that if x is an extremal point of $d(w,1)$ then either $\text{supp}(x)$ is a singleton or $|\text{supp}(x)| > k_0$. To achieve our main result, Theorem 2.2, our proof is based on the following three steps:

We fix an arbitrary $T \in \text{Isom}d(w,1)$ and we show that:

Step 1: For all $n, m \in \mathbb{N}$, $\text{supp}T(e_n)$ and $\text{supp}T(e_m)$ are either equal or disjoint;

Step 2: The cardinalities $|\text{supp}T(e_n)|$ are all equal (and finite);

Step 3: For all $n, m \in \mathbb{N}$, $|\text{supp}T(e_n)| = |\text{supp}T^{-1}(e_m)| = 1$.

Theorem 2.2. *The spaces $d(w,1)$ and $d_*(w,1)$ have standard group of isometries.*

Sketch of the proof: Because T is an isometry, $\mathbb{N} = \bigcup \text{supp}(T(e_n))$ and from step 3 we obtain that $T(e_n) = \varepsilon_{j_n} e_{j_n}$ with $\varepsilon_{j_n} = \pm 1$. Thus, $\pi(j_n) = n$ defines a permutation such that:

$$T(x_n) = (\varepsilon_n x_{\pi(n)})_{n \geq 1}$$

Then $d(w,1)$ has standard group of isometries and this property is inherited by $d_*(w,1)$ via the isomorphism in Proposition 1.1. □

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DISJOINT P-CONVERGENCE OF OPERATORS AND ADJOINT OPERATORS

V. C. C. MIRANDA.[1], G. BOTELHO[1] & L. A. GARCIA[2]

Faculdade de Matemática, UFU, MG, Brasil[1], Instituto de Matemática e Estatística, USP, SP, Brasil[2]

colferaiv@gmail.com[†] botelho@ufu.br[‡] luisgarcia@ime.usp.br[§]

Abstract

In this talk we present recent results obtained by the authors concerning operators and adjoint operators between Banach lattices that are disjoint p-convergents.

1 Introduction

Recall that a linear operator between Banach spaces is said to be completely continuous, or a Dunford-Pettis operator, if it sends weakly null sequences to norm null sequences. Notions alike have been considered in the mathematical literature. For instance, in the Banach lattice context the so-called almost Dunford-Pettis operators were introduced in [5]: a linear operator from a Banach lattice to a Banach space is almost Dunford-Pettis if it sends disjoint weakly null sequences to norm null sequences; or equivalently, if it sends positive disjoint weakly null sequences to norm null sequences. Around the summability properties, the notion of p-convergent operators, or a Dunford-Pettis operator of order p, was introduced in [4]. Letting $1 \leq p < \infty$, a linear operator between Banach spaces is said to be p-convergent if it sends weakly p-summable sequences to norm null sequences. The lattice counterpart of this class was introduced in [6]: a linear operator from a Banach lattice to a Banach space is disjoint p-convergent if it sends disjoint weakly p-summable sequences to norm null sequences; or equivalently if it sends positive disjoint weakly p-summable sequences to norm null sequences. This class of operators on Banach lattices was also studied in the recent paper of Alikhani [1] under the name of almost p-convergent operators.

2 Main Results

Since weakly p-summable sequences are weakly null, every almost Dunford-Pettis is disjoint p-convergent. Nevertheless, the reciprocal is not true, and so a natural question is under which conditions on a Banach lattice E, is every disjoint p-convergent operator from E into any Banach space X almost Dunford-Pettis? In order to answer this question, we introduced the following property: a Banach lattice E is said to have the disjoint property of order p, for $1 \leq p < \infty$, if every norm bounded disjoint sequence in E is weakly p-summable. For example, c_0 has the disjoint property of order p for all $1 \leq p < \infty$.

In particular, we provide a positive answer to our question in the following result:

Theorem 2.1. *Let E be a Banach lattice. If E has the disjoint property of order p ($1 \leq p < \infty$) or E^* has cotype p ($2 \leq p < \infty$), then every almost Dunford-Pettis operator on E is disjoint p-convergent.*

It is a classical topic in the theory of operators between Banach lattices to study when the adjoint of a positive operator between Banach lattices belongs to some class of operators. For instance, this problem was addressed to the class of almost Dunford-Pettis operators in [2], and to the class of disjoint p-convergent operators in [1]. In order to study this problem, Alikhani introduced the class of almost weak p-convergent operators. Following [1], an operator $T : E \to F$ between two Banach lattices is said to be almost weak p-convergent if for every weakly null sequence $(x_n)_n \subset E$ and for every disjoint weakly p-summable sequence $(y_n^*)_n \subset F^*$, we have that $y_n^*(Tx_n) \to 0$.

Theorem 5.11 in [1] states that the adjoint of every positive operator which is almost weak p-convergent and p-convergent $T : E \to F$ is disjoint p-convergent if and only if E^* has order continuous norm or F^* has the positive Schur property of order p (meaning that the identity operator I_{F^*} is disjoint p-convergent). In our preprint [3], we dropped the p-convergence condition in T, and proved the following:

Theorem 2.2. *The adjoint of every almost weak p-convergent positive operator $T : E \to F$ is disjoint p-convergent if and only if E^* has order continuous norm or F^* has the positive Schur property of order p.*

We observe that the equivalent condition given in [1, Theorem 5.11] is a better approach than our Theorem 2.1 in order to obtain that E^* has order continuous norm or F^* has the positive Schur property of order p. Nevertheless, in most times, we already have one of these conditions, and so to obtain that the adjoint of a positive operator is disjoint p-convergent, our Theorem 2.1 is more usefully than [1, Theorem 5.11] as we see in the next example.

Example 2.1. *The identity operator $I_{\ell_3} : \ell_3 \to \ell_3$ is an almost weak $\frac{3}{2}$-convergent operator which is not $\frac{3}{2}$-convergent, and so [1, Theorem 5.11] cannot be applied to this operator. Nevertheless, since $\ell_3^* = \ell_{3/2}$ has order continuous norm, we can apply Theorem 2.1 to obtain that $I_{\ell_3}^*$ is a disjoint $\frac{3}{2}$-convergent operator.*

As an application of Theorem 2.1, we have the following:

Corollary 2.1. *The adjoint of every almost weak p-convergent operator $T : E \to E$ is disjoint p-convergent if and only if E^* has order continuous norm.*

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(UNIFORM) EXPANSIVITY FOR OPERATORS ON LOCALLY CONVEX SPACES.

VINÍCIUS V. FÁVARO[1,†]

Faculdade de Matemática, UFU, MG, Brasil[1].

vvfavaro@ufu.br[†]

Abstract

In this work we introduce and investigate the notions of topological expansivity and uniform topological expansivity for operators on locally convex spaces. We prove that uniformly topologically expansive operators on Hausdorff locally convex spaces are never Li-Yorke chaotic.

1 Introduction

A fundamental concept in dynamical systems is that of expansivity, which was introduced by Utz [5]. Expansive and uniformly expansive operators on Banach spaces were studied by several authors (see for instance [3] and references therein). One of the main results obtained in [3] is the following:

- A uniformly expansive operator on a Banach space is never Li-Yorke chaotic.

On the other hand, many important linear dynamical systems are defined on spaces that are not normable. For instance, the translation operators of Birkhoff and the differentiation operator of MacLane, which are the classical examples of chaotic operators, are defined on the (non-normable) Fréchet space of all entire functions on $\mathbb{C}$. Our main goal in the present work is to initiate an investigation on a notion of expansivity for operators on Fréchet spaces (or in the more general setting of locally convex spaces).

2 Basic Definitions and Main Result

Given a topological vector space X over $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$, we denote by $GL(X)$ the set of all continuous linear operators on X that have a continuous inverse. Given a seminorm $\|\cdot\|$ on a vector space X, we define the *unit sphere* of $\|\cdot\|$ by

$$S_{\|\cdot\|} := \{x \in X : \|x\| = 1\}.$$

If X is a normed space with norm $\|\cdot\|$, we also write S_X instead of $S_{\|\cdot\|}$.

Given a metric space M with metric d, a homeomorphism $h : M \to M$ is said to be *expansive* if there is a constant $c > 0$ such that, for every $x, y \in M$ with $x \neq y$, there exists $n \in \mathbb{Z}$ with $d(h^n(x), h^n(y)) \geq c$.

Below we will propose a notion of (uniform) expansivity for operators on locally convex spaces, which is motivated by the following simple characterizations in the case of invertible operators on normed spaces, as observed in [3, Proposition 19]:

- T is expansive $\Leftrightarrow \sup_{n\in\mathbb{Z}} \|T^n x\| = \infty$ for every nonzero $x \in X$.
- T is uniformly expansive $\Leftrightarrow S_X = A \cup B$ where $\lim_{n\to\infty} \|T^n x\| = \infty$ uniformly on A and $\lim_{n\to\infty} \|T^{-n} x\| = \infty$ uniformly on B.

Definition 2.1. *Let X be a locally convex space over $\mathbb{K}$ whose topology is induced by a directed family $(\|\cdot\|_\alpha)_{\alpha\in I}$ of seminorms. We say that an operator $T \in GL(X)$ is* topologically expansive *if the following condition holds:*

(E) For each nonzero $x \in X$, there exists $\alpha \in I$ such that $\sup_{n\in\mathbb{Z}} \|T^n x\|_\alpha = \infty$.

We say that the operator T is uniformly topologically expansive *if:*

(UE) For every $\alpha \in I$, there exists $\beta \in I$ such that we can write $S_{\|\cdot\|_\alpha} = A_\alpha \cup B_\alpha$, where

$$\|T^n x\|_\beta \to \infty \text{ uniformly on } A_\alpha \text{ as } n \to \infty$$

and

$$\|T^{-n} x\|_\beta \to \infty \text{ uniformly on } B_\alpha \text{ as } n \to \infty.$$

The classical notion of Li-Yorke chaos was introduced for continuous maps on metric spaces. This notion was extended to group actions on Hausdorff uniform spaces in [1]. In the case of a continuous linear operator T on a Hausdorff topological vector space X, the definition reads as follows: the operator T is said to be *Li-Yorke chaotic* if there is an uncountable set $S \subset X$ such that each pair (x, y) of distinct points in S is a *Li-Yorke pair* for T, in the sense that the following conditions hold:

(LY1) For every neighborhood V of 0 in X, there exists $n \in \mathbb{N}$ such that $T^n x - T^n y \in V$.

(LY2) There exists a neighborhood U of 0 in X such that $T^n x - T^n y \notin U$ for infinitely many values of n.

Now we are in position to enunciate the main result of this work.

Theorem 2.1. *A uniformly topologically expansive operator on a Hausdorff locally convex space cannot be Li-Yorke chaotic.*

For the proof of this result we use a characterization of Li-Yorke chaos by the existence of semi-irregular vector. For a continuous linear operator T on a Fréchet space X, the following equivalence was obtained in [2]:

(i) T is Li-Yorke chaotic;

(ii) T admits a *semi-irregular vector*, that is, a vector $x \in X$ such that the sequence $(T^n x)_{n\in\mathbb{N}}$ does not converge to zero but has a subsequence converging to zero.

It was observed in [4] that this equivalence remains true in this more general setting, where x *semi-irregular* for T means that the sequence $(T^n x)_{n\in\mathbb{N}}$ does not converge to zero but has a subnet converging to zero.

Final comments. The results of this work are based on a joint work with N. C. Bernardes Jr., B. M. Caraballo, U. B. Darji and A. Peris. The author is supported by FAPEMIG - APQ-01853-23

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LOCAL BANACH-SPACE DICHOTOMIES AND ERGODIC SPACES

W. CUELLAR CARRERA.[1], N. DE RANCOURT[2] & V. FERENCZI[1]

Instituto de Matemática e Estatìstica, USP, SP, Brasil[1], U. Lille, França[2]

cuellar@ime.usp.br[†] noe.derancourtdemimerandvernyuniv-lille.fr[‡] ferenczi@ime.usp.br[§]

Abstract

We prove a local version of Gowers' Ramsey-type theorem [2], as well as local versions both of the Banach space first dichotomy (the "unconditional/HI" dichotomy) of Gowers [2] and of the third dichotomy (the "minimal/tight" dichotomy) due to Ferenczi–Rosendal [1]. As a consequence we obtain new information on the number of subspaces of non-Hilbertian Banach spaces, making some progress towards the "ergodic" conjecture of Ferenczi–Rosendal and towards a question of Johnson.

1 Introduction

A Banach space is said to be *homogeneous* if it is isomorphic to all of its (closed, infinite-dimensional) subspaces. A famous problem due to Banach, and known as the *homogeneous space problem*, asked whether, up to isomorphism, ℓ_2 is the only homogeneous Banach space. The answer turned out to be positive; this problem was eventually solved in the 1990's by a combination of results by Gowers–Maurey [3], Komorowski–Tomczak-Jaegermann [4], and Gowers [2].

The homogeneous space characterization of the Hilbert space shows that, as soon as a separable Banach space X is non-Hilbertian, it should have at least two non-isomorphic subspaces. Thus, the following general question was asked by Godefroy:

Question. (Godefroy) How many different subspaces, up to isomorphism, can a separable, non-Hilbertian Banach space have?

This question seems to be very difficult in general, although good lower bounds for several particular classes of spaces are now known. The seemingly simplest particular case of Godefroy's question is the following question by Johnson:

Question. (Johnson) Does there exist a separable Banach space having exactly two different subspaces, up to isomorphism?

Even this question is still open. More generally, it is not known whether there exist a separable, non-Hilbertian Banach space with at most countably many different subspaces, up to isomorphism.

It seems to be believed that such a space does not exist. In the rest of this paper, a separable Banach space having exactly two different subspaces, up to isomorphism, will be called a *Johnson space*.

2 Main Results

We study dichotomies associated to the *Hilbertian degree*, that is, the local degree defined by the Banach Mazur distance $d_{BM}(F, \ell_2^{\dim(F)})$, for which small spaces are exactly Hilbertian spaces. We shall denote this degree d_2:

$$d_2(F) = d_{BM}(F, \ell_2^{\dim(F)}).$$

To save notation, we say that an FDD $(F_n)_{n\in\mathbb{N}}$ of a Banach space X is *d-better* if $d(X, F_n) \xrightarrow[n\to\infty]{} \infty$. A Banach space X is a d_2-HI space if it contains no direct sum of two non-Hilbertian subspaces, and d_2-minimal if it embeds into all of its non-Hilbertian subspaces ("minimal among non-Hilbertian spaces"). An FDD is d_2-tight if all non-Hilbertian spaces are tight in it. In the case of the Hilbertian degree, our two dichotomies can be summarized as follows:

Theorem 2.1. *Let X be a non-Hilbertian Banach space. Then X has a non-Hilbertian subspace Y satisfying one of the following properties:*

(1) *Y is d_2-minimal and has a d_2-better UFDD;*

(2) *Y has a d_2-better d_2-tight UFDD;*

(3) *Y is d_2-minimal and d_2-hereditarily indecomposable;*

(4) *Y is d_2-tight and d_2-hereditarily indecomposable.*

It is clear from the definitions that if a Banach space X does not contain any isomorphic copy of ℓ_2, then the $d_2 - HI$ property is just the HI property and the d_2-minimality is just classical minimality. It is also easy to check that if X is not ℓ_2-saturated, then our two local dichotomies do not provide more information than the original ones.

For the ergodicity question we have the following consequence

Theorem 2.2. *Every non-ergodic, non-Hilbertian separable Banach space contains a d_2-minimal subspace.*

The results are part of the work: Wilson Cuellar Carrera, Noé de Rancourt, Valentin Ferenczi, *Local Banach-space dichotomies and ergodic spaces*. J. Eur. Math. Soc. (2022), https://doi.org/10.4171/jems/1257.

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GLOBAL SOLUTIONS FOR A NONLINEAR DEGENERATE NONLOCAL PROBLEM

EUGENIO CABANILLAS LAPA[1,†]

Instituto de Investigación, Facultad de Ciencias Matemáticas-UNMSM, Lima-Perú [1]

cleugenio@yahoo.com[†]

Abstract

The current paper discusses the global existence and asymptotic behavior of solutions of the following new nonlocal problem

$$u_{tt} - M\Big(\int_\Omega |\nabla u|^2\,dx\Big)\triangle u + \delta u_t = |u|^{\rho-2}u\log|u| \qquad \text{in } \Omega\times]0,\infty[,$$

where

$$M(s) = \begin{cases} a - bs, & \text{for } \ s\in[0,\frac{a}{b}[,\\ 0, & \text{for } s\in[\frac{a}{b},+\infty[. \end{cases}$$

If the initial data are appropriately small, we derive existence of global strong solutions and the exponential decay of the energy.

1 Introduction

In this research we study the following nonlocal problem

$$\begin{gathered} u_{tt} - M\Big(\int_\Omega |\nabla u|^2\,dx\Big)\triangle u + \delta u_t = |u|^{\rho-2}u\log|u| \quad \text{in } \Omega\times]0,\infty[,\\ u=0 \quad \text{on } \Gamma\times]0,\infty[,\\ u(x,0)=u^0(x), u_t(x,0)=u^1(x) \quad x\in\Omega \end{gathered} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ,

$$M(s) = \begin{cases} a - bs, & \text{for } \ s\in[0,\frac{a}{b}[,\\ 0, & \text{for } s\in[\frac{a}{b},+\infty[, \end{cases} \tag{2}$$

$a, b > 0, \rho > 2$. Equations with logarithmic nonlinearity have a lot of applications in the fields of geophysics, quantum mechanics, inflationary cosmology and so on. A number of results on the solutions to problem (1) with polynomial nonlinearities instead of the logarithmic source have been established by many researches through various approaches and assumptive conditions; see [1, 4] and references therein. Concerning nonlinear wave equations with logarithmic nonlinearities also there is many literature; see [2, 5] and references therein. In [6] Yin et al. investigated the existence and multiplicity of nontrivial solution for the new nonlocal problem

$$\begin{gathered} -\Big(a - b\int_\Omega |\nabla u|^2\,dx\Big)\triangle u = |u|^{\rho-2}u \quad \text{in } \Omega,\\ u=0 \quad \text{on } \Gamma. \end{gathered} \tag{3}$$

Also see [3, 7] for generalizations of (3). Inspired by the aforementioned studies, we are concerned with the global solvability of (1) with the nonlocal operator given in (3). In this sense, it is worth noting that to handle the logarithmic term we do not use the famous logarithmic Sobolev inequality and when $a = 0 = \delta$ the equation (1) becomes the quasilinear non well posed problem.

2 Main Results

Theorem 2.1. *Let $N = 3$ and $\frac{8}{3} < \rho < 4$. Assume further that $\{u_0, u_1\} \in (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ is sufficiently small. Then problem (1) admits a unique global solution*

$$u \in C([0, +\infty[; H_0^1(\Omega) \cap H^2(\Omega)) \cap C^1([0, +\infty[; H_0^1(\Omega)) \cap C^2([0, +\infty[; L^2(\Omega)),$$

and the energy satisfies

$$E(t) \leq Ce^{-kt} \qquad \textit{for} \;\; t \geq 0, \tag{1}$$

with some constant $C > 0$.

Proof. First, we prove a local existence result. Then, we establish global existence of solutions by using of Tartar method combined with suitable a priori estimates including $|\triangle u(t)|$ and $|\nabla_t u(t)|$ in addition to the usual energy estimate. □

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AN APPROACH FOR SOLVING STOCHASTIC DIFFERENTIAL EQUATIONS WITH NONSTANDARD COEFFICIENTS

HUGO DE LA CRUZ[1,†] & MARIO MUÑOS[2,‡]

Escola de Matemática Aplicada, FGV/EMAp, RJ, Brasil[1], Universidad de Concepción, Chile[2]

hugo.delacruzfgv.br[†] mmunoz@ing-mat.udec.cl[‡]

Abstract

Results on the convergence and stability of widely used integrators for stochastic differential equations (SDEs) are typically obtained assuming global Lipschitz assumptions on the coefficients of the equation. Therefore, in principle, these methods cannot be reliably applied to a number of models that do not meet these conditions. In this work, we first introduce a random conjugacy between a class of SDEs and random differential equations to then propose an approach to devise numerical methods for systems of SDEs with nonstandard coefficients and multiplicative noise. Simulation studies, including a comparative analysis with other integrators commonly used in applications, confirm the advantages of the proposed methods. This work is based on some ideas from our paper [1].

1 Introduction

Stochastic Differential Equations (SDEs) have become a fundamental tool for the mathematical modeling of many phenomena in which noise plays an important role. Currently, there is a wide variety of numerical methods available for the computational integration of stochastic systems, many of which have been motivated by the need to integrate particular types of SDEs in applications. However, results on important issues related to convergence, stability and long-time behavior of widely used methods are typically obtained assuming restrictive assumptions on the coefficients of the equation which may not be satisfied by many models [3]. In fact, the standard literature in stochastic numerics concentrates on numerical integrators under the hypothesis of globally Lipschitz coefficients, and when this condition is violated they can diverge or show high instability [2]. Therefore, in principle, these methods cannot be reliably applied to a number of models that do not meet these conditions. The low performance of numerical approximations is even more severe when the simulation of the system is required on long-time intervals and it is necessary that the integrator replicates, as best as possible, meaningful long-term properties of the modeled system. In this work we propose an approach that allows to devise numerical integrators for systems of SDEs with non global Lipschitz drift coefficient $f = (f_1, \dots, f_d)^\intercal$ and multiplicative noise of the form

$$\begin{aligned} dX_1 &= f_1(t, X_1, X_2, \dots, X_d)dt + (\sigma_{11}X_1 + \alpha_1)\, dW_t \\ dX_2 &= f_2(t, X_1, X_2, \dots, X_d)dt + (\sigma_{21}X_1 + \sigma_{22}X_2 + \alpha_2)\, dW_t \\ &\vdots \\ dX_d &= f_d(t, X_1, X_2, \dots, X_d)dt + (\sigma_{d1}X_1 + \sigma_{d2}X_2 + \cdots + \sigma_{dd}X_d + \alpha_d)\, dW_t \end{aligned} \tag{1}$$

where $(W_t)_{t\in\mathbb{R}^+}$ is a standard Wiener processes. SDEs with this type of structure appear in several important models in applications (see e.g., [2]).

The approach we develop consists in finding an appropriate invertible continuous transformation, linking the solution of the SDE (1) to the solution of an auxiliary random differential equation (RDE) that has the stationary Ornstein-Uhlenbeck process as the only input parameter of the system. In this way, new pathwise numerical schemes can be constructed without the need to assume global Lipschitz conditions for f. Remarkably, this framework allows to devise integrators able to approximate, with high stability, meaningful probabilistic features of the continuous system, including its stationary distribution and ergodicity.

2 Proposed approach and main results

Consider the random transformation

$$Y_k := e^{-U_k(t)} \left(\omega_1^{(k)} X_1 + \omega_2^{(k)} X_2 + \cdots +_{k-1}^{(k)} X_{k-1} + X_k + \gamma_k \right), \quad k = 1, \dots d \tag{2}$$

with

$$\begin{pmatrix} \sigma_{11} & \sigma_{21} & \cdots & \sigma_{k-1,1} \\ & \sigma_{22} & \cdots & \sigma_{k-1,2} \\ & & \ddots & \vdots \\ & & & \sigma_{k-1,k-1} \end{pmatrix} \begin{pmatrix} \omega_1^{(k)} \\ \omega_2^{(k)} \\ \vdots \\ \omega_{k-1}^{(k)} \end{pmatrix} = \begin{pmatrix} \sigma_{k,1} \\ \sigma_{k,2} \\ \vdots \\ \sigma_{k,k-1} \end{pmatrix}, \ \gamma_k = \sigma_{kk}^{-1} \sum_{i=1}^{k} \alpha_i \omega_i \ \text{ and } \ dU_k(t) = -U_k(t)dt + \sigma_{kk} dW_t.$$

The Ito-formula applied to Y_k leads to a RDE of the form

$$Y'(t) = F(t, Y(t), U(t)), \text{ with } Y(t) = (Y_1(t), \dots, Y_d(t))^\intercal \text{ and } U(t) = (U_1(t), \dots, U_d(t))^\intercal,$$

which can be studied pathwise without the need of stochastic calculus for its treatment. This allows to construct explicit numerical integrators of the form $\hat{Y}_{n+1} = \hat{Y}_n + \phi\left(t_n, \hat{Y}_n, U(t_n), h\right)$ (where h is the stepsize) for this RDE, that do not need to satisfy such a restrictive assumptions as in the case of SDEs. Then, from (2), we readily get the numerical integrator $\left(\hat{X}_n\right)_{n=1,\dots}$ for (1) defined by

$$\begin{pmatrix} \omega_1^{(1)} & 0 & \cdots & 0 \\ \omega_1^{(2)} & \omega_2^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^{(d)} & \omega_2^{(d)} & \cdots & \omega_d^{(d)} \end{pmatrix} \hat{X}_{n+1} = \begin{pmatrix} e^{U_1(t_n)} \left(\hat{Y}_{n+1}^{(1)} - \gamma_1 \right) \\ e^{U_2(t_n)} \left(\hat{Y}_{n+1}^{(2)} - \gamma_2 \right) \\ \vdots \\ e^{U_d(t_n)} \left(\hat{Y}_{n+1}^{(d)} - \gamma_d \right) \end{pmatrix}.$$

We have the following theorem concerning the convergence of the method.

Theorem 2.1. *If the numerical integrator $\left(\hat{Y}_n\right)_{n=1,\dots}$ is stable and pathwise convergent with rate of convergence p, and $U_k(t_0)$ is selected $\sim N(0, \frac{\sigma_{kk}^2}{2})$, then the corresponding numerical integrator $\left(\hat{X}_n\right)_{n=1,\dots}$ for (1) is stable and pathwise convergent with the same order p.*

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A LAGRANGIAN-EULERIAN APPROACH FOR COUPLING HYPERBOLIC-TRANSPORT AND ELLIPTIC DARCY-PRESSURE-VELOCITY EQUATIONS IN HIGH-CONTRAST POROUS MEDIA FLOW

JEAN FRANÇOIS[1], PAOLA FERRAZ[2] & EDUARDO ABREU[3]

FECLI, UECE, CE, Brasil[1], CNPEM, SP, Brasil[2], IMECC, UNICAMP, SP, Brasil[3]

jean.francoisuece.br† paola.ferraz@lnls.br‡ eabreu@ime.unicamp.br§

Abstract

A genuinely multidimensional Semi-Discrete Lagrangian-Eulerian scheme for solving initial value problems for scalar models and systems of conservation laws [2], based on the concept of no-flow curves [1] is presented. The scheme is positivity preserving and Riemann solver free. The rigorous numerical analysis is carried out in [2]. We provide numerical examples considering coupling of two-phase and three-phase fluid flow problems with discontinuous porous media for verifying the theory and illustrating the capabilities of the approach being presented.

1 Semi-Discrete Lagrangian-Eulerian formulation

The semi-discrete Lagrangian-Eulerian scheme construction is started considering the 1D scalar problem

$$\frac{\partial u}{\partial t}+\frac{\partial H(u)}{\partial x}=0, \quad x\in\mathbb{R}, \quad t>0, \qquad u(x,0)=u_0(x), \quad u_0(x)\in L^{\infty}(R) \quad \text{where } H\in C^2(\Omega),\ H:\Omega\to\mathbb{R}, \tag{1}$$

and $u=u(x,t):\mathbb{R}\times\mathbb{R}^{+}\longrightarrow\ \Omega\subset\mathbb{R}$. Following [1, 2], we obtain the fully discrete Lagrangian–Eulerian scheme,

$$u_j^{n+1}=u_j^n-\frac{\Delta t}{\Delta x}\left[F\left(u_j^n,u_{j+1}^n\right)-F\left(u_{j-1}^n,u_j^n\right)\right], \textit{ with a numerical flux function given by } F\left(u_j^n,u_{j+1}^n\right)= \tag{2}$$

$$\frac{1}{4}\left[\frac{\Delta x}{\Delta t}\left(u_j^n-u_{j+1}^n\right)+\Delta x\frac{f_j^n+f_{j+1}^n}{\Delta x_j}\left(u_j^n+u_{j+1}^n\right)+\frac{\Delta x^2}{4}\frac{f_j^n+f_{j+1}^n}{\Delta x_j}\left((u_x)_j^n-(u_x)_{j+1}^n\right)+\frac{\Delta x^2}{4\Delta t}\left((u_x)_j^n+(u_x)_{j+1}^n\right)\right]. \tag{3}$$

By the aid of the *no-flow property* $\left[\frac{\Delta x}{\Delta t}\right]\propto[O(H(u)/u)]$, (see [1] and u and $H(u)$ given by (1)), we can remove the blow-up singularity of the numerical flux $F(u_j^n,u_{j+1}^n)$ in (2)–(3) replacing $\frac{\Delta x}{\Delta t}$ with a stability condition that depends on $O((H(u)/u))$, which allows us to have $\Delta t\to 0^{+}$ and produce an accurate approximation of the local speeds. This *no-flow property* allows to obtain a suitable function

$$b_{j+\frac{1}{2}}=b_{j+\frac{1}{2}}\left(f_j,f_{j+1}\right), \quad f_j\equiv\frac{H(u_j)}{u_j}\approx\frac{H(u)}{u} \quad \text{for each } j\in\mathbb{Z}\ \textit{per time step} \quad [t^n,t^{n+1}]. \tag{4}$$

Thus, the new class of SDLE schemes for hyperbolic-transport initial value problems (1) is given by $\left[\frac{d}{dt}u_j(t)=-\frac{1}{\Delta x}\left[\mathcal{F}\left(u_j,u_{j+1}\right)-\mathcal{F}\left(u_{j-1},u_j\right)\right], \quad \mathcal{F}\left(u_j,u_{j+1}\right)=\frac{1}{4}\left[b_{j+\frac{1}{2}}\left(u^-_{j+\frac{1}{2}}-u^+_{j+\frac{1}{2}}\right)+\left(f_j+f_{j+1}\right)\left(u^-_{j+\frac{1}{2}}+u^+_{j+\frac{1}{2}}\right)\right],\right]$ where $\lim_{\Delta t\longrightarrow 0}\mathcal{F}\left(u_j^n,u_{j+1}^n\right)\neq\infty$, with $u^-_{j+\frac{1}{2}}=u_j+\frac{\Delta x}{4}\left((u_x)_j\right)$ and $u^+_{j+\frac{1}{2}}=u_{j+1}-\frac{\Delta x}{4}\left((u_x)_{j+1}\right)$. The formal 2D extension of the semi-discrete scheme is straightforward and given by

$$\frac{d}{dt}u_{j,k}(t)=-\frac{\mathcal{F}_{j+1/2,k}-\mathcal{F}_{j-1/2,k}}{\Delta x}-\frac{\mathcal{G}_{j,k+1/2}-\mathcal{G}_{j,k-1/2}}{\Delta y}, \quad \text{for the following scalar conservation law} \tag{5}$$

$$u_t + H(u)_x + G(u)_y = 0, \quad u(x,y,0) = u_0(x,y), \quad \text{where} \quad H, G, \in C^2, u_0(x,y) \in L^\infty_{loc}(\mathbb{R}^2). \tag{6}$$

The corresponding multidimensional numerical fluxes in the $x-$ and $y-$directions are, respectively, given by

$$\begin{aligned}\mathcal{F}_{j+\frac{1}{2},k} =& \frac{1}{4}\left[b^x_{j+\frac{1}{2},k}\left(u^-_{j+\frac{1}{2},k} - u^+_{j+\frac{1}{2},k}\right) + (f_{j,k} + f_{j+1,k})\left(u^-_{j+\frac{1}{2},k} + u^+_{j+\frac{1}{2},k}\right)\right] \quad \text{and} \\ \mathcal{G}_{j,k+\frac{1}{2}} =& \frac{1}{4}\left[b^y_{j,k+\frac{1}{2}}\left(u^-_{j,k+\frac{1}{2}} - u^+_{j,k+\frac{1}{2}}\right) + (g_{j,k} + g_{j,k+1})\left(u^-_{j,k+\frac{1}{2}} + u^+_{j,k+\frac{1}{2}}\right)\right],\end{aligned} \tag{7}$$

where the discretized multi-D (2D) space-time no-flow curves [1], given by (u, $H(u)$, $G(u)$ as defined in (6))

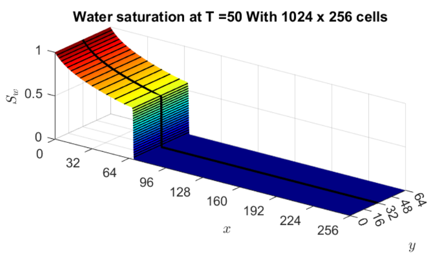
$$f_{j,k} = \frac{H(u_{jk})}{u_{jk}} \quad \text{and} \quad g_{j,k} = \frac{G(u_{jk})}{u_{jk}}, \quad \text{with} \quad \left[\frac{\Delta x}{\Delta t}\right] \propto [O(H(u)/u)] \quad \text{and} \quad \left[\frac{\Delta y}{\Delta t}\right] \propto [O(G(u)/u)]. \tag{8}$$

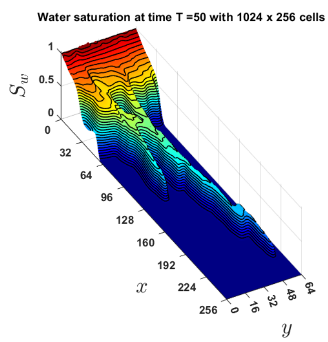
More details can be seen at [2]. An immiscible and incompressible displacement two-phase flow model of water (w) and oil (o) in heterogeneous porous media coupled with the hyperbolic conservation law is given by

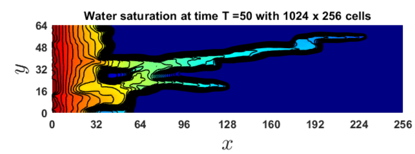
$$\nabla \cdot \mathbf{v} = 0, \quad \mathbf{v} = -\lambda(S)\mathbf{K}(\mathbf{x})\nabla \cdot p, \quad \mathbf{x} \in \Omega, \quad p \text{ the pressure}, \tag{9}$$

$$\frac{\partial S}{\partial t} + \nabla \cdot (H(S)\mathbf{v}) = 0, \quad \mathbf{x} \in \Omega, \quad t > 0, \text{ or } \quad \frac{\partial S}{\partial t} + \frac{\partial}{\partial x}(v_x H(S)) + \frac{\partial}{\partial y}(v_y H(S)) = 0, \tag{10}$$

with no effects of capillarity and gravity. Here $\mathbf{v}$ is the total seepage velocity, $S = S_w$ is the water saturation and $S_o = 1 - S_w$ the oil saturation. The scalar hyperbolic-transport model (10) is handled by our semi-discrete scheme with the water fractional flow function $H(S) = \dfrac{k_{rw}(S)/\mu_w}{\lambda(S)}$ linked to the prescribed initial data, $S(x,y,0) =: S_0(x,y) = 1$ if $x \leq 0$ and $S_0(x,y) = 0$. We consider quadratic relative permeability curves $k_{rw}(S) = S^2$ and $k_{ro}(S) = (1-S)^2$ with $\mu_w = 1$, $\mu_o = 2$ and the total mobility $\lambda(S)$ as $\lambda(S) = \dfrac{k_{rw}(S)}{\mu_w} + \dfrac{k_{ro}(S)}{\mu_o}$. The elliptic-pressure-velocity model (9) is treated by the hybrid mixed finite element discretization approach on lower index $H(div)$ Raviart-Thomas spaces [3] where the permeability fields set initially to be layers of the 3D SPE10 field with boundary conditions set as a zero pressure in the boundary $y = Y$ and pressure equal to one in $y = 0$. The absolute permeability $\mathbf{K}(\mathbf{x}) \equiv 1$ in homogeneous case.







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A LAGRANGIAN-EULERIAN APPROACH FOR COUPLING HYPERBOLIC-TRANSPORT AND ELLIPTIC DARCY-PRESSURE-VELOCITY EQUATIONS IN HIGH-CONTRAST POROUS MEDIA FLOW

SÔNIA M. GOMES.[1], KAROLINNE O. COELHO[2] & PHILIPPE R. B. DEVLOO[2]

IMECC, Unicamp, SP, Brasil[1], LabMeC-FECFAU, Unicamp, SP, Brasil[2]

soniagunicamp.br[†] karolinneoc@gmail.com[‡] phil@unicamp.br[§]

Abstract

The Scaled Boundary Finite Element Method (SBFEM) stands out for a Galerkin method where the approximation spaces are constructed using a semi-analytical approach. They are based on partitions of the computational domain by polygonal/polyhedral subregions (called S-elements), where the shape functions approximate local Dirichlet problems with piecewise polynomial trace data. The design of SBFEM was motivated by the numerical treatment of mechanical problems with singularities and non-smooth solutions, such as in the presence of re-entrant corners, cracks, V-notches, and free edges formed by dissimilar materials, for which poor performance occurs by classical Finite Element Methods. In the absence of body loads, the application of SBFEM to this class of problems has shown optimal rates of convergence, due to its ability to mimic the properties of the analytical solution. We extend the SBFEM applicability to non-homogeneous problems, by considering enriched spaces by some properly chosen energy-orthogonal bubble functions. We prove optimal convergence rates, which are verified by implemented versions of different 2D and 3D SBFEM geometry, using trianglular, tetrahedral, and pyramidal subpartitions of the S-elements.

1 Introduction

SBFEM approximations are based on partitions of the domain Ω into general polygonal/polyhedral subregions S, also called S-elements. These elements need to obey the star-shaped requirement, which means that any point at ∂S is directly visible from a center point inside S (named scaling center). Based on such element geometry, the local SBFEM approximation spaces in S for homogenous problems are composed of two components [1]:

1. A trace finite element (FE) discretization over ∂S.

2. A radial extension into S by expansions in terms of eigenfunctions and eigenvalues, obtained by an approach similar to the separation of variables.

Therefore, polynomial approximations are only adopted by SBFEM at ∂S, whilst inside S the fields are constructed by numerically solving Dirichlet local versions of the model problem.

The incorporation of analytic knowledge about the local behavior of the exact solution in the approximation spaces is the main property of SBFEM. For instance, when applying SBFEM to Poisson's equation, a semi-analytical approximation of Laplace's equation is computed in the interior of the S domain by approximate radial harmonic extensions of surface components. Because of that, SBFEM is considered a semi-analytical approach. In fact, as pointed out in [2], SBFEM can also be viewed as an operator adapted method, following principles closely related with the Partition of Unity Method or with the Virtual Element Method, according to [3] and [4], respectively, or to citations therein enclosed. On the other hand, the requirement of incorporating polynomial approximations in SBFEM spaces only at the level of the element skeleton leads to a more flexible mesh generation. It means that the meshes can be partitioned into S-elements with an arbitrary number of sides. Polygonal surface meshes, quadtrees surface meshes and octrees surface meshes can be used naturally.

1.1 The main contributions

SBFEM has been successfully applied to singular mechanical problems with vanishing force terms. In this contribution, we propose and analyze a novel procedure to approximate non-homogeneous systems of partial differential equations using a SBFEM-bubble function extension, whose main aspects are:

1. The SBFEM spaces are described in the context of composite Duffy approximation spaces in S allowing to explore the advantage of an extended energy-orthogonality relation. The orthogonality properties were initially proven in [2] for harmonic approximations. They are extended here for homogeneous elasticity problems, where local SBFEM spaces are orthogonal to a generic bubble approximation space of Duffy type in each S-element.

2. This energy-orthogonal property suggests the employment of orthogonal bubble functions to enrich the SBFEM formulation of non-homogeneous problems: some properly chosen bubble functions are added to the scaled boundary shape functions to recover convergence rates of SBFEM approximations when applied to problems with non-vanishing source terms.

3. Optimal convergence rates are demonstrated when using SBFEM for approximations of non-homogeneous Poisson and elasticity in 2D and 3D problems.

4. The decomposition of the formulation into boundary and bubble contributions allows the partition of the resulting SBFEM stiffness matrix into two uncoupled submatrices associated with the boundary and bubble functions, reducing the computational cost.

5. The bubbles are simple linear combination of Duffy shape functions. The integration of stiffness terms associated with the bubble functions can also be done semi-analytically, similarly to the stiffness matrix associated with the boundary shape functions.

6. High-order simulations for 2D meshes and different mesh configurations for 3D problems were performed using different subpartitions for the S-elements, such as triangles, tetrahedrons, and pyramids. Verification experiments illustrate numerically the expected convergence rates for examples of the model Poisson and elasticity problems in 2D and 3D.

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FUCHSIAN DIFFERENTIAL EQUATIONS AND HYPERBOLIC GEOMETRY IN THE $C_{2,8}$ CHANNEL QUANTIZATION

ANDERSON J. DE OLIVEIRA[1,†2]

Federal University of Alfenas - UNIFAL-MG, Alfenas - MG, Brazil[1]

Work developed by the Quantum and Hyperbolic Codes study group: Cátia Regina de Oliveira Quilles Queiroz - UNIFAL-MG, Giuliano Gadioli La Guardia - UEPG, Clarice Dias de Albuquerque - UFCA, Leandro Bezerra de Lima - UFMS, Vandenberg Lopes Vieira - UEPB, Reginaldo Palazzo Jr. - UNICAMP.

anderson.oliveira@unifal-mg.edu.br[†].

Abstract

It will be considered the steps to be followed in the analysis and interpretation of the quantization problem related to the $C_{2,8}$ channel, where the Fuchsian differential equations and the generators of the Fuchsian groups associated with the specific case $g = 3$ are presented. In order to obtain these results, it is necessary to determine the genus g of each surface which this channel may be embedded. After that, the procedure is to determine the algebraic structure (Fuchsian group generators) associated with the fundamental region of each surface. For this, there exists an associated linear second order Fuchsian differential equation whose linearly independent solutions provide the generators of this Fuchsian group. The aim of this work is to present a specific case, related to the genus $g = 3$, due some specificities, by using a second order Fuchsian differential equation and some structures of the hiperbolic geometry.

1 Introduction

The goals of designing a new digital communication system are to construct more reliable and less complex systems than previously known ones, [1].

Massey [4] has shown that under the error probability criterion, the performance of a binary digital communication system using soft-decision in the demodulator, for instance, an 8-level quantizer, leading to a binary input, 8-ary output symmetric channel, denoted by $C_{2,8}$, achieves a gain of up to 2 dB when compared to the performance of a binary digital communication system using hard-decision (a 2-level quantizer, leading to a binary symmetric channel (BSC), denoted by $C_{2,2}$).

One way to carry out these analyzes is to utilize an important type of ordinary differential equations, in the complex plane, the so-called Fuchsian differential equations and the Fuchsian group generators, by using the hyperbolic geometry.

According to [3], Fuchsian differential equations represent an important class of linear ordinary differential equations whose main characteristic is that every singular point in the extended complex plane is regular. These differential equations are widely used in mathematical physics problems. The most studied cases involve equations with three regular singular points, such as the hypergeometric, Legendre, and Tchebychev equations.

The genus g of a compact orientable surface $\mathbb{M}$ is determined by the number of handles connected to a sphere or the number of "holes" in $\mathbb{M}$. As the Euler characteristic of a surface, the genus is a topological invariant. We can identify surfaces with genus $g = 1$ with the torus, and the geometry associated with these surfaces is Euclidean geometry. For surfaces of genus $g \geq 2$, g-tori, the geometry to be considered is the hyperbolic geometry, [2].

In [5], relevant connections are made among Fuchsian differential equations, Riemann surfaces and Fuchsian groups, in order to analyze the uniformizing process of algebraic curves of the form $y^2 = z^{2g+1} \pm 1$.

The aim of this work is to present the Fuchsian groups generators by the polygon side-pairings, related to the cases $g = 3$, in order to analyze the channel $C_{2,8}$ quantization problem, which can be embedded in surfaces of genus $0 \leq g \leq 3$, through a second order Fuchsian differential equation.

2 Main Results

Let us consider the hyperelliptic curves with genus 3, given by $y^2 = z^7 - 1$. There is a bijective correspondence between the set of solutions of $z^7 - 1$ and the values -3, -2, -1, 0, 1, 2 and 3, since the roots of the unit divide the circumference into seven equal parts. The associated Fuchsian differential equation and the generators of the Fuchsian group will be presented in the sequence, in order to show the characterization of the channel quantization process, related to the case $g = 3$. The Fuchsian differential equation is given by:

$$(z^7 - 14z^5 + 49z^3 - 36z)y'' + \left[(z^7 - 14z^5 + 49z^3 - 36z)\left(\frac{2}{z+1} + k_1\right)\right] y' + \left[(z^7 - 14z^5 + 49z^3 - 36z).k_2]y\right] = 0, \quad k_1, k_2 \in \mathbb{C}. \tag{1}$$

The linearly independent solutions of Eq. (1) result in elliptic transformations of the form:

$$S_i(t) = \frac{a_i t + b_i}{c_i t + d_i}, \quad \text{with} \quad |a_i + d_i| = 0, \quad \text{for all} \quad 1 \leq i \leq 7. \tag{2}$$

By fixing one of these transformations and multiplying it by the remaining ones, the generators of the Fuchsian group (Γ_{4g} or Γ_{4g+2}) are determined. The generators associated with the analyzed case are presented below; they are given by S_1S_2, S_1S_3, S_1S_4, S_1S_5, S_1S_6 and S_1S_7:

$$S_1S_2 = \begin{pmatrix} 1.400969 + 0.3197621i & 1.0060962 - 0.2296349i \\ 1.0060962 + 0.2296349i & 1.400969 - 0.3197621i \end{pmatrix}, \quad S_1S_3 = \begin{pmatrix} 1.62349 - 0.1423075i & 1.0060962 + 0.802335i \\ 1.0060962 - 0.802335i & 1.62349 + 0.1423075i \end{pmatrix},$$

$$S_1S_4 = \begin{pmatrix} 1.1234898 - 0.2564293i & -2.220D - 16 + 0.5727001i \\ -2.220D - 16 - 0.5727001i & 1.1234898 + 0.2564293i \end{pmatrix},$$

$$S_1S_5 = \begin{pmatrix} 1.1234898 + 0.2564293i & 0.447755 - 0.3570727i \\ 0.447755 + 0.3570727i & 1.1234898 - 0.2564293i \end{pmatrix},$$

$$S_1S_6 = \begin{pmatrix} 1.62349 + 0.1423075i & 1.2545815 + 0.28635i \\ 1.2545815 - 0.28635i & 1.62349 - 0.1423075i \end{pmatrix}, \quad S_1S_7 = \begin{pmatrix} 1.400969 - 0.3197621i & 0.447755 + 0.9297727i \\ 0.447755 - 0.9297727i & 1.400969 + 0.3197621i \end{pmatrix}.$$

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LYAPUNOV TECHNIQUES FOR INTEGRAL EQUATIONS IN THE SENSE OF KURZWEIL.

EDUARD TOON[1,†] & FERNANDA ANDRADE DA SILVA[2,‡]

Departamento de Matemática, UFJF, MG, Brasil[1], ICMC, USP, SP, Brasil[2]

eduard.toon@ufjf.br[†] ffeandrade@gmail.com[‡]

Abstract

The theory of generalized ordinary differential equations (generalized ODEs, for short) has been shown to be a very powerful theory once several types of equations can be regarded as them. In the present lecture, we introduce a new concept of stability, called decreasing stability, and deal with some Lyapunov techniques on decreasing and exponential stability.

1 Introduction

In recent years, there has been an increasing interest in the theory of generalized ordinary differential equations, shortly known as generalized ODEs. This is due to the fact that generalized ODEs encompass other types of equations, as for instance, to mention the least, measure functional differential equations, dynamic equations on time scales and integral equations. Generalized ODEs are based on the non-absolute integration theory due to Jaroslav Kurzweil and Ralph Henstock. The Kurzweil-Henstock integral is known to handle well not only many discontinuities but also highly oscillating functions. The well-known real-valued Kurzweil-Henstock integral is equivalent to the Perron and to the restricted Denjoy integrals which, on the other hand, encompass the Newton, Riemann and Lebesgue integrals as well as their improper integrals. In the framework of generalized ODEs, an "integral equation" appears containing an integral in Kurzweil's sense. This enables one to deal with integrands which are of unbounded variation, for instance. Moreover, Stieltjes-type integrals, often used to describe differential equations involving measures can be handled naturally, once the Kurzweil-Henstock integral contains the Perron-Stieltjes integral. This is why measure functional differential equations are clearly a particular case of generalized ODEs.

Motivated by these features, we are interested here in in investigating stability criteria for generalized ODEs. Consider the generalized ODE

$$\frac{dx}{d\tau} = DF(x,t), \tag{1}$$

where $F : X \times [t_0, +\infty) \to X$ belongs to an appropriate class of functions, $t_0 \geq 0$ and X is Banach space. We assume that the generalized ODE (1) admits the trivial solution. At first, we recall the concept of exponential stability presented in [2] and we introduce a new concept of stability, called decreasing stability, which generalizes exponential stability.

Definition 1.1. *Let $s_0 \geq t_0 \geq 0$, $x_0 \in X$ and $x : [s_0, +\infty) \to X$ be the global forward solution of the generalized ODE (1) with initial condition $x(s_0) = x_0$. The trivial solution of the generalized ODE (1) is called*

1. *exponentially stable, if there exist positive constants ρ, α, β such that*

$$\|x(t)\| = \|x(t, s_0, x_0)\| < \alpha e^{-\beta(t-s_0)}, \quad \textit{for all } t \in [s_0, +\infty),$$

whenever $\|x_0\| < \rho$;

2. *decreasingly stable, if there exist $\delta > 0$ and a decreasing function $\sigma : [0, +\infty) \to \mathbb{R}^+$ such that $\sigma(0) < \infty$ and, if $\|x_0\| < \delta$, then $\|x(t)\| = \|x(t, s_0, x_0)\| < \sigma(t - s_0)$ for all $t \in [s_0, +\infty)$.*

2 Main Results

In this section, we present our main results concerning decreasing stability. The first result is a direct Lyapunov theorem.

Theorem 2.1. *If there exists a functional* $V : [t_0, +\infty) \times X \to \mathbb{R}$ *such that*

1. $V(t, z) \geq 0$ *for every* $(t, z) \in [t_0, +\infty) \times X$*;*

2. *the mapping* $[s_0, +\infty) \ni t \mapsto V(t, x(t))$ *is nonincreasing along every solution* $x : [s_0, +\infty) \to X$ *of the generalized ODE* (1)*;*

3. *there exists a positive constant* γ *such that* $\gamma\|z\| \leq V(t, z)$ *for all* $(t, z) \in [t_0, +\infty) \times X$*;*

then the trivial solution of the generalized ODE (1) *is decreasingly stable.*

The second main result is a converse Lyapunov theorem for decreasing stability.

Theorem 2.2. *If the trivial solution of the generalized ODE* (1) *is decreasingly stable, then there exist* $\delta > 0$ *and a functional* $V : [t_0, +\infty) \times B_\delta \to \mathbb{R}^+$, $B_\delta = \{x \in X; \|x\| < \delta\}$, *such that*

1. $V(t, y) \geq 0$ *for all* $(t, y) \in [t_0, +\infty) \times B_\delta$*;*

2. $V(\cdot, y) : [t_0, +\infty) \to \mathbb{R}^+$ *is left-continuous on* $(t_0, +\infty)$ *for all* $y \in B_\delta$*;*

3. *there exists a monotonically increasing and continuous function* $a : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\|y\| \leq V(t, y) \leq a(\|y\|)$ *for all* $(t, y) \in [t_0, +\infty) \times B_\delta$*;*

4. *the mapping* $[s_0, +\infty) \ni t \mapsto V(t, x(t))$ *is nonincreasing along every solution* $x : [s_0, +\infty) \to B_\delta$ *of the generalized ODE* (1)*.*

This lecture is mainly based on the submitted paper [1].

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ATTRACTORS ON IMPULSIVE DYNAMICAL SYSTEMS.

EVERALDO M. BONOTTO[1,†]

Instituto de Ciências Matemáticas e de Computação, ICMC/USP, SP, Brasil[1].

ebonotto@icmc.usp.br[†]

Abstract

A continuous dynamical system can be related to an impulsive dynamical system and to a discrete dynamical system. In this work, we present a relationship among the attractors of these systems.

1 Introduction

Let (X, d) be a metric space. A *semidynamical system* on X, denoted by (X, π), is a family of maps $\{\pi(t)\colon t \in \mathbb{R}_+\}$ acting from X to X such that $\pi(0) = I$ ($I\colon X \to X$ is the identity operator), $\pi(t+s) = \pi(t)\pi(s)$ for all $t, s \in \mathbb{R}_+$, and $\mathbb{R}_+ \times X \ni (t, x) \mapsto \pi(t)x \in X$ is a continuous map.

A nonempty closed subset $M \subset X$ is called an *impulsive set* if for every $x \in M$ there exists $\epsilon_x > 0$ such that

$$\bigcup_{t \in (0, \epsilon_x)} \{\pi(t)x\} \cap M = \emptyset.$$

Let (X, π) be a semidynamical system and M be an impulsive set. The *impact function* $\phi\colon X \to (0, \infty]$ is given by

$$\phi(x) = \begin{cases} s, & \text{if} \quad \pi(s)x \in M \quad \text{and} \quad \pi(t)x \notin M \quad \text{for} \quad 0 < t < s, \\ \infty, & \text{if} \quad \pi(t)x \notin M \quad \text{for all} \quad t > 0. \end{cases}$$

Note that, if $\phi(x) < \infty$ then this value represents the smallest positive time such that the trajectory of x meets M.

An *impulsive dynamical system* (X, π, M, I) consists of a semidynamical system (X, π), an impulsive set $M \subset X$ and a continuous function $I\colon M \to X$ called impulsive function. The *impulsive positive trajectory* of a point $x \in X$ in (X, π, M, I) is represented by a map $\tilde{\pi}(\cdot)x\colon J_x \to X$ defined on some interval $J_x \subseteq \mathbb{R}_+$ containing 0, given inductively by the following rule: if $\phi(x) = \infty$ then $\tilde{\pi}(t)x = \pi(t)x$ for all $t \in \mathbb{R}_+$. But, if $\phi(x) < \infty$, then we set $x = x_0^+$ and we define $\tilde{\pi}(\cdot)x$ on $[0, \phi(x_0^+)]$ by

$$\tilde{\pi}(t)x = \begin{cases} \pi(t)x_0^+, & \text{if} \quad 0 \leqslant t < \phi(x_0^+), \\ I(\pi(\phi(x_0^+))x_0^+), & \text{if} \quad t = \phi(x_0^+). \end{cases}$$

Now, we write $s_0 = \phi(x_0^+)$, $x_1 = \pi(s_0)x_0^+$ and $x_1^+ = I(\pi(s_0)x_0^+)$. Since $s_0 < \infty$, the previous process can go on, but now starting at x_1^+. If $\phi(x_1^+) = \infty$ then we define $\tilde{\pi}(t)x = \pi(t - s_0)x_1^+$ for all $t \geq s_0$. But, if $s_1 = \phi(x_1^+) < \infty$ i.e., $x_2 = \pi(s_1)x_1^+ \in M$ then we define $\tilde{\pi}(\cdot)x$ on $[s_0, s_0 + s_1]$ by

$$\tilde{\pi}(t)x = \begin{cases} \pi(t - s_0)x_1^+, & \text{if} \quad s_0 \leqslant t < s_0 + s_1, \\ I(x_2), & \text{if} \quad t = s_0 + s_1. \end{cases}$$

Here, denote $x_2^+ = I(x_2)$. This process ends after a finite number of steps if $\phi(x_n^+) = \infty$ for some $n \in \mathbb{N}$, or it may proceed indefinitely, if $\phi(x_n^+) < \infty$ for all $n \in \mathbb{N}$ and, in this case, $\tilde{\pi}(\cdot)x$ is defined in the interval $[0, T(x))$, where $T(x) = \sum_{i=0}^{\infty} s_i$ can be finite or infinite. From now on, we shall assume that $T(x) = \infty$ for all $x \in X$. See [1, 2, 3, 4].

In the next definition, d_H denotes the Hausdorff semidistance between two nonempty sets.

Definition 1.1. *A nonempty set $\tilde{\mathcal{A}} \subset X$ is called a global attractor for (X, π, M, I) if $\tilde{\mathcal{A}}$ is pre-compact and $\tilde{\mathcal{A}} = \overline{\tilde{\mathcal{A}}} \backslash M$, $\tilde{\mathcal{A}}$ is $\tilde{\pi}$-invariant ($\tilde{\pi}(t)\tilde{\mathcal{A}} = \tilde{\mathcal{A}}$ for all $t \in \mathbb{R}_+$), and $d_H(\tilde{\pi}(t)B, \tilde{\mathcal{A}}) \overset{t\to\infty}{\longrightarrow} 0$ for every bounded set $B \subset X$.*

If $M = \emptyset$ in Definition 1.1, then $\tilde{\mathcal{A}}$ is the global attractor of the continuous semidynamical system (X, π), which will be denoted by $\mathcal{A}$.

Let $\hat{X} = \{x \in I(M) \colon \phi(x_k^+) < \infty \text{ for all } k \in \mathbb{N}\}$ and assume that $\hat{X}$ is nonempty. The map $g \colon \hat{X} \to \hat{X}$ given by

$$g(x) = I(\pi(\phi(x))x),$$

maps $\hat{X}$ to $\hat{X}$ and, hence, defines a discrete dynamical system on $\hat{X}$, represented by $(\hat{X}, g)$, which is associated with the impulsive dynamical system (X, π, M, I). Note that $g^0(x) = x$ and $g^n(x) = x_n^+$ for all $x \in \hat{X}$ and $n \in \mathbb{N}$. Consequently, $g(x_n^+) = x_{n+1}^+$ for all $x \in \hat{X}$ and $n \in \mathbb{N}$.

Definition 1.2. *A nonempty set $\hat{\mathcal{A}} \subset \hat{X}$ is called a discrete global attractor for $(\hat{X}, g)$ if $\hat{\mathcal{A}}$ is compact, $\hat{\mathcal{A}}$ is g-invariant ($g(\hat{\mathcal{A}}) = \hat{\mathcal{A}}$), and $d_H(g^n(\hat{B}), \hat{\mathcal{A}}) \overset{n\to\infty}{\longrightarrow} 0$ for every bounded set $\hat{B} \subset \hat{X}$.*

Condition (T): If $x \in M$, $\{z_n\}_{n\in\mathbb{N}} \subset X$ is a sequence that converges to z and $t > 0$ are such that $\pi(t)z_n \overset{n\to\infty}{\longrightarrow} x$, then there exist a subsequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ and a sequence $\{\alpha_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$, $\alpha_k \overset{k\to\infty}{\longrightarrow} 0$, such that $t + \alpha_k \geq 0$ and $\pi(t + \alpha_k)z_{n_k} \in M$.

Condition (T) plays an important role in obtaining a well-behaved evolution of an impulsive dynamical system, see [1] for more details.

2 Main Results

Let $\mathcal{A}$ be the global attractor of (X, π) and $\hat{\mathcal{A}}$ be the discrete global attractor of $(\hat{X}, g)$. In Theorem 2.1, we exhibit sufficient conditions for the existence of the global attractor $\tilde{\mathcal{A}}$ of (X, π, M, I). By $\mathcal{B}(X)$ we mean the set of all bounded subsets from X.

Theorem 2.1. *Assume that (X, π) admits a global attractor $\mathcal{A}$ with $\mathcal{A} \cap M = \emptyset$, $(\hat{X}, g)$ has a global attractor $\hat{\mathcal{A}}$, (X, π, M, I) is dissipative (i.e., there exists a set $B_0 \in \mathcal{B}(X)$, such that for every $B \in \mathcal{B}(X)$ there exists a time $T_B \geq 0$ such that $\tilde{\pi}(t)B \subset B_0$ for all $t \geq T_B$) and $\phi(x) < \infty$ for all $x \in I(M)$. If $I(M) \cap M = \emptyset$, there exists $\xi > 0$ such that $\phi(x) \geq \xi$ for every $x \in I(M)$, and condition (T) holds, then (X, π, M, I) admits a global attractor $\tilde{\mathcal{A}}$ given by*

$$\tilde{\mathcal{A}} = \mathcal{A} \cup \left(\bigcup_{a\in\hat{\mathcal{A}}} \pi([0, \phi(a)))a \right).$$

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STABILITY FOR GENERALIZED STOCHASTIC EQUATIONS

EVERALDO M. BONOTTO.[1], FERNANDA ANDRADE DA SILVA[1] & MÁRCIA FEDERSON[1]

Instituto de Ciências Matemáticas e de Computação, USP, SP, Brasil[1]

ebonotto@icmc.usp.br† ffeandrade@usp.br‡ federson@icmc.usp.br§

Abstract

Similarly to generalized ODEs that comprise various types of classical deterministic equations, generalized stochastic equations (GSEs) were created to contain equations involving stochastic processes. The main goal of this work is to investigate several types of stability and boundedness for non-autonomous GSEs by means of Lyapunov functionals. We also established existence-uniqueness results for global forward solutions of GSEs.

1 Introduction

Generalized ODEs are based on the non-absolute integration theory due to Jaroslav Kurzweil and Ralph Henstock and are known to cover many types of equations, as for instance, measure functional differential equations, dynamic equations on time scales and integral equations. By considering belated partial divisions in the classic Kurzweil integral, the authors of [1] introduced a new integral which contains the Itô-Henstock integral for functions taking values in spaces of Hilbert-Schmidt operators. Furthermore, they defined a new class of equations, called generalized stochastic equations (GSEs), in such a way that classic stochastic differential equations fall into special cases of GSEs.

It is well-known that stability conditions for solutions of differential equations can be obtained using an appropriate Lyapunov functional. Moreover, the construction of different Lyapunov functionals allows obtaining different stability conditions and the reciprocal is true. Recently, it has been an increasing investigation into these types of results for generalized ODEs. In the framework of stochastic differential equations we can mention the works of V. Kolmanovskii and L. Shaikhet.

In the present paper, we are interested in establishing a stability theory for GSEs by means of Lyapunov functionals. To this end, we discuss when the trivial solution of a GSE is p-stable, asymptotically p-stable, exponentially p-stable and stochastically stable (or stable in probability). Our main goal is to prove Lyapunov-type Theorems involving all these concepts.

2 Main Results

Definition 2.1. *Let $I \subset \mathbb{R}$ be a non-degenerate interval, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in I}, \mathbb{P})$ be a filtering probability space and $F\colon L^p(\Omega, V) \times J \to \mathfrak{F}(\Omega, V)$ be an operator, where $J \subset I$ is a subinterval. A $\{\mathcal{F}_t\}$-adapted process $X = \{X_t : t \in J\}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in I}, \mathbb{P})$, with $X_t \in L^p(\Omega, V)$, for all $t \in J$, is a solution of the GSE*

$$X_t = X_s + \int_s^t F(X_r, \tau), \quad t, s \in J, \tag{1}$$

on J, whenever $X_t(\omega) \in V$ for every $t \in J$ and $\mathbb{P}$-almost every $\omega \in \Omega$ and the integral equation (1) holds, where the integral is in the sense of the Kurzweil-belated integral with $G(r, \tau) = F(X_r, \tau)$. When I is unbounded and $J = [s_0, +\infty) \subset I$, we say that X is a global forward solution with initial condition X_{s_0} and, if $X_t \equiv 0$ for all $t \in J$, then X is called the trivial solution.

The next result deals with the existence and uniqueness of global forward solutions of the GSE (1).

Theorem 2.1. *Let $I = [t_0, +\infty)$, $h\colon I \to \mathbb{R}$ be a nondecreasing continuous function and $F\colon L \times I \to \mathfrak{F}(\Omega, V)$ be an operator belonging to the class $\mathcal{G}(L \times I, h)$, where $L \subset L^p(\Omega, V)$. If $X = \{X_t : t \in [s_0, +\infty)\}$ is a maximal solution of the GSE (1), then, for every compact set $K \subset L \times I$, there exists $t_K \in [s_0, \nu)$ such that $(X_t, t) \notin K$, for all $t \in (t_k, \nu)$ and some $\nu \leq +\infty$. In particular, for all $s_0 \geq t_0$ and all $\widetilde{X} \in L^p(\Omega, V)$, there exists a unique global forward solution $X = \{X_t : t \in [s_0, +\infty)\}$ of the GSE (1) with $X_{s_0} = \widetilde{X}$, whenever $L \subset F^p(\Omega, V)$ is compact.*

In what follows, we present results which show that certain stability conditions for a GSE can be stated in term of Lyapunov functions.

Theorem 2.2. *Let $1 \leq p < \infty$ and $V\colon [t_0, +\infty) \times L^p(\Omega, V) \to \mathbb{R}^+$ be a positive definite functional such that for any solution $X = \{X_t : t \geq s_0\}$ of the GSE (1), $s_0 \geq t_0$, the following inequalities hold:*

i there exists a constant $c > 0$ such that $\mathbb{E}[V(t, X_t)] \geq c\mathbb{E}[\|X_t\|^p_{L^p}]$, for all $t \geq s_0$;

ii there exists $\sigma \in \mathcal{K}$ such that $\mathbb{E}[V(s_0, X_{s_0})] \leq \sigma\left(\|X_{s_0}\|^p_{L^p}\right)$;

iii there exists a constant $d > 0$ such that for all $t > s \geq s_0$, we have

$$\mathbb{E}[V(t, X_t) - V(s, X_s)] \leq -d \int_s^t \mathbb{E}[V(\tau, X_\tau)] d\tau.$$

Then, the trivial solution of the GSE (1) is asymptotically p-stable.

Theorem 2.3. *Let $1 \leq p < \infty$ and $V\colon [t_0, +\infty) \times L^p(\Omega, V) \to \mathbb{R}^+$ be a positive definite functional such that, for any solution $X = \{X_t : t \geq s_0\}$ of the GSE (1), with $s_0 \geq t_0$, we have*

i there exists $\alpha > 0$ for which

$$\mathbb{E}[V(t, X_t) - V(s, X_s)] \leq -\alpha \int_s^t \mathbb{E}[V(\tau, X_\tau)] d\tau,$$

for all $t, s \geq s_0$ with $s < t$;

ii there exists a constant $c > 0$ such that $\mathbb{E}[V(t, X_t)] \geq c\mathbb{E}[\|X_t\|^p_{L^p}]$, for all $t \geq s_0$;

iii there exists $\sigma \in \mathcal{K}$ such that $\mathbb{E}[V(s_0, X_{s_0})] \leq \sigma\left(\|X_{s_0}\|^p_{L^p}\right)$.

Then, the trivial solution of the GSE (1) is exponentially p-stable.

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HIERARCHIC CONTROL FOR THE BURGERS EQUATION VIA STACKELBERG-NASH STRATEGY

F. D. ARARUNA[1,†], E. FERNÁNDEZ-CARA[2,‡] & L. C. DA SILVA[3,§]

Departamento de Matemática, UFPB, PB, Brasil[1], EDAN and IMUS, US, Sevilla, Espanha[2], IFRN, RN, Brasil[3]

fagner@mat.ufpb.br[†] cara@us.es[‡] luciano.cipriano@ifrn.edu.br[§]

Abstract

We present some results on the null controllability of the Burgers equation. We analyze the hierarchic control through Stackelberg–Nash strategy, where we consider one leader and two followers. To each leader we associate a Nash equilibria corresponding to a bi-objective optimal control problem; then we look for a leader that solves the null controllability problem. We prove linear case and we use a fixed point method to solve the semilinear problem.

1 Introduction

Let us consider $T > 0$, the set $Q := (0,1) \times (0,T)$ and the nonempty open sets $\mathcal{O}$, $\mathcal{O}_1$, $\mathcal{O}_2 \subset (0,1)$, with $0 \notin \overline{\mathcal{O}}$. We introduce the Burgers' system:

$$\begin{cases} y_t - y_{xx} + y\,y_x = f 1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2}, & (x,t) \in Q, \\ y(0,t) = y(1,t) = 0, & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,1), \end{cases} \tag{1}$$

where the controls are f, v^1, v^2, the state is y and with the notation 1_A we denote the characteristic function of the set A.

The objective this paper is, follwing the ideas in [4] [1],[2] and [3], analyze the null controllability of the system (1) in the context of the hierarchic control applying the Stackelberg–Nash strategy with leader f and two followers v^1 and v^2. In this sense, we consider the cost functionals for the followers:

$$J_i\left(f;\, v^1,\, v^2\right) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d}\times(0,\,T)} |y - y_{i,d}|^2\, dx\, dt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i \times (0,\,T)} |v^i|^2\, dx\, dt, \tag{2}$$

and main functional

$$J(f) := \frac{1}{2} \iint_{\mathcal{O}\times(0,\,T)} |f|^2\, dx\, dt,$$

where $\mathcal{O}_{i,d} \subset (0,1)$ is a open nonempty set, α_i, $\mu_i > 0$ are constants and $y_{i,d} = y_{i,d}(x,t)$ are given functions in $L^2(\mathcal{O}_{i,d} \times (0,\,T))$.

For each leader control f choosing we look for an pair $\left(v^1,\, v^2\right)$ that minimize, simultaneously, the functionals J_i, that is,

$$J_1\left(f;\, v^1,\, v^2\right) = \min_{\hat{v}^1} J_1\left(f;\, \hat{v}^1,\, v^2\right) \quad \text{and} \quad J_2\left(f;\, v^1,\, v^2\right) = \min_{\hat{v}^2} J_2\left(f;\, v^1,\, \hat{v}^2\right). \tag{3}$$

An pair $\left(v^1,\, v^2\right) = \left(v^1(f),\, v^2(f)\right)$ that satisfies (3) is called Nash equilibrium for the functionals J_i. Finally, we prove that there exists $f \in L^2(\mathcal{O} \times (0,\,T))$ such that

$$J(f) = \min_{\hat{f}} J\big(\hat{f}\big) \tag{4}$$

and

$$y(\cdot,\,T) = 0 \quad \text{in} \quad (0,1). \tag{5}$$

2 Main results

The local null controllability with the initial data in $H_0^1(0,1)$ given by the result:

Theorem 2.1. *Suppose that $\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset$, $i = 1,\, 2$. Assume that one of the following conditions holds:*

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d} := \mathcal{O}_d, \tag{6}$$

or

$$\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}. \tag{7}$$

If the constants $\mu_i > 0$ $(i = 1,\, 2)$ are large enough, there exists a positive function $\rho = \rho(t)$, which decay exponentialy to 0 when $t \to T^-$, with the following property:

$$\iint_{\mathcal{O}_{i,d}\times(0,\,T)} \rho^{-2}\left|y_{i,d}\right|^2 dx\,dt \,\leq\, r^2, \quad \textit{for some} \quad r > 0, \qquad i = 1,\, 2, \tag{8}$$

then, for any $y^0 \in H_0^1(0,1)$ with $\left|\left|y^0\right|\right|_{H_0^1(0,1)} \leq r$ there exist a control $f \in L^2(\mathcal{O} \times (0,\,T))$ and a associated Nash equilibrium $\left(v^1(f),\, v^2(f)\right)$ such that has one (4) and the corresponding solution to (1) satisfies (5).

Theorem 2.2. *Suppose μ_i and $\mathcal{O}_{i,d}$ as in Theorem 2.1 and $T \geq T(r)$. There exists a function $\rho = \rho(t)$, which decay exponentialy to 0 when $t \to T^-$, with the following property: if (7) holds, for any $y^0 \in L^2(0,1)$ with $||y^0||_{L^2(0,1)} \leq r$, there exists a control $f \in L^2(\mathcal{O} \times (0,\,T))$ and a associated Nash equilibrium $\left(v^1(f),\, v^2(f)\right)$ such that the corresponding solution to (1) satisfies (5).*

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PERIODIC SOLUTIONS OF MEASURE FUNCTIONAL DIFFERENTIAL EQUATIONS

S. M. AFONSO[1,†], E. M. BONOTTO[2,‡] & MÁRCIA R. DA SILVA[3,§]

Instituto de Geociências e Ciências Exatas, UNESP[1], Instituto de Ciências Matemáticas e de Computação, USP[2]
Centro de Ciências da Natureza, UFSCar[3]

s.afonso@unesp.br[†] ebonotto@icmc.usp.br[‡] marciars@ufscar.br[§]

Abstract

Our goal is to investigate the existence of periodic solutions for measure functional differential equations of the form

$$x(t) = x(0) + \int_0^t f(s, x_s)ds + \int_0^t g(s, x_s)du(s),$$

defined for every $t \in \mathbb{R}$, under suitable assumptions on f, g and u, where the integrals on the right–hand side exist in the Perron and Perron–Stieltjes sense, respectively. We make use of a topological transversality theorem to obtain the main result. Some examples are presented to illustrate the developed theory. Moreover, we apply the results obtained in the context of measure functional differential equations to establish the existence of periodic solutions for a class of impulsive functional differential equations.

1 Introduction

This work concerns the study of periodic solutions for measure functional differential equations (measure FDEs) of type:

$$x(t) = x(0) + \int_0^t f(s, x_s)ds + \int_0^t g(s, x_s)du(s), \tag{1}$$

defined for every $t \in \mathbb{R}$, where $f, g\colon \mathbb{R} \times G_T^-(\mathbb{R}, \mathbb{R}^n) \to \mathbb{R}^n$ are T–periodic functions with respect to the first variable, $T > 0$, and $G_T^-(\mathbb{R}, \mathbb{R}^n)$ denotes the space of all T–periodic regulated left–continuous functions $\phi\colon \mathbb{R} \to \mathbb{R}^n$, with the supremum norm $\|\phi\|_T = \sup_{\theta\in[0,T]} |\phi(\theta)|$. For a given function $x \in G_T^-(\mathbb{R}, \mathbb{R}^n)$, the Krasovsky notation $x_t \in G_T^-(\mathbb{R}, \mathbb{R}^n)$, $t \in \mathbb{R}$, is used to denote the function $x_t(\theta) = x(t + \theta), \quad \theta \in \mathbb{R}$.

The theory of measure differential equations was introduced in by W. Schmaedeke in the control theory. These equations cope very well with the description of phenomena whose evolution is interrupted by abrupt changes of state. In the literature, the Leray–Schauder degree theory is used very often to obtain results on the existence of periodic solutions for different types of differential equations. In contrast to the theory of topological degree, which requires sophisticated tools, we have the theory of topological transversality that requires nothing more than the Urysohn Lemma in a metric space and some arguments of compactness. In this sense, we can say that the latter allows us to study our problem in a simpler way. It is worth mentioning that the topological transversality theory is a variant of the degree-theoretic method.

We establish in this work more general conditions to guarantee the existence of periodic solutions for equations of type (1). We make use of the Granas topological transversality theorem and, through the theory of regulated functions and the theory of Perron–Stieltjes (Perron) integration, we also consider functions that are not necessarily continuous. Section 2 deals with the main results. Using a topological transversality theorem, we establish in Theorem 2.1 sufficient conditions to obtain the existence of periodic solutions for measure FDEs of type (1). Finally, we apply the results obtained in Section 2, to prove the existence of periodic solutions for a class of impulsive FDEs.

2 Main Results

The aim of this section is to present sufficient conditions to obtain regulated periodic solutions for the measure FDE(1). As mentioned in the sequel, we shall assume that u is a left–continuous regulated function on $\mathbb{R}$. In this way, for a given $T > 0$, we shall consider $s_0 \in [0, T]$ as being a point of continuity of u. We shall assume the following general conditions:

(H1) $u\colon \mathbb{R} \to \mathbb{R}$ is a left–continuous regulated function on $\mathbb{R}$, continuous at $s_0 \in [0, T]$, and there exists $c \in \mathbb{R}$ such that $u(t + T) = u(t) + c$ for all $t \in \mathbb{R}$;

(H2) $f, g\colon \mathbb{R} \times G^-(\mathbb{R}, \mathbb{R}^n) \to \mathbb{R}^n$ are T–periodic functions with respect to the first variable such that, for each $x \in G_T^-(\mathbb{R}, \mathbb{R}^n)$, the map $t \mapsto f(t, x_t)$ is locally Perron integrable over $\mathbb{R}$ and the map $t \mapsto g(t, x_t)$ is locally Perron–Stieltjes integrable over $\mathbb{R}$ with respect to u;

(H3) there exists a function $h \in G^-(\mathbb{R}, \mathbb{R})$ continuous at s_0 such that, for any $\mu > 0$, one can obtain $N_\mu > 0$ satisfying $\left\| \int_{t_1}^{t_2} f(s, x_s) ds \right\| \leq N_\mu |h(t_2) - h(t_1)|$ and $\left\| \int_{t_1}^{t_2} g(s, x_s) du(s) \right\| \leq N_\mu |h(t_2) - h(t_1)|$, whenever $t_1, t_2 \in [s_0, s_0 + T]$ and $x \in G_T^-(\mathbb{R}, \mathbb{R}^n)$ with $\|x\|_T \leq \mu$;

(H4) given $\mu > 0$, there exist a non–negative locally Perron integrable function $M_\mu\colon \mathbb{R} \to \mathbb{R}$ and a non–negative locally Perron–Stieltjes integrable function $L_\mu\colon \mathbb{R} \to \mathbb{R}$ with respect to u such that, for all $t_1, t_2 \in [s_0, s_0 + T]$, $t_1 \leq t_2$,
$$\left\| \int_{t_1}^{t_2} [f(s, x_s) - f(s, y_s)] ds \right\| \leq \int_{t_1}^{t_2} M_\mu(s) \|x - y\|_T ds$$
and
$$\left\| \int_{t_1}^{t_2} [g(s, x_s) - g(s, y_s)] du(s) \right\| \leq \int_{t_1}^{t_2} L_\mu(s) \|x - y\|_T du(s),$$
for all x, $y \in G_T^-(\mathbb{R}, \mathbb{R}^n)$ such that $\|x\|_T \leq \mu$ and $\|y\|_T \leq \mu$.

We consider the following family of measure functional differential equations
$$x(t) = \lambda x(s_0) + \int_{s_0}^{t} \lambda f(s, x_s) ds + \int_{s_0}^{t} \lambda g(s, x_s) du(s), \quad \lambda \in (0, 1], \tag{1}$$
defined for every $t \in \mathbb{R}$.

Theorem 2.1. *Assume that conditions (H1)–(H4) hold. If there exists a constant $\beta > 0$ such that $\|x(t)\| < \beta$ for all $t \in \mathbb{R}$ whenever $x(t)$ is a T–periodic solution of (1), with $\lambda \in (0, 1]$, then the measure FDE (1) has at least one T–periodic solution.*

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ANALYSIS OF A COVID-19 MODEL WITH DELAY.

LUCIANO A. MAGRINI[2,‡] & MARTA C. GADOTTI[1,†]

Instituto de Geociências e Ciências Exatas , UNESP, Rio Claro, SP, Brasil[1], IFSP, São Paulo, SP, Brasil[2]

mc.gadotti@unesp.br[†] magrini@ifsp.edu.br[‡]

Abstract

In this work, we study the existence and stability of the equilibrium points of an SVIR model with delay. Analysing the characteristic equation of the linearized problem around the endemic equilibrium point, we prove the occurrence of Hopf-bifurcation.

1 Introduction

In [1], the authors study the following system with delay:

$$
\begin{aligned}
S'(t) =& \Lambda + (1-p)A - \frac{\beta SI}{N} - \mu S + \psi S(t-\tau) + \theta V \\
V'(t) =& \psi S(t-\tau) - \frac{\sigma\beta VI}{N} - (\mu+\theta)V \\
I'(t) =& pA + \frac{\beta SI}{N} + \frac{\sigma\beta VI}{N} - (\mu S + \alpha)I \\
R'(t) =& \alpha I - \mu R,
\end{aligned}
$$

where $S(t), V(t), I(t)$ and $R(t)$ represent to the susceptible, vaccination, infected and recovery respectively. They assume the existence of a constant flow, $A > 0$, of new members arriving into the population in unit time with the fraction p of A arriving infected $(0 \leqslant p < 1)$. The equilibrium points of the model are obtained and the stability analysis is performed based on the results of [2]. Furthermore, a result is presented about the occurrence of Hopf-bifurcation, when the delay $\tau > 0$ is considered as a parameter.

In this work, we modify this model and study the delayed effect of taking the vaccination against a COVID-19 model pandemic. We introduce the $V(t-\tau)$ and we also consider that recoveries may become susceptible again. So, we consider the system SVIR (τ), given by

$$
\begin{aligned}
S'(t) =& \Lambda + (1-p)A - \frac{\beta SI}{N} - \mu S + \delta R - \psi S(t-\tau) + \epsilon V \\
V'(t) =& \psi S(t-\tau) - \frac{\sigma\beta V(t-\tau)I}{N} - (\mu+\epsilon)V \\
I'(t) =& pA + \frac{\beta SI}{N} + \frac{\sigma\beta V(t-\tau)I}{N} - (\mu+\gamma)I \\
R'(t) =& \gamma I - (\mu+\delta)R.
\end{aligned}
$$

The goal is study the equilibrium points of this system and to stablish a result about stability and occurrence of Hopf-bifurcation.

2 Main Results

For $p = 0$, we prove that $E_0 = (\bar{S}, \bar{V}, 0, 0)$, where $\bar{S} = \dfrac{(\Lambda + A)(\epsilon + \mu)}{\epsilon\mu + \mu^2 + \mu\psi} > 0$ and $\bar{V} = \dfrac{(\Lambda + A)\psi}{\epsilon\mu + \mu^2 + \mu\psi} > 0$, is a endemic equilibrium point.

We make the change variables $\xi(t) = S(t) - \bar{S}$, $\eta(t) = V(t) - \bar{V}$ and we consider the linear system around the E_0,

$$X' + AX + BX(t - \tau) = 0, \qquad \text{where} \quad X(t) = \begin{pmatrix} \xi(t) \\ \eta(t) \\ I(t) \\ R(t) \end{pmatrix}.$$

Therefore, the characteristic equation is given by

$$\Delta(\lambda, \tau) = \det(\lambda I + A + e^{-\tau\lambda} B).$$

If $\tau = 0$, $E_0 = (\bar{S}, \bar{V}, 0, 0)$ is an unstable point if $K_1 = -\frac{\beta}{N}(\bar{S} - \sigma\bar{V}) + (\mu + \gamma) > 1$ and E_0 is stable if $K_1 < 1$.

The occurrence of Hopf-bifurcation near the endemic equilibrium point is studied. Note that $\Delta(0, \tau) \neq 0$. We can show that there exists a pair (iw_0, τ_0), $\tau_0 > 0$ such that $\Delta(w_0 i, \tau_0) = 0$. In fact, we have a sequence (τ_j)

$$\tau_j = \frac{1}{w} \tan^{-1} \frac{w}{-\psi(\epsilon + \mu)} + \frac{j\pi}{w}, \quad j = 0, 1, 2, 3...$$

satisfying $\Delta(wi, \tau_j) = 0$, with $w = \pm w_0$. Finally, we prove that

$$\left[\frac{Re(\lambda(\tau))}{d\tau}\right]_{\tau=\tau_0} \neq 0.$$

Remark 2.1. *The system SVIR(τ) has other equilibrium points, whose coordinates are all positive. We intend to study the stability of such points and also, if possible, verify the occurrence of Hopf-bifurcation. We also intend to do some numerical simulations.*

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FRACTIONAL NAVIER-STOKES EQUATIONS AND THE LIMIT PROBLEM.

PAULO M. CARVALHO NETO[1,†] & GABRIELA PLANAS[2,‡]

Centro de Ciências Físicas e Matemáticas, UFSC, SC, Brasil[1], IMECC, UNICAMP, SP, Brasil[2]

paulo.carvalho@ufsc.br[†] silva@ime.unicamp.br[‡]

Abstract

This work studies the incompressible Navier-Stokes equations with Caputo time-fractional derivative. We discuss the existence, uniqueness, and regularity of solutions to the equations. We then introduce the limit problem, which is the problem of studying the convergence of the solutions of the fractional Navier-Stokes equations to the solutions of the classical Navier-Stokes equations as the order of the fractional derivative approaches 1.

1 Introduction

Consider the incompressible Navier-Stokes equations with fractional time derivative

$$\begin{array}{rcll} cD_t^\alpha u - \nu\Delta u + (u\cdot\nabla)u + \nabla p &=& f & \text{in } \mathbb{R}^N,\ t>0,\\ \nabla\cdot u &=& 0 & \text{in } \mathbb{R}^N,\ t>0,\\ u(x,0) &=& u_0 & \text{in } \mathbb{R}^N. \end{array} \tag{1}$$

Above we have that cD_t^α is the Caputo fractional derivative of order $\alpha\in(0,1)$, $u=(u_1(x,t),\ldots,u_N(x,t))$ represents the fluid velocity vector, $\nu>0$ the viscosity coefficient, $p=p(x,t)$ the associated pressure, $u_0=u_0(x)$ the initial velocity vector and $f=(f_1(x,t),\ldots,f_N(x,t))$ a given external force.

We rewrite (1) in its abstract form to obtain

$$\begin{array}{rcl} cD_t^\alpha u &=& -A_r u + F(u) + Pf, \quad t>0,\\ u(0) &=& u_0 \in L^r_\sigma, \end{array} \tag{2}$$

where $P: L^r\to L^r_\sigma$ is the Helmholtz-Leray projection, $A_r: D(A_r)\subset L^r_\sigma\to L^r_\sigma$ is the Stokes operator, and $F(u)=F(u,u)$, with $F(u,v)=-P(u\cdot\nabla)v$. We also assume, in order to simplify the ideas presented, that $f=0$.

In view of the above considerations, we are able to discuss the existence, uniqueness, and regularity of solutions to (2), which are summarized in the following theorems:

Theorem 1.1. *For $\alpha\in(0,1)$ e $u_0\in L^N_\sigma$, there exists $\lambda>0$ such that if $\|u_0\|_{L^N}\leq\lambda$, then (2) has a unique global mild solution $u:[0,\infty)\to L^N_\sigma$. Moreover,*

$$t^{\alpha[1-(N/q)]/2}u\in C_b([0,\infty);L^q_\sigma),\qquad \text{for } 2\leq N\leq q\leq\infty,$$

$$t^{\alpha[1-(N/2q)]}\nabla u\in C_b([0,\infty);L^q_\sigma),\qquad \text{for } 2\leq N\leq q<\infty,$$

both of which are zero at $t=0$ except when $q=N$ in the first sentence, in which case $u(0)=u_0$.

Theorem 1.2. *For $\alpha\in(0,1)$ and $u_0\in L^N_\sigma$, there exists $\lambda>0$ such that if $\|u_0\|_{L^N}\leq\lambda$, it holds that:*

i) For $2\leq N\leq 2/\alpha$, problem (2) has a unique global mild solution that belongs to $L^r(0,\infty;L^q_\sigma)$, with

$$\frac{1}{r}=\alpha\Big(1-\frac{N}{q}\Big)/2 \quad \text{and} \quad N<q<\infty;$$

ii) For $N > 2/\alpha$, problem (2) has a unique global mild solution that belongs to $L^r(0,\infty;L^q_\sigma)$, with

$$\frac{1}{r} = \alpha\Big(1 - \frac{N}{q}\Big)/2 \quad \text{and} \quad N < q < \frac{\alpha N^2}{\alpha N - 2}.$$

With the aim of studying the Navier-Stokes equations, here we introduce an initial answer to the following question: if $\alpha \in (0,1]$, X is a Banach space and we consider the abstract differential equation

$$\begin{array}{rcl} cD_t^\alpha u &=& \mathcal{A}u + f(t,u), \quad t > 0, \\ u(0) &=& u_0 \in X, \end{array} \tag{P_α}$$

with cD_t^α representing the Caputo fractional derivative of order when $\alpha \in (0,1)$ and cD_t^1 representing the classical derivative, $\mathcal{A}$ is a (at least) closed and densely defined operator, and f is a suitable function, is it possible to prove the convergence of the solution of (P_α) to the solution of (P_1), when $\alpha \to 1^-$?

In summary, we can consider the following as a first interesting result that answers this question:

Theorem 1.3. *Consider the Cauchy problem (P_α) with $\alpha \in (0,1]$ and assume that the function $f : [0,\infty) \times X \to X$ is continuous, locally Lipschitz in the second variable, uniformly with respect to the first variable, and bounded. Then, if $\phi_\alpha(t)$ is a mild local (or global) solution of (P_α) defined on its maximal domain of existence $[0,\omega_\alpha)$ (or $[0,\infty)$), there exists $t_* > 0$ (independent of α) such that*

$$\lim_{\alpha \to 1^-} \|\phi_\alpha(t) - \phi_1(t)\| = 0,$$

for every $t \in [0,t^*]$. The uniform convergence also occurs in compact subsets of $(0,t^*]$.

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GEOMETRIC ASPECTS OF YOUNG INTEGRAL: DECOMPOSITION OF FLOWS

PAULO RUFFINO.[1], PEDRO CATUOGNO[1] & LOURIVAL LIMA[2]

IMECC, UNICAMP, SP, Brasil[1], Escuela Superior Politècnica del Litoral, Ecuador[2]

`ruffino@unicamp.br`[†] `pedrojc@unicamp.br`[‡] `lourodri@espol.edu.ec`[§]

Abstract

In this paper we study geometric aspects of dynamics generated by Young differential equations (YDE) driven by α-Hölder trajectories with $\alpha \in (1/2, 1)$. We present a number of properties and geometrical constructions in this context: Young Itô geometrical formula, horizontal lift in principal fibre bundles, parallel transport, covariant derivative, development and anti-development, among others. Our main application here is a geometrical decomposition of flows generated by YDEs according to diffeomorphisms generated by complementary distributions (integrable or not). The proof of existence of this decomposition is based on an Itô-Wentzel type formula for Young integration along α-Hölder paths proved by Castrequini and Catuogno (Chaos Solitons Fractals, 2022).

1 Introduction

In this paper we study geometric aspects of dynamics generated by Young differential equations (YDE) driven by α-Hölder trajectories with $\alpha \in (1/2, 1]$. More precisely, given a smooth manifold M, we focus on geometrical properties of equations of the type:

$$dx_t = X(x_t)\, dZ_t, \tag{1}$$

with initial condition $x_0 \in M$ at $t = 0$, where $x \to X(x) \in \mathcal{L}(\mathbb{R}^d, T_xM)$ is a smooth assignment of d vector fields on M and $Z \in C^\alpha([0,T], \mathbb{R}^d)$ is an α-Hölder continuous trajectory in $\mathbb{R}^d$. Similar to the theory of semimartingales on manifolds, we say that a path $x : [0,T] \to M$ is a solution of equation (1) if for all test function $f \in C^\infty(M; \mathbb{R})$ we have that

$$f(x_t) = f(x_0) + \int_0^t Xf(x_s)\, dZ_s, \tag{2}$$

where Xf is a short term for $\sum Df(x)X(x)e_i$, with e_i's the elements of the canonical basis of $\mathbb{R}^d$. The last term of equation (2) is an integral in the Young sense, see e.g. the classical [7], or more recent Hairer and Friz [3], Gubinelli et al. [4], Castrequini and Catuogno [1], Cong [2], Ruzmaikina [6], among many others.

Here, in a scenario of low regularity of trajectories, the Itô type formula in the context of Young integration, Theorem 2.1 opens the possibility for many basic geometric constructions on this dynamics. These topics are exploited in the next section, where we prove the existence of horizontal lifts in principal fiber bundles with an affine connection. In particular, considering a Riemannian manifold and its orthonormal bundle, parallel transport and covariant derivatives can be established along α-Hölder trajectories. Development and anti-development can also be constructed.

2 Main Results

Theorem 2.1 (Itô formula)**.** *Let M and N be Riemannian manifolds. Consider $x \in \mathcal{C}^\alpha([0,T], M)$ and a smooth function $F : M \to N$. Then*

$$dF(x_t) = DF(x_s)\; dx_s. \tag{1}$$

Mind that formula (1) above means that if β is a 1-form in N then

$$\int_0^t \beta \; dF(x_s) = \int_0^t (dF(x_s))^* \beta \; dx_s. \tag{2}$$

In particular, if N is an Euclidean space:

$$F(x_t) = F(x_0) + \int_0^t DF(x_s) dx_s. \tag{3}$$

Let $\{P, M, G, \pi\}$ be a principal fibre bundle with base M, structure group G and total space P.

Theorem 2.2 (Horizontal lifts)**.** *Given an α-Hölder continuous path $x : [0,T] \to M$ and an element u in the fibre $\pi^{-1}(x_0)$, there exists (up to an explosion time) a unique horizontal lift $\widetilde{x} : [0,T] \to P$ with $\tilde{x}_0 = u$.*

Proof See the full version of the paper.

From this result, development and anti-development can be obtained along α-Hölder trajectories on Riemannian manifolds. For these and other results on decomposition of flows generated by YDE on manifolds, please see the full version of the paper to appear in Mediterranean Journal of Mathematics.

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GLOBAL BIFURCATION RESULTS FOR A DELAY DIFFERENTIAL SYSTEM REPRESENTING A CHEMOSTAT MODEL.

PIERLUIGI BENEVIERI[1,†] & PABLO AMSTER[2,‡]

IME, Universidade de São Paulo, USP, Brasil[1], Departamento de Matemática, FCEyN - Universidad de Buenos Aires & IMAS-CONICET, Argentina[2]

pluigi@ime.usp.b[†] pamster@dm.uba.ar[‡]

Abstract

We present a global bifurcation result for a first order system of delay differential equations depending on a real parameter. The system represent a chemostat model. The approach is topological and is based of a topological degree theory for nonlinear Fredholm maps between Banach spaces.

1 Introduction

In the paper we prove a global bifurcation result for periodic solutions to the following delayed first order system, depending on a real parameter $\lambda \geq 0$,

$$\begin{cases} s'(t) = Ds^0(t) - Ds(t) - \dfrac{\lambda}{\gamma}\,\mu(s(t))x(t) & t \geq 0 \\ x'(t) = x(t)\big[\lambda\mu(s(t-\tau)) - D\big] & t \geq 0 \\ s(\theta) = \phi(\theta) \quad \text{and} \quad x(0) = x_0 & \text{if } \ \theta \in [-\tau, 0], \end{cases} \tag{1}$$

in which the following conditions hold:

(a) $s^0 : \mathbb{R} \to \mathbb{R}$ is continuous, positive and ω-periodic, where $\omega > 0$ is given,

(b) $\mu : [0, +\infty) \to [0, +\infty)$ is C^1 and verifies $\mu(0) = 0$ and $\mu'(s) > 0$, for any $s \in [0, +\infty)$,

(c) D, γ and the delay τ are positive constants,

(d) $\phi : [-\tau, 0] \to \mathbb{R}$ is continuous.

For a given λ, a *solution* of (1) is defined as a pair (s, x) of maps $s : [-\tau, +\infty) \to \mathbb{R}$ and $x : [0, +\infty) \to \mathbb{R}$, such that s is continuous, its restriction to $[0, +\infty)$ is C^1, x is C^1 and both satisfy the equations in (1) as well as the boundary conditions.

System (1) has been studied in [1] and it represents a chemostat model, with a delay. The chemostat is a continuous bioreactor with a constant volume, in which one or more microbial species are cultivated in a liquid medium containing a set of resources with, in particular, a specific nutrient. The maps $s(t)$ and $x(t)$ are, respectively, the densities of the nutrient and of the microbial species at time t. The device receives continuously an input of liquid volume, described by $s^0(t)$, containing a variable concentration of the specific nutrient. It expulses continuously towards the exterior an output of liquid volume containing a mixing of microbial biomass and nutrient. The model described by the system (1) assumes that the consumption of the nutrient has no immediate effects on the microbial growth, but we have a time interval $[t - \tau, t]$ in which the microbial species metabolize(s) the nutrient.

If (s, x) is any solution of (1) such that x vanishes at some t_0, then x turns out to be identically zero. Thus, the first equation in system (1) becomes linear and appears in the form

$$v'(t) = Ds^0(t) - Dv(t). \tag{2}$$

Among the infinite solutions of (2), one and only one is ω-periodic, it is positive and can be written as

$$v^*(t) = \int_{-\infty}^{t} e^{-D(t-r)} D s^0(r)\, dr$$

For a sake of simplicity, assume that

$$\frac{1}{\omega}\int_0^{\omega} \mu(v^*(t))\, dt = D.$$

In general, the relation between the average of $\lambda(\mu \circ v^*)$ and D is crucial. In [1], the authors prove that

(a) *if* $\lambda < 1$ *(resp.* $\lambda > 1$*) and* (s, x) *is an* ω*-periodic solution, different from* $(v^*, 0)$*, then,* $x(t) < 0$ *(resp.* $x(t) > 0$*) for all* $t \in \mathbb{R}$*;*

(b) *if* $\lambda = 1$*, no* ω*-periodic solution is different from* $(v^*, 0)$*.*

Solutions with $x(t) < 0$ are not interesting form a biological point of view, as x is the densitiy of the microbial species at time t. Observing items (a) and (b) above, it is quite natural to ask if $(v^*, 0)$ is a bifurcation point for ω-periodic solutions of (1) with x positive, as well as it is important to investigate the global behaviour of the connected components of such solutions whose closures (in a suitable topology) contain $(v^*, 0)$, analogously to the classical bifurcation results of Rabinowitz in [3].

2 Main Results

We need some notation. If (s, x) is an ω-periodic solution of (1) for a given λ, the triple (λ, s, x) will be called *ω-triple*. An ω-triple (λ, s, x) such that $\lambda \neq 1$ and $(s, x) \neq (v^*, 0)$ will be called *nontrivial*. We consider an ω-triple as an element of $E := \mathbb{R} \times C^1_\omega \times C^1_\omega$, where

$$C^1_\omega = \{u \in C^1([0, \omega], \mathbb{R}) : u(0) = u(\omega) \text{ and } u'(0) = u'(\omega)\},$$

is a Banach space with the usual norm. Our main result is the following global bifurcation theorem.

Theorem 2.1. *There exist in* E *exactly two connected components* $\mathcal{C}_+$ *and* $\mathcal{C}_-$ *of nontrivial* ω*-triples, which are unbounded, contain* $(1, v^*, 0)$ *in their closure and are such that every* $(\lambda, s, x) \in \mathcal{C}_+$ *verifies* $\lambda > 1$*,* $0 < s < v^*$ *and* $x > 0$*, while every* $(\lambda, s, x) \in \mathcal{C}_-$ *verifies* $\lambda < 1$*,* $s > v^*$ *and* $x < 0$*.*

The proof is given by a topological approach based on a concept of degree introduced in [2] for Fredholm maps of index zero between Banach spaces or smooth Banach manifolds. This degree is based on a notion of topological orientation for nonlinear Fredholm maps of index zero. The degree coincides with the Brouwer degree for C^1 maps between finite dimensional oriented manifolds of Euclidean spaces; in the infinite dimensional case and for C^1 compact vector fields, it coincides with the Leray–Schauder degree.

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FRACTIONAL $(P, Q)-$ LAPLACIAN OPERATOR IN BOUNDED DOMAINS

THIAGO RODRIGUES CAVALCANTE[1,†] & EDCARLOS D. SILVA[2,‡]

Matemática Campus Arraias, UFT, TO, Brazil[1], Instituto de Matemática e Estatistica, UFG, GO, Brazil[2]

thiago.cavalcante@mail.uft.edu.br[†] edcarlos@ufg.br[‡]

Abstract

In this we establish existence of solutions for nonlocal elliptic problems driven by the fractional $(p, q)-$ Laplacian.. More specifically, we shall consider the following nonlocal elliptic problem :

$$\begin{cases} (-\Delta)_p^{s_1} u - \mu(-\Delta)_q^{s_2} = \lambda |u|^{r-2} u \text{ in } \Omega \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where$N > s_1 p$, $N > s_2 q$, $s_1 > s_2$ and $r > p > q$. The main feature is to find sharp parameters $\lambda > 0$ and $\mu > 0$ where the Nehari method can be applied finding the largest positive number $\mu^* > 0$ such that our main problem admits at least two distinct solutions for each $\mu \in (0, \mu^*)$.

1 Introduction

In the present work we shall consider nonlocal elliptic problems driven by the fractional $(p, q)-$ Laplacian defined in bounded domanin. Namely, we shall consider the following nonlocal elliptic problem

$$\begin{cases} (-\Delta)_p^{s_1} u - \mu(-\Delta)_q^{s_2} = \lambda |u|^{r-2} u \text{ in } \Omega \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{P_μ}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N > s_1 p$, $N > s_2 q$, $s_1 > s_2$ and $r > p > q$.

In order to do that we employ the nonlinear Rayleigh quotient together a fine analysis on the fibering maps associated to the energy functional. It is important to mention also that for each parameters $\lambda > 0$ and $\mu > 0$ there exist degenerate points in the Nehari set which give serious difficulties.

2 Main Results

The working space is defined by $X = \{u \in W^{s,p}(\mathbb{R}^N);\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$ and the energy functional $J : X \to \mathbb{R}$ associated to Problem (P_μ) is given by

$$J_{\lambda,\mu}(u) = \frac{1}{p}[u]_p^p - \frac{\mu}{q}[u]_q^q - \frac{\lambda}{r}\|u\|_r^r,$$

where

$$[u]_p^p := [u]_{s,p}^p = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\, dxdy,\ u \in X.$$

In this work, we study the fibers maps of two functionals based on the parameter μ. The first defined for the case where $J'_{\lambda,\mu}(u) = 0$ and the second, considering μ, for which $J(u) = 0$. In short, we consider the functionals $R_n, R_e : X \setminus \{0\} \to \mathbb{R}$ associated with the parameter $\mu > 0$ in the following form

$$R_n(u) = \frac{[u]_p^p - \lambda\|u\|_r^r}{[u]_q^q} \quad \text{and} \quad R_e(u) = \frac{\frac{q}{p}[u]_p^p - \lambda\frac{q}{r}\|u\|_r^r}{[u]_q^q}, u \in X \setminus \{0\},$$

the nonlinear Rayleigh quotients. In association with these, based on work [3], we define the following coefficients

$$\mu^* := \inf_{u\in X\setminus\{0\}} \inf_{t>0} R_n(tu) \qquad \text{and} \qquad \mu^{**} := \inf_{u\in X\setminus\{0\}} \inf_{t>0} R_e(tu). \tag{1}$$

The subset of X, in which the $J_{\lambda,\mu}$ function will be minimized, well known and studied in recent years for Nehari is

$$\mathcal{N}_{\lambda,\mu} = \{u \in X,\, u \neq 0 \;:\; \langle J'_{\lambda,\mu}(u), u\rangle = 0\}.$$

Under these conditions, by using the same ideas considered in [2], we shall split the Nehari manifold $\mathcal{N}_{\lambda,\mu}$ into three disjoint subsets in the following way:

$$\begin{aligned}
\mathcal{N}^+_{\lambda,\mu} &= \{u \in \mathcal{N}_{\lambda,\mu} : J''_{\lambda,\mu}(u)(u,u) > 0\},\\
\mathcal{N}^-_{\lambda,\mu} &= \{u \in \mathcal{N}_{\lambda,\mu} : J''_{\lambda,\mu}(u)(u,u) < 0\},\\
\mathcal{N}^0_{\lambda,\mu} &= \{u \in \mathcal{N}_{\lambda,\mu} : J''_{\lambda,\mu}(u)(u,u) = 0\}.
\end{aligned}$$

We shall state our first main result as follows:

Theorem 2.1. *Suppose $\mu \in (0, \mu^*)$, where μ^* follows from (1). Then there are two solutions $u_1, u_2 \in X \setminus \{0\}$ that satisfy the following statements:*

i) $J''_{\lambda,\mu}(u_1, u_1) < 0$, *that is,* $u_1 \in \mathcal{N}^-_{\lambda,\mu}$;

ii) $J''_{\lambda,\mu}(u_2, u_2) > 0$, *that is,* $u_2 \in \mathcal{N}^+_{\lambda,\mu}$;

iii) $J_{\lambda,\mu}(u_2) < 0$, *for all* $\mu \in (0, \mu^*)$.

Moreover, the weak solution $u_2 \in X$ satisfies the following assertions:

a) *For each* $0 < \mu < \mu^{**}$, *we obtain* $J_{\lambda,\mu}(u_2) > 0$;

b) *For* $\mu = \mu^{**}$ *it follows that* $J_{\lambda,\mu}(u_2) = 0$

c) *For each* $\mu^{**} < \mu < \mu^*$ *we obtain also that* $J_{\lambda,\mu}(u_2) < 0$

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SYMMETRY AND SYMMETRY BREAKING FOR HÈNON-TYPE SYSTEMS INVOLVING THE P-LAPLACIAN OPERATOR

ANDERSON DOS S. GONZAGA.[1], LUIZ F. O. FARIA[1] & ANDERSON L. A. DE ARAUJO[2]

Instituto de Ciências Exatas, UFJF, MG, Brasil[1], Departamento de Matemática, UFV, MG, Brasil[2]

andersongonzaga640@gmail.com[†] luiz.faria@ufjf.edu.br[‡] anderson.araujo@ufv.br[§]

Abstract

We study, in this work, for a positive strictly parameter, the existence of at least one radially symmetric solution for the Hénon-type system involving the p-Laplacian operator and, for the sufficiently large parameter, we show the existence of symmetry breaking by presence of non-radial solutions.

1 Introduction

In [4], Hénon proposed the model

$$-\Delta u = |x|^{\theta}|u|^{q-2}u \quad \text{in} \quad \Omega \subset \mathbb{R}^{N}, \tag{1}$$

where $\theta > 0, q > 2$ and $N > 2$, to statistically examine the stability of roughly spherical clusters of stars known as spherical steady in astrophysics. Since that time, a number of scholars have investigated various generalizations of this equation. Motivated by [1, 2, 3], we investigated the existence and the breakdown of the radial solution for the following system of the Hénon-type

$$\begin{cases} -\text{div}(|\nabla w|^{p-2}\nabla w) = |x|^{\theta}|w|^{\alpha-2}w|z|^{\beta} & \text{in} \quad B \\ -\text{div}(|\nabla z|^{p-2}\nabla z) = |x|^{\theta}|w|^{\alpha}|z|^{\beta-2}z & \text{in} \quad B \\ \qquad w > 0, z > 0 & \text{in} \quad B \\ \qquad w = 0, z = 0 & \text{on} \quad \partial B, \end{cases} \tag{2}$$

where $B \subset \mathbb{R}^N$ is the unit ball, $N \geq 3$, $\theta > 0$ and $p < \alpha + \beta < p_\theta^* := \frac{p(N+\theta)}{N-p}$. Since (1) is not variational, we consider the following auxiliary problem

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) = \alpha|x|^{\theta}|u|^{\alpha-2}u|v|^{\beta} & \text{in} \quad B \\ -\text{div}(|\nabla v|^{p-2}\nabla v) = \beta|x|^{\theta}|u|^{\alpha}|v|^{\beta-2}v & \text{in} \quad B \\ \qquad u > 0, v > 0 & \text{in} \quad B, \\ \qquad u = 0, v = 0 & \text{on} \quad \partial B. \end{cases} \tag{3}$$

2 Main Results

Theorem 2.1. *Suppose that $N \geq 3$, $\theta > 0$ and $p < \alpha + \beta < p_\theta^*$. Then there exists a nontrivial radially symmetric solution of* (1).

In the second part of the work, we address the question of extending the results of [2] for (1). More specifically, we study symmetry breaking for (1). For this, we define

$$p^*(N) := \begin{cases} \dfrac{(N+2)p}{N-2p+2} & \text{if N is even;} \\ \dfrac{\left(\left[\frac{N}{2}\right]+2\right)p}{\left[\frac{N}{2}\right]-p+2} & \text{if N is odd,} \end{cases}$$

where $[a]$ denotes the integer part of $a \in \mathbb{R}$ and $p \geq 1$.

Theorem 2.2. *Suppose that $N \geq 4$ and $p < \alpha + \beta < p^*(N)$. Then there exists $\theta_0 > 0$ such that (1) has a non-negative and non-radial solution, for $\theta > \theta_0$.*

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ON A NEUMANN PROBLEM WITH DISCONTINUOUS NONLINEARITIES.

AUGUSTO C. R. COSTA[1,†] & ANDREIA G. PINHEIRO[2,‡]

Instituto de Ciências Exatas e Naturais, UFPA, PA, Brasil[1], Instituto Federal do Pará, IFPA, Bragança, Brasil[2]

augufpa.br[†] andreia.pinheiro@ifpa.edu.br[‡]

Abstract

In this work we study the existence of positive solution for a class of elliptic problems with two critical exponents, Neumann boundary condition and discontinuous nonlinearities involving the $p(x)-$Laplace operator.

1 Introduction

Let us consider the following problem

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u &= f(u) + |u|^{r(x)-2}u\left[\int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx\right]^\alpha \text{ in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} &= g(u) + |u|^{q(x)-2}u\left[\int_{\partial\Omega} \frac{1}{q(x)}|u|^{q(x)}d\Gamma\right]^\beta \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 2$, $p, r \in C(\overline{\Omega})$, $q \in C(\partial\Omega)$, α and β are positive parameters, $f, g : \mathbb{R} \to \mathbb{R}$ has an uncountable set of discontinuity points, $\dfrac{\partial u}{\partial \nu}$ is the outer unit normal derivative, $d\Gamma$ denotes the boundary measure and $\Delta_{p(x)}$ is the $p(x)$-Laplace operator, that is,

$$\Delta_{p(x)}u = \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(|\nabla u|^{p(x)-2}\frac{\partial u}{\partial x_i}\right), \; 1 < p(x) < N.$$

We will consider the sets

$$\mathcal{A}_1 := \{x \in \partial\Omega; q(x) = p_*(x)\} \;\; and \;\; \mathcal{A}_2 := \{x \in \overline{\Omega}; r(x) = p^*(x)\},$$

disjoint and not empty. Furthermore, we assume

$$p^+ < \min\left\{\frac{(\alpha+1)(r^-)^{\alpha+1}}{(r^+)^\alpha}, \frac{(\beta+1)(q^-)^{\beta+1}}{(q^+)^\beta}, p^*, p_*\right\}. \tag{2}$$

We assume the following hypotheses for $f : \mathbb{R} \to \mathbb{R}$:

(f_1) For all $t \in \mathbb{R}$, there exist $C_1 > 0$ and $s_1 \in C(\overline{\Omega})$ with $p(x) < s_1(x) < p^*(x)$ such that

$$|f(t)| \leq C_1(1 + |t|^{s_1(x)-1}).$$

(f_2) For all $t \in \mathbb{R}$, there exists $\theta_1 \in (p^+, p^*)$ such that

$$0 \leq \theta_1 F(t) = \theta_1 \int_0^t f(\sigma)d\sigma \leq t\underline{f}(t),$$

where

$$\underline{f}(t) := \lim_{\epsilon \to 0^+} ess \inf_{|t-\sigma|<\epsilon} f(\sigma) \text{ and } \overline{f}(t) := \lim_{\epsilon \to 0^+} ess \sup_{|t-\sigma|<\epsilon} f(\sigma)$$

are N-measurable.

(f_3) There exists $a_1 > 0$, which will be defined later by checking

$$H(t - a_1) \leq f(t), \ \ \forall t \in \mathbb{R},$$

where H is the function of Heaviside, that is,

$$H(t) = \begin{cases} 0, \text{ if } t \leq 0, \\ 1, \text{ if } t > 0. \end{cases}$$

(f_4) $\limsup\limits_{t \to 0^+} \dfrac{f(t)}{t^{s_1(x)-1}} = 0$ and $f(t) = 0$ if $t \leq 0$.

The hypotheses for the $g : \mathbb{R} \to \mathbb{R}$; (g_1), (g_2), (g_3) and (g_4), are of the types of the f.

2 Main Result

Theorem 2.1. *Assume that $(f_1) - (f_4)$ and $(g_1) - (g_4)$ hold. Then, the problem (1) has a positive solution. Also, if $u \in W^{1,p(x)}(\Omega)$ is a solution to the problem (1), then $|\{x \in \Omega; u(x) > a_1\} \cup \{x \in \partial\Omega; u(x) > a_2\}| > 0$.*

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REGULARIZING EFFECT FOR A CLASS OF MAXWELL-SCHRÖDINGER SYSTEMS.

AYANA PINHEIRO DE CASTRO SANTANA[1,†] & LUÌS HENRIQUE DE MIRANDA[1,‡]

Departamento de Matemática, UnB, Brasilia - DF, Brasil[1]

A.P.C.Santana@mat.unb.br[†] l.h.miranda@mat.unb.br[‡]

Abstract

In this paper we prove the existence and regularity of weak solutions for the following system

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + h(x,u,v) = f \ \text{ in } \ \Omega \\ -\operatorname{div}(M(x)\nabla v) = g(x,u,v) \ \text{ in } \ \Omega \\ \quad u = v = 0 \ \text{ on } \ \partial\Omega \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N \geqslant 3)$, $f \in L^m(\Omega)$ with $m \geqslant (r+\theta+1)'$ and $h,\ g$ be two Carathèodory functions. We will show that under certain conditions imposed on the functions f and g, the system gives rise to a regularizing effect producing the existence of finite solutions for some suffeciently small positive θ and $r > 1$.

1 Introduction

In the present paper, we are interested to investigate the regularity properties for the positive solutions to the system

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + h(x,u,v) = f \ \text{ in } \ \Omega \\ -\operatorname{div}(M(x)\nabla v) = g(x,u,v) \ \text{ in } \ \Omega \\ \quad u = v = 0 \ \text{ on } \ \partial\Omega \end{cases} \tag{1}$$

Where unless otherwise $\Omega \subset \mathbb{R}^N$ is open and bounded, with $N \geqslant 3$, $f \in L^m(\Omega)$ with $m \geqslant 1$, $r > 1$, $0 < \theta < \frac{4}{N-2}$ and $h,\ g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be two Carathéodory functions with following properties:

(a) there exists $c_1,\ c_2 > 0$ such that $c_2|t|^{r-1}|s|^{\theta+1} \leqslant |h(x,t,s)| \leqslant c_1|t|^{r-1}|s|^{\theta+1}$;

(b) $h(x,t,s)t \geqslant 0 \ \ \forall\ (t,s) \in \mathbb{R}\times\mathbb{R}$, a.e. x in Ω;

(c) there exists $d_1,\ d_2 > 0$ such that $d_2|t|^r|s|^\theta \leqslant |g(x,t,s)| \leqslant d_1|t|^r|s|^\theta$;

(d) $g(x,t,s)t \geqslant 0 \ \ \forall\ (t,s) \in \mathbb{R}\times\mathbb{R}$, a.e. x in Ω.

(e) $M(x)$ is a symmetric measurable matrix such that there exists $\alpha, \beta \in \mathbb{R}^+$ satisfying

$$\alpha|\xi|^2 \leqslant M(x)\xi\cdot\xi\ ,\ |M(x)| \leqslant \beta \ \text{ for every } \ \xi \in \mathbb{R}^N.$$

Under appropriate conditions, encompasses the so-called Maxwell-Schrödinger system. The general idea regarding these systems is that due to the strong coupling between both equations, solutions have zones where they are more regular than expected from standard regularity theory. This phenomenon was discovered in [2] by L. Boccardo and since then has been addressed by D. Arcoya, L. Orsina, and R. Durastani, among others, where we refer the reader to [1, 3, 4, 5, 6] and the references therein.

2 Main Results

To emphasize the regularity gains obtained in the solutions u and v of our problem, we will patent a definition that will establish conditions necessary for us to gain regularity better than that stressed by the classical Stamppachia theory.

Definition 2.1. *Let $F \in L^m(\Omega)$ where $1 \leqslant m < \frac{N}{2}$. Consider w a distributional solution of*

$$-div(M(x)\nabla w) = F(x). \tag{1}$$

(i) If $w \in L^s(\Omega)$ where $s > m^{**}$ we say u is Lebesgue regularized.

(ii) If $w \in W_0^{1,t}(\Omega)$ where $t > m^*$ we say that w is Sobolev regularized.

Theorem 2.1. *Let $f \geqslant 0$ be a function in $L^m(\Omega)$ with $m \geqslant (r+\theta+1)'$, $r > 1$, and $0 < \theta < \frac{4}{N-2}$. Then there exists a weak solution (u, v) of system (1), with $u \in W_0^{1,2}(\Omega) \cap L^{r+\theta+1}(\Omega)$, $u \geqslant 0$ a.e. in Ω and $v \in W_0^{1,2}(\Omega)$, $v \geqslant 0$ a.e in Ω.*

In the next results, for the sake of clarity, by using Definition 2.1 we detail the gain of regularity in Lebesgue spaces or Sobolev spaces, for the solutions of our system given by Theorem 2.1.

Corollary 2.1. *Let (u, v) be the weak solution of (1), given by Theorem 2.1.*

(A) If $r + \theta + 1 > 2^*$ and $(r + \theta + 1)' \leqslant m < (2^*)'$, then u is Lebesgue and Sobolev regularized.

(B) If $r + \theta + 1 > 2^*$ and $(2^*)' \leqslant m < \frac{N(r+\theta+1)}{N+2(r+\theta+1)}$ then u is Lebesgue regularized.

(C) If $r + \theta + 1 < \left(\frac{2^*}{\theta+1}\right)'$ then v is Sobolev regularized.

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SOME ASYMMETRIC SEMILINEAR ELLIPTIC INTERFACE PROBLEMS

BRÁULIO B. V. MAIA[1], CRISTIAN MORALES-RODRIGO[2] & ANTONIO SUÁREZ[2]

Universidade Federal Rural da Amazônia, UFRA, PA, Brasil[1], Universidad de Sevilla, Espanha[2]

braulio.maia@ufra.edu.br†

Abstract

Neste trabalho, estudaremos dois problemas elípticos semilineares de interface que surgem da dinâmica populacional. Em ambos os problemas, cada população vive em um subdomínio e eles interagem em uma fronteira comum, que age como uma barreira geográfica. Analisamos a existência e unicidade de soluções positivas. Para isto, precisamos estudar os problemas de autovalores associados aos problemas.

1 Introdução

Neste trabalho, esatremos interessados em estudar os seguintes problemas de interface:

$$\begin{cases} -\Delta u_i = \lambda m_i(x) u_i (1-u_i) & \text{em } \Omega_i, \\ \partial_\nu u_i = \gamma_i (u_2 - u_1) & \text{sobre } \Sigma, \\ \partial_\nu u_2 = 0 & \text{sobre } \Gamma, \end{cases} \tag{1}$$

e

$$\begin{cases} -\Delta u_i + c_i(x) u_i = \lambda m_i(x) u_i - u_i^{p_i} & \text{em } \Omega_i, \\ \partial_\nu u_i = \gamma_i (u_2 - u_1) & \text{sobre } \Sigma, \\ \partial_\nu u_2 = 0 & \text{sobre } \Gamma, \end{cases} \tag{2}$$

onde Ω é um domínio limitado de $\mathbb{R}^N$ de forma que

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Sigma,$$

com Ω_i sendo subdomínios, com interface interna $\Sigma = \partial\Omega_1$, e $\Gamma = \partial\Omega_2 \setminus \Sigma$. Também, denotamos por ν_i o vetor normal exterior Ã Ω_i, e definiremos $\nu := \nu_1 = -\nu_2$ (veja Figura 1 onde temos um exemplo ilustrado de Ω).

Nos problemas (1) e (2) as funções c_i, m_i são limitadas em Ω_i, $\lambda \in \mathbb{R}$, $g_i > 0$, $p_i > 1$ e $\partial_\nu u_i = \nabla u_i \cdot \nu$.

O problema (1) surge da genética de populações, onde u_i e $1-u_i$ são as frequências de A e a, respectivamente, os dois alelos dos genes considerados (veja por ex. [2], [4] e suas referênicas para uma explicação biológica do modelo).

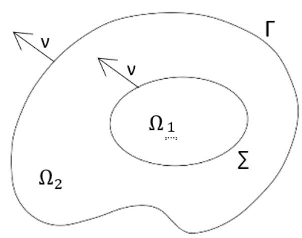

Figure 1: Uma configuração possível do domínio $\Omega = \Omega_1 \cup \Omega_2 \cup \Sigma$.

Por outro lado, o problema (2) é um modelo geral de dinâmica populacional, onde cada espécie u_i vive em um domínio Ω_i seguindo a lei logística, e ambas espécies interagem em uma fronteira comum Σ. Em ambos os modelos, a condição de interface é chamada de Kedem-Katchaslky.

Para estudar existência e unicidade de soluções positivas para (1) e (2) é necessário um estudo detalhado dos seguintes problemas de autovalor

$$\begin{cases} -\Delta u_i + c_i(x)u_i = \lambda u_i & \text{em } \Omega_i, \\ \partial_\nu u_i = \gamma_i(u_2 - u_1) & \text{sobre } \Sigma, \\ \partial_\nu u_2 = 0 & \text{sobre } \Gamma, \end{cases} \tag{3}$$

e

$$\begin{cases} -\Delta u_i + c_i(x)u_i = \lambda m_i(x) u_i & \text{em } \Omega_i, \\ \partial_\nu u_i = \gamma_i(u_2 - u_1) & \text{sobre } \Sigma, \\ \partial_\nu u_2 = 0 & \text{sobre } \Gamma. \end{cases} \tag{4}$$

Estes problemas foram analisados em [4] (ver também [3]) no caso particular $\gamma_1 = \gamma_2$, que torna nossa problema autoadjunto. Assim, segue da mythria espectral a existência de uma sequência autovalores reais (Λ_n) de (3) com $\Lambda_n \to \infty$.

Em nosso trabalho consideraremos o caso assimétrico, isto é, $\gamma_1 \neq \gamma_2$, o que implica que os problemas (1.3) e (1.4) não são autoadjuntos. Para contornar essa dificuldade,usamos o mythrema de Krein-Rutmann para provar a existência de autovalor para (1.3), denotado por $\Lambda_1(c_1; c_2)$.

Denotaremos os autovalores de (1.4) são os zeros da aplicação

$$F(\lambda) := \Lambda_1(c_1 - \lambda m_1, c_2 - \lambda m_2).$$

Estudamos essa aplicação no qual o comportamento depende fortemente de c_i e do sinal de m_i, e podemos concluir que, sob condições em c_i, existe dois autovalores principais em (4).

Uma vez estudado os problemas de autovalores, é possível provar a existência de soluções de (1) e (2) usando o método de sub-supersoluções. A unicidade é provada usando mythrema de ponto fixo em (1) e o resultado de Brezis-Oswald [1] em (2).

2 Principais resultados

Theorem 2.1. *Assuma que $p_i > 1$. (2) possui ao menos uma solução positiva se e só se $F(\lambda) < 0$. Além disso, se tal solução positiva exista, ela é única.*

Theorem 2.2. *Assuma que $F(\lambda) < 0$ e $F(-\lambda) < 0$, então o problema (1) possui uma única solução positiva (u_1, u_2) tal que $0 < u_1, u_2 < 1$.*

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GLOBAL EXISTENCE OF SOLUTIONS FOR BOUSSINESQ SYSTEM WITH ENERGY DISSIPATION

CHARLES AMORIM[1,†], EDER MATEUS[2,‡] & MARCELO FERNANDES DE ALMEIDA[3,§]

Departamento de Matemática de Itabaiana-DMAI, UFS, SE, Brasil[1] Departamento de Matemática-DMA, UFS, SE, Brasil[2].

amorimcb@academico.ufs.br[†] edermateus@mat.ufs.br[‡] marcelo@mat.ufs.br[§]

Abstract

In this paper we are interested to show global existence of solutions $[u,\theta]$ for Boussinesq system coupled by bilinear energy dissipation $\Phi(u) = 2\mu\,\mathcal{E}(u)\cdot\mathcal{E}(u)$ on smooth bounded domain $\Omega \subseteq \mathbb{R}^n$ or whole space $\mathbb{R}^n$, $n \geq 3$, when the initial data $[u_0,\theta_0] \in \mathbf{X}_0 = W^{1,n/2}_\sigma(\Omega) \times L^{n/2}(\Omega)$ is sufficiently small and the external force $F(\theta) = \varrho f\theta e_n$ has low regularity in the sense $t^{1/2-b/2n} f \in L^\infty\left((0,T) : L^b(\Omega)\right)$ or $f \in L^s((0,T) : L^b(\Omega))$, where $\frac{2}{s} = 1 - \frac{n}{b}$ and $b \in [n,\infty)$.

1 Introduction

There is a vast literature on the studies of local and global existence of weak and strong solutions for viscous heat-conductive Navier-Stokes equations when the fluid is incompressible [3] or compressible [2]. We refer [3] for global existence of strong solutions and for a good review about this model. In the case when external forces dependent of temperature, $F = F(\theta)$, the viscous heat-conductive Navier-stokes equations can be described by Boussinesq approximation system with viscous dissipation of energy, namely,

$$\begin{cases} u_t - \operatorname{div}(\mu\nabla u) + (u\cdot\nabla)u + \nabla p = F(\theta) & \text{in} \quad \Omega\times(0,T) \\ \theta_t + u\cdot\nabla\theta - \operatorname{div}(\kappa\nabla\theta) = \Phi(u), & \text{in} \quad \Omega\times(0,T) \\ \operatorname{div} u = 0, & \text{in} \quad \Omega\times(0,T) \end{cases} \tag{1}$$

and

$$\begin{cases} u(x,0) = u_0 \quad \operatorname{div}\, u_0 = 0 \quad \text{and} \quad \theta(x,0) = \theta_0 \quad \text{on} \quad \Omega\times\{t=0\} \\ u|_{\partial\Omega} = 0 \quad \text{and} \quad \theta|_{\partial\Omega} = 0 \quad \text{on} \quad \partial\Omega\times(0,T). \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a smooth bounded domain or $\Omega = \mathbb{R}^n$ itself, and $p(x,t) \in \mathbb{R}$ denotes the pressure derived from *stress tensor* T_{ij} acting on unknown viscous fluid $u = (u_1, u_2, \cdots, u_n)$. The gains and losses of energy of $u = u(x,t)$ is described by $(1)_2$, where $\kappa > 0$ denotes the coefficient of heat conductivity and

$$\Phi(u) = 2\mu\,\mathcal{E}(u)\cdot\mathcal{E}(u) \quad \text{and} \quad \mathcal{E}_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i,j \in \{1,2,\cdots,n\}$$

denotes the dissipation of energy derived from viscosity $\mu > 0$, here we assume μ and κ constant. The Boussinesq approximation, designed to simplify classical heat-conductive Navier-Stokes equations, essentially says there are flows where the fluctuations of density ρ can be ignored except the external forces $F(\theta) = \varrho f\theta e_n$ (buoyancy term) which is assumed proportional to temperature, $\theta = \theta(x,t)$ of the fluid. Here, ϱ is the coefficient of volume expansion (considered sufficiently small) and f denotes the acceleration of gravity.

2 Main Results

Definition 2.1 (mild solution). *Let* $Y_j \hookrightarrow \mathcal{D}'\Omega)$, $j = 1, 2$, *be Banach spaces and* $T > 0$. *The function* $[u, \theta] \in L^{s_1}((0,T) : Y_1) \times L^{s_2}((0,T) : Y_2)$ *is called a mild solution to* (1), *if it satisfies the integral equation*

$$\begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} = e^{-tL} \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} - \int_0^t e^{-(t-s)L} \begin{bmatrix} \mathbb{P}(u \cdot \nabla)u \\ u \cdot \nabla\theta \end{bmatrix}(s) ds + \int_0^t e^{-(t-s)L} \begin{bmatrix} \mathbb{P}(\varrho\theta f) \\ \Phi(u) \end{bmatrix}(s) ds, \tag{3}$$

for all $t \in (0,T)$ *and* $[u(t), \theta(t)] \rightharpoonup [u_0, \theta_0]$ *in sense of distributions as* $t \to 0^+$. *Here, if* $s_1 = s_2 = \infty$ *then* $t^{\gamma/2} f \in L^\infty\left((0,T) : L^b(\Omega)\right)$ *is taken such that*

$$\|f\|_{\widehat{\mathbf{X}}^{\gamma}_{b,T}} = \sup_{0<t<T} t^{\frac{\gamma}{2}} \|f(\cdot,t)\|_{L^b(\Omega)} < \infty \quad \textit{with} \quad \gamma = 1 - \frac{n}{b} \quad \textit{and} \quad b \in [n, \infty) \tag{4}$$

and for $1 < s_1, s_2 < \infty$ *we consider* $f \in L^s((0,T) : L^b(\Omega))$ *with* $\frac{2}{s} = 1 - \frac{n}{b}$ *and* $b \in (n, \infty)$.

Theorem 2.1. *Let* $\Omega \subset \mathbb{R}^n$ *be smooth bounded domain or whole space* $\mathbb{R}^n$ *itself,* $n \geq 3$. *For* $T > 0$ *denote by* $\mathbf{X}^{\alpha,\beta}_{q,r,T}$ *the space of distributions* $[u(t), \theta(t)] \in \mathbf{X}_0 = \dot{W}^{1,n/2}_{\sigma}(\Omega) \times L^{n/2}(\Omega)$ *such that*

$$\|[u, \theta]\|_{\widehat{\mathbf{X}}^{\alpha,\beta}_{q,r,T}} := \sup_{0<t<T} t^{\frac{\alpha}{2}} \|u(\cdot,t)\|_{\dot{W}^{1,q}_{\sigma}(\Omega)} + \sup_{0<t<T} t^{\frac{\beta}{2}} \|\theta(\cdot,t)\|_{L^r(\Omega)} < \infty, \tag{5}$$

where $\alpha = 2 - n/q$ *and* $\beta = 2 - n/r$. *Assume* $\max\{2, n/2\} < q < r < n$ *and* $n \leq b < \infty$ *be such that* $1/b + 1/r > 1/q$.

(A) (Local existence and uniqueness) *For any* $f \in \widehat{\mathbf{X}}^{\gamma}_{b,T}$ *and* $[u_0, \theta_0] \in \mathbf{X}_0$, *there is* $T > 0$ *and a mild solution* $[u, \theta]$ *to the problem* (1)-(2) *in the class* $[u(t), \theta(t)] \in \mathbf{X}_0$ *such that*

$$\|\nabla u(\cdot,t)\|_{L^q(\Omega)} \lesssim t^{-\frac{\alpha}{2}} \quad \textit{and} \quad \|\theta(\cdot,t)\|_{L^r(\Omega)} \lesssim t^{-\frac{\beta}{2}} \quad \textit{for all} \quad t \in (0,T). \tag{6}$$

The previous solution $[u, \theta]$ *is unique in the closed ball* $\bar{B}(0, \frac{2\varepsilon}{1-\zeta}) \subset \mathbf{X}^{\alpha,\beta}_{q,r,T}$, *for some* $\varepsilon > 0$ *and* $\zeta \in (0,1)$. *Moreover, the mild solution* $[u, \theta]$ *is stable with respect to* $[u_0, \theta_0] \in \mathbf{X}_0$ *in the following sense: There are constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$\left\| \begin{bmatrix} u \\ \theta \end{bmatrix} - \begin{bmatrix} \tilde{u} \\ \tilde{\theta} \end{bmatrix} \right\|_{\mathbf{X}^{\alpha,\beta}_{q,r,T}} \leq \left(1 - \zeta - \frac{4\varepsilon(c_1 + c_2)}{1 - \zeta}\right)^{-1} \left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} - \begin{bmatrix} \tilde{u}_0 \\ \tilde{\theta}_0 \end{bmatrix} \right\|_{\mathbf{X}_0}, \tag{7}$$

whenever $[\tilde{u}, \tilde{\theta}]$ *is another solution of the integral equation* (3) *with initial data* $[\tilde{u}_0, \tilde{\theta}_0] \in \mathbf{X}_0$.

(B) *The previous mild solution* $[u, \theta]$ *of* (1)-(2) *satisfies*

$$t^{\frac{\alpha}{2}} u(t) \in BC([0,T) : \dot{W}^{1,q}_{\sigma}(\Omega)) \quad \textit{and} \quad t^{\frac{\beta}{2}} \theta(t) \in BC([0,T) : L^r(\Omega)). \tag{8}$$

Moreover, $[t^{\frac{\alpha}{2}} u(t), t^{\frac{\beta}{2}} \theta(t)] \to 0$ *in* $\dot{W}^{1,q}_{\sigma}(\Omega) \times L^r(\Omega)$ *and* $[u(t), \theta(t)] \to [u_0, \theta_0]$ *in* $\mathbf{X}_0$ *as* $t \to 0^+$.

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BREZIS-KAMIN TYPE RESULTS INVOLVING LOCALLY INTEGRABLE WEIGHTS

DIEGO FERRAZ.[1], AILTON R. DA SILVA[1] & PEDRO UBILLA[2]

Departamento de Matemática, UFRN, Natal, Brasil[1],
Departamento de Matematica, Universidad de Santiago de Chile, Santiago, Chile[2]

diego.ferraz.br@gmail.com[†] ardsmat@gmail.com[‡] pedro.ubilla@usach.cl[§]

Abstract

We study existence and asymptotic behavior of entire positive bounded solutions for the following class of semilinear elliptic problem

$$\begin{cases} L(u) = \varrho(x) f(x,u) & \text{in } \mathbb{R}^N,\ N \geq 3, \\ \quad u > 0, & \text{in } \mathbb{R}^N, \end{cases}$$

where $0 \leq \varrho \in L^p_{loc}(\mathbb{R}^N)$, for some $N < p \leq \infty$. Here L is a local uniform elliptic operator and $f(x,s)$ is a nonlinearity with sublinear behavior at zero and at $+\infty$. This type of result has already been studied in the celebrated work by H. Brezis and S. Kamin for the case when $L = -\Delta$ and $\varrho \in L^\infty_{loc}(\mathbb{R}^N)$. Our approach allows us to include for instance $-div\,((1+|x|^\mu)^\nu \nabla u) = u^q(|x|^\alpha + |x|^\beta)^{-1}$ with suitable $\alpha,\ \beta > 0,\ \mu,\ \nu \in \mathbb{R}$ and $0 < q < 1$. Here we include two local uniform elliptic situations: $\mu > 0$ with $\nu = 1$ or $\nu = -1$.

1 Introduction

We study existence and behavior of entire positive bounded solutions of the following problem

$$\begin{cases} L(u) = \varrho(x) f(x,u) & \text{in } \mathbb{R}^N,\ N \geq 3. \\ \quad u > 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{P}$$

where $L(u)$ is a local elliptic operator in divergence form, that is, for any compact set $\Omega \subset \mathbb{R}^N$, there are $\lambda_\Omega, \Lambda_\Omega > 0$ such that

$$L(u) = -\sum_{i,j=1}^{N} D_j(a_{ij}(x) D_i(u)), \quad a_{ij} = a_{ji} \in C^{0,1}_{loc}(\mathbb{R}^N),$$

$$\lambda_\Omega |\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \leq \Lambda_\Omega |\xi|^2,\ \forall\, x \in \Omega,\ 0 < \lambda_\Omega \leq \Lambda_\Omega < \infty.$$

and $C^{0,1}_{loc}(\mathbb{R}^N)$ denotes the space of locally Lipschitz functions. The study of Problem ($\mathcal{P}_\Phi$) was motivated by the following well known problem

$$\begin{cases} -\Delta u = \varrho(x) u^q & \text{in } \mathbb{R}^N,\ N \geq 3. \\ \quad u > 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{BK}$$

where $0 < q < 1$ and ϱ is a nonnegative locally bounded weight. The celebrated work of H. Brezis and S. Kamin [1], proved the existence and asymptotically behavior for classical solutions of (BK). Their argument to prove the existence of bounded positive solutions of (BK) was strongly based on the linear problem

$$-\Delta u = \varrho(x), \tag{LBK}$$

more precisely, they proved that:

Theorem 1.1. *[1, Theorem 1] Problem (BK) has a bounded solution if and only if, the linear problem (LBK) has a bounded solution. Moreover, there is a minimal positive solution of (BK).*

It is important to observe that it is always assumed that ϱ is a locally bounded function. At this point a natural question arises: Is it possible to prove a Brezis-Kamin type result in the sense of Theorem 1.1 considering a more general weight $\varrho \in L^p_{loc}(\mathbb{R}^N)$, $1 \leq p < \infty$, that is not necessarily locally bounded? Our main objective in our work is to give a positive answer to this question, where we approach Problem ($\mathcal{P}_\Phi$) by means of a Brezis-Kamin type result.

2 Main Results

Theorem 2.1. *Assume*

(ϱ_1) $\varrho(x) \geq 0$ a.e. (i.e. almost everywhere) in $\mathbb{R}^N$ and there exists a bounded set $\mathcal{O}$, with positive measure, such that $\varrho > 0$ in $\mathcal{O}$.

(ϱ_2) $\varrho \in L^p_{loc}(\mathbb{R}^N)$, for some $p > N$.

(f_1) $\displaystyle\lim_{t\to 0_+} \frac{f(x,t)}{t} = +\infty$,

(f_2) either one of the following conditions hold:

1. *given $[0,a]$ there exists $M_a > 0$ such that $f(x, Mt) \leq M$, for all $M > M_a$, $t \in [0,a]$, and*

$$\lim_{t\to +\infty} \frac{f(x,t)}{t} = 0. \tag{1}$$

2. *there exists a continuous nondecreasing function $f_0 : [0,\infty) \to [0,\infty)$ satisfying (1) and $f(x,s) \leq f_0(s)$ for each $s \in [0,\infty)$ (a.e. $x \in \mathbb{R}^N$).*

(f_3) the function $t \mapsto \dfrac{f(x,t)}{t}$ *is decreasing on $(0,\infty)$,*

(f_4) for each $[0,l]$, there is a constant $K > 0$ such that $t \mapsto f(x,t) + Kt$ is nondecreasing, for a.e. $x \in \mathbb{R}^N$.

and that the equation $L(U) = \varrho(x)$ in $\mathbb{R}^N$, has a solution $U \in E_p(\mathbb{R}^N)$. Then Problem ($\mathcal{P}_\Phi$) has a positive solution $u \in E_p(\mathbb{R}^N)$, such that $u \leq CU$ in $\mathbb{R}^N$, for some $C > 0$, where $E_p(\mathbb{R}^N) = L^\infty(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N)$.

Corollary 2.1. *Suppose (ϱ_1) and (ϱ_2). Also that, $\varrho(x) = \varrho(|x|)$, with*

$$\lim_{r\to 0} r^2 \varrho(r) \in \mathbb{R} \text{ and } \int_0^\infty r\varrho(r)dr < +\infty.$$

Then

$$\begin{cases} -\Delta u = \varrho(x) u^q & \text{in } \mathbb{R}^N,\ N \geq 3,\ 0 < q < 1, \\ u > 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{BK}$$

has a positive solution $u \in E_p(\mathbb{R}^N)$.

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QUASILINEAR ELLIPTIC PROBLEMS WITH GENERAL CONCAVE-CONVEX NONLINEARITIES

J. RODRIGUES[1], EDCARLOS D. SILVA[2] & M. L. M. CARVALHO[2]

DM, UFPA, PA, Brazil[1], Instituto de Matemática e Estatistica, UFG, GO, Brazil[2]

edcarlos@ufg.br[†] marcos_leandro_carvalho@ufg.br[‡] jrsantosjr.ufpa@gmail.com[§]

Abstract

In this talk we establish existence and multiplicity of solution for the following class of quasilinear elliptic problems

$$\begin{cases} -\Delta_\Phi u = \lambda a(x)|u|^{q-2}u + |u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$, is a smooth bounded domain, $1 < q < \ell \leq m < p < \ell^*$ and $\Phi : \mathbb{R} \to \mathbb{R}$ is suitable N-function. The main feature here is to determine whether the Nehari method can be applied finding the largest positive number $\lambda^* > 0$ such that our main problem admits at least two distinct solutions for each $\lambda \in (0, \lambda^*)$.

1 Introduction

This talk is devoted to establish existence and multiplicity of solutions to the following class of quasilinear elliptic problem

$$\begin{cases} -\Delta_\Phi u = \lambda a(x)|u|^{q-2}u + |u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$ is a smooth bounded domain, $1 < q < \ell \leq m < p < \ell^*, \ell^* = N\ell/(N-\ell), N \geq 2$. Recall that $\Delta_\Phi u := \operatorname{div}(\phi(|\nabla u|)\nabla u)$ is the Φ-Laplacian operator. Throughout this work we shall consider $\Phi : \mathbb{R} \to \mathbb{R}$ an even function defined by

$$\Phi(t) = \int_0^{|t|} s\phi(s)\, \mathrm{d}s, t \in \mathbb{R}. \tag{1}$$

2 Main Results

In this work we shall assume the following hypotheses: $\phi : (0, +\infty) \to (0, +\infty)$ is a C^1-function satisfying the following assumptions:

(ϕ_1) $t\phi(t) \mapsto 0$, as $t \mapsto 0$ and $t\phi(t) \mapsto \infty$, as $t \mapsto \infty$;

(ϕ_2) $t\phi(t)$ is strictly increasing in $(0, \infty)$;

(ϕ_3) $\displaystyle\int_0^1 \frac{\Phi^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\mathrm{d}\tau < \infty$ and $\displaystyle\int_1^{+\infty} \frac{\Phi^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\mathrm{d}\tau = \infty$;

(ϕ_4) There holds

$$-1 < \ell - 2 := \inf_{t>0} \frac{\phi'(t)t}{\phi(t)} \leq \sup_{t>0} \frac{\phi'(t)t}{\phi(t)} := m - 2 < \infty.$$

Furthermore, we assume also that

(H_1) $1 < q < \ell \leq m < p < \ell^*, \ell^* = N\ell/(N-\ell)$;

(H_2 The function $a : \Omega \to \mathbb{R}$ belongs to $L^{\ell/(\ell-q)}(\Omega)$ and $a(x) \geq a_0 > 0$ for each $x \in \Omega$;

(H_3) The function

$$t \mapsto \frac{(2-q)\phi(t) + \phi'(t)t}{t^{p-2}}$$

is strictly decreasing for each $t > 0$.

Throughout this work we shall consider the Nehari set as follows

$$\mathcal{N}_\lambda = \left\{u \in W_0^{1,\Phi}(\Omega) \setminus \{0\} : J'_\lambda(u)u = 0\right\}, \lambda > 0.$$

The Nehari set can be separated in the following form:

$$\begin{aligned}
\mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : J''_\lambda(u)(u,u) > 0\}, \\
\mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : J''_\lambda(u)(u,u) < 0\}, \\
\mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : J''_\lambda(u)(u,u) = 0\}.
\end{aligned}$$

The main feature in the present work is to ensure that the minimization problems

$$c_{\mathcal{N}^+} := \inf\{J_\lambda(u) : u \in \mathcal{N}^+\}, c_{\mathcal{N}^-} := \inf\{J_\lambda(u) : u \in \mathcal{N}^-\} \tag{2}$$

are attained by some specific functions.

These functionals give us sharp conditions taking into account the Nehari method for existence of solutions $u \in X$ for our main problem for each $\lambda \in (0, \lambda^*)$. Namely, we consider the nonlinear Rayleigh quotient:

$$\lambda_* := \inf_{u \in W_0^{1,\Phi}(\Omega)\setminus\{0\}} \Lambda_e(u), \lambda^* := \inf_{u \in W_0^{1,\Phi}(\Omega)\setminus\{0\}} \Lambda_n(u). \tag{3}$$

Theorem 2.1. *Suppose that assumptions $(\phi_1) - (\phi_4)$ and $(H_1) - (H_3)$ are satisfied. Then $0 < \lambda_* < \lambda^* < \infty$. Furthermore, Problem (P_λ) admits at least one positive ground state solution $u \in X$ for each $\lambda \in (0, \lambda^*)$ satisfying the following properties: There holds $J''_\lambda(u)(u,u) > 0, J_\lambda(u) < 0$.*

Theorem 2.2. *Suppose that assumptions $(\phi_1) - (\phi_4)$ and $(H_1) - (H_3)$ are satisfied. Then $0 < \lambda_* < \lambda^* < \infty$. Furthermore, Problem (P_λ) admits at least one positive solution $v \in X$ for each $\lambda \in (0, \lambda^*)$ satisfying the following properties: There holds $J''_\lambda(v)(v,v) < 0$ and $J_\lambda(v) > 0$ for each $\lambda \in (0, \lambda_*)$. Furthemore, we obtain $J_\lambda(v) = 0$ for $\lambda = \lambda_*$ and $J_\lambda(v) < 0$ for each $\lambda \in (\lambda_*, \lambda^*)$.*

Corollary 2.1. *Suppose that assumptions $(\phi_1) - (\phi_4)$ and $(H_1) - (H_3)$ are satisfied. Then $0 < \lambda_* < \lambda^* < \infty$. Furthermore, Problem (P_λ) has at least two positive solution provided that $\lambda \in (0, \lambda^*)$.*

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GLOBAL STRONG SOLUTIONS FOR NAVIER-STOKES EQUATIONS WITH DENSITY VARIABLE AND MASS DIFFUSION IN THIN DOMAINS

FELIPE W. CRUZ[1,†], MIRELLE M. SOUSA[1,‡] & MARKO A. ROJAS-MEDAR[2,§]

Universidade Federal de Pernambuco, PE, Brazil[1], Universidad de Tarapacá, Chile[2]

felipe.wcruz@ufpe.br[†] mirelle.moura@ufpe.br[‡] mmedar@academicos.uta.cl[§]

Abstract

We prove existence of global in time strong solutions for the Navier-Stokes equations in the presence of mass diffusion in 3-dimensional thin domains of the form $\Omega \overset{def}{=} \mathbb{R}^2 \times (0, \epsilon)$, where $0 < \epsilon \leq 1$. Moreover, we show that, when $\epsilon \to 0^+$, the velocity and the gradient of density tend to vanish away from the initial time.

1 Introduction

In [1] a model has been proposed to describe the motion of a two component fluid taking into consideration the diffusion between its parts. Here, this model is studied in a thin domain Ω_ϵ with viscosity $\mu > \lambda/2 \operatorname{osc}(\rho_0)$, where λ is the diffusion coefficient and ρ_0 is the initial density taken strictly positive. The thin domain can be characterized as follows: given a suitable domain $\Omega \subset \mathbb{R}^n$, a corresponding thin domain is a set of the form

$$\Omega_\epsilon = \{(\mathbf{x}, y) \in \mathbb{R}^{n+1};\ \mathbf{x} \in \Omega \text{ and } \ 0 < y < g(\mathbf{x}, \epsilon)\}$$

where $\epsilon \in (0, \epsilon_0]$. Here ϵ_0 is a parameter and $g : \bar{\Omega} \times [0, \epsilon_0] \to \mathbb{R}$ is a function of class C^3 such that

$$g(\mathbf{x}, 0) = 0, \quad \frac{\partial g}{\partial \epsilon}(\mathbf{x}, 0) > 0, \quad g(\mathbf{x}, \epsilon) > 0, \quad \forall \mathbf{x} \in \bar{\Omega},\ \forall \epsilon \in (0, \epsilon_0)$$

The simplest case is when $g(\mathbf{x}, \epsilon) = \epsilon$ and therefore $\Omega_\epsilon = \Omega \times (0, \epsilon)$. For simplicity, this will be the case that we will consider throughout this work. Consider a set of the form $\Omega_\epsilon \times (0, \infty)$ where $\Omega_\epsilon = \Omega \times (0, \epsilon)$ with $\Omega = \mathbb{R}^2$ and $\epsilon \in (0, 1)$. The equations governing the motion of a two-component fluid with diffusion obtained in [1] are

$$\begin{cases} \rho \mathbf{u}_t + (\rho \mathbf{u} \cdot \nabla)\mathbf{u} - \mu \Delta \mathbf{u} - \lambda(\mathbf{u} \cdot \nabla)\nabla \rho - \lambda(\nabla \rho \cdot \nabla)\mathbf{u} + \nabla P = 0 \\ \rho_t + \mathbf{u} \cdot \nabla \rho - \lambda \Delta \rho = 0 \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \tag{1}$$

with initial conditions

$$\begin{cases} \rho(\cdot, 0) = \rho_0(\cdot), \quad \mathbf{u}(\cdot, 0) = \mathbf{u}_0(\cdot) \ \text{ in } \ \Omega_\epsilon \\ \mathbf{u} = 0 \ \text{ on } \ \Sigma_\epsilon = \Gamma_\epsilon \times (0, \infty) \\ \dfrac{\partial \rho}{\partial \mathbf{n}} = 0 \ \text{ on } \ \Sigma_\epsilon \end{cases} \tag{2}$$

where $\Gamma_\epsilon \overset{def}{=} \{(x_1, x_2, x_3);\ (x_1, x_2) \in \mathbb{R}^2, x_3 = 0 \ \text{ or } \ x_3 = \epsilon\}$. Existence and uniqueness results can be found in [1].

Let us introduce the following functional spaces

$$\mathcal{V} = \{\mathbf{u} \in \mathbf{C}_0^\infty(\Omega_\epsilon);\ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega_\epsilon\}, \quad H_N^k(\Omega_\epsilon) = \{\rho \in H^k(\Omega_\epsilon);\ \nabla \rho = 0 \text{ on } \Gamma_\epsilon\},\ k = 2, 3,$$

and denote by $\mathbf{H}$ and $\mathbf{V}$ the closure of $\mathcal{V}$ in $\mathbf{L}^2(\Omega_\epsilon)$ and $\mathbf{H}_0^1(\Omega_\epsilon)$, respectively.

2 Auxiliary results and complementary remarks

In this work we prove that the unique solution of the problem (1)-(2) exists globally in time, if the initial velocity and density satisfies

$$\epsilon^{\frac{1}{2}}(\|\nabla \mathbf{u}_0\|^2 + \|\Delta\rho_0\|^2)^{\frac{1}{2}} \le c_0 \tag{3}$$

with $c_0 > 0$ a small enough constant. The next theorem is a compilation of Poincare's type inequalities suitable for this work.

Theorem 2.1. *(Poincare's inequalities)*

i) If $\mathbf{u} \in \mathbf{H}^2(\Omega_\epsilon) \cap \mathbf{H}_0^1(\Omega_\epsilon)$ *then* $\|\mathbf{u}\| \le \epsilon\|D_3\mathbf{u}\| \le \epsilon\|\nabla\mathbf{u}\|$ *and* $\|\nabla\mathbf{u}\| \le \epsilon\|\Delta\mathbf{u}\|$.

ii) If $\rho \in H_N^3(\Omega_\epsilon)$ *then* $\|\nabla\rho\| \le \epsilon\|\partial_3\nabla\rho\| \le \epsilon\|\Delta\rho\|$ *and* $\|\Delta\rho\| \le \epsilon\|\nabla\Delta\rho\|$.

3 Main Results

Let

$$\begin{aligned}
\mathcal{U}(t) &\overset{def}{=} \|(\mathbf{u},\nabla\rho)\|^2_{L_t^\infty(\mathbf{L}^2(\Omega_\epsilon))} + \|(\nabla\mathbf{u},\Delta\rho)\|^2_{L_t^2(\mathbf{L}^2(\Omega_\epsilon))},\\
\mathcal{V}(t) &\overset{def}{=} \|(\nabla\mathbf{u},\Delta\rho)\|^2_{L_t^\infty(\mathbf{L}^2(\Omega_\epsilon))} + \|(\mathbf{u}_t,\nabla\rho_t,\Delta\mathbf{u},\nabla\Delta\rho)\|^2_{L_t^2(\mathbf{L}^2(\Omega_\epsilon))},\\
\mathcal{W}(t) &\overset{def}{=} \|(\mathbf{u}_t,\nabla\rho_t,\Delta\mathbf{u},\nabla\Delta\rho,\nabla p)\|^2_{L_t^\infty(\mathbf{L}^2(\Omega_\epsilon))} + \|(\nabla\mathbf{u}_t,\Delta\rho_t)\|^2_{L_t^2(\mathbf{L}^2(\Omega_\epsilon))}
\end{aligned}$$

and $\mathcal{U}_0 \overset{def}{=} \|\mathbf{u}_0\|^2 + \|\nabla\rho_0\|^2$, $\mathcal{V}_0 \overset{def}{=} \|\nabla\mathbf{u}_0\|^2 + \|\Delta\rho_0\|^2$, $\mathcal{W}_0 \overset{def}{=} \|\Delta\mathbf{u}_0\|^2 + \|\nabla\Delta\rho_0\|^2$.

Theorem 3.1. *Assume that the initial data* ρ_0 *and* $\mathbf{u_0}$ *satisfies the conditions* (3) *and*

$$0 < m \le \rho_0(\mathbf{x}) \le M < \infty, \quad \mathbf{u}_0 \in \mathbf{V}, \quad \rho_0 \in H_N^2(\Omega_\epsilon).$$

Then there exists $\lambda_0 >$ *depending on* m *and* M *such that if* $\lambda/\mu < \lambda_0$*, problem* (1)-(2) *has a unique global in time strong solution* $(\rho,\mathbf{u},p)$ *and ther exists positive constants* $C = C(m,M,\mu,\lambda)$ *and* $\tilde{C} = \tilde{C}(m,M,\mu,\lambda,c_0)$ *such that, for all* $t \in (0,\infty)$,

$$\begin{aligned}
&m \le \rho(\mathbf{x},t) \le M, \quad \mathcal{U}(t) \le C\mathcal{U}_0, \quad \mathcal{V}(t) \le \tilde{C}\mathcal{V}_0,\\
&\|(\mathbf{u},\nabla\rho)\|^2_{L_t^\infty(\mathbf{L}^2(\Omega_\epsilon))} \le C\mathcal{U}_0 e^{-\min\{\mu/2M,\lambda\}\epsilon^{-2}t},\\
&\|(\nabla\mathbf{u},\Delta\rho)\|^2_{L_t^\infty(\mathbf{L}^2(\Omega_\epsilon))} \le \tilde{C}\mathcal{V}_0 e^{-\min\{m\mu/16M^2,\lambda/4\}\epsilon^{-2}t}.
\end{aligned}$$

Futhermore, if $\mathbf{u}_0 \in \mathbf{V} \cap \mathbf{H}^2(\Omega_\epsilon)$ *and* $\rho_0 \in H_N^3(\Omega_\epsilon)$*, then*

$$\mathcal{W}(t) \le \tilde{C}\mathcal{W}_0, \quad \|(\Delta\mathbf{u},\nabla\Delta\rho)\|^2_{L_t^\infty(\mathbf{L}^2(\Omega_\epsilon))} \le \tilde{C}\mathcal{W}_0 e^{-\min\{m\mu/16M^2,\lambda/4\}\epsilon^{-2}t}.$$

In particular, for all $t_* > 0$*, we have*

$$\lim_{\epsilon\to 0^+}(\mathbf{u},\nabla\rho) = (\mathbf{0},\mathbf{0}) \quad \textit{uniformly in} \quad C([t_*,\infty),\mathbf{H}^2(\Omega_\epsilon)).$$

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FRACTIONAL POWERS APPROACH OF OPERATORS FOR ABSTRACT EVOLUTION EQUATIONS OF THIRD ORDER IN TIME

FLANK D. M. BEZERRA[1,†] & LUCAS A. SANTOS[2,‡]

Departamento de Matemática, UFPB, PB, Brasil[1], IFPB - Campus João Pessoa, PB, Brasil[2]

flank@mat.ufpb.br[†] lucas.santos@ifpb.edu.br[‡]

Abstract

In this work we consider approximations of a class of third order in time linear evolution ill-posed equations governed by fractional powers. We explicitly calculate the fractional powers of matricial operators associated with evolution equations of third order in time. We show for which values of the exponent we transform the original problem into a well-posed one. The results presented here form part of my doctoral thesis and constitutes an article entitled '*Fractional powers approach of operators for abstract evolution equations of third order in time*' by myself and Flank D. M. Bezerra published by the Journal of Differential Equations.

1 Introduction

In this work we consider the following abstract linear evolution equation of third order in time

$$\partial_t^3 u + Au = 0 \tag{1}$$

with initial conditions given by

$$u(0) = u_0 \in X^{\frac{2}{3}},\ \partial_t u(0) = u_1 \in X^{\frac{1}{3}},\ \partial_t^2 u(0) = u_2 \in X, \tag{2}$$

where X is a separable Hilbert space and $A : D(A) \subset X \to X$ is a linear, closed, densely defined, self-adjoint and positive definite unbounded operator. We wish to study the fractional powers of Λ, the matricial operator obtained by rewriting (1)-(2) as a first order abstract system as follows:

We will consider the phase space

$$Y = X^{\frac{2}{3}} \times X^{\frac{1}{3}} \times X$$

which is a Banach space equipped with the norm given by

$$\|\cdot\|_Y^2 = \|\cdot\|_{X^{\frac{2}{3}}}^2 + \|\cdot\|_{X^{\frac{1}{3}}}^2 + \|\cdot\|_X^2$$

and we write the problem (1)-(2) as a Cauchy problem on Y, letting $v = \partial_t u$, $w = \partial_t^2 u$ and $\mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ and the initial value problem

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} + \Lambda \mathbf{u} = 0,\ t > 0 \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{3}$$

where the unbounded linear operator $\Lambda : D(\Lambda) \subset Y \to Y$ is defined by

$$D(\Lambda) = D(A) \times D(A^{\frac{2}{3}}) \times D(A^{\frac{1}{3}}), \tag{4}$$

and

$$\Lambda \mathbf{u} = \begin{bmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ A & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} := \begin{bmatrix} -v \\ -w \\ Au \end{bmatrix},\ \forall \mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in D(\Lambda). \tag{5}$$

2 Main Results

In this first result we prove that the operator Λ under study is of K-type positive, this means that we can consider its fractional powers Λ^α for $\alpha \in (0,1)$. Moreover, we explicitly calculate the matrix representing the fractional power of this operator. Finally, we give information about some spectral characteristics of the operator Λ^α as having the compact resolvent and the explicit calculation of its eigenvalues.

Theorem 2.1. *If A and Λ are as mentioned above, then we have all the following.*

i) Fractional powers Λ^α can be defined for $\alpha \in (0,1)$ by the Balakrishnan formula

$$\Lambda^\alpha = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1} \Lambda(\lambda I + \Lambda)^{-1} d\lambda. \tag{1}$$

ii) Given any $\alpha \in (0,1)$ we have $\Lambda^\alpha : D(\Lambda^\alpha) \subset Y \to Y$ is given by

$$\Lambda^\alpha = \begin{bmatrix} k_{\alpha,0}A^{\frac{\alpha}{3}} & -k_{\alpha,2}A^{\frac{\alpha-1}{3}} & k_{\alpha,1}A^{\frac{\alpha-2}{3}} \\ -k_{\alpha,1}A^{\frac{\alpha+1}{3}} & k_{\alpha,0}A^{\frac{\alpha}{3}} & -k_{\alpha,2}A^{\frac{\alpha-1}{3}} \\ k_{\alpha,2}A^{\frac{\alpha+2}{3}} & -k_{\alpha,1}A^{\frac{\alpha+1}{3}} & k_{\alpha,0}A^{\frac{\alpha}{3}} \end{bmatrix} \tag{2}$$

where

$$D(\Lambda^\alpha) = D(A^{\frac{\alpha+2}{3}}) \times D(A^{\frac{\alpha+1}{3}}) \times D(A^{\frac{\alpha}{3}}),$$

and

$$k_{\alpha,j} = \frac{1}{3}\left(2\cos\tfrac{2\pi(\alpha+j)}{3} + 1\right), \quad \text{for } j \in \{0,1,2\}. \tag{3}$$

iii) Let $\alpha \in (0,1]$. Then $0 \in \rho(\Lambda^\alpha)$. If, in addition A has compact resolvent, then Λ^α has compact resolvent.

iv) For each $\alpha \in (0,1]$ the spectrum of $-\Lambda^\alpha$ is such that the point spectrum consisting of eigenvalues

$$\left\{(\mu_n)^{\frac{\alpha}{3}} e^{i\pi} : n \in \mathbb{N}\right\} \cup \left\{(\mu_n)^{\frac{\alpha}{3}} e^{i\frac{\pi(3-2\alpha)}{3}} : n \in \mathbb{N}\right\} \cup \left\{(\mu_n)^{\frac{\alpha}{3}} e^{i\frac{\pi(3+2\alpha)}{3}} : n \in \mathbb{N}\right\} \tag{4}$$

where $\{\mu_n\}_{n\in\mathbb{N}}$ denotes the ordered sequence of eigenvalues of A including their multiplicity.

The next myth is the main result of this paper. We prove that though the negative of Λ is not an infinitesimal generator of a strongly continuous semigroup, i.e. the problem (1) is ill-posed, we can establish the maximum subinterval of $(0,1)$ where α is taken such that the negative of Λ^α is a generator, namely $-\Lambda^\alpha$ generates a strongly continuous semigroup on Y if and only if $0 < \alpha \leq \frac{3}{4}$ and it generates strongly continuous analytic semigroup on the open interval $0 < \alpha < \frac{3}{4}$.

Theorem 2.2. *The negative of the operator Λ^α in (2) is the generator of a strongly continuous semigroup on Y if and only if $\alpha \in \left(0, \frac{3}{4}\right]$. Moreover $-\Lambda^\alpha$ generates a strongly continuous analytic semigroup on Y for $\alpha \in \left(0, \frac{3}{4}\right)$.*

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ELLIPTIC SYSTEMS OF HÉNON TYPE INVOLVING ONE-SIDED CRITICAL GROWTH

EUDES BARBOZA[1,†] & BRUNO RIBEIRO[2,‡]

Departamento de Matemática, UFRPE, PE, Brasil[1], Departametto de Matemática, UFPB, PB, Brasil[2]

eudes.barboza@ufrpe.br[†] bhcr@academico.ufpb.br[‡]

Abstract

In this work, our goal is to study the a class of systems of Hénon type problems with a nonlineaty is assumed to be in the critical level with subcritical perturbation. Under appropriate hypotheses, we prove the existence of at least two radial solutions for this problem using variational methods.

1 Introduction

This work is based on paper [1]. Here we search for two non-trivial radially symmetric solutions of the Dirichlet problem involving a Hénon-type system of the form

$$\begin{cases} -\Delta u = a(x)u + b(x)v + |x|^\alpha K_u(u_+, v_+) + f_1(x) & \text{in} \quad B_1, \\ -\Delta v = b(x)u + c(x)v + |x|^\alpha K_v(u_+, v_+) + f_2(x) & \text{in} \quad B_1, \\ u = 0, \ v = 0 & \text{on} \quad \partial B_1, \end{cases} \tag{1}$$

where $\alpha \geq 0$, B_1 is a unity ball centered at the origin of $\mathbb{R}^N$ $(N \geq 3)$ and $K(s,t) = H(s,t) + G(s,t)$ with

$$H(s,t) = a_1 s^{2^*_\alpha} + a_{k+1} t^{2^*_\alpha} + \sum_{i=2}^{k} a_i s^{\beta_i} t^{\gamma_i}, \tag{2}$$

where β_i and γ_1 are positive numbers such that $\beta_i + \gamma_i = 2^*_\alpha = 2(N+\alpha)/(N-2)$ with $\beta_i, \gamma_i > 1$. We require that $a_i \geq 0$, for $1 \leq i \leq k+1$ with $a_i \neq 0$ for some i and that $G(s,t)$ is a C^1 function in $\mathbb{R}^+ \times \mathbb{R}^+$ which is assumed to be in the subcritical growth range.

1.1 Hypotheses

We explore the interaction of eigenvalues of the matrix

$$A(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & c(x) \end{pmatrix} \in C(\overline{B}_1, M_{2\times 2}(\mathbb{R}))$$

with respect to the spectrum of $(-\Delta, H^1_{0,\mathrm{rad}}(B_1))$. We rewrite (1) in its vectorial form and will be parametrized considering these different assumptions on $A(x)$. Here, we suppose that A is constant. For this, fix

$$F(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \in L^\tau(B_1) \times L^\tau(B_1)$$

and define

$$F_T(x) = P(x) + Te_1(x), \tag{3}$$

where e_1 denotes the first positive and normalized eigenfunction associated to $-\Delta$ in $H^1_{0,\mathrm{rad}}(B_1)$,

$$P(x) = \begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix} \in L^\tau(B_1) \times L^\tau(B_1), \quad \text{and} \quad T = \begin{pmatrix} s \\ t \end{pmatrix}. \tag{4}$$

Then, we consider the following problem:

$$\begin{cases} -\Delta U \;=\; A(x)U + |x|^{\alpha}\nabla\big(H(U_+) + G(U_+)\big) + F_T(x) & \text{in} \quad B_1, \\ U \;=\; 0 & \text{on} \quad \partial B_1, \end{cases} \tag{5}$$

and we will discuss conditions on the bidimensional parameter T to achieve multiplicity of solutions.

Let us establish the conditions on matrix $A(x)$. Denote by $\mu_1(x), \mu_2(x)$ the eigenvalues of $A(x)$ for all $x \in B_1$. We assume two cases, more precisely,

($\mathbf{A_1}$) A is constant and $\mu_1 \le \mu_2 < \lambda_1$; or

($\mathbf{A_2}$) A is constant and there exists $k \ge 1$ such that $\lambda_k < \mu_1 \le \mu_2 < \lambda_{k+1}$;

Before stating our main results, we shall also introduce the following assumptions on the nonlinearity G:

($\mathbf{G_1}$) $G \in C^1(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+)$ and $G, G_u, G_v \ge 0$;

($\mathbf{G_2}$) $G(U) > 0$ if $|U_+| > 0$ and G is $p-$homogeneous. More precisely, for all $\lambda > 0$ we assume that $G(\lambda u, \lambda v) = \lambda^p G(u, v)$ with

$$\begin{cases} 2^*_\alpha - \dfrac{2N-8}{3N-8} < p < 2^*_\alpha & \text{for} \quad N \ge 5; \\ (4+\alpha) - \dfrac{2}{5} < p < 4+\alpha = 2^*_\alpha & \text{for} \quad N = 4; \\ (6+2\alpha) - \dfrac{2}{5} < p < 6+2\alpha = 2^*_\alpha & \text{for} \quad N = 3, \end{cases}$$

($\mathbf{G_3}$) $G_u(0,1) = G_v(1,0) = 0$.

2 Main Results

Our first theorem reads

Theorem 2.1. *Suppose (G_1). Then, for each $N \ge 3$, we have*

1. *Assuming (A_1) or (A_2), there exist two lines α_1, α_2 dividing the plane $\mathbb{R}^2$ in four unbounded regions such that if $T = (s,t)$ lies in one of them, then Problem (5) admits a radial negative solution, denoted by Φ_T.*

Theorem 2.2. *Assume the existence of a radial nonnegative solution $\Phi = (\phi_1, \phi_2)$ of (5), $(G_1) - (G_3)$ and (A_1). Then, (5) possesses a second radial solution provided that $F \in L^\tau(B_1) \times L^\tau(B_1)$ with $\tau \ge 12$ if $N = 3$, $\tau \ge 8$ if $N = 4$ and $\tau > N$ if $N \ge 5$.*

Theorem 2.3. *Assume the existence of a radial nonnegative solution $\Phi = (\phi_1, \phi_2)$ of (5), $(G_1) - (G_3)$, (A_2) and*

(G_4) a_1 and a_{k+1}, given in (2), are strictly positive.

Then, (5) possesses a second radial solution provided that $F \in L^\mu(B_1) \times L^\mu(B_1)$ with $\mu \ge 12$ if $N = 3$, $\mu \ge 8$ if $N = 4$ and $\mu > N$ if $N \ge 5$.

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A PERTURBED ELLIPTIC PROBLEM INVOLVING THE $P(X)$-KIRCHHOFF TYPE TRIHARMONIC OPERATOR

GABRIEL RODRÍGUEZ V[1], WILLY BARAHONA M.[1], LUIS MACHA C.[2] & JESÚS LUQUE R.[1]

Instituto de Investigación, GI-EDOACBI, Facultad de Ciencias Matemáticas-UNMSM, Lima-Perú,[1],
Instituto de Investigación, GI-ECUKI, Facultad de Ciencias Matemáticas-UNMSM, Lima-Perú[2]

grodriguezv@unmsm.edu.pe[†], wbarahonam@unmsm.edu.pe[‡], lmachac@unmsm.edu.pe[§], jluquer@unmsm.edu.pe[§§].

Abstract

This paper examines the existence of weak solutions for a nonlinear boundary value problem of $p(x)$-Kirchhoff type involving the p(x)-Kirchhoff type triharmonic operator and perturbed external source terms. We establish our results by using the Degree theory of (S_+) type mappings in the framework of variable exponent Sobolev spaces.

1 Introduction

The purpose of this work is to investigate the existence of weak solutions for the following nonlinear elliptic problem involving the p(x)-Kirchhoff type triharmonic operator, with Navier boundary conditions

$$\begin{array}{c} -M(L(u))\Delta^3_{p(x)}u = f_\lambda(x,u,\nabla u,\Delta u,\nabla\Delta u) \text{ en } \Omega, \\ u = \Delta u = \Delta^2 u = 0 \text{ en } \partial\Omega, \end{array} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and $N \geq 3$, $p \in C(\overline{\Omega})$ for any $x \in \overline{\Omega}$; $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, $L(u) = \int_\Omega \frac{1}{p(x)} |\nabla\Delta u|^{p(x)}\, dx$, $\Delta^3_{p(x)}u := div\left(\Delta\left(|\nabla\Delta u|^{p(x)-2}\nabla\Delta u\right)\right)$ is the so-called $p(x)$-triharmonic operator, $f_\lambda = f_1 + \lambda f_2$, where f_1, f_2 are continuous functions, $\lambda > 0$, $p \in C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\}$ and

$$1 < p^- := \min_{\overline{\Omega}} p(x) \leq p^+ := \max_{\overline{\Omega}} p(x) < N \quad \text{for every } p \in C_+(\overline{\Omega}).$$

In recent years, Kirchhoff type equations involving variable exponent have attracted an increasing attention and many results have been obtained (see for example [3, 6]); however, there are few contributions to the study of the triharmonic problems with reaction term $f(x,u,\nabla u,\Delta u,\nabla\Delta u)$. We can cite [1, 2, 5]. Recently, Mehraban et al. [4] considered the existence and multiplicity of solutions for the problem (1), with $M(t) = 1, f(x,u,\nabla u,\Delta u,\nabla\Delta u) := \lambda f(x,u) + \mu g(x,u)$. Due to the presence of $\nabla u, \triangle u$ and $\nabla\triangle u$ in f the most usual variational techniques can not used to study it; so we adapt a topological tool: the degree theory for (S_+) type mappings. It is worth noting that, in this work, f does not satisfies typical growth conditions.

2 Assumptions and Main Result

Throughout this paper, let

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k\}$$

with the norm

$$\|u\|_{k,p(x)} \equiv \|u\|_{W^{k,p(x)}(\Omega)} = \sum_{|\alpha|\leq k} |D^\alpha u|_{L^{p(x)}(\Omega)}.$$

The space $W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. We denote by

$$X = W_0^{1,p(x)}(\Omega) \cap W^{3,p(x)}(\Omega)$$

and define a norm $\|.\|_X$ by

$$\|u\|_X = \|u\|_{1,p(x)} + \|u\|_{2,p(x)} + \|u\|_{3,p(x)}.$$

Suppose that M and f satisfy the following hypotheses:

(M_0) $M : [0, +\infty[\to [m_0, +\infty[$ is a continuous and nondecreasing function with $m_0 > 0$.

(F_1) $f_i \in C\left(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n; \mathbb{R}\right)$, $i = 1, 2$ and there exists a positive constants c_1 such that

$$|f_i(x, s, \xi, t, \zeta)| \le c_1(\sigma_i(x) + |s|^{\eta_i(x)} + |\xi|^{\delta_i(x)} + |t|^{\delta_i(x)} + |\zeta|^{\delta_i(x)}), \quad \forall x \in \Omega,$$

$$\forall s, t \in \mathbb{R}, \zeta, \xi \in \mathbb{R}^n, \text{ where } \eta_i, \delta_i \in C(\overline{\Omega}),\ q \in C_+(\overline{\Omega}), \frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

$$\sigma_i \in L^{p'(x)}(\Omega),\ 0 \le \eta_1(x) < p(x) - 1,\ 0 \le \delta_i(x) < \frac{p(x) - 1}{p'(x)}, i = 1, 2;\ p^- + 1 \le \eta_2(x) < p^+ + 1 \text{ for } x \in \overline{\Omega}.$$

Theorem 2.1. *Assume that hypotheses (M_0) and (F_1) hold. If $\lambda > 0$ is small enough, then (1) has a weak solution in X.*

Proof. First, we solve the problem for $\lambda = 0$, via the Degree theory of (S_+) type mappings. Then we use the continuity and boundedness and the Nemytskii operator to get our result. □

Remark 2.1. *With additional hypotheses on the function f we can get uniqueness of solutions.*

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THE CHEEGER CONSTANT AS LIMIT OF SOBOLEV-TYPE CONSTANTS

GREY ERCOLE[1,†]

Departamento de Matemática, UFMG, MG, Brasil[1].

grey@ufmg.br[†].

Abstract

Let Ω be a bounded, smooth domain of $\mathbb{R}^N$, $N \geq 2$. For $1 < p < N$ and $0 < q(p) < p^* := \frac{Np}{N-p}$ let

$$\lambda_{p,q(p)} := \inf \left\{ \int_\Omega |\nabla u|^p : u \in W_0^{1,p}(\Omega) \text{ and } \int_\Omega |u|^{q(p)} \, \mathrm{d}x = 1 \right\}.$$

We prove that if $\lim_{p\to 1^+} q(p) = 1$, then $\lim_{p\to 1^+} \lambda_{p,q(p)} = h(\Omega)$, where $h(\Omega)$ denotes the Cheeger constant of Ω. Moreover, we study the behavior of the minimizers $u_{p,q(p)}$ corresponding to $\lambda_{p,q(p)}$, as $p \to 1^+$. Our results extend those by Kawohl and Fridman (2003), where $q(p) = p$.

1 Introduction

Let Ω be a smooth, bounded domain of $\mathbb{R}^N$, $N \geq 2$. For $1 < p < N$ and $0 < q \leq p^* := \frac{Np}{N-p}$ let

$$\lambda_{p,q} := \inf \left\{ \|\nabla u\|_p^p : u \in W_0^{1,p}(\Omega) \text{ and } \|u\|_q = 1 \right\}, \tag{1}$$

where

$$\|u\|_r := \left(\int_\Omega |u|^r \, \mathrm{d}x \right)^{\frac{1}{r}}, \quad r > 0.$$

We recall that $\|\cdot\|_r$ is the standard norm of the Lebesgue space $L^r(\Omega)$ if $r \geq 1$, but it is not a norm if $0 < r < 1$.

Kawohl and Fridman proved in [2] that

$$\lim_{p\to 1^+} \lambda_{p,p} = h(\Omega) \tag{2}$$

where $h(\Omega)$ is the Cheeger constant of Ω.

We recall that

$$h(\Omega) := \inf \left\{ \frac{P(E)}{|E|} : E \subset \overline{\Omega} \text{ and } |E| > 0 \right\},$$

where $P(E)$ stands for the perimeter of E in $\mathbb{R}^N$ and $|E|$ stands for the N-dimensional Lebesgue volume of E.

The Cheeger problem consists of finding a subset $E \subset \overline{\Omega}$ such that $h(\Omega) = \frac{P(E)}{|E|}$. Such a subset E is called Cheeger set of Ω.

In this paper we suppose that q varies with p along a more general path, $q = q(p)$ for $p \in (1, p^*)$, and study the behavior of $\lambda_{p,q(p)}$ when $p \to 1^+$ and $q(p) \to 1$.

2 Main Results

Our main results are stated as follows.

Theorem 2.1. *If $0 < q(p) < p^*$ and $\lim_{p\to 1^+} q(p) = 1$, then*

$$\lim_{p\to 1^+} \lambda_{p,q(p)} = h(\Omega)$$

and

$$\lim_{p\to 1^+} \left\|u_{p,q(p)}\right\|_1 = 1 = \lim_{p\to 1^+} \left\|u_{p,q(p)}\right\|_\infty^{q(p)-p}.$$

Moreover, any sequence $\left(u_{p_n,q(p_n)}\right)$, *with* $p_n \to 1^+$, *admits a subsequence that converges in* $L^1(\Omega)$ *to a nonnegative function* $u \in L^1(\Omega) \cap L^\infty(\Omega)$ *such that:*

(a) $\|u\|_1 = 1$,

(b) $\dfrac{1}{|\Omega|} \leq \|u\|_\infty \leq \dfrac{h(\Omega)^N}{|\Omega|\, h(\Omega^\star)^N}$, *where* $\Omega^\star$ *denotes the ball centered at the origin and such that* $|\Omega^\star| = |\Omega|$, *and*

(c) for almost every $t \geq 0$, *the* t*-superlevel set* $E_t := \{x \in \Omega : u(x) > t\}$ *is a Cheeger set.*

Theorem 2.2. *Let* $w_{p,q(p)} \in W_0^{1,p}(\Omega)$ *be a positive weak solution to the Lane-Emden type problem*

$$\begin{cases} -\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) = |v|^{q-2}v & \text{in} \quad \Omega \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

with $p < q(p) < p^*$. *Then, either*

$$\limsup_{p\to 1^+} \left\|w_{p,q(p)}\right\|_{q(p)}^{q(p)-p} = +\infty$$

or

$$\lim_{p\to 1^+} \left\|w_{p,q(p)}\right\|_{q(p)}^{q(p)-p} = h(\Omega) = \lim_{p\to 1^+} \left\|w_{p,q(p)}\right\|_\infty^{q(p)-p}. \tag{2}$$

Remark 2.1. *It is well known that (1) has a unique positive weak solution when* $0 < q(p) < p$. *As this solution is given by* $\lambda_{p,q(p)}^{\frac{1}{q(p)-p}} u_{p,q(p)}$ *the limits in (2.2) follow directly from Theorem 2.1. In the particular case* $q(p) \equiv 1$ *they had already been obtained in [1] by Bueno and Ercole, without using (2).*

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SOME WEIGHTED SOBOLEV EMBEDDING INVOLVING FUNCTIONS THAT VANISHING ONLY IN A DIRECTION AND APPLICATION

J. A. CARDOSO[1], J. L. CARVALHO[1] & E. S. DE MEDEIROS[2]

Department of Mathematics, Federal University of Sergipe 49100-000, São Cristóvão-SE, Brazil[1],
Department of Mathematics, Federal University of Paraíba 58051-900, João Pessoa-PB, Brazil[2]

anderson@mat.ufs.br[†] jonison@mat.ufs.br[‡] everaldo@mat.ufpb.br[§]

Abstract

In this talk we discuss about some weighted Sobolev embedding involving functions that vanishing only in a direction. In this setting we prove a weighted Trudinger-Moser type inequality and as an application, we addressed the existence of solutions to a class of nonlinear equation elliptic of the form divergence, where the nonlinearity f has exponential critical growth in sense of Trudinger-Moser.

1 Introduction

In this talk, we are concerned with divergence elliptic equations of the form

$$-\mathrm{div}(a(x)\nabla u) + V(x)u = K(x)f(u) \quad \text{in} \quad \mathbb{R}^2, \tag{$\mathcal{P}$}$$

where the potential V and the weight functions a, K satisfy some growth conditions and the nonlinearity $f(s)$ is a continuous function with exponential critical growth in sense of Trudinger-Moser. Before describe our main results, we need first to introduce some functional space. For $\gamma, \sigma \in \mathbb{R}$, we denote by $E^{1,\gamma,\sigma}$ the weighted Sobolev space defined as the completion of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|u\|_{E^{1,\gamma,\sigma}} := \left(\int_{\mathbb{R}^2} \left[\left(1 + x_2^2\right)^{\gamma} |\nabla u|^2 + \frac{u^2}{\left(1 + x_2^2\right)^{\sigma}} \right] \mathrm{d}x \right)^{1/2}.$$

2 Main Results

In order to deal with equation ($\mathcal{P}_\Phi$) we shall need new embeddings from $E^{1,\gamma,\sigma}$ into weighted Lebesgue spaces. Precisely, we shall state the following embedding result:

Theorem 2.1 (Sobolev)**.** *Let $\sigma, \gamma \in \mathbb{R}$ be such that $2\sigma \leq 1 \leq \gamma$. Then, for any $2 \leq p < \infty$ and $2\beta \geq 1$ there exists a constant $C = C(p, \sigma) > 0$ such that, for all $u \in E^{1,\gamma,\sigma}$, we have*

$$\int_{\mathbb{R}^2} \frac{|u|^p}{\left(1 + x_2^2\right)^{\beta}} \mathrm{d}x \leq C \left(\int_{\mathbb{R}^2} \left[(1 + x_2^2)^{\gamma} |\nabla u|^2 + \frac{u^2}{\left(1 + x_2^2\right)^{\sigma}} \right] \mathrm{d}x \right)^{p/2}. \tag{1}$$

By using an argument free of symmetry we are able to state the weighted Trudinger-Moser type inequality:

Theorem 2.2 (Trudinger-Moser)**.** *Let $\sigma, \gamma \in \mathbb{R}$ be such that $2\sigma \leq 1 \leq \gamma$. Then, for any $2\beta \geq 1$, $\alpha > 0$ and $u \in E^{1,\gamma,\sigma}$, we have that $(1 + x_2^2)^{-\beta} \left(e^{\alpha u^2} - 1\right) \in L^1(\mathbb{R}^2)$ and there exists $\alpha^* > 0$ such that*

$$L(\gamma, \sigma, \beta, \alpha) := \sup_{\{u \in E^{1,\gamma,\sigma} : \|u\|_{E^{1,\gamma,\sigma}} \leq 1\}} \int_{\mathbb{R}^2} \frac{1}{\left(1 + x_2^2\right)^{\beta}} \left(e^{\alpha u^2} - 1\right) \mathrm{d}x < \infty,$$

whenever $0 < \alpha < \alpha^*$. Furthermore, there exists $\alpha^{**} > \alpha^*$ such that $L(\gamma, \sigma, \beta, \alpha) = \infty$, for all $\alpha > \alpha^{**}$.

Now, we shall assume the following basic assumptions on the potential V and the weight functions a, K:

(H_1) $V:\mathbb{R}^2\to\mathbb{R}$ is a measurable function and there are $V_0>0$, $2\sigma\leq 1$ such that $V_0/\left(1+x_2^2\right)^\sigma\leq V(x)$ for a.e. $x\in\mathbb{R}^2$;

(H_2) $a:\mathbb{R}^2\to\mathbb{R}$ is measurable and there are $a_0>0$, $\gamma\geq 1$ such that $a_0\left(1+x_2^2\right)^\gamma\leq a(x)$ for a.e. $x\in\mathbb{R}^2$;

(H_3) $K:\mathbb{R}^2\to\mathbb{R}$ is measurable and there are $K_0>0$, $2\beta\geq 1$ such that $0<K(x)\leq K_0/\left(1+x_2^2\right)^\beta$ for a.e.$x\in\mathbb{R}^2$.

We consider $(\mathcal{P}_\Phi)$ when the nonlinearity is a continuous function $f:\mathbb{R}^+\to\mathbb{R}^+$ with exponential critical growth in the sense of Trudinger-Moser, that is, there exists $\alpha_0>0$ such that

$$\lim_{|s|\to+\infty}\frac{|f(s)|}{e^{\alpha s^2}}=\begin{cases}0, & \text{if}\quad \alpha>\alpha_0,\\ \infty, & \text{if}\quad \alpha<\alpha_0.\end{cases}\tag{2}$$

We also require that the function $f(s)$ satisfies the following structural conditions:

(f_1) $f(s)=o(s)$ as $s\to 0^+$;

(f_2) there exist $\lambda>0$ and $q>2$ such that $F(s)\geq\lambda|s|^q$, for all $s\in\mathbb{R}$;

(f_3) the function $s\to f(s)/s^3$ is increasing for $s\in(0,\infty)$.

Our main existence result for equation $(\mathcal{P}_\Phi)$ is state as follows.

Theorem 2.3. *Assume that $(H_1)-(H_3)$, (2) and $(f_1)-(f_3)$ hold. In addition, suppose that the weight function K satisfies*

$$\int_{\mathbb{R}^2}K^2(x)\left(1+x_2^2\right)^\beta\,\mathrm{d}x<\infty.\tag{3}$$

Then, there exists $\widetilde{\lambda}>0$ such that if, (f_2) holds, for $\lambda\geq\widetilde{\lambda}$, then the equation $(\mathcal{P}_\Phi)$ has a nonnegative ground state solution.

Based on a Moser's iterative we obtain the asymptotic behavior for nonnegative weak solutions of $(\mathcal{P}_\Phi)$.

Theorem 2.4. *Let w be a nonnegative weak solution of $(\mathcal{P}_\Phi)$. Then, w is bounded in strips, that is, there exists $C>0$ such that*

$$w(x_1,x_2)\leq C\left(1+x_2^2\right)^\beta,$$

for any $x=(x_1,x_2)\in\mathbb{R}^2$. Moreover, suppose that there exists $\theta>0$ such that

$$8\theta^2a(x)\leq V(x),\qquad K(x)\leq\frac{1}{(1+x_2^2)^{2\theta+\beta}},\quad \text{for a.e. } x\in\mathbb{R}^2,\tag{4}$$

and

$$\int_{\mathbb{R}^2}K^2(x)\left(1+x_2^2\right)^{4\theta+\beta}\,\mathrm{d}x<\infty.\tag{5}$$

Then, $w\in L^\infty(\mathbb{R}^2)$.

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EXISTENCE OF SOLUTIONS FOR A QUASILINEAR THIRD-ORDERBOUNDARY VALUE PROBLEMS

JESÚS LUQUE R.[1], WILLY BARAHONA M.[1], ROCÍO DE LA CRUZ M.[2] & GABRIEL RODRÍGUEZ V.[1]

Instituto de Investigación, GI-EDOACBI, Facultad de Ciencias Matemáticas-UNMSM, Lima-Perú[1],
Instituto de Investigación, GI-ECUKI, Facultad de Ciencias Matemáticas-UNMSM, Lima-Perú[2]

jluquer@unmsm.edu.pe† wbarahonam@unmsm.edu.pe‡ rdelacruz@unmsm.edu.pe§ grodriguezv@unmsm.edu.pe§§

Abstract

This article establishes the existence of solutions for a for a quasilinear third-order boundary value problem. The main tools in the proof are the general Lax Milgram Lemma and the Schauder's fixed point theorem.

1 Introduction

This work deals with the existence of solutions for the quasilinear boundary value problem

$$\begin{aligned} -\frac{d}{dx}\left(a(u)\frac{d^2u}{dx^2}\right) &= f, \quad \text{in } [0,1] \\ u(0) = 0,\ u'(0) &= 0 = u'(1) \end{aligned} \tag{1}$$

where
(A_1) $a : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous and $0 < \lambda \leq a(s) \leq \Lambda, \forall s$, for some constants λ, Λ.
(A_2) $f : [0,1] \longrightarrow \mathbb{R}$ is the continuous continuous.

Third-order boundary value problems (BVPs) were discussed in many papers in recent years, for instance, see [1, 2, 3, 4] and references therein. Investigation of the existence of solutions for this type of BVP is often related to the construction of corresponding Green's functions. Thus, Green's functions play an important role in the theory of boundary value problems. In our case it is difficult to implement explicit Green functions, so to obtain the existence of solutions we use the general Lax Milgram Lemma and the Schauder's fixed point theorem.

2 Main Results

We will use the space $H = \{u \in H_0^2(0,1) : u(0) = 0\, u'(0) = 0 = u'(1)\}$. It is known that H is a Hilbert space.

Theorem 2.1. *Assume conditions (A_1) and (A_2) are fulfilled. Then boundary value problem (1) has at least one solution $u \in H$.*

Proof The proof is based on the general Lax Milgram Lemma and the Schauder's fixed point theorem, combined with variational arguments. □

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QUASILINEAR EQUATION WITH CRITICAL EXPONENTIAL GROWTH IN THE ZERO MASS CASE

J. CARVALHO[1,†] & J. C. DE ALBUQUERQUE[2,‡]

Departamento de Matemática, UFS, SE, Brasil[1], Departamento de Matemática, UFPE, PE, Brasil[2]

jonison@mat.ufs.br[†] josecarlos.melojunior@ufpe.br[‡]

Abstract

In this work we study the existence of solutions for the following class of quasilinear elliptic equations

$$-\text{div}\left(A(|x|)|\nabla u|^{N-2}\nabla u\right) = Q(|x|)f(u), \quad \text{in } \mathbb{R}^N,$$

where $N \geq 2$ and f has critical exponential growth. We establish conditions on the non-homogeneous weights A and Q to introduce a suitable function space where we are able to apply Variational Methods to obtain weak solutions. The main key is a Hardy type inequality for radial functions. Our approach is based on a new Trudinger-Moser type inequality, a version of the Symmetric Criticality Principle and Mountain Pass Theorem.

1 Introduction

In the last decades, many researchers have worked in the following class of problems

$$-\text{div}\left(A(x)|\nabla u|^{p-2}\nabla u\right) + B(x)|u|^{p-2}u = f(x,u), \quad \text{in } \mathbb{R}^N, \tag{1}$$

where $N \geq 2$ and $1 < p \leq N$. For the case $B \not\equiv 0$ and $A \equiv 1$, that is, when the differential operator is the p-laplacian, we refer the readers to the seminal works [3, 14, 2, 19, 1, 1]. The case in which the potential A is not trivial, i.e. $A \not\equiv 1$, together with the condition $B \not\equiv 0$, has been considered in [11, 6, 4]. The case $B \equiv 0$ is known in the literature as the zero-mass case and such condition on B presents more difficulty, since $W^{1,p}(\mathbb{R}^N)$ is not the natural space to look for solutions. The main purpose of this work is establishing conditions to study a zero-mass problem in the borderline case, that is, when $p = N$, precisely

$$-\text{div}\left(A(|x|)|\nabla u|^{N-2}\nabla u\right) = Q(|x|)f(u), \quad \text{in } \mathbb{R}^N, \tag{$\mathcal{P}$}$$

where $N \geq 2$ and f has critical exponential growth. Our main difficulties rely on the fact that in the borderline case, $p = N$, the space $D^{1,N}(\mathbb{R}^N)$ is not "well defined" and in generally, the Hardy inequality is false, see for example [12]. For this reason, we establish conditions to introduce a new space where we are able to study Problem ($\mathcal{P}_\Phi$) variationally.

2 Main Results

Theorem 2.1. *(Hardy type inequality) Let $N \geq 2$, $l > 0$ and $a = l - N$. There exists $C = C(N,l) > 0$ such that*

$$\int_{\mathbb{R}^N} |x|^a |u|^N \, dx \leq C \int_{\mathbb{R}^N} |x|^l |\nabla u|^N \, dx, \quad \text{for all } u \in C^1_{0,\text{rad}}(\mathbb{R}^N), \tag{1}$$

where $C^1_{0,\text{rad}}(\mathbb{R}^N)$ denotes the space of radial functions that belong to $C^1(\mathbb{R}^N)$ with compact support. Furthermore, the exponent a is optimal, in the sense that if $a \neq l - N$, then (1) is false.

Theorem 2.2. *Considering some hypotheses the Problem ($\mathcal{P}_\Phi$) admits a nonzero weak solution.*

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SCHRÖDINGER-POISSON SYSTEM WITH ZERO MASS IN $\mathbb{R}^2$ INVOLVING $(2, Q)$-LAPLACIAN

JONISON CARVALHO[1], JOSÉ CARLOS DE ALBUQUERQUE[2] & EDCARLOS D. SILVA[3]

Departamento de Matemàtica, UFS, SE, Brasil[1], Departamento de Matemàtica, UFPE, PE, Brasil[2], Instituto de Matemàtica e Estatìstica, UFG, GO, Brasil[3]

jonison@mat.ufs.br† josecarlos.melojunior@ufpe.br‡ edcarlos@ufg.br§

Abstract

It is established the existence of positive least energy solution for the following class of planar elliptic systems in the zero mass case

$$\begin{cases} -\Delta u - \Delta_q u + \phi|u|^{r-2}u = \lambda|u|^{p-2}u, & \text{in } \mathbb{R}^2, \\ \Delta\phi = 2\pi|u|^r, & \text{in } \mathbb{R}^2, \end{cases}$$

where $\lambda \geq 0$, $1 < q < 2$, $q^* := 2q/(2-q) < r < \infty$ and $p \geq 2r$. Due to the nature of the problem, we deal with the logarithmic integral kernel. Furthermore, we study the asymptotic behavior of the solutions whenever the parameter λ goes to zero or infinity. Finally, we study regularity of the solutions.

1 Introduction

In this work we study the existence, asymptotic behavior and regularity of solutions for the following class of problems

$$\begin{cases} -\Delta u - \Delta_q u + \phi|u|^{r-2}u = \lambda|u|^{p-2}u, & \text{in } \mathbb{R}^2, \\ \Delta\phi = 2\pi|u|^r, & \text{in } \mathbb{R}^2. \end{cases} \tag{$\mathcal{S}_\lambda$}$$

System ($\mathcal{S}_\lambda$) is reduced into the following equivalent integro-differential equation

$$-\Delta u - \Delta_q u + \phi_u|u|^{r-2}u = \lambda|u|^{p-2}u, \quad \text{in } \mathbb{R}^2, \tag{$\mathcal{P}_\lambda$}$$

where

$$\phi_u = \phi_{2,u,r}(x) := (\Gamma_2 * |u|^r)(x) = \int_{\mathbb{R}^2} \Gamma_2(x-y)|u(y)|^r \, dy \quad \text{and} \quad \Gamma_2(x) = \frac{1}{2\pi}\log|x|.$$

The first difficulty in studying ($\mathcal{P}_\Phi$) (zero mass case) variationally is finding a suitable function space to look for weak solutions, once $H^1(\mathbb{R}^2)$ and $W^{1,q}(\mathbb{R}^2)$ are not suitable. We consider E^q defined as the completion of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|u\|_{E^q} := \left(\int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \left(\int_{\mathbb{R}^2} |\nabla u|^q \, dx \right)^{2/q} \right)^{1/2}.$$

Due to the presence of the sign-changing and unbounded logarithmic integral kernel, inspired by [3, 5], we introduce the space W defined as the completion of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|u\|_W := \left(\|u\|_{E^q}^2 + \left(\int_{\mathbb{R}^2} \log(1+|x|)|u|^r \, dx \right)^{2/r} \right)^{1/2}.$$

We also consider the radial space $E^q_{\text{rad}} := \{u \in E^q : u \text{ is radial}\}$. The following embedding result will be crucial in the course of this work, whose proof can be found in [4, Theorem 2.1] and [2, Proposition 2.1].

Theorem 1.1. *The embedding $E^q \hookrightarrow L^r(\mathbb{R}^2)$ is continuous, for any $q^* \leq r < \infty$. Furthermore, the embedding $E^q_{\text{rad}} \hookrightarrow L^r(\mathbb{R}^2)$ is compact, for any $q^* < r < \infty$.*

We say that $u \in W$ is a weak solution for Problem ($\mathcal{P}_\Phi$) if satisfies

$$\int_{\mathbb{R}^2} \left(\nabla u \nabla \varphi + |\nabla u|^{q-2} \nabla u \nabla \varphi\right) \mathrm{d}x + \int_{\mathbb{R}^2} \phi_u |u|^{r-2} u \varphi \, \mathrm{d}x = \lambda \int_{\mathbb{R}^2} |u|^{p-2} u \varphi \, \mathrm{d}x, \quad \text{for all } \varphi \in W.$$

The energy functional associated to ($\mathcal{P}_\Phi$) is given by

$$I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^2} |\nabla u|^q \, \mathrm{d}x + \frac{1}{2r} \int_{\mathbb{R}^2} \phi_u |u|^r \, \mathrm{d}x - \frac{\lambda}{p} \int_{\mathbb{R}^2} |u|^p \, \mathrm{d}x.$$

Thus, if $u \in W$ is a critical point of I_λ, then u is a weak solution of ($\mathcal{P}_\Phi$). We say that a nontrivial solution $u \in W$ of ($\mathcal{P}_\Phi$) is a least energy solution if $I_\lambda(u) \leq I_\lambda(v)$, for any other nontrivial solution $v \in W$ of ($\mathcal{P}_\Phi$). In order to look for solutions in W_{rad}, inspired by [1], we prove a version of the Principle of Symmetric Criticality due to Palais.

2 Main Results

Theorem 2.1. ***(Existence)*** *Suppose that $1 < q < 2$, $q^* := 2q/(2-q) < r < \infty$ and $p \geq 2r$. Then, for any $\lambda \geq 0$, ($\mathcal{P}_\Phi$) possesses a nonnegative least energy solution $u_\lambda \in W_{\mathrm{rad}} \setminus \{0\}$. Furthermore, the pair $(u_\lambda, \phi_{u_\lambda}) \in W_{\mathrm{rad}} \times W^{2,s}_{\mathrm{loc}}(\mathbb{R}^2)$, for any $s > 1$, is a weak solution of ($\mathcal{S}_\lambda$).*

Theorem 2.2. ***(Asymptotic Behavior)*** *Assume the conditions of Theorem 2.1 and let u_λ be the solution of ($\mathcal{P}_\Phi$). Then:*

(i) the following convergences are satisfied: $\lim_{\lambda \to +\infty} \|u_\lambda\|_{E^q} = 0$ *and* $\lim_{\lambda \to +\infty} \|u_\lambda\|_W = 0$;

(ii) $u_\lambda \rightharpoonup v$ in W_{rad}, as $\lambda \to 0^+$, where v is a least energy solution of ($\mathcal{P}_\Phi$), with $\lambda = 0$.

Theorem 2.3. ***(Regularity)*** *Assume the conditions of Theorem 2.1 and let u_λ be the solution of ($\mathcal{P}_\Phi$). There exists $\widetilde{r} = \widetilde{r}(q,p) > 0$ such that, if $r \geq \widetilde{r}$, then u_λ is positive and belongs to $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, for some $\alpha \in (0,1)$. Furthermore, $\phi_{u_\lambda} \in C^2(\mathbb{R}^2)$.*

Remark 2.1. *Although we are inspired by [3], our approach is significantly different that, in the sense that we do not use either the periodicity of the energy functional or Lions' vanishing-nonvanishing arguments. For this reason, our approach could be adapted to deal with some variations of ($\mathcal{P}_\Phi$) when the associated energy functional is not invariant under translations with respect to $\mathbb{Z}^2$.*

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GLOBAL SOLUTIONS FOR A FRACTIONAL DIFFUSION EQUATION WITH GRADIENT NONLINEARITY

ARLÚCIO VIANA[1,†] & HALLEY GOMES[2,‡]

Departamento de Matemática, UFS, SE, Brasil[1] Departamento de Matemática, UFRN, RN, Brasil[2]

arlucioviana@academico.ufs.br[†] professorhalleygomes@gmail.com[‡]

Abstract

We give sufficient conditions to obtain the existence and asymptotic behaviors of global solutions of a fractional reaction–diffusion equation with power-type and gradient nonlinearities. In particular, we obtain results of the fractional viscous Hamilton-Jacobi equation.

1 Introduction and the main result

We give sufficient conditions to obtain global solutions of

$$u_t(t,x) = \partial_t \int_0^t g_\alpha(s)\Delta u(t-s,x)ds + c_1|u(t,x)|^{\rho-1}u(t,x) + c_2|\nabla u(t,x)|^q, \qquad \text{in } (0,\infty)\times\mathbb{R}^N \tag{1}$$

$$u(0,x) = u_0(x), \qquad \text{in } \mathbb{R}^N, \tag{2}$$

with initial data in the critical Lebesgue space $L^p(\mathbb{R}^N)$, with $p = \frac{\alpha N}{2}(\rho-1)$, as well as we study time decays of the solutions. Here, $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, for $t>0$ and $0<\alpha<1$, $c_1, c_2 \in \mathbb{R}$, $\rho>1$, and $q = \frac{2\rho}{2+\alpha(\rho-1)}$.

We observe that, for $\alpha = 1$, global existence, was studied by Snoussi, Tayachi and Weissler [3]. The recent work by Tayachi and Zaag [5] brings together the state of the art of (1)–(2), with $\alpha = 1$, as well as they prove nice blow-up results for that problem, when $q = \frac{2\rho}{\rho+1}$. Such critical exponent is consistent with $q = \frac{2\rho}{2+\alpha(\rho-1)}$, which is given by the scaling computation in (1). In [4], one can find a presentation of (1) as a model to describe population dynamics of biological species living on a territory represented by a domain $\Omega \subset \mathbb{R}^N$, where deaths depend on the predator's density given by $|\nabla u|^{q-1}$. On the other hand, when $\alpha = 1$ and $c_1 = 0$, (1) becomes the equation known as the viscous Hamilton-Jacobi equation. Then, we call (1) with $\alpha \in (0,1)$ and $c_1 = 0$ by the fractional viscous Hamilton-Jacobi equation. We refer the reader to [1] for a local theory in Lebesgue spaces.

An important feature brought by the combination of the power and gradient nonlinearities in (1)–(2) is that, even for $\alpha = 1$ (see [3]), the existence of solutions cannot be obtained exactly as in the case $c_2 = 0$ on account of the constraints produced by the linear estimates would form an empty set. Thus, we shall use the Gagliardo-Nirenberg inequality to overcome this.

Now, define the Banach space X_β of all Bochner integrable functions $u:(0,\infty)\to W^{1,r}(\mathbb{R}^N)$ such that $t^\beta u$ and $t^{\beta+\frac{\alpha}{2}}\nabla u$ are bounded continuous functions in $L^r(\mathbb{R}^N)$, for $t>0$, whose norm is given by

$$\|u\|_{X_\beta} = \max\left\{\sup_{t>0}\|t^\beta u(t,\cdot)\|_{L^r}, \sup_{t>0}\|t^{\beta+\frac{\alpha}{2}}\nabla u(t,\cdot)\|_{L^r}\right\}, \tag{3}$$

where $1<p<r<\infty$ and $\beta = \frac{\alpha N}{2}\left(\frac{1}{p}-\frac{1}{r}\right)$.

By formally integrating and applying the Laplace and Fourier transforms in (1)-(2), we obtain

$$u(t,x) = S_\alpha(t)u_0(x) + \int_0^t S_\alpha(t-s)c_1|u(t,x)|^{\rho-1}u(s,x)ds + \int_0^t S_\alpha(t-s)c_2|\nabla u(s,x)|^q ds, \tag{4}$$

where $S_\alpha(t)\varphi$ is the unique solution for

$$\begin{aligned} u_t(t,x) =& \partial_t \int_0^t g_\alpha(s)\Delta u(t-s,x)ds \\ u(0,x) =& \varphi(x). \end{aligned}$$

We can see that

$$S_\alpha(t)\varphi = \int_0^\infty \Phi_\alpha(s) e^{st^\alpha \Delta}\varphi ds, \tag{5}$$

for every distribution φ, where $(e^{t\Delta})_{t\geq 0}$ stands for the heat semigroup associated to the Laplacian. Here Φ_α denotes the Mainardi function. It is known that $(S_\alpha(t))_{t\geq 0}$ is not a semigroup and this brings some new difficulties in the analysis, comparing with [3].

The representation (4) motivates the following definition.

Definition 1.1. *A continuous function $u : (0,\infty) \to W^{1,r}(\mathbb{R}^N)$ that satisfies (4) is called a mild solution of the problem (1)–(2).*

The parameters α, N, ρ, q, r, m, with $m, r > 1$, are assumed to satisfy

$$\rho > 1 + 2\frac{1-\alpha}{\alpha}, \tag{H1}$$

$$1 < p < \alpha r < \alpha q p, \tag{H2}$$

$$\max\left\{r, \frac{\rho N r}{N+r}, \rho\right\} < m < \rho p. \tag{H3}$$

Theorem 1.1 ([2])**.** *Assume that* (H1)–(H3) *hold. If $R > 0$ satisfies $C_1 R^{\rho-1} + C_2 R^{q-1} < 1$, where C_1 and C_2 are positive constants, then it is possible to choose $\mu > 0$ such that if $\|u_0\|_{L^p} \leq \mu$, then there exists a unique global solution $u \in B_{X_\beta}(R)$ of the problem (1)-(2), which depends continuously on the initial data. In particular,*

$$t^{\beta+\frac{\alpha}{2}}\|\nabla u(t,\cdot)\|_{L^r} + t^\beta \|u(t,\cdot)\|_{L^r} \to 0, \quad as \quad t\to\infty. \tag{6}$$

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SPECTRUM OF ELLIPTIC DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS IN DIMENSION N ON REAL SCALES OF LOCALIZED SOBOLEV SPACES

LUÍS M. SALGE[1,†] & TIAGO H. PICON[2,‡]

Instituto de Matemática, UFRJ, RJ, Brasil[1], FFCLRP, USP, SP, Brasil[2]

lmsalge@im.ufrj.br[†] picon@ffclrp.usp.br[‡]

Abstract

In this work, we extended the study of spectrum started in [4] for elliptic homogeneous differential operatators with constant coefficients to the n dimensional case in the real scale of localized Sobolev Spaces. This is quite different from what we find in the literature, where all the relevant results are concerned with spectrum on Banach spaces.

Our aim is to understand the behavior of the spectrum using the closure of the operator. In particular we show that there is no complex number in the resolvent set of such operators, which suggest a new way to define spectrum if we want to reproduce the classical theorems of the Spectral Theory in Fréchet spaces.

1 Introduction

In this work we present a study of the closure of elliptic differential operator on an open set $\Omega \subset \mathbb{R}^n$ given by

$$a(D) : H_0^{s+m}(\Omega) \subset H_{loc}^s(\Omega) \longrightarrow H_{loc}^s(\Omega), \qquad s \in \mathbb{R}.$$

Here, the Sobolev space $H_{loc}^s(\Omega)$ is endowed with the topology generated by a family of seminorms $\left(p_j^{(s)}\right)_{j\in\mathbb{N}}$ given by $p_j^{(s)}(f) \doteq \|\varphi_j f\|_{H^s}$, for each $f \in H_{loc}^s(\Omega)$, where Ω_j is sequence of open sets such that $\overline{\Omega_j} \subset \Omega_{j+1}$, with $\Omega = \bigcup_{j\in\mathbb{N}} \Omega_j$, and $\varphi_j \in C_c^\infty(\Omega_{j+1})$ satisfies $\varphi_j = 1$ in $\overline{\Omega_j}$. We denote $H_0^{s+m}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $H^{s+m}(\Omega)$ with the induced topology from $H_{loc}^{s+m}(\Omega)$.

2 Main Results

2.1 Closure of a Differential Operator on a Fréchet Space

Here we determine the closure of an elliptic differential operator with constant coefficients $a(D)$ of order $m \geq 1$ on $H_{loc}^s(I)$. That will allow us to obtain an analysis of the spectrum.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $s \in \mathbb{R}$. Given a function $u \in H_{loc}^s(\Omega)$, consider the natural extension

$$u_e(x) = \begin{cases} u(x), & \text{if } x \in \Omega \\ 0, & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Now let $(\Omega_j)_{j\in\mathbb{N}}$ a sequence of open bounded sets with $\Omega = \bigcup_{j\in\mathbb{N}} \Omega_j$, $\overline{\Omega_j} \subset \Omega_{j+1}$ and $d(\Omega_j, \mathbb{R}^n \backslash \Omega) \geq 2/j$. Consider $g_j(x) = \chi_{\Omega_j}(x) \cdot u_e(x)$, where χ_{Ω_j} is the characteristic function of Ω_j, and $u_j = \phi_j \star g_j$, where $\phi_j(x) = j^n \phi(jx)$ with $\phi \in C_c^\infty(B_1(0))$, $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi(x)dx = 1$. Clearly $g_j \in H^s(\mathbb{R}^n)$ and $u_j \in C_c^\infty(\Omega)$.

Theorem 2.1. *Let $s \in \mathbb{R}$ and $u \in H_{loc}^s(\Omega)$. For each $j \in \mathbb{N}$ the sequence $u_j \doteq \phi_j \star (\chi_{\Omega_j} u_e) \in C_c^\infty(\Omega)$ converges to u in $H_{loc}^s(\Omega)$.*

Using the result above it's possible to calculate the closure of an elliptic differential operator with constant coefficients as follows.

Theorem 2.2. *Let* $a(D) : H_0^{s+m}(\Omega) \subset H_{loc}^s(\Omega) \to H_{loc}^s(\Omega)$ *be an elliptic differential operator with* $s \in \mathbb{R}$ *and* $m \in \mathbb{N}$. *Then its closure is given by*

$$\overline{a(D)} : H_{loc}^{s+m}(\Omega) \subset H_{loc}^s(\Omega) \longrightarrow H_{loc}^s(\Omega)$$

with $\overline{a(D)} = a(D)$.

The main result in this paper is the following:

Theorem 2.3. *Given* $s \in \mathbb{R}$ *and* $m \in \mathbb{N}$, *consider* $a(D) : H_0^{s+m}(\Omega) \subset H_{loc}^s(\Omega) \to H_{loc}^s(\Omega)$ *an elliptic differential operator with constant coefficients. Then we have*

$$\sigma(a(D)) = \sigma\left(\overline{a(D)}\right) = \sigma_p\left(\overline{a(D)}\right) = \mathbb{C}$$

and the spectrum does not depend of s.

2.2 Spectrum of the Laplace operator on a Fréchet Space

In this section we apply the results obtained in the previous section to the Laplacian operator.

Corollary 2.1. *The Laplace operator, for each* $s \in \mathbb{R}$, *seen as a pseudodiferencial operator* $\Delta : H_0^{s+2}(\Omega) \subset H_{loc}^s(\Omega) \longrightarrow H_{loc}^s(\Omega)$ *and its closure* $\overline{\Delta} : H_{loc}^{s+2}(\Omega) \subset H_{loc}^s(\Omega) \longrightarrow H_{loc}^s(\Omega)$, *both have resolvent set empty and their spectra are the whole plane:* $\sigma_(\Delta) = \sigma\left(\overline{\Delta}\right) = \sigma_p\left(\overline{\Delta}\right) = \mathbb{C}$.

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BLOW-UP SOLUTIONS FOR THE INHOMOGENEOUS NONLINEAR SCHRÖDINGER EQUATION

MYKAEL CARDOSO[1,†] & LUIZ GUSTAVO FARAH[2,‡]

Departamento de Matemática, UFPI, PI, Brasil[1] ICex, UFMG, MG, Brasil[2]

mykael@ufpi.edu.br[†] farah@mat.ufmg.br[‡]

Abstract

1 Introduction

In this work we consider the initial value problem (IVP) for the inhomogeneous nonlinear Schrödinger (INLS) equation

$$\begin{cases} i\partial_t u + \Delta u + |x|^{-b}|u|^{2\sigma}u = 0, \;\; x \in \mathbb{R}^N,\, t > 0, \\ u(0) = u_0, \end{cases} \tag{1}$$

for $N \geq 3$ and some $b, \sigma > 0$. This model is a generalization of the classical nonlinear Schrödinger (NLS) equation and also has applications in laser beam propagation upon a nonlinear optical medium [4] and [6].

Over the last few years, the INLS equation has been the subject of a great deal of mathematical research. This is part of a recently growing interest in the global dynamics of NLS type equations lacking the usual symmetries. In the present case, the translation invariance is not present and there is a space-dependent singular coefficient in the nonlinearity. Several results concerning well-posedness theory, existence and concentration of blow-up solutions, stability of solitary waves and asymptotic behavior of global solutions have been recently obtained for the INLS.

We are mainly interested in the intercritical regime. To understand this terminology, we recall that if $u(x,t)$ solves (1) so does $u_\lambda(x,t) = \lambda^{\frac{2-b}{2\sigma}} u(\lambda x, \lambda^2 t)$ and also $\|u_\lambda(0)\|_{\dot{H}^{s_c}} = \|u_0\|_{\dot{H}^{s_c}}$ where $s_c = \frac{N}{2} - \frac{2-b}{2\sigma}$. The mass-supercritical and energy-subcritical regime is such that $0 < s_c < 1$ and we can reformulate this condition as

$$\frac{2-b}{N} < \sigma < \frac{2-b}{N-2}. \tag{2}$$

Finite time solutions in H^1 also enjoy other important properties. For instance, from the H^1 local Cauchy theory obtained by [5], if $T^* < \infty$, then

$$\|\nabla u(t)\|_{L^2} \to \infty, \;\; \text{as} \;\; t \to T^*. \tag{3}$$

Moreover, the recent work by [1] proved that these solutions obey the lower bound

$$\|\nabla u(t)\|_{L^2} \geq \frac{c}{(T^* - t)^{\frac{1-s_c}{2}}}.$$

2 Main Results

In [2] and [3], we investigate the existence of solutions with a finite maximal time of existence, as well as the behavior of the critical norm when the time nears the maximal time of existence. For the intercritical INLS equation, we have obtained the following results.

Theorem 2.1. *Let $N \geq 3$, $0 < b < \min\{\frac{N}{2}, 2\}$ and $\frac{2-b}{N} < \sigma < \min\left\{\frac{2-b}{N-2}, \frac{2}{N}\right\}$. Let $u_0 \in H^1$ and assume that the maximal time of existence $T^* > 0$ for the corresponding solution $u \in C([0,T^*) : H^1)$ of (1) is finite. Define $\beta = \frac{2-\sigma N}{b}$, then there exists a universal constant $C = C(u_0, \sigma, b, N) > 0$ such that the following space-time upper bound holds*

$$\int_t^{T^*} (T^* - \tau)\|\nabla u(\tau)\|_{L^2}^2 \, d\tau \leq C(T^* - t)^{\frac{2\beta}{1+\beta}}, \tag{4}$$

for t close enough to T^*.

Theorem 2.2. *Let $\sigma_c = \frac{2N\sigma}{2-b}$ such that $\dot{H}^{s_c} \subset L^{\sigma_c}$. Assume $N \geq 3$, $0 < b < \min\{\frac{N}{2}, 2\}$ and $\frac{2-b}{N} < \sigma < \min\left\{\frac{2-b}{N-2}, \frac{2}{N}\right\}$. Given $u_0 \in H^1$ so that the maximal time of existence $T^* > 0$ for the corresponding solution $u \in C([0,T^*) : H^1)$ of (1) is finite. Then there exist positive constants c_0 and c_1 depending only on N, σ and b such that*

$$\liminf_{t \to T^*} \int_{|x| \leq c_{u_0} \|\nabla u(t)\|_{L^2}^{-\frac{2-\sigma N}{b}}} |u(x,t)|^{\sigma_c} \, dx \geq c_0, \tag{5}$$

where $c_{u_0} = c_1 \max\left\{\|u_0\|_{L^2}, \|u_0\|_{L^2}^{\frac{2\sigma+2-\sigma N}{b}}\right\}$.

Theorem 2.3. *Let $N \geq 3$, $0 < b < \min\{\frac{N}{2}, 2\}$ and $\frac{2-b}{N} < \sigma < \min\left\{\frac{2-b}{N-2}, \frac{2}{N}\right\}$. If $u_0 \in \dot{H}^{s_c} \cap \dot{H}^1$ and $E(u_0) \leq 0$, then the maximal time of existence $T^* > 0$ of the corresponding solution $u(t)$ to (1) is finite.*

Theorem 2.4. *Let $\sigma_c = \frac{2N\sigma}{2-b}$ such that $\dot{H}^{s_c} \subset L^{\sigma_c}$. Assume $N \geq 3$, $0 < b < \min\{\frac{N}{2}, 2\}$ and $\frac{2-b}{N} < \sigma < \min\left\{\frac{2-b}{N-2}, \frac{2}{N}\right\}$. Given $u_0 \in \dot{H}^{s_c} \cap \dot{H}^1$ so that the maximal time of existence $T^* > 0$ of the corresponding solution u to (1) is finite and satisfies*

$$\|\nabla u(t)\|_{L^2} \geq \frac{c}{(T^* - t)^{\frac{1-s_c}{2}}}, \tag{6}$$

for some constant $c = c(N, \sigma)$ and t close enough to T^*. Then there exists $\gamma = \gamma(N, \sigma, b) > 0$ such that

$$c\|u(t)\|_{\dot{H}^{s_c}} \geq \|u(t)\|_{L^{\sigma_c}} \geq |\log(T^* - t)|^{\gamma}, \quad \text{as } t \to T^*. \tag{7}$$

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ASYMPTOTIC PROPERTIES FOR A GENERALIZED PLATE EQUATION UNDER EFFECTS OF ROTATIONAL INERTIA AND A LOGARITHMIC TYPE DISSIPATION

FÉLIX PEDRO Q. GÓMEZ[1,†] & RUY COIMBRA CHARÃO[2,‡]

Departamento de Matemática, UTFPR, PR, Brasil[1], MTM-CFM, UFSC, SC, Brasil, ruy.charao@ufsc.br[2]

felixgomez@utfpr.edu.br[†] ruy.charao@ufsc.br[‡]

Abstract

We introduce a new model of the logarithmic type of wave like plate equation with a non-local logarithmic damping mechanism. We consider the Cauchy problem for this new model in $\mathbb{R}^n$, and study the asymptotic profile and optimal decay rates of solutions as $t \to \infty$ in L^2-sense.

1 Introduction

We consider in this work a new model of evolution equations based on an operator L_θ, that combines the composition of logarithm function with the Laplace operator as follows,

$$\partial_t^2 u + (-\Delta)^\delta \partial_t^2 u + (-\Delta)^\alpha u + L_\theta \partial_t u = 0, \quad (t,\ x) \in]0,\ \infty[\times \mathbb{R}^n, \tag{1}$$

$$u(0,\ x) = u_0(x), \quad u_t(0,\ x) = u_1(x), \quad x \in \mathbb{R}^n, \tag{2}$$

where the linear operator

$$L_\theta : D(L_\theta) \subsetneq L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad \theta > 0,$$

is defined by

$$D(L_\theta) := \left\{ f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \left(\log\left(1 + |\xi|^{2\theta}\right)\right)^2 \left|\hat{f}(\xi)\right|^2 d\xi < +\infty \right\},$$

and for $f \in D(L_\theta)$,

$$L_\theta\, f := \mathcal{F}^{-1}_{\xi \to x}\left[\log\left(1 + |\xi|^{2\theta}\right)\, \widehat{f}(\xi)\right] \qquad \text{iff} \qquad \mathcal{F}[L_\theta f](\xi) = \log\left(1 + |\xi|^{2\theta}\right) \widehat{f}(\xi).$$

Here, one has just denoted the Fourier transform $\mathcal{F}_{x\to\xi}[f](\xi)$ of $f(x)$ as usual with $i := \sqrt{-1}$, and $\mathcal{F}^{-1}_{\xi\to x}$ expresses its inverse Fourier transform.

As for the existence of the unique solution to problem (1)–(2) we discuss, by employing the standard Lumer-Phillips Theorem, similarly in the paper of Gómez & Charão (2021) [4].

We study the equation (1) only from the purely mathematical point of view. The model equation itself is strongly inspired from the related paper due to Dharmawardane et. al. (2012) [2].

In order to introduce our main results we should define function spaces with respect to the logarithmic Laplace operator L such that for $\delta \geq 0$

$$Y^\delta = \left\{ f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n_\xi} \left(1 + \log\left(1 + |\xi|^2\right)\right)^\delta |\widehat{f}(\xi)|^2 \, d\xi < \infty \right\}$$

with its natural norm

$$\|f\|_{Y^\delta} := \left(\int_{\mathbb{R}^n_\xi} \left(1 + \log\left(1 + |\xi|^2\right)\right)^\delta \left|\widehat{f}(\xi)\right|^2 d\xi \right)^{1/2} \quad \text{for all} \quad f \in Y^\delta, \tag{3}$$

and its corresponding inner product.

Remark 1.1. Due to the fact that $\log\left(1+|\xi|^2\right) \leq |\xi|^2$ for all $\xi \in \mathbb{R}^n$, one notices

$$H^\delta(\mathbb{R}^n) \subsetneq Y^\delta \subsetneq L^2(\mathbb{R}^n) \quad \textit{for} \quad \delta \geq 0.$$

Furthermore, we set

$$I_{0,\,l} := \sqrt{\|u_0\|^2_{1,\,1} + \|u_1\|^2_{1,\,1} + \|u_0\|^2_{Y^{l+1}} + \|u_1\|^2_{Y^l}}, \quad l \geq 0 \quad \text{and} \quad P_j := \int\limits_{\mathbb{R}^n} u_j(x)\,dx, \quad j = 0,\ 1.$$

2 Main Results

Theorem 2.1. *Let $n \geq 4$ and $l = \frac{n}{2} - 1$. If $(u_0,\ u_1) \in \left(L^{1,\,1}(\mathbb{R}^n) \cap Y^{l+1}\right) \times \left(L^{1,\,1}(\mathbb{R}^n) \cap Y^l\right)$, then there exists a constant $C > 0$, which is independent of $t,\ u_0,\ u_1$ such that the weak solution $u(t,\ x)$ to problem (1)–(2) satisfies*

$$\left\| u(t,\ \cdot) - \mathcal{F}^{-1}_{\xi\to x}[\varphi(t,\ \xi)]\,(\cdot) \right\|_2 \leq C\, I^2_{0,\,l}\, t^{-\frac{n+2}{4}}$$

for $t \gg 1$, where

$$\varphi(t,\ \xi) := \varphi_1(t,\ \xi) + \varphi_2(t,\ \xi).$$

Remark 2.1. In Theorems above one has assumed $l \geq 1$ because in this case the problem (1)–(2) has a unique weak solution in the class

$$u \in C\left([0,\ +\infty[\ ;\ Y^2\right) \cap C^1\left([0, +\infty[\ ;\ Y^1\right) \cap C^2\left([0,\ +\infty[\ ;\ L^2(\mathbb{R}^n)\right).$$

Remark 2.2. The value $l^*(n)$ defined by $l^*(n) := \frac{n}{2} - 1$ expresses a kind of critical number on the regularity $l \geq 1$, which divides the property of the solution $u(t,\ x)$ into three types: one is diffusive-like, the other is wave-like and the remaining is both of them (Theorem 2.1).

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STRONG SINGULAR PROBLEM IN NONREFLEXIVE FRACTIONAL ORLICZ-SOBOLEV SPACE

MARCOS L. M. CARVALHO[1], LUANA DE C. MACIEL[1], CARLOS A. P. SANTOS[2] & MAXWELL L. SILVA[1]

Instituto de Matemática e Estatística, UFG, GO, Brasil[1],
Departamento de Matemática, Universidade de Brasìlia, Brasìlia, DF, Brasil[2],

marcos_leandro_carvalho@ufg.br† luanamaciel@discente.ufg.br‡ csantos@unb.br§ maxwell@ufg.br§§

Abstract

In this work, we deal with existence and uniqueness of positive solution u for the strongly singular quasilinear problem $(-\Delta_\Phi)^s u = u^{-\gamma}$ in the nonreflexive fractional Orlicz-Sobolev $W^s_{loc}L^\Phi(\Omega)$ for $0 < s < 1 < \gamma$. The main difficulties encountered were the lack of homogeneity of the fractional Φ-Laplacian operator, the lack of reflexivity of the natural workspaces and the presence of the strongly singular term. To overcome these difficulties we applied Galerkin's method to a truncated problem and we proved a new comparison principle even for the p-Laplacian operator.

1 Introduction

In this work, let us prove existence and uniqueness of weak solutions for the following class of strongly singular problems ($\gamma > 1$)

$$(-\Delta_\Phi)^s u = u^{-\gamma}, \quad \text{in} \quad \Omega, \ u > 0, \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $N \geq 1$, $h \in L^1(\Omega)$ is a non-negative function, $\gamma > 1$. Moreover,

$$(-\Delta_\Phi)^s u(x) \quad = \quad (1-s) \lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^N \setminus B_\varepsilon(0)} \phi\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{u(x)-u(y)}{|x-y|^s} \frac{\mathrm{d}y}{|x-y|^N},$$

with $\phi : (0,\infty) \longrightarrow (0,\infty)$ is of class C^1 and satisfies:

(ϕ_1) **(i)** $t\phi(t) \longrightarrow 0$, when $t \to 0$ and **(ii)** $t\phi(t) \longrightarrow +\infty$ when $t \to +\infty$;

(ϕ_2) $t\phi(t)$ is strictly increasing in $(0,\infty)$;

(ϕ_3) there exist $\ell, \ m \in [1,\infty)$ such that $1 \leq \ell := \inf_{t \geq 0} \frac{t^2\phi(t)}{\Phi(t)} \leq \sup_{t \geq 0} \frac{t^2\phi(t)}{\Phi(t)} =: m < \infty.$

2 Main Results

In our main result, we shall use the auxiliary convex function

$$\Gamma(t) = \int_0^t \Phi^{-1}(\tau^{\gamma-1})\mathrm{d}\tau, \ t \geq 0.$$

Definition 2.1. *A positive function $u \in W^s_{loc}L^\Phi(\Omega)$ is a weak solution for Problem (1) if $\Gamma(u) \in W^s_0L^\Phi(\Omega)$, $h(x)u^{-\gamma} \in L^1_{loc}(\Omega)$ and*

$$\iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \phi(|D_s u|) D_s u D_s v \mathrm{d}\mu = \int\limits_\Omega h(x)u^{-\gamma}\mathrm{d}x, \ \forall v \in C^\infty_c(\Omega).$$

Definition 2.2. *Let u be such that $u = 0$ in $\mathbb{R}^N \backslash \Omega$. We say that $u \leq 0$ in $\partial\Omega$ if for all $\epsilon > 0$ we have $(u-\epsilon)^+ \in W_0^s L^\Phi(\Omega)$. We say that $u = 0$ in $\partial\Omega$ if u is non-negative and $u \leq 0$ in $\partial\Omega$.*

Definition 2.3. *We say that u has zero Dirichlet boundary if $u = 0$ in the sense of Definition 2.2.*

Theorem 2.1. *Assume that $(\phi_1) - (\phi_3)$, $h \in L^1(\Omega)$, $h \geq 0$ and $\gamma > 1$ hold. Then, there is a unique solution $u \in W^s_{loc} L^\Phi(\Omega)$ of the Problem (1), that is,*

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \phi(|D_s u|) D_s u D_s v \mathrm{d}\mu = \int_\Omega h(x) u^{-\gamma} v \mathrm{d}x, \qquad \forall v \in C_c^\infty(\Omega).$$

Moreover, $u \geq Cd$ a.e. in Ω for some $C > 0$ and u satisfies the zero Dirichlet boundary condition.

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BOUNDARY WEAK HARNACK ESTIMATES AND REGULARITY FOR ELLIPTIC PDE IN DIVERGENCE FORM

MAYRA SOARES[1], FIORELLA RENDÓN[2] & BOYAN SIRAKOV[2]

Departamento de Matemática, UnB, Brasília, Brasil[1], Departamento de Matemática, PUC-Rio, RJ, Brasil[2]

`mayra.soares@unb.br`[†] `fiorellareg@gmail.com`[‡] `bbsirakov@gmail.com`[§]

Abstract

We obtain a Boundary Point Hopf Lemma and a Boundary Weak Harnack Inequality for a nonnegative supersolution of a general uniform elliptic equation in divergence form under the weakest assumptions on the leading coefficients and on the boundary of the domain. Our main tool is the use of suitable barrier functions, which are solutions of auxiliaries problems. Furthermore, we provide an application showing how to use these results in order to deduce a priori upper bounds and multiplicity of solutions for a class of quasilinear elliptic problems.

1 Introduction

We prove the extension of the interior weak Harnack inequality up to the boundary for a general divergence-type elliptic equation under strongly weakened assumptions. More specifically, we consider nonnegative weak supersolutions of the problem

$$Lu = f(x),\ x \in \Omega, \tag{1}$$

for operator L given by

$$Lu := -\mathrm{div}(A(x)Du) + b(x)\cdot Du + c(x)u,\ x \in \Omega, \tag{2}$$

where $\Omega \subset \mathbb{R}^n$, for $n \geq 2$, is a bounded domain satisfying the interior $C^{1,\mathcal{D}}$-paraboloid condition. Moreover, operator L is uniformly elliptic and $A(x) = (a_{i,j}(x))$ a symmetric matrix, satisfying

$$a_{i,j} \in C^{0,\mathcal{D}}(\Omega) \quad \text{for all } i,j = 1,...,n \quad \text{and} \quad \vartheta I_n \leq A(x) \leq \vartheta^{-1} I_n \qquad \text{in } \Omega, \tag{3}$$

for some $\vartheta > 0$, where $I_n : \mathbb{R}^n \to \mathbb{R}^n$ is the identity operator. In addition, for the low order coefficients we require

$$|b| \in L^q(\Omega),\ c \in L^q(\Omega) \text{ and } c \geq 0 \text{ in } \Omega, \tag{4}$$

and assume also $f \in L^q(\Omega)$ for some $q > n$.

Inspired by [3, 4], we seek to show that the well known interior weak Harnack inequality has a boundary extension, in terms of the distance to the boundary d, called Boundary Weak Harnack Inequality, given by

$$\inf_{\Omega} \frac{u}{d} \geq C\left(\int_{\Omega}\left(\frac{u}{d}\right)^{\varepsilon}\right)^{\frac{1}{\varepsilon}} - C||f||_{L^q(\Omega)},\ \text{for some } \varepsilon > 0. \tag{5}$$

Such result is an important tool for the regularity theory, it is applied, for instance, to establish uniform a priori bounds, as well to obtain multiplicity of solution, as we exemplify in the last section of this work.

Our main interest is to extend the interior weak Harnack inequality, introduced by de Giorgi, up to the boundary for divergence-type equations with low regularity on the coefficients and on the domain. However, this estimate is deeply related to the Boundary Point Hopf Lemma, hence, it is also necessary to keep this result in mind and make a parallel between them. In fact, the Boundary Weak Harnack Inequality (5) is a quantification of the Boundary Point Hopf Lemma, since for the homogeneous equation, the Boundary Weak Harnack Inequality is given by

$$\inf_{\Omega} \frac{u}{d} \geq C \left(\int_{\Omega} \left(\frac{u}{d} \right)^{\varepsilon} \right)^{\frac{1}{\varepsilon}}, \tag{6}$$

which, passing to the limit, clearly implies the result of the Boundary Point Hopf Lemma. Furthermore, the expression (5) quantify how the loss of superharmonicity, by a non null function f, preserves the integrability of u/d, but rectifies the boundary point Hopf estimate with a L^q-norm of f.

2 Main Results

We prove a Hopf Lemma based on [1, 2]. We consider the full operator L as in [2], however we are under more general assumptions on the leading coefficients. Hence, we apply similar arguments to [1], although we need to perform one step more. Indeed, we apply a fixed point theorem to ensure the existence of a solution $\tilde{v}$ to the homogeneous problem with the full operator L, namely, when $c \not\equiv 0$ and subsequently we estimate Du making a comparison between it and such a solution $D\tilde{v}$. We highlight our particular interest in proving this Boundary Hopf Point Lemma, since it allows to generalize several other results to our setting.

Lemma 2.1 (Boundary Point Hopf Lemma)**.** *Let $u \in C^1(\overline{\Omega})$ be a weak supersolution of ($\mathcal{P}_\Phi$), where $\Omega \subset \mathbb{R}^n$ is a bounded domain with the boundary $\partial\Omega$ satisfying the interior $C^{1,\mathcal{D}}$-paraboloid condition, L be a uniformly elliptic operator given by (2) satisfying (3)-(4), for some $q > n$ and also $f \equiv 0$. Suppose that u attends its minimum at a point $x_0 \in \partial\Omega$ with $u(x_0) \leq 0$, if under a $C^{1,\mathcal{D}}$-regular change of variables there exists a ball $B \subset \Omega$ such that $x_0 \in \partial B$ and $u(x) > u(x_0)$ for all $x \in B$, then $\dfrac{\partial u}{\partial \eta}(x_0) < 0$, where η is the outward normal on $\partial\Omega$.*

In what follows, we enunciate our main result, the global version of the weak Harnack inequality. Namely, the statement for an arbitrary $\Omega \subset \mathbb{R}^N$, whose proof follows as a consequence of the $C^{1,\mathcal{D}}$-regularity required on the domain and version for cubes.

Theorem 2.1 (Boundary Weak Harnack Inequality)**.** *Let $\Omega \subset \mathbb{R}^n$, for $n \geq 2$, be a bounded domain. Assume that u is a nonnegative weak supersolution of problem ($\mathcal{P}_\Phi$) in Ω, where Ω and L are under the hypotheses of Lemma 2.1 and $f \in L^q(\Omega)$ is non-positive in Ω. Then for any $x_0 \in \partial\Omega$ there exist constants $\overline{R} > 0$, $\varepsilon > 0$ and $C > 0$ such that for all $R \in (0, \overline{R}]$,*

$$\inf_{B_R(x_0)\cap\Omega} \frac{u(x)}{d(x)} \geq C \left(\int_{B_R(x_0)\cap\Omega} \left(\frac{u(x)}{d(x)} \right)^{\varepsilon} dx \right)^{\frac{1}{\varepsilon}} - C||f||_{L^q(\Omega)}.$$

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QUASILINEAR ELLIPTIC PROBLEMS WITH EXPONENTIAL GROWTH VIA THE NEHARI MANIFOLD METHOD: EXISTENCE OF NONNEGATIVE AND NODAL SOLUTIONS

SANDRA I. MOREIRA[1], GIOVANY M. FIGUEIREDO[2] & RICARDO RUVIARO[2]

Departamento de Matemática e Informática, Universidade Estadual do Maranhão, UEMA, MA, Brasil[1],
Departamento de Matemática, Universidade de Brasília, UnB, DF, Brasil[2]

ymaculada@gmail.com† giovany@unb.br‡ ruviaro@unb.br§

Abstract

In this work we will be concerned with the problem

$$-\Delta u - \frac{1}{2}\Delta(a(x)u^2)u + V(x)u = f(u), \quad x \in \mathbb{R}^2,$$

where V is a potential continuous and $f : \mathbb{R} \to \mathbb{R}$ is a superlinear continuous function with exponential subcritical or exponential critical growth. We use as a main tool the Nehari manifold method in order to show existence of nonnegative solutions and existence of nodal solutions. Our results complement the classical results due to Jia-quan Liu, Ya-qi Wang and Zhi-Qiang Wang in [2].

1 Introduction

In this work we prove existence of nonnegative and nodal solutions for the following class of quasilinear problems with nonlinearities involving a superlinear continuous function with exponential growth and V is a potential continuous. More precisely, we consider

$$-\Delta u - \frac{1}{2}\Delta(a(x)u^2)u + V(x)u = f(u), \quad x \in \mathbb{R}^2. \tag{1}$$

Under the following nonlinearities hypotheses on the functions a, V and f are the following:

(a_0) $a \in C(\mathbb{R}^2, \mathbb{R})$ and there exist $0 < a_0, a_1$ such that $a_0 < a(x) < a_1$, for all $x \in \mathbb{R}^2$;

(V_0) $V \in C(\mathbb{R}^2, \mathbb{R})$;

(V_1) $0 < V_0 \leq \inf_{x\in\mathbb{R}^2} V(x)$;

(V_2) for all $M > 0$, there holds $measure(\{x \in \mathbb{R}^2 : V(x) \leq M\}) < \infty$;

(f_1) consider $f : \mathbb{R} \to \mathbb{R}$ is of class C^0 and there exists $\alpha_0 \geq 0$ such that the function $f(t)$ satisfies

$$\lim_{|t|\to\infty} \frac{f(t)}{\exp(\alpha|t|^2)} = 0 \text{ for } \alpha > \alpha_0 \text{ and } \lim_{|t|\to\infty} \frac{f(t)}{\exp(\alpha|t|^2)} = \infty \text{ for } \alpha < \alpha_0;$$

(f_2) the following limit holds: $\lim_{|t|\to 0} \frac{f(t)}{t^3} = 0$;

(f_3) the function $t \to \frac{f(t)}{|t|^3}$ is increasing in $\mathbb{R} \setminus \{0\}$;

(f_4) there are $r > 4$ and $\tau > \tau^*$ such that $sgn(t)f(t) \geq \tau|t|^{r-1}$, for all $t \geq 0$, where

$$\tau > \tau^* := \max\left\{1, \left(\frac{4\alpha_0}{\pi} c_r \frac{r-2}{r-4}\right)^{(r-2)/2}\right\},$$

where c_r will be defined throughout the proof of results.

The class of problems (1) has been extensively studied during the last years via variational methods. The main technique used is change of variable. For example, the authors in [3] used a dual approach to reduce the quasilinear equation to a semilinear one and used an Orlicz space framework. The same method was used in [1], where the normal Sobolev space was used as the working space. Since in our problem there is a function $a(x)$, the change of variable used in [1] or [3] cannot be considered here. Then, it is necessary to work directly on the associate functional in spite of its lack of smoothness. To the best of our knowledge, the first existence result involving variational methods was obtained in [4]. In that paper the authors apply variational techniques to prove the existence of standing wave solutions for quasilinear Schrödinger equations containing strongly singular nonlinearities which include derivatives of the second order. Another pioneering work in sense is [2], in this paper the authors consider the existence of both one sign and nodal ground states of soliton type solutions for problem (1). Under certain conditions on the potential function $V(x)$, the authors show that the problem has at least one weak positive solution as well as one sign-changing solution with exactly two nodal domains, respectively. The problem was treated, for both the positive solution and the sign-changing solution, by a more unified approach, that is the Nehari method.

2 Main Results

Theorem 2.1. *Assume that a satisfies (a_0), V satisfies $(V_0)-(V_2)$, f satisfies (f_1) with $\alpha_0 = 0$ and $(f_2)-(f_3)$ and $f(t) = 0$, for $t \leq 0$ hold. Then, problem (1) has a nonnegative solution with minimal energy.*

Theorem 2.2. *Suppose that a satisfies (a_0), V satisfies $(V_0)-(V_2)$, f satisfies (f_1) with $\alpha_0 = 0$ and $(f_2)-(f_3)$. Then problem (1) possesses a least energy nodal solution, which has precisely two nodal domains.*

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GROUND STATE SOLUTION FOR LINEARLY COUPLED SYSTEMS INVOLVING CRITICAL EXPONENTIAL GROWTH

UBERLANDIO B. SEVERO[1], JOSÉ CARLOS DE ALBUQUERQUE[2] & EDJANE O. DOS SANTOS[3]

Universidade Federal da Paraíba, Departamento de Matemática, CEP 58051-900, João Pessoa - PB, Brasil [1],
Universidade Federal de Pernambuco, Departamento de Matemàtica, CEP 50670-901, Recife - PE, Brasil [2],
Universidade Catòlica de Pernambuco, Escola de Educação e Humanidades, CEP 50050-900, Recife - PE, Brasil [3]

uberlandio@mat.ufpb.br † josecarlos.melojunior@ufpe.br‡ edjane.oliveira@unicap.br §

Abstract

In this work we study the following class of linearly coupled systems in the plane:

$$\begin{cases} -\Delta u + u = f_1(u) + \lambda v, & \text{in} \quad \mathbb{R}^2, \\ -\Delta v + v = f_2(v) + \lambda u, & \text{in} \quad \mathbb{R}^2, \end{cases} \tag{1}$$

where f_1, f_2 are continuous functions with critical exponential growth in the sense of Trudinger-Moser inequality and $0 < \lambda < 1$ is a parameter. For any $\lambda \in (0,1)$, by using minimization arguments and minimax estimates we prove the existence of a positive ground state solution. This class of systems can model phenomena in nonlinear optics and in plasma physics.

1 Introduction

Due to its broad field of applications in several physical situations such as in nonlinear optics, in double Bose–Einstein condensates, in plasma physics, among others (see for instance [1] and reference therein), the class of linearly coupled systems

$$\begin{cases} -\Delta u + u = f_1(u) + \lambda v, & \text{in} \quad \mathbb{R}^N, \\ -\Delta v + v = f_2(v) + \lambda u, & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{2}$$

($N \geq 2$) has been considered under many different assumptions. Our goal here is to extend the study of System (2) to the critical case in the plane, that is, we are concerned with System (1), when the nonlinearities f_1 and f_2 have critical exponential growth in the sense of Trudinger–Moser inequality. Inspired by [2, 3, 4, 5], we study the following minimization problem:

$$\mathcal{A}_\lambda = \inf_{(u,v)\in E\setminus\{(0,0)\}} \left\{ \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + |\nabla v|^2)\,\mathrm{d}x \; ; \; \int_{\mathbb{R}^2} G_\lambda(u,v)\,\mathrm{d}x = 0 \right\}, \tag{3}$$

where

$$G_\lambda(u,v) = F_1(u) + F_2(v) - \frac{u^2}{2} - \frac{v^2}{2} + \lambda uv, \quad F_i(s) := \int_0^s f_i(\tau)\,\mathrm{d}\tau, \quad i = 1,2, \tag{4}$$

$\lambda \in (0,1)$ and the space $E := H^1_{rad}(\mathbb{R}^2) \times H^1_{rad}(\mathbb{R}^2)$ is endowed with the inner product

$$\langle (u,v),(w,z)\rangle = \int_{\mathbb{R}^2} (\nabla u \nabla w + uw + \nabla v \nabla z + vz)\,\mathrm{d}x$$

and its correspondent norm $\|(u,v)\| = \sqrt{\langle (u,v),(u,v)\rangle}$. The minimization problem (3) is related to the existence of ground states for System (1). We suppose that f_1, f_2 belongs to $C(\mathbb{R},\mathbb{R})$ and satisfy the following assumptions[1]:

[1] Hereafter, for the sake of simplicity, the index "i" is understood as $i = 1,2$.

(H_1) $\lim_{s\to 0^+} \frac{f_i(s)}{s} = 0$;

(H_2) $\lim_{s\to +\infty} \frac{f_i(s)}{e^{\alpha s^2}} = 0$, if $\alpha > 4\pi$ and $\lim_{s\to +\infty} \frac{f_i(s)}{e^{\alpha s^2}} = +\infty$, if $\alpha < 4\pi$;

(H_3) $f_i(s)s - 2F_i(s) \geq 0$, for all $s \geq 0$;

(H_4) There exists $\theta > 0$ such that $\liminf_{s\to +\infty} \frac{F_1(s)}{s^\theta e^{4\pi s^2}} =: \beta_0 > 0$.

2 Main Results

Associated to System (1), we have the energy functional $I_\lambda : E \to \mathbb{R}$ defined by

$$I_\lambda(u,v) = \frac{1}{2}\|(u,v)\|^2 - \int_{\mathbb{R}^2} [F_1(u) + F_2(v)]\,\mathrm{d}x - \lambda \int_{\mathbb{R}^2} uv\,\mathrm{d}x.$$

Under our conditions, we may check that I_λ is well defined. Moreover, one can see that $I_\lambda \in C^1(E,\mathbb{R})$. We say that $(u,v) \in E$ is a weak solution of System (1) if the equality

$$\langle (u,v),(w,z)\rangle = \int_{\mathbb{R}^2} [f_1(u)w + f_2(v)z]\,\mathrm{d}x + \lambda \int_{\mathbb{R}^2} (uz + vw)\,\mathrm{d}x,$$

holds for all $(w,z) \in C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2)$. Thus, after applying the Palais Symmetric Criticality Principle, critical points of I_λ are weak solutions of System (1). A pair (u,v) is said to be a nonnegative (positive) solution for (1) when $u(x), v(x) \geq 0$ ($u(x), v(x) > 0$) for all $x \in \mathbb{R}^2$. We recall that $(u,v) \in E\backslash\{(0,0)\}$ is said to be a ground state (least energy) solution for System (1), if (u,v) is a solution of (1) and its energy is the lowest among all nontrivial solutions of (1), i.e. $I_\lambda(u,v) \leq I_\lambda(w,z)$ for any other nontrivial solution $(w,z) \in E$.

The main results of this work are stated as follows:

Proposition 2.1. *If $\lambda \in (0,1)$ and $(H_1) - (H_4)$ are satisfied, then the infimum $\mathcal{A}_\lambda$ is attained.*

Theorem 2.1. *If $\lambda \in (0,1)$ and $(H_1) - (H_4)$ are satisfied, then System (1) has a positive ground state solution.*

Although there are some papers concerned with the existence of ground states for linearly coupled systems in the plane, not much has been done considering weaker conditions. Our work extends the results from [2, 3, 4] since we are considering a system of equations that are coupled by a linear term. It is important to say the we are not considering usual assumptions such as the well known Ambrosetti–Rabinowitz condition and the monotonicity of $f(s)/s$.

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A VARIATIONAL INEQUALITY WITH COMPETING OPERATORS OF $P(X)$-KIRCHHOFF TYPE AND CONVECTION TERM

WILLY BARAHONA M[1], EUGENIO CABANILLAS L.[2], ROCÍO DE LA CRUZ M.[1] & GABRIEL RODRÍGUEZ V[1]

Instituto de Investigación, GI-EDOACBI, Facultad de Ciencias Matemáticas-UNMSM, Lima-Perú[1],
Instituto de Investigación, GI-ECUKI, Facultad de Ciencias Matemáticas-UNMSM, Lima-Perú[2]

wbarahonam@unmsm.edu.pe[†] cleugenio@yahoo.com[‡] rdelacruz@unmsm.edu.pe[§] grodriguezv@unmsm.edu.pe[§§]

Abstract

In this research, we show the existence of solutions in a generalized sense for a competing $(p(x), q(x))$-Kirchhoff type variational inequality. The deficit of ellipticity, monotonicity and variational structure prevents the use of any known variational method. We obtain the existence of this solutions by means of the penalty method and Galerkin's approximation, working in the context of the variable exponent Lebesgue-Sobolev spaces.

1 Introduction

The purpose of this work is to investigate the existence of weak solutions for the following variational elliptic inequality of Kirchhoff type

$$-M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)\,\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + M\Big(\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}\,dx\Big)\,\mathrm{div}(|\nabla u|^{q(x)-2}\nabla u) \geq f(x,u,\nabla u) \quad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, and $N \geq 1$, $p, q \in C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\}$; M, f are given fucntions and

$$1 < p^- := \min_{\overline{\Omega}} p(x) \leq p^+ := \max_{\overline{\Omega}} p(x) < N \quad \text{for every } p \in C_+(\overline{\Omega}).$$

In recent years, many results have been obtained on problems related with (1). In [3], with $M(t) = 1, p(x) = p, q(x) = q, 1 < q < p < +\infty$, the author investigated the existence of both a generalized solution and of a strong generalized solution; similar approach is presented in [4]. Variational inequalities as the development and extension of classic variational problems, are a very useful tool to research PDEs, optimal control, physics, mechanics, engineering and elliptic inequalities. In [5, 1, 2], the authors studied variational inequalities of Kirchhoff type but without convection term and just one operator. We can not apply directly variational methods for problem (1), so to overcome this difficulty, we will employ the penalty method and Galerkin's approximation.

2 Assumptions and Main Result

We will work in the well-known generalized Lebesgue space $L^{p(x)}(\Omega)$ and the Sobolev space $W_0^{1,p(x)}(\Omega)$.

We will assume

(A_0) $M : [0, +\infty[\to [m_0, +\infty[$ is a continuous and increasing function; $m_0 > 0$.

(F_0) $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy the Carathéodory condition in the sense that $f(., u, \xi)$ is measurable for all $(u, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x, ., .)$ is continuous for almost all $x \in \Omega$.

(F_1) $|f(x,u,\xi)| \leq k(x) + |u|^{\eta(x)} + |\xi|^{\delta(x)}$ a.e. $x \in \Omega$, all $(u,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where $k : \mathbb{R} \to \mathbb{R}^+$, $k \in L^{p'(x)}(\Omega)$ and $0 \leq \eta(x) < p(x) - 1, 0 \leq \delta(x) < (p(x)-1)/p'(x)$.

(F_2) There exist constant $0 < C_0 < 1$ and $1 < \alpha(x) \leq p^-$ such that

$$f(x,s,\xi)s \leq C_0|\xi|^{p(x)} + C_1\left(|s|^{\alpha(x)} + 1\right) \quad \text{for a.e. } x \in \Omega, \text{all } s \in \mathbb{R}, \xi \in \mathbb{R}^N.$$

Introducing previously the notion of generalized solution to problem (1), we present our main result.

Theorem 2.1. *Suppose (A_0), (F_0) - (F_2) hold. Then problem (1) has a generalized solution $u \in W_0^{1,p(x)}(\Omega)$.*

Proof We use the penalty method combined with the Galerkin approximation procedure to get such a solution. □

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ESTIMATE FOR CONCENTRATION LEVEL OF THE ADAMS FUNCTIONAL AND EXTREMALS FOR ADAMS-TYPE INEQUALITY

ABIEL C. MACEDO[1,†] & JOSÉ F. A. DE OLIVEIRA[2,‡]

IME, UFG, GO, Brasil[1], Departamento de Matemàtica, UFPI, PI, Brasil[2]

abielcosta@ufg.br[†] jfoliveira@ufpi.edu.br[‡]

Abstract

This paper is mainly concerned with the existence of extremals for the Adams inequality. We first establish an upper bound for the classical Adams functional along of all concentrated sequences in the higher order Sobolev space with homogeneous Navier boundary conditions $W_{\mathcal{N}}^{m,\frac{n}{m}}(\Omega)$, which in particular includes the classical Sobolev space $W_0^{m,\frac{n}{m}}(\Omega)$, where Ω is a smooth bounded domain in Euclidean n-space. Secondly, based on the Concentration-compactness alternative due to Do Ó and Macedo, we prove the existence of extremals for the Adams inequality under Navier boundary conditions for second order derivatives at least for higher dimensions when Ω is an Euclidean ball.

1 Introduction

Let Ω be a smooth domain in $\mathbb{R}^n$, $n \geq 2$, with n-measure $|\Omega| < \infty$, and $W_0^{m,\frac{n}{m}}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ in $W^{m,\frac{n}{m}}(\Omega)$, for positive integer $m < n$. Given $u \in C_0^\infty(\Omega)$ we will denote

$$\nabla^m u = \begin{cases} \Delta^{m/2} u, & \text{if } m \text{ is even} \\ \nabla \Delta^{(m-1)/2} u, & \text{if } m \text{ is odd.} \end{cases}$$

Adams in [1] proved that

$$\sup_{\substack{u \in W_0^{m,\frac{n}{m}}(\Omega), \\ \|\nabla^m u\|_{\frac{n}{m}} \leq 1}} \int_\Omega e^{\beta |u|^{\frac{n}{n-m}}} dx < \infty, \text{ if and only if } \beta \leq \beta_0, \tag{1}$$

where

$$\beta_0 = \beta_0(m,n) = \begin{cases} \frac{n}{\omega_{n-1}} \left[\frac{\pi^{\frac{n}{2}} 2^m \Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{n-m+1}{2}\right)} \right]^{n/(n-m)}, & \text{if } m \text{ is odd,} \\ \frac{n}{\omega_{n-1}} \left[\frac{\pi^{\frac{n}{2}} 2^m \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)} \right]^{n/(n-m)}, & \text{if } m \text{ is even,} \end{cases} \tag{2}$$

in which $\Gamma(x) = \int_0^1 (-\ln t)^{x-1}\, dt$, $x > 0$ is the gamma Euler function and ω_{n-1} is the area of the surface of the unit n-ball.

Tarsi [3] extends (1) to functions with homogeneous Navier boundary conditions. More precisely, it was proved that

$$\sup_{\substack{u \in W_{\mathcal{N}}^{m,\frac{n}{m}}(\Omega), \\ \|\nabla^m u\|_{\frac{n}{m}} \leq 1}} \int_\Omega e^{\beta |u|^{\frac{n}{n-m}}} dx < \infty, \text{ if and only if } \beta \leq \beta_0, \tag{3}$$

where

$$W_{\mathcal{N}}^{m,\frac{n}{m}}(\Omega) := \{u \in W^{m,\frac{n}{m}}(\Omega) : u_{|\partial\Omega} = \Delta^j u_{|\partial\Omega} = 0 \text{ in the sense of trace}, 1 \leq j < m/2\}.$$

We are interested in finding extremal function for the Adams Inequality. In this direction we provide the following estimate for Adams functional along of all concentrated sequences:

Theorem 1.1. *Let m, n be positive integers, $n \geq 2$ and $n > m$, and Ω be a smooth bounded domain in $\mathbb{R}^n$. Let $(u_i) \subset W_{\mathcal{N}}^{m,\frac{n}{m}}(\Omega)$, with $\|\nabla^m u_i\|_{\frac{n}{m}} = 1$ be a sequence concentrating at $x_0 \in \overline{\Omega}$, i.e.,*

$$\lim_{i\to\infty} \int_{\Omega\setminus B_r(x_0)} |\nabla^m u_i|^{\frac{n}{m}} dx = 0, \quad \text{for any } r > 0.$$

Then

$$\limsup_i \int_\Omega e^{\beta_0 |u_i|^{\frac{n}{n-m}}} dx \leq |\Omega| \left(1 + e^{\psi(\frac{n}{m})+\gamma}\right),$$

where $\gamma = \lim_{n\to\infty}\left(\sum_{j=1}^n (1/j) - \ln n\right)$ is the Euler-Mascheroni constant and $\psi(x) = \frac{d}{dx}(\ln\Gamma(x))$ is the classical Psi-function.

Since $W_0^{m,\frac{n}{m}}(\Omega)$ is a subspace of $W_{\mathcal{N}}^{m,\frac{n}{m}}(\Omega)$, as a direct consequence of the Theorem 2.1, we can highlight the following:

Corollary 1.1. *For γ and ψ as in Theorem 2.1, we have*

$$\limsup_i \int_\Omega e^{\beta_0 |u_i|^{\frac{n}{n-m}}} dx \leq |\Omega| \left(1 + e^{\psi(\frac{n}{m})+\gamma}\right), \tag{4}$$

for any $(u_i) \subset W_0^{m,\frac{n}{m}}(\Omega)$ under the same hypotheses of Theorem 2.1.

Although Corollary 1.1 is an easy consequence of Theorem 2.1 it is new and has merit itself. Indeed, from the Concentration-Compactness alternative [2, Theorem 1], in order to ensure the existence of extremal functions for the classical Adams inequality (1), it is now sufficient to show that there are test functions $u \in W_0^{m,n/m}(\Omega)$ such that

$$\|\nabla^m u\|_{\frac{n}{m}} = 1 \text{ and } \int_\Omega e^{\beta_0 |u|^{\frac{n}{n-m}}} dx > |\Omega| \left(1 + e^{\psi(\frac{n}{m})+\gamma}\right).$$

With this approach, we state

Theorem 1.2. *Let B_R be the unit ball with radius $R > 0$ centered at $0 \in \mathbb{R}^n$. Then, there exists $u_0 \in W_{\mathcal{N}}^{2,\frac{n}{2}}(B_R)$, with $\|\Delta u_0\|_{\frac{n}{2}} \leq 1$ such that*

$$C_{\beta_0}(B_R) = \sup_{\substack{u \in W_{\mathcal{N}}^{2,\frac{n}{2}}(B_R) \\ \|\Delta u\|_{\frac{n}{2}} \leq 1}} \int_{B_R} e^{\beta_0 |u|^{\frac{n}{n-2}}} dx = \int_{B_R} e^{\beta_0 |u_0|^{\frac{n}{n-2}}} dx \tag{5}$$

provided that $n \geq 2T_0$, where T_0 is the smallest positive integer such that

$$T_0 \geq 1 + \frac{1+36\sigma}{17-24\gamma} + \left[1 + \left(\frac{1+36\sigma}{17-24\gamma}\right)^2 + \frac{72\sigma}{17-24\gamma}\right]^{\frac{1}{2}} \approx 51.9233$$

where $\sigma = 1 + 2/\sqrt{3}$ and γ is the Euler-Mascheroni constant.

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ON UNIQUENESS RESULTS FOR SOLUTIONS OF THE BENJAMIN EQUATION

ALYSSON CUNHA[1,†]

Instituto de Matemàtica e Estatìstica, UFG, GO, Brasil[1].

alysson@ufg.br[†]

Abstract

We prove that the uniqueness results obtained in [1] for the Benjamin equation, cannot be extended for any pair of non-vanishing solutions. On the other hand, we study uniqueness results of solutions of the Benjamin equation. With this purpose, we showed that for any solutions u and v defined in $\mathbb{R}\times[0,T]$, if there exists an open set $I\subset\mathbb{R}$ such that $u(\cdot,0)$ and $v(\cdot,0)$ agree in I, $\partial_t u(\cdot,0)$ and $\partial_t v(\cdot,0)$ agree in I, then $u\equiv v$. A better version of this uniqueness result is also established. To finish, this type of uniqueness results were also proved for the nonlocal perturbation of the Benjamin-Ono equation (npBO).

1 Introduction

In this work, we study the initial-value problem (IVP) concerning the Benjamin equation

$$\begin{cases} u_t+\mathcal{H}\partial_x^2 u+\partial_x^3 u+uu_x=0, & x,t\in\mathbb{R}\\ u(x,0)=\phi(x), \end{cases} \tag{1}$$

where $u=u(t,x)$ is a real-valued function and $\mathcal{H}$ stands for the Hilbert transform defined as

$$\begin{aligned}\mathcal{H}f(x)&=\frac{1}{\pi}\lim_{\epsilon\downarrow 0}\int_{|y|\geq\epsilon}\frac{f(x-y)}{y}dy\\ &=-i(\operatorname{sgn}(\xi)\hat{f}(\xi))^{\vee}(x).\end{aligned}$$

The integral-differential equation (1) is a mathematical model to describe a class of the intermediate waves in the stratified fluid. It was deduced by Benjamin [2], to study gravity-capillary surface waves of solitary type on deep water.

We also studied the IVP for a nonlocal perturbation of the Benjamin Ono equation (npBO)

$$\begin{cases} u_t+\mathcal{H}u_{xx}+\mu(\mathcal{H}u_x+\mathcal{H}u_{xxx})+uu_x=0, & t>0, \quad x\in\mathbb{R},\\ u(x,0)=\phi(x), \end{cases} \tag{2}$$

where $\mu>0$. See [3].

Here, to obtain our results, we use techniques present in [4], [5], [6] and [7].

2 Main Results

Theorem 2.1. *Let u and v solutions of the IVP (1) with initial data ϕ and φ, respectively. Suppose that $\phi,\varphi\in Z_{9,4}$ satisfies $\phi\neq\varphi$,*

$$\|\phi\|=\|\varphi\|, \tag{3}$$

$$\int\phi(x)dx=\int\varphi(x)dx \tag{4}$$

and

$$\int x\phi(x)dx = \int x\varphi(x)dx. \tag{5}$$

Then $u \neq v$ and for all $T > 0$

$$u - v \in L^{\infty}([-T, T]; Z_{9^-,4}). \tag{6}$$

In the next result, we show a better version of the Theorem 2.1 for the more low regularity $s \geq 2r$.

Theorem 2.2. *Let $\theta \in (0, 1/2)$ and u, v solutions of the IVP (1) with initial data ϕ and φ, respectively. Suppose that $\phi, \varphi \in Z_{8+2\theta,4+\theta}$ satisfies $\phi \neq \varphi$, (3), (4) and (5), above.*

Then $u \neq v$ and for all $T > 0$

$$u - v \in L^{\infty}([-T, T]; Z_{8+2\theta,4+\theta}). \tag{7}$$

Theorem 2.3. *Let u and v be real solutions of the IVP (1) (or IVP (2)) such that*

$$u, v \in C([0, T]; H^s(\mathbb{R})) \cap C^1((0, T); H^{s-3}(\mathbb{R})), \quad s > 7/2. \tag{8}$$

If there exist an open set $I \subset \mathbb{R}$ such that

$$u(x, 0) = v(x, 0) \quad \text{and} \quad \partial_t u(x, 0) = \partial_t v(x, 0), \quad \text{for all} \quad x \in I, \tag{9}$$

then

$$u \equiv v. \tag{10}$$

In particular, if $u \equiv 0$ in $I \times \{0\}$ and $\partial_t u(x, 0) = 0$, $\forall x \in I$, then

$$u \equiv 0. \tag{11}$$

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SMOOTHING AND FINITE-DIMENSIONALITY OF UNIFORM ATTRACTORS IN BANACH SPACES

ARTHUR CUNHA[1], ALEXANDRE CARVALHO.[2], HONGYONG CUI[3] & JOSÈ LANGA[4]

Instituto de Matemàtica e Estatìstica, UFBA, BA, Brazil[1]
Instituto de Ciências Matemáticas e de Computação, USP - São Carlos, SP, Brazil[2],
School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, China[3],
Departamento de Ecuaciones Diferenciales y Anàlisis Numèrico, Universidad de Sevilla, Sevilla, Spain[4].

cunha.arthurc@gmail.com[†] andcarva@icmc.usp.br[‡] h.cui@outlook.com[§] langa@us.es[§§]

Abstract

The aim of this talk is to find an upper bound for the fractal dimension of uniform attractors in Banach spaces. The main technique is essentially based on a compact embedding of some auxiliary Banach space into the phase space and a corresponding smoothing effect between these spaces. Our bounds on the fractal dimension of uniform attractors are given in terms of the dimension of the symbol space and the Kolmogorov entropy number of the embedding. A dynamical analysis on the symbol space is also given, showing that the finite-dimensionality of the hull of a time-dependent function is fully determined by the tails of the function, which allows us to consider more general non-autonomous terms than quasi-periodic functions. As application, we show that the uniform attractor of a reaction-diffusion equation is finite-dimensional in L^2 and in L^p, with $p > 2$.

1 Introduction

Let (Ξ, d_Ξ) be a complete metric space and let $\{\theta_s\}_{s\in\mathbb{R}}$ be a group of continuous operators acting on Ξ, i.e., $\theta_0\sigma = \sigma$ and $\theta_t(\theta_s\sigma) = \theta_{t+s}\sigma$ for all $\sigma \in \Xi$, $t, s \in \mathbb{R}$, and for each $s \in \mathbb{R}$, $\theta_s : \Xi \to \Xi$ is a continuous map in Ξ. Let $\Sigma \subseteq \Xi$ be a *compact* subset of Ξ which is invariant under $\{\theta_s\}_{s\in\mathbb{R}}$, that is, $\theta_s\Sigma = \Sigma$ for all $s \in \mathbb{R}$.

Considering evolution processes $\{U_\sigma(t, s)\}_{\sigma\in\Sigma}$ (i.e., for each $\sigma \in \Sigma$ the two-parameter family $\{U_\sigma(t, s) : t \geq s\}$ in a Banach space X satisfies $U(s, s) = Id_X$ and $U(t, \tau)U(\tau, s) = U(t, s)$ for all $t, \tau, s \in \mathbb{R}$ with $t \geq \tau \geq s$) it will be called a *system* if the *translation-identity* is satisfied:

$$U_{\theta_h\sigma}(t, s) = U_\sigma(t + h, s + h), \quad \forall\sigma \in \Sigma,\ t \geqslant s,\ h \in \mathbb{R}.$$

Definition 1.1. *A compact set $\mathcal{A}_\Sigma \subset X$ is the* uniform (w.r.t. $\sigma \in \Sigma$) attractor *of a system $\{U_\sigma(t, s)\}_{\sigma\in\Sigma}$ if*

(i) $\mathcal{A}_\Sigma$ *is* uniformly attracting, *i.e., for any bounded set $E \subset X$ it holds*

$$\lim_{t\to\infty}\left[\sup_{\sigma\in\Sigma} dist_X\big(U_\sigma(t, 0)E, \mathcal{A}_\Sigma\big)\right] = 0,$$

where for non-empty sets $A, B \subset X$ we denote the Hausdorff semi-distance $dist_X(A, B) := \sup_{a\in A}\inf_{b\in B}\|a - b\|_X$.

(ii) (Minimality) If $\mathcal{A}'_\Sigma \subset X$ is a closed uniformly attracting set, then $\mathcal{A}_\Sigma \subset \mathcal{A}'_\Sigma$.

2 Main Results

In this section we give our main results on the dimensionality of uniform attractors. Let us start with the definition of fractal dimension.

Definition 2.1. *Let A be a non-empty precompact subset of X. The fractal dimension of A in X is defined as*

$$dim_F(A;X) := \limsup_{\varepsilon\to 0^+} \frac{\ln N_X[A;\varepsilon]}{-\ln\varepsilon},$$

where $N_X[A;\varepsilon]$ denotes the minimum number of open ε-balls in X that are necessary to cover A.

Now we give a criterion for the uniform attractor $\mathcal{A}_\Sigma$ of $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ to be finite-dimensional. The main idea is an (X,Y)-smoothing property combined with assumptions on the dimension of the symbol space. Let $\mathcal{B}\subset X$ be a closed bounded uniformly absorbing set of $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ (it has necessarily to satisfy $\mathcal{A}_\Sigma\subseteq\mathcal{B}$) and suppose that:

(H_1) The symbol space Σ has finite fractal dimension in space Ξ, i.e., $\dim_F(\Sigma;\Xi)<\infty$.

(H_2) $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ is *uniformly* (X,Y)*-smoothing* on the absorbing set $\mathcal{B}$, i.e., there is an auxiliary Banach space Y *compactly embedded* in the Banach space X and for any $t>0$ there exists a $\kappa(t)>0$ such that

$$\sup_{\sigma\in\Sigma}\|U_\sigma(t,0)u-U_\sigma(t,0)v\|_Y\leqslant\kappa(t)\|u-v\|_X,\quad \forall u,v\in\mathcal{B}.$$

(H_3) $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ is $(\Sigma\times X,X)$*-continuous*. Moreover, $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ is (Σ,X)*-Lipschitz* within $\mathcal{B}$, i.e.,

$$\|U_{\sigma_1}(t,0)u-U_{\sigma_2}(t,0)u\|_X\leqslant L(t)d_\Xi(\sigma_1,\sigma_2),\quad \forall t\geq 1,\ \sigma_1,\sigma_2\in\Sigma,\ u\in\mathcal{B},$$

where $1\leq L(t)\leq c_1e^{\beta t}$ for some positive constants $c_1,\beta>0$ for $t\geq 1$.

Theorem 2.1. *Let $\{U_\sigma(t,s)\}_{\sigma\in\Sigma}$ be a system in X with uniform attractor $\mathcal{A}_\Sigma$. If conditions (H_1) - (H_3) hold, then the uniform attractor $\mathcal{A}_\Sigma$ has finite fractal dimension in X with*

$$dim_F\big(\mathcal{A}_\Sigma;X\big)\leqslant\ln N_X\big[B_Y(0,1);1/(2e\kappa)\big]+(\beta+1)\,dim_F\big(\Sigma;\Xi\big),$$

for some $\kappa=\kappa(T_\mathcal{B})$ depending on $\mathcal{B}$ with $T_\mathcal{B}\geq 1$ an absorption time after which $\mathcal{B}$ uniformly absorbs itself.

About the symbol space we also provide the following result in order to construct finite-dimensional examples.

Theorem 2.2. *Let $\mathcal{M}$ be a complete metric space and g_+, $g_-\in\Xi=\mathcal{C}(\mathbb{R};\mathcal{M})$ with finite-dimensional hulls $\mathcal{H}(g_+)$ and $\mathcal{H}(g_-)$ in Ξ, respectively, where $\mathcal{H}(\xi)=\overline{\{\xi(\cdot+s):s\in\mathbb{R}\}}^{\Xi}$ for $\xi\in\Xi$. Suppose that $g\in\Xi$ is Lipschitz continuous from $\mathbb{R}$ to $\mathcal{M}$ and that g converges forwards to g_+ and backwards to g_- eventually exponentially. Then the hull $\mathcal{H}(g)$ of g is compact and finite-dimensional in Ξ with*

$$\dim_F\big(\mathcal{H}(g);\Xi\big)\leq\max\Big\{1,\dim_F\big(\mathcal{H}(g_+);\Xi\big),\dim_F\big(\mathcal{H}(g_-);\Xi\big)\Big\}.$$

As an application of our theoretical results we investigate the reaction-diffusion equation

$$\begin{cases} v_t+\lambda v-\Delta v=f(v)+\sigma(x,t),\\ v(x,t)|_{t=\tau}=v_\tau(x),\quad v(x,t)|_{\partial\mathcal{O}}=0,\quad x\in\mathcal{O},\ t\geq\tau,\end{cases}\tag{1}$$

where $\mathcal{O}\subset\mathbb{R}^N$, $N\in\mathbb{N}$, is a bounded smooth domain and $\lambda>0$. The nonlinearity $f(\cdot)\in\mathcal{C}^1(\mathbb{R};\mathbb{R})$ satisfies

$$f(s)s\leq-\alpha_1|s|^p+\beta_1\quad,\quad|f(s)|\leq\alpha_2|s|^{p-1}+\alpha_2\quad,\quad|f'(s)|\leq\kappa_2|s|^{p-2}+l_2\quad\text{and}\quad f'(s)\leq-\kappa_1|s|^{p-2}+l_1,$$

where $p\geq 2$ and all the coefficients are positive. The non-autonomous symbol σ is in a symbol space Σ constructed as the hull $\mathcal{H}(g)$ of a given non-autonomous function $g\in\Xi:=\mathcal{C}\big(\mathbb{R};L^2(\mathcal{O})\big)$, i.e., for $\theta_rg(\cdot):=g(\cdot+r)$ we have

$$\Sigma=\mathcal{H}(g):=\overline{\{\theta_rg:r\in\mathbb{R}\}}.$$

If $\dim_F\left(\Sigma;\Xi\right)<\infty$ we prove that the uniform attractor has finite fractal dimension in L^2 and in L^p, with $p>2$.

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THE HARDY PARABOLIC PROBLEM WITH INITIAL DATA IN UNIFORMLY LOCAL LEBESGUE SPACES

BRANDON CARHUAS DE LA TORRE[1], MIGUEL LOAYZA[1] & RICARDO CASTILLO[2]

Departamento de Matemática, UFPE, PE, Brasil[1], Departamento de Matemática, UBB, Concepción, Chile[2]

brandon.carhuas@ufpe.br† miguel.loayza@ufpe.br‡ rcastillo@ubiobio.cl§

Abstract

We consider the singular nonlinear equation $u_t - \Delta u = |\cdot|^{-\gamma} f(u)$ in $\mathbb{R}^N \times (0,T)$ with $\gamma > 0$ and $f \in C(\mathbb{R})$ non-decreasing. This equation is known in the literature as a Hardy parabolic equation. We establish local existence result for $u_0 \in L^r_{ul}(\mathbb{R}^N)$, $1 \leq r < \infty$. In particular, we obtain necessary and sufficient conditions for the existence in the case that $f \in C([0,\infty))$, $u_0 \geq 0$, improving results given in the context of Lebesgue spaces.

1 Introduction

Let $f \in C(\mathbb{R})$ be a non-decreasing function, $T > 0$ and $\gamma > 0$. We consider the parabolic problem with singular potential

$$\begin{cases} u_t - \Delta u &= |\cdot|^{-\gamma} f(u) \quad \text{in } \mathbb{R}^N \times (0,T), \\ u(0) &= u_0 \quad \text{in } \mathbb{R}^N. \end{cases} \tag{1}$$

The first equation of (1) is also known in the literature as Hardy's parabolic equation and it has been considered by many authors for specific functions f and initial data in different spaces.

For $\gamma > 0$ and $f(u) = |u|^{p-1}u$ the well-posedness of problem (1) was studied by Slimene et al. [2] for continuous and in the Lebesgue spaces initial data. The case with $f \in C([0,\infty))$ and $u_0 \in L^r(\mathbb{R}^N)$, $u_0 \geq 0$ was analyzed in [4], and it is shown that there exists a critical value

$$p^*_\gamma = 1 + \frac{(2-\gamma)r}{N} \tag{2}$$

but the results obtained by them do not provide a complete characterization on the local existence in the sense of [1]; also, they did not treat the case $\gamma = N/r$.

Our objective is to improve the results given in [4]. To do this, we consider the uniformly local Lebesgue space $L^r_{ul,\rho}(\mathbb{R}^N)$ which is defined as follows. For $1 \leq r \leq \infty$,

$$L^r_{ul,\rho}(\mathbb{R}^N) = \left\{ u \in L^1_{loc}(\mathbb{R}^N); \|u\|_{L^r_{ul,\rho}(\mathbb{R}^N)} < \infty \right\}, \text{ where}$$

$$\|u\|_{L^r_{ul,\rho}(\mathbb{R}^N)} := \begin{cases} \sup_{y \in \mathbb{R}^N} \left(\int_{B_\rho(y)} |u(x)|^r \, dx \right)^{1/r} & \text{if } 1 \leq r < \infty, \\ \operatorname{esssup}_{y \in \mathbb{R}^N} \|u\|_{L^\infty(B_\rho(y))} & \text{if } r = \infty, \end{cases}$$

and $B_\rho(y) \subset \mathbb{R}^N$ denotes the open ball centered at y with radius $\rho > 0$. It is clear that $L^\infty_{ul,\rho}(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$.

We denote by $\mathcal{L}^r_{ul,\rho}(\mathbb{R}^N)$ the closure of the space of bounded uniformly continuous functions $BUC(\mathbb{R}^N)$ in the space $L^r_{ul,\rho}(\mathbb{R}^N)$, that is, $\mathcal{L}^r_{ul,\rho}(\mathbb{R}^N) := \overline{BUC(\mathbb{R}^N)}^{\|\cdot\|_{L^r_{ul,\rho}(\mathbb{R}^N)}}$.

To reduce notation, we write $L^r_{ul}(\mathbb{R}^N)$ and $\mathcal{L}^r_{ul}(\mathbb{R}^N)$ if $\rho = 1$.

2 First Result

The notion of solution used in the work is the following.

Definition 2.1. *Let* $\gamma > 0, u_0 \in L^r_{ul}\left(\mathbb{R}^N\right)$, $1 \leq r < \infty$ *and* $f \in C(\mathbb{R})$. *We say that* $u \in L^\infty\left((0,T), L^r_{ul}\left(\mathbb{R}^N\right)\right) \cap L^\infty_{loc}\left((0,T), L^\infty\left(\mathbb{R}^N\right)\right)$, *for some* $T > 0$, *is a solution of the problem (1) if it verifies*

$$u(t) := S(t)\, u_0 + \int_0^t S(t-\sigma)\, |\cdot|^{-\gamma} f(u(\sigma))\, d\sigma$$

a.e. in $\mathbb{R}^N \times (0,T)$, *where* $\{S(t)\}_{t\geq 0}$ *denotes the heat semigroup.*

Theorem 2.1. *Suppose that* $f \in C(\mathbb{R})$ *is a nondecreasing function,* $0 < \gamma < \min\{2, N\}$, p^*_γ *defined by (2), and one of the following conditions hold:*

1. $u_0 \in L^1_{ul}(\mathbb{R}^N)$ *and*

$$\int_1^\infty \sigma^{-(1+(2-\gamma)/N)} \tilde{F}(\sigma) d\sigma < \infty, \quad \text{where} \quad \tilde{F}(t) := \sup_{1\leq|\sigma|\leq t} \frac{f(\sigma)}{\sigma}. \tag{3}$$

2. $r > 1$ *and*

$$\limsup_{|t|\to\infty} |t|^{-p^*_\gamma} |f(t)| < \infty, \quad \text{if } u_0 \in \mathcal{L}^r_{ul}(\mathbb{R}^N), \tag{4}$$

$$\lim_{|t|\to\infty} |t|^{-p^*_\gamma} |f(t)| = 0, \quad \text{if } u_0 \in L^r_{ul}(\mathbb{R}^N). \tag{5}$$

Then, problem (1) has a solution u *defined on some interval* $(0,T)$. *Moreover,* $t^{N/2r}\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq C$ *for some* $C > 0$ *and all* $t \in (0,T)$.

3 Main Result

When we consider non-negative initial data we have the following.

Theorem 3.1. *Let* $f : [0,\infty) \to [0,\infty)$ *be a continuous and non-decreasing function, and let* $0 < \gamma < \min\{2, N\}$. *Problem (1) has a local non-negative solution for every* $u_0 \in \mathcal{L}^r_{ul}(\mathbb{R}^N), u_0 \geq 0, r \geq 1$ *if and only if*

$$\begin{cases} \int_1^\infty \sigma^{-[1+(2-\gamma)/N]} F(\sigma) d\sigma < \infty & \text{if } r = 1, \\ \limsup_{t\to\infty} t^{-p^*_\gamma} f(t) < \infty & \text{if } r > 1, \end{cases} \tag{6}$$

where $F(t) = \sup_{1\leq\sigma\leq t} f(\sigma)/\sigma$, $t > 0$.

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DYNAMICS OF THE ENERGY-CRITICAL INHOMOGENEOUS HARTREE EQUATION

CARLOS M. GUZMÀN[1,†] & CHENGBIN B. XU[2,‡]

Instituto de Matemática e Estatìstica, UFF, RJ, Brasil[1],
School of Mathematics and Statistics, Qinghai Normal University, China[2]

carlosguzman@idd.uff.br[†] xcbsph@163.com[‡]

Abstract

In this talk, we discuss the dynamic of the energy critical inhomogeneous nonlinear Hartree equation in $3D$

$$i\partial_t u + \Delta u + |x|^{-b}(I_\alpha * |\cdot|^{-b}|u|^p)|u|^{p-2}u = 0, \quad x \in \mathbf{R}^3,$$

where $p = 3 + \alpha - 2b$. We study the global well-posedness and scattering below the ground state threshold with general initial data in $\dot{H}^1$. To this end, we exploit the decay of the nonlinearity, which together with the Kenig–Merle roadmap, allows us to treat the non-radial case as the radial case. The inhomogeneous model presents some new challenges arising from the broken translation symmetry. Here, we overcome that and also discuss some open problems.

1 Introduction

We consider the initial value problem (IVP), for the focusing inhomogeneous generalized Hartree equation (which also call inhomogeneous Choquard equation)

$$\begin{cases} i\partial_t u + \Delta u + |x|^{-b}(I_\alpha * |\cdot|^{-b}|u|^p)|u|^{p-2}u = 0, & t \in \mathbb{R},\ x \in \mathbb{R}^3, \\ u(0,x) = u_0(x) \in \dot{H}^1(\mathbb{R}^3) \end{cases} \tag{1}$$

where $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$, $p = 3 + \alpha - 2b$. The inhomogeneous term is $|\cdot|^{-b}$ for some $b > 0$. The Riesz-potential is defined on $\mathbb{R}^3$ by

$$I_\alpha := \frac{\Gamma(\frac{3-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{3}{2}}2^\alpha|\cdot|^{3-\alpha}} := \frac{\mathcal{K}}{|\cdot|^{3-\alpha}}, \quad 0 < \alpha < 3.$$

The nonlinearity in (1) makes the equation a focusing, energy-critical model. In fact, the $\dot{H}^1(\mathbb{R}^3)$ norm is invariant under the standard scaling

$$u_\lambda(t,x) = \lambda^{\frac{1}{2}}u(\lambda^2 t, \lambda x),$$

and $u_\lambda(t,x)$ is also the solution of the equation (1). Moreover, equation (1) conserves the energy, defined as the sum of the kinetic and potential energies:

$$E(u) := \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{1}{p}(I_\alpha * |\cdot|^{-b}|u|^p)|x|^{-b}|u(x)|^p dx.$$

In this work, we study the energy-critical case, that is, the critical index Sobolev $s_c = 1$. We establish the global well-posedness and scattering for (1.1) assuming general initial data.

2 Main Result

Theorem 2.1. *Let* $0 < b \leq \min\{\frac{1+\alpha}{3}, \frac{\alpha}{2}\}$. *Suppose* $u_0 \in \dot{H}^1(\mathbb{R}^3)$ *satisfies*

$$E(u_0) < E(W) \quad \text{and} \quad \|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}. \tag{1}$$

Then the solution u *to* (1) *is global in time and scatters in* $\dot{H}^1(\mathbb{R}^3)$. *Here,* W *denotes the ground state, i.e., the solution to the elliptic equation*

$$\Delta W + (I_\alpha * |\cdot|^{-b}|W|^p)|x|^{-b}|W|^{p-2}W = 0.$$

Some consequences[1]

Corollary 2.1. *Let* $b = 0$ *and* $u_0 \in \dot{H}^1(\mathbb{R}^3)$ *radial such that*

$$E(u_0) < E(W) \quad \text{and} \quad \|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}. \tag{2}$$

Then the solution u *is global and scatters both in time.*

Corollary 2.2. *Let* $0 < b \leq \min\{\frac{1+\alpha}{3}, \frac{\alpha}{2}\}$. *For any* $u_0 \in \dot{H}^1(\mathbb{R}^3)$, *there exists a unique global solution to*

$$i\partial_t u + \Delta u - |x|^{-b}(I_\alpha * |\cdot|^{-b}|u|^p)|u|^{p-2}u = 0,$$

with initial data u_0. *Furthermore, this global solution scatters both in time.*

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A MODEL OF SUSPENSION BRIDGE IN LAMINATED BEAMS

CARLOS ALBERTO RAPOSO[1,†] & DUCIVAL CARVALHO PEREIRA[2,‡]

Departamento de Matemática, UFBA, BA, Brasil[1], UEPA, PA, Brasil[2]

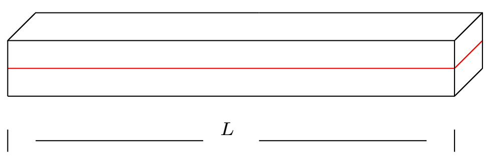
carlos.raposo@ufba.br[†] ducival@uepa.br[‡]

Abstract

This manuscript introduces a suspension bridge system where laminated beams model the deck. The action of frictional damp- ing is considered. Well-posedness is proved using the Lumer-Phillips theorem, and the exponential stability is obtained by applying the Gearhart-Huang-PrÃ¼ss theorem

1 Introduction

We introduce a model of suspension bridge given as mechanical structure that carries vertical loads through the main cables modeled by an elastic string $u = u(x, t)$, which is coupled to the deck employing suspension cables, where x denotes the distance along the center line of the deck in its equilibrium configuration and t the time variable. Considering that the deck has negligible transversal section dimensions compared to the length (span of the bridge), it is modeled in Timoshenko's theory [5] as a laminated beam system of length L, proposed by Hansen and Spies [1, 2] for two-layered beams in which a slip can occur at the interface of contact.



Denoting by $\varphi = \varphi(x, t)$ the displacement of the cross-section on the point $x \in (0, L)$, by $\psi = \psi(x, t)$ the rotation angle of the cross-section, and by $s = s(x, t)$ the slip along the interface of contact (in red at figure above), we have the following coupled system

$$u_{tt} - \alpha u_{xx} - \lambda(\varphi - u) = 0, \tag{1}$$

$$\rho\varphi_{tt} + G(\psi - \varphi_x)_x + \lambda(\varphi - u) = 0, \tag{2}$$

$$I_\rho(3S_{tt} - \psi_{tt}) - D(3S_{xx} - \psi_{xx}) - G(\psi - \varphi_x) = 0, \tag{3}$$

$$I_\rho 3S_{tt} - D3S_{xx} + 3G(\psi - \varphi_x) + 4\gamma_0 S + 4\delta_0 S_t = 0, \tag{4}$$

The positive parameters ρ, I_ρ, G, D, and γ_0, are the density, mass moment of inertia, shear stiffness, flexural rigidity, and adhesive stiffness, respectively. The non-negative parameter δ_0 is called the adhesive damping, and s_t is a structural damping of the system.

In [7] was proved that the structural damping S_t created by the interfacial slip alone is not enough to stabilize this system of laminated beam exponentially to its equilibrium state. Naturally, the question arises of studying the action of additional stabilizing mechanisms for the model (1)-(4). We consider the action frictional dampings on each component, as in [5], and prove the well-posedness and exponential stability by Semigroup Theory [2, 2].

2 Main Results

We introduce the vector function

$$U = (u, w, \varphi, \phi, \psi, \eta, S, z)^T,$$

where $w = u_t$, $\phi = \varphi_t$, $\eta = \psi_t$ and $z = S_t$.

The system (1)-(4) can be written as

$$\begin{cases} U_t - \mathcal{A}U = 0, \\ U(x,0) = U_0(x), \end{cases} \tag{1}$$

where the linear operator

$$\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$$

is defined on energy space $\mathcal{H} = [H_0^1(0,L) \times L^2(0,L)]^4$ and $D(\mathcal{A}) = [H_0^1(0,L) \cap H^2(0,L) \times H_0^1(0,L)]^4$. The well-posedness is ensured by the following theorem.

Theorem 2.1. *For $U_0 \in \mathcal{H}$, there exists a unique weak solution U of (1) satisfying*

$$U \in C^0((0,\infty); \mathcal{H}). \tag{2}$$

Moreover, if $U_0 \in D(\mathcal{A})$, then

$$U \in C^0((0,\infty); D(\mathcal{A})) \cap C^1((0,\infty); \mathcal{H}). \tag{3}$$

Our main result is:

Theorem 2.2. *The semigroup $S(t) = e^{\mathcal{A}t}$, $t \geq 0$, generated by $\mathcal{A}$ is exponentially stable.*

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STABILITY OF EQUILIBRIUM SOLUTIONS OF A LOGARITHMIC HEAT EQUATION IN COMMUTATIVE BANACH ALGEBRAS

CÈSAR ADOLFO HERNÀNDEZ MELO[1,†] & LAURA MASSUDA CREMA[2,‡]

Departamento de Matemática, UEM, PR, Brasil[1], Departamento de Matemática, UEM, PR, Brasil[2]

cahmelo@uem.br[†] pg404031@uem.br[‡]

Abstract

In this work, it will be mainly discussed the existence and stability of equilibrium solutions of a nonlinear heat equation where the unknown takes values in a commutative Banach algebra and the nonlinearity is of logarithmic type.

1 Introduction

The general reaction diffusion equation,

$$u_t = u_{xx} + f(u), \tag{1}$$

where u is a function from $\mathbb{R} \times \mathbb{R}^+$ to a commutative Banach algebra $\mathbb{X}$ and f denotes a function defined from an open set of $\mathbb{X}$ to $\mathbb{X}$ has received considerable attention from the scientific community in the last decades. For a wide discussion about applications and mathematical problems associated to the equation (1) when $\mathbb{X} = \mathbb{R}$, see for instance [1] and the references therein. An equilibrium solution of the equation (1) is a solution of the equation that is independent of the variable $t \in \mathbb{R}^+$. Thus, $u(x,t) = \phi(x) \in \mathbb{X}$ is an equilibrium solution of the equation (1) on the open interval J, if ϕ satisfies the following second order ordinary differential equation

$$\phi''(x) + f(\phi(x)) = 0 \in \mathbb{X}, \qquad \text{for all } x \in J. \tag{2}$$

Let Y be Banach space where the Cauchy problem

$$\begin{cases} u_t = u_{xx} + f(u), \\ u(0) = u_0 \in Y \end{cases} \tag{3}$$

is well posed. An equilibrium solution $\phi \in Y$ is said to be stable in $Y = (Y, ||\cdot||_Y)$, if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{If} \quad ||u_0 - \phi||_Y < \delta, \quad \text{then} \quad ||u(t) - \phi||_Y < \epsilon, \quad \text{for all } t > 0.$$

Here, u denotes the solution of the Cauchy problem in (6) with $u(0) = u_0 \in Y$. Otherwise, the equilibrium solution ϕ is said to be unstable. A classical method to study the stability/instability of an equilibrium solution ϕ is based on the analysis of the spectral properties of the following linear operator

$$\mathcal{L}_\phi = \frac{d^2}{dx^2} + f'(\phi) \tag{4}$$

which is defined on a certain Banach space. Roughly speaking, if the spectrum of the operator $\mathcal{L}_\phi$ intercepts the set $\{z \in \mathbb{C} : 0 < \text{Re}(z)\}$, then the equilibrium solution is unstable. On the other hand, if the spectrum of the operator $\mathcal{L}_\phi$ is contained in the set $\{z \in \mathbb{C} : \text{Re}(z) \leq b\}$ for some $b < 0$, then the equilibrium solution is stable. See theorems 5.1.1 and 5.1.3 in [1] for details. In this work, it is studied the problem of existence and stability of equilibrium solutions of the equation (1) when $\mathbb{X}$ is a commutative Banach algebra with identity $e \in \mathbb{X}$ and

$f(u) = wu + \text{Log}(u^2)u$ with $w \in \mathbb{X}$ and Log denotes the logarithmic function defined on the open ball $B(e,1) \subset \mathbb{X}$ by the following power series

$$\text{Log}(p) = -\sum_{k=1}^{\infty} \frac{(e-p)^k}{k} = -(e-p) - \frac{(e-p)^2}{2} - \frac{(e-p)^3}{3} - \cdots \tag{5}$$

That is to say, the logarithmic heat equation

$$u_t = u_{xx} + wu + \text{Log}(u^2)u. \tag{6}$$

2 Main Results

In this section, it will be described the principal results about the stability theory of equilibrium solutions associated to the equation (6). Our findings were based on the results discussed in [2], where the problem of existence and stability of equilibrium solution to the equation (6) was treated in the case that $\mathbb{X} = \mathbb{R}$.

Theorem 2.1 (Existence of equilibriums)**.** *Let $\mathbb{X}$ be a commutative Banach algebra with identity e and $w \in \mathbb{X}$ satisfying $||e - w|| < \log(2)$. For each $r \in \mathbb{R}$, the function $\phi_r : \mathbb{R} \to \mathbb{X}$ given by*

$$\phi_r(x) = Exp\left(\frac{e-w}{2}\right)\exp\left(\frac{-(x+r)^2}{2}\right) \tag{1}$$

is an equilibrium solution of the logarithmic heat equation (6) on $J = \mathbb{R}$. That is, ϕ_r satisfies the second order differential equation $\phi''(x) + w\phi(x) + Log(\phi^2(x))\phi(x) = 0 \in \mathbb{X}$ for all $x \in \mathbb{R}$.

When $f(u) = wu + \text{Log}(u^2)u$ and $\phi = \phi_r$ is given by the formula in (1) then the linear operator $\mathcal{L}_\phi$ takes the explicit form

$$\mathcal{L}_r(g) = \frac{d^2}{dx^2}g + 3g - (x+r)^2 g, \qquad g \in D(\mathcal{L}_r) \subset C_b(\mathbb{R}, \mathbb{X}), \tag{2}$$

where $C_b(\mathbb{R}, \mathbb{X})$ denotes the Banach algebra of bounded functions from $\mathbb{R}$ to $\mathbb{X}$ and $D(\mathcal{L}_r)$ denotes the domain of the liner operator $\mathcal{L}_r$. About the spectrum of the linear operator $\mathcal{L}_r$, it is obtained the following result:

Theorem 2.2 (Spectral properties of $\mathcal{L}_r$)**.** *The point spectrum of the linear operator $\mathcal{L}_r$ is given by the sequence of eigenvalues $\lambda_n = -2(n-1)$, $n = 0, 1, 2, \ldots$, that is*

$$\{2, 0, -2, -4, \ldots, -2(n-1), \ldots\} = \sigma_p(\mathcal{L}_r) \subset \sigma(\mathbb{R}). \tag{3}$$

Furthermore, the eigenspace associated to each one of the eigenvalues λ_n is generated by just one eigenfunction.

Since, $\lambda_0 = 2$ is a positive eigenvalue of $\mathcal{L}_r$, it is deduced that the equilibrium solution ϕ given in (1) is unstable. Actually, it is possible to prove an stronger result, namely

Theorem 2.3 (Nonlinear instability)**.** *Any nontrivial equilibrium solution ϕ of (6) is unstable. More exactly, exists $\epsilon > 0$, a sequence of functions g_n and a sequence of times $t_n > 0$ such that*

$$g_n \to \phi, \qquad \textit{and} \qquad ||u_n(\cdot, t_n) - \phi(\cdot)|| > \epsilon.$$

Here, u_n denotes a sequence of solutions of the LHE such that $u_n(\cdot, 0) = g_n(\cdot)$.

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ON THE FRACTIONAL NAVIER-STOKES-CORIOLIS EQUATION IN FOURIER-BESOV-MORREY SPACES

EDER MATEUS[1], BRUNO DE ANDRADE[2] & CHARLES AMORIM[3]

Departamento de Matemàtica de Itabaiana-DMAI, UFS, SE, Brasil[1], Departamento de Matemàtica-DMA, UFS, SE, Brasil[2], Departamento de Matemàtica de Itabaiana-DMAI, UFS, SE, Brasil[3]

edermateus@mat.ufs.br† ‡ amorimcb@academico.ufs.br§

Abstract

1 Introduction

In this work we are interested in studying the initial value problem for the fractional equations

$$\begin{cases} \partial_t^\alpha u + \nu(-\Delta)^\beta u + \Omega e_3 \times u + (u\cdot\nabla)u + \nabla p = g\omega e_3, & (t,x)\ \mathbb{R}\times\mathbb{R}^3, \\ \partial_t^\alpha \theta + \mu(-\Delta)^\beta\theta + (u\cdot\nabla)\theta = -\mathcal{N}^2 u_3, & (t,x)\ \mathbb{R}^+\times\mathbb{R}^3, \\ \nabla\cdot u = 0, & \\ u(0,x) = u_0(x), & x\in\mathbb{R}^3, \end{cases} \tag{1}$$

where ∂_t^α is the Caputo's fractional derivative of order $\alpha \in (0,1]$ and $\frac{1}{2} \leq \beta < \frac{5}{2}$. When $\alpha = 1$, these equations represent the 3D fractional Boussinesq-Coriolis equations with stratification. In this context μ is the viscosity, $p = p(x,t)$ is the pressure of the fluid and θ is a scalar function that represents the buoyancy density in the fluid (in the case of the ocean this function depends temperature and salinity, and in the case of the atmosphere it depends on temperature). The initial data $u_0 = u_0(x) = (u_0^1(x), u_0^2(x), u_0^3(x))$ denotes the initial velocity field satisfying the compatibility condition $\nabla\cdot u = 0$. The constants ν, μ and g are related to viscosity, diffusivity and gravity, respectively. The constant $\Omega \neq 0$ represents the speed of rotation around the vertical unit vector $e_3 = (0,0,1)$ and is called the Coriolis parameter. The stratification parameter $\mathcal{N}$ is a non-negative constant that represents the frequency of the Brunt-Väisälä wave. The proportion $P = \frac{\mu}{\nu}$ is known as the Prandtl number and $B = \frac{\Omega}{\mathcal{N}}$ is essentially the Burger number of geophysics.

Considering $N = \mathcal{N}\sqrt{g}, v = (v^1, v^2, v^3, v^4) = (u^1, u^2, u^3, \sqrt{g}\theta/\mathcal{N})$, $v_0 = (v_0^1, v_0^2, v_0^3, v_0^4) = (u_0^1, u_0^2, u_0^3, \sqrt{g}\theta_0/\mathcal{N})$, and $\tilde{\nabla} = (\partial_1, \partial_2, \partial_3, 0)$, we can convert the above system as

$$\begin{cases} \partial_t^\alpha v + \mathcal{A}v + \mathcal{B}v + \tilde{\nabla}p = -(v\cdot\tilde{\nabla})v, \text{ in } \mathbb{R}^3\times(0,\infty) \\ \tilde{\nabla}\cdot v = 0, \text{ in } \mathbb{R}^3\times(0,\infty) \\ v(x,0) = v_0(x), \text{ in } \mathbb{R}^3 \end{cases}$$

where

$$\mathcal{A} = \begin{pmatrix} \nu(-\Delta)^\beta & 0 & 0 & 0 \\ 0 & \nu(-\Delta)^\beta & 0 & 0 \\ 0 & 0 & \nu(-\Delta)^\beta & 0 \\ 0 & 0 & 0 & k(-\Delta)^\beta \end{pmatrix} \text{ and } \mathcal{B} = \begin{pmatrix} 0 & -\Omega & 0 & 0 \\ \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -N \\ 0 & 0 & N & 0 \end{pmatrix}.$$

A mild solution of this problem is a function that verifies the integral equation

$$v(t) = E_\alpha(t)v_0 - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s)\mathbb{P}(v\cdot\tilde{\nabla})v\,ds, \quad t\geq 0, \tag{2}$$

where $\mathbb{P}$ is the Leray Projector. The main objectives of this work are to guarantee the existence of the Mittag-Leffler families $\{E_\alpha(t)\}_{t\geq 0}$ and $\{E_{\alpha,\alpha}(t)\}_{t\geq 0}$ for equation (1.1), establish their behavior on the scale of Fourier-Besov-Morrey spaces and obtain asymptotic estimates for such families.

2 Main Results

Fourier-Besov-Morrey spaces are constructed by mean of a kind of localization procedure in Fourier variables on the well-known Morrey spaces, see [3]. Consider the ring $\mathcal{C} = \left\{\xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\}$ and φ a smooth function supported in $\mathcal{C}$ satisfying $0 \leq \varphi \leq 1$ and

$$\sum_{j\in\mathbb{Z}} \varphi_j(\xi) = 1, \text{ for all } \xi \neq 0, \text{ where } \varphi_j(\xi) = \varphi\left(2^{-j}\xi\right).$$

Let $1 \leq q < \infty, 0 \leq \mu < n, 1 \leq r < \infty$ and $s \in \mathbb{R}$. The Fourier-Besov-Morrey space $\mathcal{FN}^s_{q,\mu,r}$ is the set of all distributions $f \in \mathcal{S}'/\mathcal{P}$, where $\mathcal{P}$ is the set of all polynomials in $\mathbb{R}^n$, such that $\varphi_j \hat{f}$ belongs to the Morrey space $\mathcal{M}_{q,\mu}$, for all $j \in \mathbb{Z}$, and

$$\|f\|_{\mathcal{FN}^s_{q,\mu,r}} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} \left(2^{js} \left\|\varphi_j \hat{f}\right\|_{q,\mu}\right)^r\right)^{1/r} < \infty, 1 \leq r < \infty \\ \sup_{j\in\mathbb{Z}} 2^{js} \left\|\varphi_j \hat{f}\right\|_{q,\mu} < \infty, r = \infty. \end{cases}$$

The pair $\left(\mathcal{FN}^s_{q,\mu,r}, \|\cdot\|_{\mathcal{FN}^s_{q,\mu,r}}\right)$ is a Banach space. Our main result establishes the smoothing effect of the Mittag-Leffler families $\{E_\alpha(t)\}_{t\geq 0}$ and $\{E_{\alpha,\alpha}(t)\}_{t\geq 0}$ in Fourier-Besov-Morrey spaces.

Theorem 2.1. *Let $\alpha \in (0,1]$, $I = (0,+\infty)$, $1 \leq q_1 < q_2 \leq \infty$, $\max\left\{0, 3 - \frac{q_1 q_2}{q_1 - q_2}\right\} < \mu < 3$, $1 \leq r \leq \infty$ and $s \in \mathbb{R}$. For each par $(\Omega, \mathcal{N}) \in (\mathbb{R} - \{0\})^2$, consider $L = \max\left\{2, \frac{|\Omega|}{\mathcal{N}\sqrt{g}}, \frac{\mathcal{N}\sqrt{g}}{|\Omega|}\right\}$. Then, there exists a constant $C > 0$ such that*

$$\|E_\alpha(t)v_0\|_{\mathcal{FN}^s_{q_2,\mu,r}} \leq CL(\nu t)^{-\frac{\alpha}{2\beta}\left(\frac{3-\mu}{q_2} - \frac{3-\mu}{q_1}\right)} \|v_0\|_{\mathcal{FN}^s_{q_1,\mu,r}},$$

and

$$\|E_{\alpha,\alpha}(t)v_0\|_{\mathcal{FN}^s_{q_2,\mu,r}} \leq KL\alpha(\nu t)^{-\frac{\alpha}{2\beta}\left(\frac{3-\mu}{q_2} - \frac{3-\mu}{q_1}\right)} \|v_0\|_{\mathcal{FN}^s_{q_1,\mu,r}},$$

for all $v_0 \in \mathcal{FN}^s_{q_1,\mu,r}$ and $t \in I$.

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ON A NAVIER-STOKES-VOIGT SYSTEM

GERALDO M. DE ARAÚJO[1,†]

Instituto de Ciências exatas e Naturais-Faculdade de Matemática, UFPA, Pa, Brasil[1].

gera@ufpa.br[†].

Abstract

In this paper we investigate the unilateral problem of Navier-Stokes-Voigt operator L in cylindrical case, where $Lu = u_t - \alpha^2 \Delta u_t - \nu \Delta u + (u \cdot \nabla)\, u - f + \nabla p$. Using an appropriate penalization, we obtain a variational inequality for the Navier-Stokes-Voigt perturbed system.

1 Introduction

Regularized fluid equations in hydrodynamics are fundamental in understanding turbulent phenomena in science. One such regularized model was introduced by Oskolkov [1] as a model for the motion of a linear viscoelastic incompressible fluid:

$$\begin{aligned} u_t - \alpha \Delta u_t &- \nu \Delta u + (u \cdot \nabla)(u - \alpha \Delta u) + \nabla p = f \text{ in } Q \\ \nabla \cdot u &= 0 \text{ in } Q,\ u = 0 \text{ on } \Sigma,\ u(0) = u_0 \text{ in } \Omega. \end{aligned} \tag{1}$$

In [2] the author studies the equation below that describe the motion of non-Newtonian fluid to which a small quantity of polymers is added:

$$\begin{aligned} u_t - \alpha^2 \Delta u_t &- \nu \Delta u + (u \cdot \nabla) u + \nabla p = f \text{ in } Q \\ \nabla \cdot u &= 0 \text{ in } Q,\ u = 0 \text{ on } \Sigma,\ u(0) = u_0 \text{ in } \Omega. \end{aligned} \tag{2}$$

where Ω denotes the bounded domain of flow in R^n, $n = 2, 3$, with boundary $\partial\Omega$; the vector function u represents the velocity field, p denotes the pressure; $\nu > 0$ is the viscosity coefficient; α is a length scale parameter such that α^2/ν is the relaxation time of the viscoelastic fluid; f is the external forces field; and u_0 is the initial velocity. Using the Faedo-Galerkin method with a special basis of eigenfunctions of the Stokes operator, he construct a global-in-time strong solution, which is unique in both two-dimensional and three-dimensional domains. Type (1) and (1) systems are known in the literature as Navier-Stokes-Voigt (NSV), are often called the Kelvin-Voigt equations or Oskolkov's equations. The NSV model and related models of viscoelastic fluid flows have been studied extensively by different mathematicians over the past several decades starting from the pioneering papers by Oskolkov [1].

A nonlinear perturbation of problem (1) is given by

$$u_t - \alpha^2 \Delta u_t - \nu \Delta u + (u \cdot \nabla) u + \nabla p - f \geq 0 \tag{3}$$

In the present work we investigated the unilateral problem associated with this perturbation, that is, a variational inequality given for (3) (see [3]). Making use of the penalty method, and Galerkin's approximations, we establish existence and uniqueness of solutions.This inequality is satisfied in a certain sense, that is, we formulate the problem as follows. Let K a closed and convex subset of $V \cap V_2$, the variational problem is to find a solution $u(x,t)$ satisfying

$$\int_Q (u_t - \alpha^2 \Delta u_t - \nu \Delta u + (u \cdot \nabla)\, u - f)(v - u) \geq 0, \forall v \in K, \tag{4}$$

with $u(x,t) \in K$ a. e. on $[0,T]$ and talking the initial and boundary data $u=0$ on Σ, $u(.,0)=$ in Ω. In general, dynamic contact problems are characterized by nonlinear hyperbolic variational inequalities, so it is interesting to study unilateral problem.

2 Main Results

Theorem 2.1. *Suppose $n=2$ and $f, f_t \in L^2(0,T;V')$, $u_0 \in K$. Then there exists a function u such that*

$$u \in L^\infty(0,T;V) \cap L^2(0,T;V_2)\ , u_t \in L^2(0,T;V) \cap L^\infty(0,T;H),\ u(t) \in K, \quad \forall t \in [0,T],\ \textit{satisfying}\ . \tag{1}$$

$$\int_Q (u_t - \alpha^2 \Delta u_t - \nu \Delta u + (u \cdot \nabla) u - f)(v-u)dxdt \ \ \forall v \in K, \quad a.e.\ in\ t,\ ,\ u(0) = u_0. \tag{2}$$

The penalized problem associated with the variational inequalitiy (2) consists in, given $0 < \epsilon < 1$, find u_ϵ solution in Q of the mixed problem

$$\begin{aligned} & u_t^\epsilon - \alpha^2 \Delta u_t^\epsilon - \nu \Delta u^\epsilon + \sum_{i=1}^n u_i^\epsilon \frac{\partial u^\epsilon}{\partial x_i} + \frac{1}{\epsilon} \beta u^\epsilon = f - \nabla p \text{ in } Q \\ & div\, u^\epsilon = 0 \text{ in } Q,\ u^\epsilon = 0 \text{ on } \Sigma,\ u^\epsilon(x,0) = u_0^\epsilon(x) \text{ in } \Omega. \end{aligned} \tag{3}$$

The solution of this problem is given by the following theorem:

Theorem 2.2. *Assume that $n=2,3$ and $f, f_t \in L^2(0,T;V')$. Then for each $0 < \epsilon < 1$ and $u_0^\epsilon \in V$, there exists a function u^ϵ with $u^\epsilon \in L^\infty(0,T;V)$, $u_t^\epsilon \in L^2(0,T;V) \cap L^\infty(0,T;H)$ satisfying (3) in the sense of $L^2(0,T;V')$.*

Proof - First estimate We consider $w_j = u^{\epsilon m}(t)$ in the approximate problem.
Second estimate -In both side in the approximate problem we take the derivatives with respect t and consider $w_j = u_t^{\epsilon m}(t)$. These estimates are sufficient to pass the limit in the approximate penalized problem with $m \to \infty$ and consequently prove theorem 2.2. To prove theorem 2.1 we need one more estimate.
Third estimate We consider $n=2$. Let (w_ν) be the ortonormal system of $V \cap V_2$ formed by the eigenfunctions of the Laplace operator with $u^{\epsilon m}(x,0) \to u^\epsilon(x,0)$ strongly in $V \cap V_2$ and let's take $w_j = -\Delta u_m$ in the approximate problem. From the convergences obtained above and Banach-Steinhauss theorem, it follows that there exists a subnet $(u_\epsilon)_{0<\epsilon<1}$, such that it converges to u as $\epsilon \to 0$, in the weak sense, that is,

$$\begin{aligned} & u^\epsilon \longrightarrow u \text{ weak star in } L^\infty(0,T;H),\ u^\epsilon \longrightarrow u \text{ weak in } L^\infty(0,T;V),\ \beta u^\epsilon \longrightarrow \beta u \text{ weak in } L^2(0,T;V'). \\ & u_t^\epsilon \longrightarrow u_t \text{ weak in } L^2(0,T;V),\ u_t^\epsilon \longrightarrow u_t \text{ weak in } L^\infty(0,T;H),\ u^\epsilon \longrightarrow u \text{ strong in } L^2(0,T;H) \text{ and a.e. in } Q. \\ & u_i^\epsilon u_j^\epsilon \longrightarrow u_i u_j \text{ weak in } L^2(0,T;L^2(\Omega)),\ u^\epsilon \longrightarrow u \text{ weak in } L^\infty(0,T;V_2).,\ u^\epsilon \longrightarrow u \text{ strong in } L^2(0,T;V) \text{ and a.e. in } Q. \end{aligned} \tag{4}$$

The convergences above are sufficient to conclude that (2) is valid. To complete the proof of Theorem 2.1, it remains to show that $u(t) \in K$ a.e.it is, but this presents no difficulties.

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ASYMPTOTIC BEHAVIOR OF PARABOLIC PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS AND VARYING BOUNDARIES

GLEICIANE S. ARAGÃO[1,†], JOSÉ M. ARRIETA[2,‡] & SIMONE M. BRUSCHI[3,§]

Departamento de Ciências Exatas e da Terra, Universidade Federal de São Paulo, Diadema, SP, Brasil[1],
Departamento de Análisis Matemático y Matemática Aplicada, Facultad de Matemáticas, Universidad Complutense de Madrid, Madrid, Spain[2],
Departamento de Matemática, Universidade de Brasília, Brasília, DF, Brasil[3]

gleiciane.aragao@unifesp.br[†], arrieta@mat.ucm.es[‡], sbruschi@unb.br[§].

Abstract

We analyze the asymptotic behavior of parabolic problems with nonlinear Neumann boundary conditions when the boundary of the domain varies very rapidly as a parameter ϵ goes to zero. For the case where we have a Lipschitz deformation of the boundary with the Lipschitz constant uniformly bounded in ϵ, we show that the solutions of these problems converge to the solution of a limit parabolic problem of the same type, where the boundary condition has a factor that captures the oscillations of the boundary, and we prove the existence and upper semicontinuity of attractors at $\epsilon = 0$. For that it is necessary to consider a notion of convergence in varying domains. Moreover, if every equilibrium of the limit problem is hyperbolic, then we also prove the continuity of local unstable manifolds and the lower semicontinuity of attractors at $\epsilon = 0$.

1 Introduction

We will analyze the asymptotic behavior, for small ϵ, of the family of solutions of a parabolic problem with nonlinear Neumann boundary conditions of the type

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t} - \Delta u_\epsilon + u_\epsilon = f(x, u_\epsilon), & \text{in } \Omega_\epsilon \times (0,\infty) \\ \dfrac{\partial u_\epsilon}{\partial n_\epsilon} = g(x, u_\epsilon), & \text{on } \partial\Omega_\epsilon \times (0,\infty) \\ u_\epsilon(0) = u_\epsilon^0(x), & \text{in } \Omega_\epsilon \end{cases} \tag{1}$$

when the boundary of the domain presents a highly oscillatory behavior, as the parameter ϵ goes to zero. To describe the problem, we will consider a family of uniformly bounded smooth domains $\Omega_\epsilon \subset \mathbb{R}^n$, with $n \geq 2$ and $0 \leq \epsilon \leq \epsilon_0$, for some $\epsilon_0 > 0$ fixed, and we will look at this problem from the perturbation of the domain point of view and we will refer to $\Omega \equiv \Omega_0$ as the unperturbed domain and Ω_ϵ as the perturbed domains. We will assume that $\Omega_\epsilon \to \Omega$ and $\partial\Omega_\epsilon \to \partial\Omega$, as $\epsilon \to 0$, in the sense of Hausdorff. We will also assume that the boundary $\partial\Omega_\epsilon$ is expressed in local charts as a Lipschitz deformation of $\partial\Omega$ with the Lipschitz constant uniformly bounded in ϵ.

It is reasonable to expect that the family of solutions $\{u_\epsilon\}_{\epsilon \in (0,\epsilon_0]}$ of (1) will converge to the solution of an equation with a nonlinear boundary condition on $\partial\Omega$ that inherits the information about the behavior of the measure of the deformation of $\partial\Omega_\epsilon$ with respect to $\partial\Omega$. More precisely, we will show that the solutions of (1) converge in $H^1(\Omega_\epsilon)$ to the solution of the following parabolic problem with nonlinear Neumann boundary conditions

$$\begin{cases} \dfrac{\partial u_0}{\partial t} - \Delta u_0 + u_0 = f(x, u_0), & \text{in } \Omega \times (0,\infty) \\ \dfrac{\partial u_0}{\partial n} = \gamma(x) g(x, u_0), & \text{on } \partial\Omega \times (0,\infty) \\ u_0(0) = u_0^0(x), & \text{in } \Omega \end{cases} \tag{2}$$

where the function $\gamma \in L^\infty(\partial\Omega)$ is related to the behavior of the measure $(n-1)$-dimensional of the $\partial\Omega_\epsilon$. Indeed, assuming that the nonlinearities f and g satisfy growth, sign and dissipative conditions, we will prove the existence and continuity of the family of attractors of (1) and (2) at $\epsilon = 0$ in $H^1(\Omega_\epsilon)$.

The behavior of the solutions of elliptic problems with nonlinear Neumann boundary conditions and rapidly varying boundaries was studied in [2], for this case of uniformly Lipschitz deformation of the boundary. In particular, if we regard these elliptic equations as stationary equations of the parabolic evolutionary equations (1) and (2), then the continuity of the set of equilibria of (1) and (2) at $\epsilon = 0$ in $H^1(\Omega_\epsilon)$ was proved. Thus, the goal of our work [1] is to continue the analysis initiated in [2].

2 Main Results

Domain perturbation problems have been considered by several authors and one of the main difficulties when treating these problems is that the solutions live in different spaces, so it is necessary a tool to compare functions which are defined in different spaces. In [1] we use the concept of E-convergence and the extension operator $E_\epsilon : H^1(\Omega) \to H^1(\Omega_\epsilon)$ defined in [2]. Since the problems (1) and (2) have nonlinear terms on boundary, then it is necessary to use spaces with negative exponents. Thus, we present a definition of extension operators and the concept of E^*-convergence in spaces with negative exponents, and we prove that the notion of E-convergence is the same as the notion of convergence stablished in an abstract way in [3]. Consequently, we can use some results of [3] in our problem.

The technique of extension operators in spaces with negative exponents was not considered in previous works. Indeed, this definition can be used in other problems involving negative exponents, so allowing to work with asymptotic behavior of problems with nonlinear boundary conditions. In this way, it is our main contribution obtained in [1].

To obtain the E-continuity of the family of attractors of (1) and (2) at $\epsilon = 0$ in $H^1(\Omega_\epsilon)$, initially we show result on compact convergence of the resolvent operators and the E-convergence of linear semigroups. Later, we prove the E^*-convergence of the nonlinearities, E-convergence of nonlinear semigroups and we concluded the E-upper semicontinuity of attractors. Moreover, if every equilibrium of the limit problem (2) is hyperbolic, then we also prove the E-continuity of local unstable manifolds and we concluded the E-lower semicontinuity of attractors.

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MULTI-OBJECTIVE CONTROL PROBLEMS FOR PARABOLIC SYSTEM AND KDV EQUATION.

ISLANITA CECILIA ALCANTARA DE ALBUQUERQUE[1,†] & MAURICIO CARDOSO SANTOS[2,‡]

Universidade de Pernambuco, Campus Mata Norte, UPE-CMN, PE, Brasil[1], Departamento de Matematica, UFPB, PE, Brasil[2]

islanita.albuquerque@upe.br[†] mcsantos@mat.ufpb.br[‡]

Abstract

This work is dedicated to the study of some multi-objective control problems for partial differential equations. Usually, problems containing many objectives are not well-posed, since one objective may completely determine the control, turning the others objectives impossible to reach. We apply the so called Stackelberg-Nash strategy, we consider a hierarchy, in the sense that we have one control which we call the leader, and other controls which we call the followers. Once the leader policy is fixed, the followers intend to be in equilibrium according to their targets, after we determine the followers in such a way they accomplish their objectives in a optimal way, and to do that a concept of equilibrium is applied. In this work, we apply the concept of Nash Equilibrium, which correspond to a non-cooperative strategy. Two problems are solved, in the first chapter, we consider a linear system of parabolic equations and prove that the Stackelberg-Nash strategy can be applied under some suitable conditions for the coupling coefficients. In the second one, we consider the nonlinear Korteweg-de Vries (KdV) equation, which has a very different nature of parabolic equations, and the same method is applied.

1 Introduction

The first presented problem consists of a system of parabolic equations, with zero-order couplings of the structure:

$$\begin{cases} y^1_t - \Delta y^1 = \sum_{p=1}^{2} A_{1\,p} y^p + F^1 & \text{in} \quad Q, \\ y^2_t - \Delta y^2 = \sum_{p=1}^{2} A_{2\,p} y^p + F^2 & \text{in} \quad Q, \\ y^1 = y^2 = 0 & \text{in} \quad \Sigma, \\ y^1(0) = y^1_0,\ y^2(0) = y^2_0 & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where (y^1, y^2) represents the state, (F^1, F^2) are controls and $\{A_{i,j}\}_{i,j=1,2}$ are uniformly bounded functions and represent the coupling matrix of the system. Each F^i control has three objectives for the state (y^1, y^2) that are to be described below.

Through the Stackelberg Method, we write

$$F^i = f^i \mathbb{1}_{\mathcal{O}} + v^{i\,1} \mathbb{1}_{\mathcal{O}_{i\,1}} + v^{i\,2} \mathbb{1}_{\mathcal{O}_{i\,2}},\ i = 1, 2, \tag{2}$$

where the sets $\mathcal{O}$, $\mathcal{O}_{i\,1}$ and $\mathcal{O}_{i\,2}$ are open and disjoint. The $\{f^i\}_{i=1}^2$ controls will be the leaders, being responsible for the objectives of the controllability type, while $\{v^{i\,j}\}_{i,j=1}^2$ are called followers and are engaged in the minimization of the cost functionals.

In the second problem, we have considered the nonlinear Korteweg-de Vries equation (KdV) and have also applied the Stackelberg-Nash method combined with a zero control objective for the leader:

$$\begin{cases} y_t + (1+y) y_x + y_{xxx} = f \mathbb{1}_{\mathcal{O}} + v^1 \mathcal{X}_{\mathcal{O}_1} + v^2 \mathcal{X}_{\mathcal{O}_2} & \text{in} \quad Q, \\ y\,(0,\cdot) = y\,(L,\cdot) = y_x\,(L,\cdot) = 0 & \text{in} \quad (0,T)\,, \\ y\,(x,\cdot) = y^0 & \text{in} \quad (0,1)\,, \end{cases} \tag{3}$$

where $y = y(x,t)$ is the state and y^0 is initial datum. In (3), the set $\mathcal{O} \subset (0,1)$ is the *leader control's domain* f while $\mathcal{O}_1, \mathcal{O}_2 \subset (0,1)$ are the *follower control's domains* v^1 and v^2 (all of which are very small and disjoint suppositions). With $\mathbb{1}_{\mathcal{O}}$ we denoted the characteristic function in $\mathcal{O}$ while $\mathcal{X}_{\mathcal{O}_i}$ are positive functions in $C_0^\infty(\mathcal{O}_i)$, $i = 1,2$.

2 Main Results

Theorem 2.1. *Given* $(y_0^1, y_0^2) \in L^2(Q)^2$ *and* $\{v^j\}_{j=1}^2$ *a Nash equilibrium for the functional cost, there exist a control* $f^1 \in L^2(\mathcal{O} \times (0,T))^2$ *such that the solution of* (1) *with* F^1 *given by* (2) *and* $F^2 = 0$ *satisfies* $(y^1(T), y^2(T)) = (0,0)$.

Proof We use Carleman estimates and the Hilbert Uniqueness Method. See [1]; [2].

Theorem 2.2. *For* $i = 1,2$, *suppose that*

$$\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset \tag{1}$$

and that μ_i *are sufficiently large. Also, assume that one of the two conditions holds:*

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d} \tag{2}$$

or

$$\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}. \tag{3}$$

Then, there exist a positive function $\hat{\rho} = \hat{\rho}(t)$ *blowing up at* $t = T$ *and* $\delta > 0$ *such that if*

$$\|z^0\|^2_{H_0^1(0,1)} + \sum_{i=1}^{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} \hat{\rho}^2 |z_{i,d}|^2 \, dx\, dt < \delta, \tag{4}$$

there exist controls $f \in L^2(\mathcal{O} \times (0,T))$ *and associated Nash equilibria* (v^1, v^2) *such that the corresponding solutions to optimality system* ($z = y - \bar{y}$) *satisfies* $y(\cdot,T) = \bar{y}(\cdot,T)$.

Proof See [3].

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ON THE FRACTIONAL CHEMOTAXIS NAVIER-STOKES SYSTEM IN THE CRITICAL SPACES.

C. CUEVAS[1], J. AZEVEDO.[2], J. DANTAS[3] & C. SILVA[4]

Universidade Federal de Pernambuco, UFPE, PE, Brasil[1], Universidade de Pernambuco, UPE, PE, Brasil[2],
Universidade Federal de Pernambuco, UFPE, PE, Brasil[3],
Universidade Federal Rural de Pernambuco, UFRPE, PE, Brasil[4].

cch@dmat.ufpe.br[†], joelma.moura@upe.br[‡], jarbas01001@gmail.com[§], clessius.silva@ufrpe.br[§§].

Abstract

We consider the fractional chemotaxis Navier-Stokes equations which are the fractional Keller-Segel model coupled with the Navier-Stokes fluid in the whole space, and prove the existence of global mild solutions with the small critical initial data in Besov-Morrey spaces. Our results enable us to obtain the self-similar solutions provided the initial data are homogeneous functions with small norms and considering the case of chemical attractant without degradation rate.

1 Introduction

In this work, we deal with the double chemotaxis model of fractional order under the effect of the Navier-Stokes fluid:

$$\begin{cases} {}^cD_t^\alpha n + u\cdot\nabla n = \Delta n - \nabla\cdot(n\nabla c) - \nabla\cdot(n\nabla v), & \text{in } \mathbb{R}^N\times(0,\infty),\\ {}^cD_t^\alpha c + u\cdot\nabla c = \Delta c - nc, & \text{in } \mathbb{R}^N\times(0,\infty),\\ {}^cD_t^\alpha v + u\cdot\nabla v = \Delta v - \gamma v + n, & \text{in } \mathbb{R}^N\times(0,\infty),\\ {}^cD_t^\alpha u + (u\cdot\nabla)u = \Delta u - \nabla p - nf, & \text{in } \mathbb{R}^N\times(0,\infty),\\ \nabla\cdot u = 0, & \text{in } \mathbb{R}^N\times(0,\infty),\\ n(x,0)=n_0(x),\ c(x,0)=c_0(x),\ v(x,0)=v_0(x),\ u(x,0)=u_0(x), & \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

where $N\geq 2$, $\gamma\geq 0$, ${}^cD_t^\alpha$ is the Caputo fractional derivative of order $\alpha\in(0,1)$. The unknown $n(x,t)$, $c(x,t)$, $v(x,t)$, $u(x,t)$ and $p(x,t)$ stand for the cell density, oxygen concentration, chemical-attractant concentration, fluid velocity, and the pressure of the fluid, respectively. The time-independent field f denotes a force field acting on the motion of the fluid that can be produced by different mechanisms, e.g., force due to the aggregation of bacteria onto the fluid generating a bouyancy-like force. The parameter $\gamma\geq 0$ denotes the degradation rate of attractant, while $n_0=n_0(x)$, $c_0=c_0(x)$, $v_0=v_0(x)$ and $u_0=u_0(x)$ denote the initial data. The system (1) with $\alpha=1$ was first proposed by Tuval et al. [1] to describe the dynamics of swimming aerobic bacteria living in an incompressible viscous fluid, which swim toward a higher concentration of oxygen and chemical attractant.

Recently Ferreira and Postigo [2] treated the global well-posedness for (1) by considering $\alpha=1$, $N\geq 2$ and small initial data in critical Besov-Morrey spaces, more precisely, they consider

$$\begin{cases} n_0\in\mathcal{N}_{q,q_1,\infty}^{\frac{N}{q}-2}(\mathbb{R}^N),\ c_0\in L^\infty(\mathbb{R}^N),\ \text{with } \nabla c_0\in\mathcal{N}_{r,r_1,\infty}^{\frac{N}{r}-1}(\mathbb{R}^N),\\ v_0\in\mathcal{S}'/\mathcal{P} \text{ with } \nabla v_0\in\mathcal{N}_{r,r_1,\infty}^{\frac{N}{r}-1}(\mathbb{R}^N),\ u_0\in\mathcal{N}_{p,p_1,\infty}^{\frac{N}{p}-1}(\mathbb{R}^N) \text{ with } \nabla\cdot u_0=0, \end{cases} \tag{2}$$

and force $f\in\mathcal{M}_{N_1}^{N}(\mathbb{R}^N)$, where the exponents p,p_1,q,q_1,r,r_1 and N_1 satisfy suitable conditions of Kozono et al. type (see [3]).

In this work we extend the results obtained recently in [2] (see also [3]) to the fractional chemotaxis-fluids framework. The solutions for the problem (1) are obtained by means of a fixed point argument in a time-dependent critical space $\mathcal{X}^\alpha$ defined as follows: Let us introduce the spaces

$$\mathcal{X}_1^\alpha = \{n \ : \ t^{-\frac{\alpha N}{2q}+\alpha} n \in B((0,\infty);\mathcal{M}_{q_1}^q)\},\ \mathcal{X}_2^\alpha = \{c \ : \ c \in B((0,\infty);L^\infty) \text{ with } t^{-\frac{\alpha N}{2r}+\frac{\alpha}{2}} \nabla c \in B((0,\infty);\mathcal{M}_{r_1}^r)\},$$
$$\mathcal{X}_3^\alpha = \{v \ : \ v(\cdot,t) \in S'/\mathcal{P} \text{ for } t>0 \text{ and } t^{-\frac{\alpha N}{2r}+\frac{\alpha}{2}} \nabla v \in B((0,\infty);\mathcal{M}_{r_1}^r)\},\ \mathcal{X}_4^\alpha = \{u \ : \ t^{-\frac{\alpha N}{2p}+\frac{\alpha}{2}} u \in B((0,\infty);\mathcal{M}_{p_1}^p)\},$$

which are Banach spaces endowed with the respective norms

$$\|n\|_{\mathcal{X}_1^\alpha} = \sup_{t>0} t^{-\frac{\alpha N}{2q}+\alpha} \|n(t)\|_{\mathcal{M}_{q_1}^q},\ \|c\|_{\mathcal{X}_2^\alpha} = \sup_{t>0} \|c(t)\|_{L^\infty} + \sup_{t>0} t^{-\frac{\alpha N}{2r}+\frac{\alpha}{2}} \|\nabla c(t)\|_{\mathcal{M}_{r_1}^r},$$
$$\|v\|_{\mathcal{X}_3^\alpha} = \sup_{t>0} t^{-\frac{\alpha N}{2r}+\frac{\alpha}{2}} \|\nabla v(t)\|_{\mathcal{M}_{r_1}^r},\ \|u\|_{\mathcal{X}_4^\alpha} = \sup_{t>0} t^{-\frac{\alpha N}{2p}+\frac{\alpha}{2}} \|u(t)\|_{\mathcal{M}_{p_1}^p}.$$

Next, let us introduce the space $\mathcal{X}^\alpha$ and $\mathcal{I}^\alpha$ as

$$\mathcal{X}^\alpha := \{(n,c,v,u) \ : \ n \in \mathcal{X}_1^\alpha, c \in \mathcal{X}_2^\alpha, v \in \mathcal{X}_3^\alpha, u \in \mathcal{X}_4^\alpha\},\ \mathcal{I}^\alpha := \{(n_0,c_0,v_0,u_0) \ : \ n_0,c_0,v_0 \text{ and } u_0 \text{ are as in (2)}\},$$

which are Banach spaces with the norms

$$\|(n,c,v,u)\|_{\mathcal{X}^\alpha} := \|n\|_{\mathcal{X}_1^\alpha} + \|c\|_{\mathcal{X}_2^\alpha} + \|v\|_{\mathcal{X}_3^\alpha} + \|u\|_{\mathcal{X}_4^\alpha},$$
$$\|(n_0,c_0,v_0,u_0)\|_{\mathcal{I}^\alpha} := \|n_0\|_{\mathcal{N}_{q,q_1,\infty}^{\frac{N}{q}-2}} + \|c_0\|_{L^\infty} + \|\nabla c_0\|_{\mathcal{N}_{r,r_1,\infty}^{\frac{N}{r}-1}} + \|\nabla v_0\|_{\mathcal{N}_{r,r_1,\infty}^{\frac{N}{r}-1}} + \|u_0\|_{\mathcal{N}_{p,p_1,\infty}^{\frac{N}{p}-1}}.$$

We got the following results for the problem (1):

2 Main Results

Theorem 2.1. *Let $N \geq 2$ and let the exponents p, p_1, q, q_1, r, r_1 and N_1 be as the conditions introduced in [3, Theorem 1] and [2, Assumption 1]. Suppose that the initial data $(n_0, c_0, v_0, u_0) \in \mathcal{I}^\alpha$ and the external force $f \in \mathcal{M}_{N_1}^N(\mathbb{R}^N)$. There exist positive constants ε, $\delta = \delta(\varepsilon)$, and K_1 such that the problem (1) has a unique global mild solution $(n,c,v,u) \in \mathcal{X}^\alpha$ satisfying $\|(n,c,v,u)\|_{\mathcal{X}^\alpha} \leq 2K_1\varepsilon$ provided that $\|(n_0,c_0,v_0,u_0)\|_{\mathcal{I}^\alpha} \leq \delta$. Moreover, the data-solution map is locally Lipschitz continuous.*

Since the space $\mathcal{X}^\alpha$ is critical with respect to the scaling, we can obtain self-similar solutions.

Corollary 2.1. *Let $N \geq 3$ and $\gamma = 0$. Assume that (n_0, c_0, v_0, u_0) and f are as in Theorem 2.1. Suppose that n_0, c_0, v_0, u_0 and f are homogeneous functions with degree $-2, 0, 0, -1$ and -1, respectively. Then, the solution (n,c,v,u) obtained through Theorem 2.1 is self-similar.*

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HIGHER-ORDER ASYMPTOTIC SOLUTIONS FOR SOME BURGERS EQUATIONS

JUSSARA M. MOREIRA.[1], GASTÃO A. BRAGA[2] & ANTÔNIO M. SILVA[3]

Departamento de Matemática, UFMG, MG, Brasil[1], Departamento de Matemática, UFMG, MG, Brasil[2], Departamento de Matemática, UFOP, MG, Brasil[3]

jmoreira@mat.ufmg.br† gbraga@mat.ufmg.br‡ antonio.marcos@ufop.edu.br§

Abstract

The Renormalization Group (RG) method has found extensive applications in modern theoretical physics and its application to the long-time asymptotic analysis of solutions to nonlinear differential equations has been developed since the early 90's. This work aims to study higher order corrections in the asymptotic behavior of solutions of a class of nonlinear time-evolution problems, particularly we study the large-time behavior of solutions to a generalized Burgers Equation, with initial zero mass data.

1 Introduction

The mathematical aspects of the Renormalization Group method for differential equations were rigorously established by Bricmont, Kupiainen and collaborators [1] in 1994. Such methodology ended up developing great success, specifically for the determination of critical exponents and universality classes and in providing the leading order long-time asymptotics of solutions to a wide class of initial value problems, both analytically and numerically. On the other hand, the method seemed inefficient when dealing with higher order corrections, as in the treatment of problems with zero mass initial data, $\int f(x)dx = 0$.

Here we show that, by implementing a modification in the Renormalization Group operator, it is possible to obtain the asymptotic behavior of solutions of the type

$$\begin{cases} u_t + uu_x = u_{xx} + \lambda F(u, u_x), \; x \in \mathbb{R}, t > 1, \\ u(x, 1) = f(x), \end{cases} \tag{1}$$

for small enough initial data and if one of the two following hypotheses is satisfied:

(H-1) $\int_{\mathbb{R}} f(x)dx = 0$ and $F(u, u_x) = \sum_{n \geq 2} c_n u^n u_x$,

(H-2) f is odd and $F(u, v) = \sum_{m,n \geq 0}^{*} c_{m,n} \; u^{2m+1} v^n$ where the $*$ excludes the pairs $(m, n) = (0, 0)$ and $(m, n) = (0, 1)$ from the sum.

We will show that the solution $u(x, t)$ to IVP (1) behaves, for $t \gg 1$, as

$$u(x, t) \approx \frac{A}{t} f_1^* \left(\frac{x}{\sqrt{t}} \right), \tag{2}$$

where A is a prefactor and

$$f_1^* (x) = -\frac{x}{2} \frac{e^{-\frac{x^2}{4}}}{\sqrt{4\pi}}. \tag{3}$$

The long time behavior (2) will emerge from the iterates of a nonlinear operator (the RG operator) whose linearization has $f_1^* (x)$, given by (3), as a fixed point. Furthermore, the time decay exponent $\alpha = 1$ and the time spread exponent $\beta = 1/2$, on the right hand side of (2), are intimately related to the definition of the RG operator. The nonlinearity $F(u, u_x)$ in **(H-1)** or **(H-2)** is such that it preserves the symmetry of the initial data along the time evolution. Also, $F(u, u_x)$ is chosen to be "irrelevant" under the RG flow so that the long time behavior (2) will be, essentially, the one given by the linearized problem.

2 Main Result

In order to state our main result, we first need to define the space for the initial data. Given $q > 1$, let

$$\|f\|_q = \sup_{\omega\in\mathbb{R}}(1 + |\omega|^q)\left(|\widehat{f}(\omega)| + |\widehat{f}'(\omega)| + |\widehat{f}''(\omega)|\right) \tag{1}$$

and

$$\mathcal{B}_q = \left\{f \in L^1(\mathbb{R}) : \widehat{f}(\omega) \in C^2(\mathbb{R}) \text{ and } \|f\|_q < +\infty\right\}. \tag{2}$$

Theorem 2.1. *Given $q > 2$, consider IVP (1) satisfying hypothesis* **(H-1)** *or* **(H-2)***, $f \in \mathcal{B}_q$ and $|\lambda| \le 1$. There are $\epsilon > 0$ and $A = A(f, F)$ such that, if $\|f\|_q < \epsilon$, then the solution u to IVP (1) satisfies*

$$\lim_{t\longrightarrow+\infty} \|tu(t^{\frac{1}{2}}\cdot, t) - Af_1^*\|_q = 0, \tag{3}$$

where f_1^* is given by (3).

The RG approach that we employ to prove the above result is basically the integration of the equation followed by a rescaling. That is, let u be a real-valued function of $(x, t) \in \mathbb{R} \times \mathbb{R}^+$. For a fixed $L > 1$, define, inductively, a sequence of rescaled functions $\{u_n\}_{n=0}^{\infty}$, by $u_0 = u$ and, for $n \ge 1$, $u_n(x, t) = L^2 u_{n-1}(Lx, L^2 t)$. If the original function u is a global solution to IVP (1), then we get that that u_n satisfies a sequence of renormalized IVPs. Then, for sufficiently small initial data f_n we prove the existence of a unique solution to each IVP, defined in a limited time interval, allowing the definition of the RG operator with $n \ge 0$:

$$\left(R_{L,n} f_n\right)(x) \equiv L^2 u_n(Lx, L^2),\ \forall x \in \mathbb{R}. \tag{4}$$

The study of the long-time asymptotics of solutions to (1) is equivalent to studying the fixed points, and their basins of attraction (i.e., universality classes), of the transformation (4).

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CONTROL RESULTS WITH OVERDETERMINATION CONDITION FOR HIGHER ORDER DISPERSIVE SYSTEM

ROBERTO DE A. CAPISTRANO–FILHO[1,†] & LUAN SOARES DE SOUSA[1,‡]

Departamento de Matemática, UFPE, PE, Brasil[1]

roberto.capistranofilho@ufpe.br[†] luan.soares@ufpe.br[‡]

Abstract

In recent years, controllability problems for dispersive systems have been extensively studied. This work is dedicated to proving a new type of controllability for a dispersive fifth order equation that models water waves, what we will now call the *overdetermination control problem*. Precisely, we are able to find a control acting at the boundary that guarantees that the solutions of the problem under consideration satisfy an overdetermination integral condition. In addition, when we make the control act internally in the system, instead of the boundary, we are also able to prove that this condition is satisfied. These results present a new way to prove boundary and internal controllability results for a fifth order KdV type equation.

1 Introduction

In this work, we will be interested in a kind of control property to the Kawahara equation when an *integral overdetermination condition* is required, namely

$$\int_0^L u(t,x)\omega(x)dx = \varphi(t),\ t \in [0,T], \tag{1}$$

with some known functions ω and φ. To present the problem, let us consider the Kawahara equation in the bounded rectangle $Q_T = (0,T) \times (0,L)$, where T and L are positive numbers with boundary function h_i, for $i = 1,2,3,4$ and h or the right-hand side f of a special form to specify latter, namely,

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} + uu_x = f(t,x) & \text{in } Q_T, \\ u(t,0) = h_1(t),\ u(t,L) = h_2(t),\ u_x(t,0) = h_3(t), & \text{in } [0,T], \\ u_x(t,L) = h_4(t),\ u_{xx}(t,L) = h(t) & \text{in } [0,T], \\ u(0,x) = u_0(x) & \text{in } [0,L]. \end{cases} \tag{2}$$

Thus, we are interested in studying two control problems, which we will call them from now on by *overdetermination control problem*. The first one can be read as follows:

Problem $\mathcal{A}$: For given functions u_0, h_i, for $i = 1,2,3,4$ and f in some appropriated spaces, can we find a boundary control h such that the solution associated to the equation (2) satisfies the integral overdetermination (1)?

The second problem of this work is concentrated to prove that for a special form of the function

$$f(t,x) = f_0(t)g(t,x), \quad (t,x) \in Q_T, \tag{3}$$

the integral overdetermination (1) is verified, in other words.

Problem $\mathcal{B}$: For given functions u_0, h_i, for $i = 1,2,3,4$, h and g in some appropriated spaces, can we find an internal control f_0 such that the solution associated to the equation (2) satisfies the integral overdetermination (1)?

2 Main Results

The first result of this work gives us an answer for Problem $\mathcal{A}$, presented at the beginning of the introduction.

Theorem 2.1. *Let $p \in [2, \infty]$. Suppose that $u_0 \in L^2(0, L)$, $f \in L^p(0, T; L^2(0, L))$, $\widetilde{h} \in \mathcal{H}$ and $h_i \in L^p(0, T)$, for $i = 1, 2, 3, 4$. If $\varphi \in W^{1,p}(0, T)$ and $\omega \in \mathcal{J}$ are such that $\omega''(L) \neq 0$ and*

$$\int_0^L u_0(x)\omega(x)dx = \varphi(0), \tag{4}$$

considering $c_0 = \|u_0\|_{L^2(0,L)} + \|f\|_{L^2(0,T;L^2(0,L))} + \|\widetilde{h}\|_{\mathcal{H}} + \|\varphi'\|_{L^2(0,T)}$, the following assertions hold true.

1. *For a fixed c_0, there exists $T_0 > 0$ such that for $T \in (0, T_0]$, then we can find a unique function $h \in L^p(0, T)$ in such a way that the solution $u \in X(Q_T)$ of (2) satisfies (1).*
2. *For each $T > 0$ fixed, exists a constant $\gamma > 0$ such that for $c_0 \leq \gamma$, then we can find a unique boundary control $h \in L^p(0, T)$ with the solution $u \in X(Q_T)$ of (2) satisfying (1).*

The next result ensures for the first time that we can control the Kawahara equation with a function f_0 supported in $[0, T]$. Precisely, we will give an affirmative answer to the Problem $\mathcal{B}$ mentioned in this introduction.

Theorem 2.2. *Let $p \in [1, \infty]$, $u_0 \in L^2(0, L)$, $h \in L^{max\{2,p\}}(0, T; L^2(0, L))$, $\widetilde{h} \in \mathcal{H}$ and $h_i \in L^p(0, T)$, for $i = 1, 2, 3, 4$. If $\varphi \in W^{1,p}(0, T)$, $g \in C([0, T]; L^2(0, L))$ and $\omega \in \mathcal{J}$ are such that $\omega''(L) \neq 0$, and there exists a positive constant g_0 such that (4) is satisfied and*

$$\left| \int_0^L g(t, x)\omega(x)dx \right| \geq g_0 > 0,$$

considering $c_0 = \|u_0\|_{L^2(0,L)} + \|h\|_{L^2(0,L)} + \|\widetilde{h}\|_{\mathcal{H}} + \|\varphi'\|_{L^1(0,T)}$, we have that:

1. *For a fixed c_0, so there exists $T_0 > 0$ such that for $T \in (0, T_0]$, exists a unique $f_0 \in L^p(0, T)$ and a solution $u \in X(Q_T)$ of (2), with f defined by (3), satisfying (1).*
2. *For a fixed $T > 0$, there exists a constant $\gamma > 0$ such that for $c_0 \leq \gamma$, we have the existence of a control input $f_0 \in L^p(0, T)$ which the solution $u \in X(Q_T)$ of (2), with f as in (3), verifies (1).*

Some of the spaces that show up at the manuscript are listed below.

i. The space of Kawahara solution consider here is denoted by

$$X(Q_T) = C([0, T]; L^2(0, L)) \cap L^2(0, T; H^2(0, L)),$$

ii. Consider

$$\mathcal{H} = H^{\frac{2}{5}}(0, T) \times H^{\frac{2}{5}}(0, T) \times H^{\frac{1}{5}}(0, T) \times H^{\frac{1}{5}}(0, T),$$

where $\widetilde{h} = (h_1, h_2, h_3, h_4)$.

iii. The function ω must be a fixed function that belongs to the following set

$$\mathcal{J} = \{\omega \in H^5(0, L) \cap H_0^2(0, L);\ \omega''(0) = 0\}.$$

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THE INITIALIZATION PROBLEM FOR BIOCONVECTIVE FLOW

LUIS FRIZ[1,†] & MARKO ROJAS-MEDAR[2,‡]

Departamento de Matemática, Universidad de Tarapacá, Chile[1], CCE-UBB, Departamento de Ciencias Básicas, Facultad de Ciencias, Universidad del Bío-Bío, Chile[2]

mmedar@academicos.uta.cl[†] lfriz@ubiobio.cl[‡]

Abstract

In this work we want to study the initialization problem for the bioconvective fluid system. That is, from certain observations, recover the initial conditions. The problem is posed as an optimal control problem.

1 Introduction

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain and represents the region of flow of fluid. It denotes $\partial\Omega$ the boundary of Ω. We consider the bioconvective flow system (see [2]):

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = -\kappa m\chi + \boldsymbol{f}, \\ div\boldsymbol{u} = 0, \\ \dfrac{\partial m}{\partial t} - \theta\Delta m + \boldsymbol{u}\cdot\nabla m + \boldsymbol{U}\dfrac{\partial m}{\partial x_2} = 0, \ in \ (0,T)\times\Omega \end{cases} \tag{1}$$

with boundary and initial conditions

$$\begin{cases} \boldsymbol{u} = 0, \ \text{on } (0,T)\times\partial\Omega, \\ \theta\dfrac{\partial m}{\partial \boldsymbol{n}} - \boldsymbol{U}mn_2 = 0, \ \text{on } (0,T)\times\partial\Omega, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \ m(0) = m_0, \ \text{in } \Omega. \end{cases} \tag{2}$$

Here $\boldsymbol{u}(x,t)$ denotes the fluid velocity, $p(x,t)$ is the hydrostatic pressure, $c = \kappa(g\rho)^{-1}m$ where $c(x,t)$ represents the concentration of microorganisms at a point $x = (x_1, x_2) \in \Omega$, and instant $t \in [0,T]$, here $0 < T < +\infty$, ν is the kinematic viscosity of the culture fluid, g is the intensity of the acceleration of gravity (assumed constant), $\boldsymbol{f}$ represents an external force given. We will suppose that $\boldsymbol{f}$ is divided into two parts,$\widetilde{\boldsymbol{f}}$ which does not depend on t and $\hat{\boldsymbol{f}}$ that depends on t, θ is a constant that indicates the rate of diffusion of microorganisms, χ is a unitary vector in the vertical direction, i.e., $\chi = (0,1)^t$. That is, coordinate system is placed so that the gravitational forces acting on vertical, $\boldsymbol{U}$ denotes the average velocity of swimming of the microorganisms in the vertical direction, ρ is a positive constant, given by $\rho = \dfrac{\rho_0}{\rho_m} - 1$, where ρ_0 and ρ_m are the density of one organism and the culture fluid density, respectively.

The functional framework is the following:

$$\begin{aligned} \boldsymbol{V}(\Omega) &= \{\boldsymbol{v} \in H_0^1(\Omega)^2 : div\boldsymbol{v} = 0\}, \\ B &= \left\{ w \in H^1(\Omega) : \int_\Omega w(x)dx = 0 \right\}. \end{aligned}$$

Let us define the trilinear maps $b_0 : \boldsymbol{V}(\Omega)\times\boldsymbol{V}(\Omega)\times\boldsymbol{V}(\Omega) \to \mathbb{R}$ $b_1 : \boldsymbol{V}(\Omega)\times H_0^1(\Omega)\times H_0^1(\Omega) \to \mathbb{R}$ given by

$$b_0(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \sum_{i,j=1}^{2}\int_\Omega u_j(x)\frac{\partial v_i}{\partial x_j}(x)w_i(x)dx, \ b_1(\boldsymbol{u},\phi,\psi) = \sum_{i=1}^{2}\int_\Omega u_j(x)\frac{\partial \phi}{\partial x_j}(x)\psi(x)dx$$

2 Main Results

We consider the initialization problem (see [1]). Let $T>0$, Z_1, Z_2 Hilbert spaces acting as observation spaces and $C:\boldsymbol{V}(\Omega)\times B\to Z_1\times Z_2$ is a linear bounded mapping. We assume $z=(z_1,z_2)\in L^2(0,T;Z_1)\times L^2(0,T;Z_2)$. Let $S:\boldsymbol{V}(\Omega)\times B\to L^2(0,T;\boldsymbol{V}(\Omega))\times L^2(0,T;B)$ be the mapping that takes the initial value $(\boldsymbol{w}_0,m_0)$ to the solution $(\boldsymbol{u},m)$, which it is possible to prove that it is continuous, among other properties. Define

$$J(\widehat{\boldsymbol{w}},\widehat{\eta}) \quad = \quad \int_0^T \|CS(\widehat{\boldsymbol{w}},\widehat{\eta})(t)-(z_1,z_2)(t)\|^2_{Z_1\times Z_2}dt+\beta\|(\widehat{\boldsymbol{w}},\widehat{\eta})\|^2_{\boldsymbol{V}(\Omega)\times B}. \tag{3}$$

We wish to solve the following optimal control problem: to find $\Psi_0\in\boldsymbol{V}(\Omega)\times B$ such that

$$J(\boldsymbol{w}_0,\eta_0) \quad = \quad \min\{J(\widehat{\boldsymbol{w}},\widehat{\eta}):(\widehat{\boldsymbol{w}},\widehat{\eta})\in\boldsymbol{V}(\Omega)\times B\}. \tag{4}$$

Theorem 2.1. *There exists a solution of the problem (4), moreover, suppose that $(\boldsymbol{w}_0,\eta_0)$ is solution of the problem (4), then there exist $\boldsymbol{w},\boldsymbol{p}\in L^2(0,T;\boldsymbol{V}(\Omega))$ and $\eta,q\in L^2(0,T;B)$ such that:*

$$\begin{gathered}\left(\frac{\partial\boldsymbol{w}}{\partial t},\boldsymbol{v}\right)+\left(\frac{\partial\eta}{\partial t},\phi\right)+\nu(\nabla\boldsymbol{w},\nabla\boldsymbol{v})+\theta(\nabla\eta,\nabla\phi)+b_0(\boldsymbol{w},\boldsymbol{w},\boldsymbol{v})+b_0(\boldsymbol{w},\boldsymbol{u}_\alpha,\boldsymbol{v})+b_0(\boldsymbol{u}_\alpha,\boldsymbol{w},\boldsymbol{v})\\ +b_1(\boldsymbol{w},\eta,\phi)+b_1(\boldsymbol{w},m_\alpha,\phi)+b_1(\boldsymbol{u}_\alpha,\eta,\phi)-U\left(\eta,\frac{\partial\phi}{\partial x_2}\right)=\kappa(\eta,\chi\cdot\boldsymbol{v})+(\hat{\boldsymbol{f}},\boldsymbol{v})\\ \boldsymbol{w}(0)=\boldsymbol{w}_0,\ \eta(0)=\eta_0,\end{gathered}$$

$$\begin{gathered}-\left(\frac{\partial\boldsymbol{p}}{\partial t},\boldsymbol{v}\right)-\left(\frac{\partial q}{\partial t},\phi\right)+\nu(\nabla\boldsymbol{p},\nabla\boldsymbol{v})+\theta(\nabla q,\nabla\phi)+b_0(\boldsymbol{v},\boldsymbol{w},\boldsymbol{p})+b_0(\boldsymbol{w},\boldsymbol{v},\boldsymbol{p})+b_0(\boldsymbol{v},\boldsymbol{u}_\alpha,\boldsymbol{p})+b_0(\boldsymbol{u}_\alpha,\boldsymbol{v},\boldsymbol{p})\\ +b_1(\boldsymbol{p},\eta,\phi)+b_1(\boldsymbol{w},\phi,q)+b_1(\boldsymbol{v},m_\alpha,q)+b_1(\boldsymbol{u}_\alpha,\phi,q)-U\left(\phi,\frac{\partial q}{\partial x_2}\right)=\kappa(\phi,\chi\cdot\boldsymbol{p})+(CS(\Psi_0)-z,C\zeta),\\ \boldsymbol{p}(T)=0,\ q(T)=0.\end{gathered} \tag{1}$$

$$2\left\{(\boldsymbol{p}(0),\boldsymbol{v})+(q(0),\phi)\right\}+\beta((\Psi_0,\zeta))+\int_0^T\left\{b_1(\boldsymbol{p}(t),\eta(t),\varphi_2(t))-b_1(\varphi_1(t),\eta(t),q(t))\right\}dt=0, \tag{2}$$

for all $(\boldsymbol{v},\phi)\in\boldsymbol{V}(\Omega)\times B$.

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A SINGULAR PARABOLIC EQUATION WITH CONCAVE NON LINEARITY.

MIGUEL LOAYZA[1,†] & MOHAMED MAJDOUB[2,‡]

Departamento de Matemática, UFPE, PE, Brasil[1], Department of Mathematics, College of Science, Imam Abdulrahman Bin Faisal University, Dammam, Saudi Arabia[2]

miguel.loayza@ufpe.br[†] mmajdoub@iau.edu.sa[‡]

Abstract

We prove existence and uniqueness of non-negative global solutions for the nonlinear heat equation $u_t - \Delta u = |x|^{-\gamma}\, u^q$, $0 < q < 1$, $\gamma > 0$ in the whole space $\mathbb{R}^N$, and for initial data $u_0 \in C_0(\mathbb{R}^N)$, $u_0 \geq 0$.

1 Introduction

Consider the singular nonlinear parabolic problem

$$\begin{cases} u_t - \Delta u &= |x|^{-\gamma}\, u^q \quad \text{in} \quad \mathbb{R}^N \times (0,\infty), \\ u(x,0) &= u_0(x) \geq 0 \quad \text{in} \quad \mathbb{R}^N, \end{cases} \tag{1}$$

where $0 < q < 1$, $\gamma > 0$, and $u_0 \in C_0(\mathbb{R}^N)$, where $C_0(\mathbb{R}^N)$ denotes the closure in $L^\infty(\mathbb{R}^N)$ of infinitely differentiable functions with compact support in $\mathbb{R}^N$. Our main interest in this paper is to analyze the existence and uniqueness of global solutions of (1) in the class $L^\infty(\mathbb{R}^N \times (0,T))$.

The initial value problem (1) has attracted considerable attention in the mathematical community. The case $\gamma = 0$ was considered in [1]. The case when $|x|^{-\gamma}$ is replaced with a function $\mathbf{a}(x)$ belonging to some Lebesgue space was studied in [3] for bounded domains.

It is worth to mention that there has been a large amount of researches on the nonlinear heat equation with convex nonlinearities (i.e. $q > 1$), and the monograph [4] cover a very extensive overview on the most established results on the subject. See also [2, 5] and the references therein.

To our knowledge, there is no previous results on existence (local/global) for $\gamma \neq 0$. The first difficulty to treat problem (1) is that the nonlinearity u^q is not a Lipschitz function. This is overcome, when $\gamma = 0$ in [1], by the fact that every non-negative (nontrivial) solution u of problem (1) verifies the following estimate from below

$$u(t) \geq [(1-q)t]^{1/(1-q)} = \underline{u}(t). \tag{2}$$

Moreover, as it is easily verified, $\underline{u}$ is a global solution of problem (1) with $\underline{u}(0) = 0$ and $\gamma = 0$. The second difficulty is the presence of the singular weight $a(x) = |x|^{-\gamma}$ with $\gamma > 0$. In this case, $\underline{u}$ is not a solution of (1), and we need to find a new estimate from below. Thus, we obtain the following estimate

$$u(x,t) \geq \eta_0 t^{1/(1-q)}(|x| + \sqrt{t})^{-\gamma/(1-q)} = w(x,t),$$

where η_0 is given by (2) below . In particular, when $\gamma = 0$ we have $w(x,t) = \underline{u}(t)$. Although w is not a solution of problem (1) it is a subsolution with initial data $u_0 = 0$; see Remark 2.1 below.

As is a standard practice, we study (1) via the associated integral equation:

$$u(t) = \mathbf{S}(t)u_0 + \int_0^t \mathbf{S}_\gamma(t-\sigma)\, u^q(\sigma)\, d\sigma, \tag{3}$$

where $\{\mathbf{S}(t)\}_{t\geq 0}$ denotes the heat semigroup and $\mathbf{S}_\gamma(t) = \mathbf{S}(t)|\cdot|^{-\gamma}$.

2 Main Results

Theorem 2.1. *Let* $u_0 \in C_0(\mathbb{R}^N), u_0 \geq 0$, $0 < q < 1$ *and* $0 < \gamma < \min\{2, N\}$. *Then, there exists a non-negative global solution for problem (1).*

Theorem 2.2. *Assume that* $0 < q < 1$, $0 < \gamma < \min\{2, N\}$, $u_0 \in C_0(\mathbb{R})^N$, $u_0 \geq 0$.

- *If* $u_0 \neq 0$, *then the solution of problem (1) is positive and verifies the following estimate*

$$u(x,t) \geq [\eta_0(1-q)]^{1/(1-q)} t^{1/(1-q)} (|x| + \sqrt{t})^{-\gamma/(1-q)} := w(x,t), \tag{1}$$

  for all $x \in \mathbb{R}^N$ *and* $t \geq 0$, *where*

$$\eta_0 = \eta_0(\gamma, q) = (4\pi)^{-N/2} \int_{\mathbb{R}^N} \exp\left(-\frac{|z|^2}{4}\right) (1 + |z|)^{-\gamma} (2 + |z|)^{-\gamma q/(1-q)} dz. \tag{2}$$

  Moreover, w *is a subsolution of problem (1) with* $u_0 = 0$.

- *For* $u_0 = 0$ *there exists a positive solution* u *which also verifies the inequality (1).*

- *There exists* $\gamma^* \in (0, \min\{2, N\})$ *so that for every* $u_0 \neq 0$, *the global solution obtained in Theorem 2.1 is unique if* $0 < \gamma < \gamma^*$.

Remark 2.1. *Here are some comments on Theorem 2.2.*

- *If* $\gamma = 0$, *then* $\eta_0 = 1$ *and* $w(x,t) = [(1-q)]^{1/(1-q)} t^{1/(1-q)}$, *that is,* w *is the bound from below given in (2).*

- *For* $\gamma = 0$ *and* $u_0 \neq 0$, *problem (1) admits a unique solution. Thus, we extend this result for a small values of* $\gamma > 0$.

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MASSERA'S THEOREMS FOR A HIGHER ORDER DISPERSIVE SYSTEM

ROBERTO DE ALMEIDA CAPISTRANO FILHO[1,†] & ISADORA MARIA DE JESUS[2,‡]

Departamento de Matemática, UFPE, PE, Brasil[1], Instituto de Matemática, UFAL, AL, Brasil[2]

roberto.capistranofilho@ufpe.br[†] isadora.jesus@im.ufal.br[‡]

Abstract

This work is devoted to presenting Massera-type theorems for the Kawahara system, a higher-order dispersive equation, posed in a bounded domain. Precisely, thanks to some properties of the semigroup and the decay of the solutions of this equation, we can prove its solutions are periodic, quasi-periodic, and almost periodic.

1 Introduction

We are interested to prove some periodic properties of the following Kawahara equation in a bounded domain

$$\begin{cases} u_t + u_{xxx} - u_{xxxxx} + uu_x = 0 & (x,t) \in I \times \mathbb{R} \\ u(0,t) = \varphi(t),\ u(1,t) = u_x(1,t) = u_x(0,t) = 0, & t \in \mathbb{R} \\ u_{xx}(1,t) = \alpha u_{xx}(0,t), & t \in \mathbb{R} \\ u(x,0) = u_0(x), & x \in I, \end{cases} \tag{1}$$

with a boundary force $\varphi(t)$ in a bounded domain $I = (0,1)$ and a damping term $\alpha u_{xx}(0,t)$, where $|\alpha| < 1$. Precisely, we are interested to understand if the system (1) has good properties when we investigate its solutions, considering the context introduced to Massera. Roughly speaking, we are interested in the study of the existence and qualitative property of recurrent solutions. This kind of property may be reformulated in the following question.

Question $\mathcal{A}$: Are there periodic solutions for the system (1)?

The first result is devoted to proving the well-posedness *via* semigroup theory, which is the key to proving the other main results of the article. Precisely, we first prove that the linear Kawahara operator generates $\{S(t)\}_{t\geq 0}$ the C_0–semigroup of contraction on $L^2(I)$.

Theorem 1.1. *There exists $\omega > 0$ such that for any $k = 0, 1, 2, 3, 4$ and 5, we can find a positive constant $C_k > 0$ which the semigroup associated to the linear Kawahara operator satisfies*

$$\|S(t)u_0\|_{H^k_\alpha(I)} \leq C_k e^{-\omega t} \|u_0\|_{H^k_\alpha(I)},$$

for all $t > 0$.

The previous theorem is the key to proving the existence of the bounded solution for the Kawahara equation (1). The next theorem, thanks to the previous one, ensures that the solutions of (1) are bounded.

Theorem 1.2. *There exists a constant $\epsilon > 0$ such that for all $\varphi \in C^1(\mathbb{R})$ satisfying $\|\varphi\|_{C^1(\mathbb{R})} \leq \epsilon$, the system (1) admits a unique solution u such that*

$$\|u\|_X \leq C\epsilon,$$

where $C > 0$ is a constant independent of ϵ.

The next three theorems give us Massera-type theorems for a higher-order dispersive system. The first one, stated below, guarantees that the solution of (1) is T–periodic.

Theorem 1.3. *Let*

$$\|\varphi\|_{C^1(\mathbb{R})} \leq \epsilon,$$

where ϵ is the constant determined by Theorem 1.2. If φ is a function T–periodic, thus u solution of (1), given by Theorem 1.2, is also a function T–periodic.

Additionally, the next Massera-type theorem gives some property of the periodicity of the solution to (1). The result can be read as follows.

Theorem 1.4. *Let*

$$\|\varphi\|_{C^1(\mathbb{R})} \leq \epsilon,$$

where ϵ is the constant determined by Theorem 1.2. If φ is a quasi-periodic function, the solution u of (1), obtained in Theorem 1.2, is also a quasi-periodic function. Moreover, if φ is $\overline{\omega}$–quasi-periodic function in t, thus the solution u of (1), obtained in Theorem 1.2, is also $\overline{\omega}$–quasi-periodic function in t.

Finally, let us present the last result of this work. Precisely, we can prove that the solutions of (1) are almost periodic.

Theorem 1.5. *Let $\|\varphi\|_{C^1(\mathbb{R})} \leq \epsilon$, where $0 < \epsilon \ll 1$ is obtained via Theorem 1.2. If φ, φ' are functions almost periodic, the solution u of (1), given by Theorem 1.2, is also an almost periodic function.*

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REGULARITY RESULTS FOR DEGENERATE WAVE EQUATIONS IN A NEIGHBORHOOD OF THE BOUNDARY

REGINALDO DEMARQUE.[1], BRUNO SÉRGIO V. ARAÚJO[2] & LUÍZ VIANA[3]

Departamento de Ciências da Natureza, UFF, RJ, Brasil[1], Unidade Acadêmica de Matemática, UFCG, PB, Brasil[2], Departamento de Análise, UFF, RJ, Brasil[3]

reginaldo@id.uff.br[†] bsergio@mat.ufcg.edu.br[‡] luizviana@id.uff.br[§]

Abstract

In this work we establish some regularity results concerning the behavior of weak solutions and very weak solutions of the degenerate wave equation near the boundary. For the nondegenerate case, the correponding results were originally obtained by Fabre and Puel [1]. This kind of results is closely related to the exact boundary controllability for the wave equation as the limit of internal controllability.

1 Introduction

In this work we are interested in studying the behavior, near the boundary point $x = 1$, of the weak and very weak solutions of the following degenerate wave equation:

$$\begin{cases} u_{tt} - (x^\alpha u_x)_x = f, & (t,x) \in Q, \\ u(t,1) = 0, & \text{in } (0,T), \\ \begin{cases} u(t,0) = 0, & \text{if } \alpha \in (0,1), \\ \text{or} & \\ (x^\alpha u_x)(t,0) = 0, & \text{if } \alpha \in [1,2), \end{cases} & t \in (0,T), \\ u(0,x) = u_0(x) \text{ and } u_t(0,x) = u_1(x) & x \in (0,1), \end{cases} \tag{1}$$

where $T > 0$, $Q = (0,T) \times (0,1)$, $\alpha \in (0,2)$ and the data (f, u_0, u_1) belongs to spaces that will determine the regularity of the solution. To develop this study, the L^2 norm of the solution will be analyzed in an ε-neighborhood of the boundary point $x = 1$.

For the nondegenerate wave equation, an analogous investigation has been considered by Fabre and Puel in [1]. Their results have played a key role in [2], where an exact boundary controllability is achieved as the limit of a sequence of internal controllability problems, set in ε- neighborhoods of the boundary, as $\varepsilon \to 0$.

In order to stablish our main results, let us define some spaces. Consider $\alpha \in (0,1)$, for the **weakly degenerate case** (WDC), or $\alpha \in [1,2)$, for the **strongly degenerate case** (SDC).

(I) For the (WDC), we set

$$H^1_\alpha := \{u \in L^2(0,1);\ u \text{ is absolutely continuous in } [0,1], x^{\alpha/2}u_x \in L^2(0,1) \text{ and } u(1) = u(0) = 0\},$$

equipped with the natural norm

$$\|u\|_{H^1_\alpha} := \left(\|u\|^2_{L^2(0,1)} + \|x^{\alpha/2}u_x\|^2_{L^2(0,1)}\right)^{1/2};$$

(II) For the (SDC),

$$H^1_\alpha := \{u \in L^2(0,1);\ u \text{ is locally absolutely continuous in } (0,1], x^{\alpha/2}u_x \in L^2(0,1) \text{ and } u(1)=0\},$$

and the norm keeps the same;

(III) In both situations, the (WDC) and the (SDC),

$$H^2_\alpha := \{u \in H^1_\alpha;\ x^{\alpha/2}u_x \in H^1(0,1)\}$$

with the norm $\|u\|_{H^2_\alpha} := \left(\|u\|^2_{H^1_\alpha} + \|(x^{\alpha/2}u_x)_x\|^2_{L^2(0,1)}\right)^{1/2}$.

Another important space in this context is $H^{-1}_\alpha = (H^1_\alpha)'$ (the dual space of H^1_α).

Now we are ready to state our main results below.

2 Main Results

Theorem 2.1. *Given $0 < \varepsilon_0 < 1$, there exists $C > 0$ such that, for all $(u_0, u_1) \in H^1_\alpha \times L^2(0,1)$ and $f \in L^1(0,T;L^2(0,1))$, if u is a weak solution to (1), then*

$$\frac{1}{\varepsilon^3}\int_0^T\int_{1-\varepsilon}^1 |u(t,x)|^2\,dx\,dt \le C\left(\|f\|^2_{L^1(0,T;L^2(0,1))} + \|u_0\|^2_{H^1_\alpha} + \|u_1\|^2_{L^2(0,1)}\right),\ \forall \varepsilon \in (0,\varepsilon_0],$$

where C only depends on ε_0, α and T.

Theorem 2.2. *Let us consider a family of functions $(h_\varepsilon, \varphi^0_\varepsilon, \varphi^1_\varepsilon) \in L^1(0,T : L^2(\Omega)) \times L^2(\Omega) \times H^{-1}_\alpha$ such that*

$$\begin{aligned} h_\varepsilon &\rightharpoonup h && \text{in } L^1(0,T : L^2(0,1)),\\ \varphi^0_\varepsilon &\rightharpoonup \varphi^0 && \text{in } L^2(0,1),\\ \varphi^1_\varepsilon &\rightharpoonup \varphi^1 && \text{in } H^{-1}_\alpha, \end{aligned}$$

and let φ_ε be the solution by transposition of problem (1) with $(f, u^0, u^1) = (h_\varepsilon, \varphi^0_\varepsilon, \varphi^1_\varepsilon)$. If

$$\frac{1}{\varepsilon^3}\int_0^T\int_{1-\varepsilon}^1 |\varphi_\varepsilon(t,x)|^2\,dx\,dt \le C,$$

where C does not depend on ε, then $\varphi_x(\cdot,1) \in L^2(0,T)$ and

$$\frac{1}{3}\|\varphi_x(\cdot,1)\|^2_{L^2(0,T)} \le \liminf_{\varepsilon\to 0^+}\left(\frac{1}{\varepsilon^3}\int_0^T\int_{1-\varepsilon}^1 |\varphi_\varepsilon(t,x)|^2\,dx\,dt\right)$$

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ATTRACTORS FOR A CLASS OF NONLOCAL ENERGY PLATE MODEL

VANDO NARCISO[1,†]

Center of Exact and Tecnological Sciences, UEMS, MS, Brasil[1].

vnarciso@uems.br[†].

Abstract

This paper is concerned with dynamics of a class of beam/plate equation with nonlocal damping that is derived from nonlocal dissipative energy models for flight structures proposed by Balakrishnan-Taylor [2].

1 Introduction

This work is concerned with the well-posedness and long-time dynamics of solutions to the initial boundary value problem of a plate equation with nonlocal nonlinear damping and source terms:

$$u_{tt} + \Delta^2 u + \gamma \left[\|\Delta u\|^2 + \|u_t\|^2 \right]^{\frac{\beta}{2}} u_t + f(u) = h \quad \text{in} \quad \Omega \times \mathbb{R}^+. \tag{1}$$

where $\gamma > 0, \beta \geq 1$, $\Omega \subset \mathbb{R}^N$ is a bounded domain of with smooth boundary $\Gamma = \partial\Omega$, f is a nonlinear function, h is an external force, and $\|\cdot\|$ stands for the norm in $L^2(\Omega)$. We consider either clamped or hinged boundary conditions, described respectively by

$$u|_{\Gamma\times\mathbb{R}^+} = \frac{\partial u}{\partial \nu}|_{\Gamma\times\mathbb{R}^+} = 0 \quad \text{or} \quad u|_{\Gamma\times\mathbb{R}^+} = \Delta u|_{\Gamma\times\mathbb{R}^+} = 0, \tag{2}$$

where ν is the unit exterior normal to Γ. The initial conditions associated to ($\mathcal{P}_\Phi$) are given by

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x), \quad x \in \Omega. \tag{3}$$

Let $V_0 = L^2(\Omega)$ and $V_1 = H_0^1(\Omega)$, and to attend the two boundary conditions in (2) we define $V_2 = H_0^2(\Omega)$ or $V_2 = H^2(\Omega) \cap H_0^1(\Omega)$. Let $\lambda_1 > 0$ the first eigenvalue of the bi-harmonic operator Δ^2 in V_2.

Assumption 1.1 f is a C^1-function on $\mathbb{R}$ satisfying

- $|f'(s)| \leq c_{f'}(1 + |s|^\rho), \quad \forall\, s \in \mathbb{R}$,
- $-c_f - \frac{\alpha}{2}s^2 \leq \hat{f}(s) := \int_0^s f(\tau)d\tau \leq f(s)s + \frac{\alpha}{2}s^2, \quad \forall\, s \in \mathbb{R}$,

where we consider $c_{f'} > 0$, $c_f \geq 0$, $0 \leq \alpha < \lambda_1$, and $\rho > 0$ if $1 \leq n \leq 4$ or $0 < \rho \leq \frac{4}{n-4}$ if $n \geq 5$.

Our analysis with respect to the global existence and long-time behavior of solutions is given on the phase space $\mathcal{H} = V_2 \times V_0$ equipped with norm $\|(u,v)\|_{\mathcal{H}}^2 = \|\Delta u\|^2 + \|v\|^2$.

2 Mathematical Results

The existence and uniqueness results of the global solutions in the space $\mathcal{H}$ are given in the following theorem.

Theorem 2.1. *Let $T > 0$ be arbitrary, $h \in W_0$, $\gamma > 0$, and $\beta \geq 1$. Under Assumption 1.1 we have: if initial data $(u_0, u_1) \in \mathcal{H}$, then problem ($\mathcal{P}_\Phi$)-(3) has a unique weak solution*

$$(u, u_t) \in C([0,T], \mathcal{H}), \quad \forall\, T > 0, \tag{4}$$

satisfying

$$u \in L^\infty(0,T;V_2), \quad u_t \in L^\infty(0,T;V_0) \quad \text{and} \quad u_{tt} \in L^2(0,T;V_2'). \tag{5}$$

Proof The principle of the proof is classical. We using the Faedo-Galerkin method associated to compactness arguments. The well-posedness of problem ($\mathcal{P}_\Phi$)-(3) given by Theorem 2.1 implies that the evolution operator $S(t) : \mathcal{H} \to \mathcal{H}$ defined by

$$S(t)(u_0, u_1) = (u(t), u_t(t)), \quad t \geq 0, \tag{6}$$

where (u, u_t) is the unique weak solution of the system ($\mathcal{P}_\Phi$)-(3), defines a nonlinear C_0-semigroup which is locally Lipschitz continuous on the phase space $\mathcal{H}$. Therewith the dynamics of problem ($\mathcal{P}_\Phi$)-(3) can be studied through the continuous dynamical system $(\mathcal{H}, S(t))$.

Our main result in the present work is the following.

Theorem 2.2. *Assume that hypotheses of Theorem 2.1 hold with* $1 \leq 2^{q-1}q < 2$. *Then, we have*

- **(Global attractor)** *the associate dynamical system* $(\mathcal{H}, S(t))$ *of problem* ($\mathcal{P}_\Phi$)-(3) *has a compact global attractor* $\mathfrak{A}$ *in* $\mathcal{H}$.
- **(Characterization)** *the global attractor* $\mathfrak{A}$ *is precisely the unstable manifold* $\mathfrak{A} = \mathrm{M}^u(\mathcal{N})$ *emanating from the set of stationary solution* $\mathcal{N}$. *In addition,* $\mathfrak{A}$ *consist of full trajectories* $\Upsilon = \{U(t) = (u(t), u_t(t)) : t \in \mathbb{R}\}$ *such that*

$$\lim_{t\to-\infty} dist_{\mathcal{H}}(U(t), \mathcal{N}) = 0 \quad and \quad \lim_{t\to+\infty} dist_{\mathcal{H}}(U(t), \mathcal{N}) = 0.$$

Proof The existence of a compact global attractor is granted once our dynamical system $(\mathcal{H}, S(t))$ is dissipative and satisfies an asymptotic smoothness property. Here we explore recent results from dynamics theory for equations of evolution of Chueshov and Lasiecka [3, 4].

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THE PARAMETER INVERSION IN COUPLED GEOMECHANICS USING BAYESIAN INFERENCE

JUAREZ S. AZEVEDO[1,†] & JARBAS A. FERNANDES[2,‡]

UFBA, ICTI - Bahia, Brazil[1] UFOB - Bahia, Brazil[2]

jdazevedo@ufba.br[†] jarbas.fernandes@ufob.edu.br[‡]

Abstract

In order to introduce uncertainty in geomechanical analysis, taking into account the random nature of soil parameters, Bayesian inference techniques are implemented in highly heterogeneous porous media. Within the framework of a coupling algorithm, these are incorporated into the inverse poroelasticity problem, with porosity, permeability and Young modulus treated as stationary random fields obtained by the moving average (MA) method. To this end, the Metropolis-Hasting (MH) algorithm was chosen to seek the geomechanical parameters that yield the lowest misfit. Numerical simulations related to injection problems and fluid withdrawal in a $3D$ domain are performed to compare the performance of this methodology. We conclude with some remarks about numerical experiments

1 Introduction

Most flow-geomechanical simulators are impacted by rock heterogeneities. They are characterized by spatial variations in the distribuition of their flow properties such as the permeability and porosity, and poromechanical parameters that control the velocity and displacement of flow beyond the change of fluid content. In consequence, there are fluctuations in Darcy velocity and stress caused by the variations in porosity, permeability field, and elastic constants that directly impacts oil recovery models [1]. In this case, obtaining a more accurate solution is very important since fingering instabilities are determined by these parameters, which are uncertain in the description of the porous medium. The poromechanical parameters must be represented correctly in order to take into account the random anture of the geomechanical simulation.

A particular interest in the present work is checking how random parameters (porosity, permeability, Young modulus) impact the fluid pressure around the reservoir. These changes in the reservoir may interfere with the production scenario; futhermore, they may create necessity for efficient reservoir modeling. Our goal is to present an inversion procedure based on Bayesian inference for a coupled $3D$ poroelasticity problem in order to estimate fields of porosity, permeability, and Young modulus, produced form a priori information of the solution in heterogeneous reservoirs.

Porosity, permeability, and Young modulus will be modeled as random fields when describing two-phase flow in porous media. We will use fluid dynamics relationships to construct inverse models of the porosity, permeability and the Young modulus. Due to the random nature of the process, we will impose in the experiments that these fields are generated independently by MA method [2]. To generate single-chains based on the Metropolis criterion, the Random Walk (RW) algorithm is commonly used to generate posterior probability distributions for all parameters [3]. It is important to note that we present a simple technique of Bayesian inference that is easily integrated into any discretization technique.

2 Stochastic Geomechanical Model

Let us consider a time interval $[0, T]$ and an open subset $D \subset \mathbb{R}^3$, with boundary ∂D and unit outward normal $\boldsymbol{n}$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Here Ω is the set of outcomes, $\mathcal{F} \subset 2^\Omega$ is the σ-algebra of events, and $P : \mathcal{F} \to [0, 1]$ is a probability measure. Let the set $\{\omega, \boldsymbol{x}, t\} \subset \Omega \times D \times [0, T]$ based on an random event, spatial position and time, respectively.

The displacement is given by $\boldsymbol{u} := \boldsymbol{u}(\omega, \boldsymbol{x}, t) \in \mathbb{R}^3$ while $p := p(\omega, \boldsymbol{x}, t)$ refer to the Biot pressure related to the fluid variables by the constitutive relation:

$$\nabla \cdot (\boldsymbol{\sigma}(\boldsymbol{u}) - \alpha p \boldsymbol{I}) = \boldsymbol{f}, \tag{1}$$

where $\boldsymbol{f} : D \times (0, T] \to \mathbb{R}^3$ is a external volumetric force, α is the Biot coefficient and $\mathbf{I}$ is the unit tensor, while the elastic variables:

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon} \quad \text{and} \quad \boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla + \nabla^T)\boldsymbol{u}. \tag{2}$$

denote the Cauchy stress tensor and infinitesimal strain tensors respectively. The fourth order stiffness tensor $\boldsymbol{C}$ is given by coefficients $C_{ijkl} = \lambda_s \delta_{ij}\delta_{kl} + \mu_s(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ where δ_{ij} is the Kronecker delta and (λ, μ) is the pair of Lamé coefficients associated with the Young modulus $E := E(\omega, \boldsymbol{x})$ through the definitions:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}, \tag{3}$$

where ν is the deterministic Poisson ratio (with $0 \le \nu < 0.5$).

In order to relate the equations that govern elasticity and fluid, a constitutive relation for porosity is defined as:

$$\phi = \phi_0 + \alpha \nabla \cdot \boldsymbol{u} + c_r(\alpha - \phi_0)(p - p_0), \tag{4}$$

in that, ϕ_0 and p_0 refer to initial values of Lagrangian porosity and pressure respectively, while c_r is the rock compressibility (for details see [4] pg. 74). Here ϕ_0 is a uncertainty parameter.

We consider a saturated porous medium where the governing equations of the two-fluid model are as follows:

$$\partial_t(\phi b_k s_k) + \nabla \cdot (b_k \boldsymbol{v}_k) = b_k q_k \quad k = w, o, \tag{5}$$

where q_k is the source term, s_k the saturation, and $\boldsymbol{v}_k$ the Darcy velocity relating for the fluxes in the porous media expressed by:

$$\boldsymbol{v}_k = -\frac{\kappa_r \boldsymbol{K}}{\mu_k}(\nabla p_k - \rho_k g \nabla z), \tag{6}$$

such that g is the gravity field and $\boldsymbol{K}$ is the absolute permeability. For ease, we will assume that $\mathbf{K} = \kappa(\mathbf{x})\mathbf{I}$, where $\mathbf{I}$ is a unit matrix and $\kappa(\mathbf{x})$ is a scalar function. Here b_k represents the ratio between volume at elevated pressure and volume at surface conditions, κ_r is the relative permeability, μ_k define the viscosity, and ρ_k denote the density of a fluid in the phase k.

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FACTORIZATION THEOREMS FOR SUMMING OPERATORS

ALEXANDRE LIMA[1,†] & JOEDSON SANTOS[2,‡]

Departamento de Matemática, UFPB, PB, Brasil[1], Departamento de Matemática, UFPB, PB, Brasil[2]

alexandre.cesar.bl.505@gmail.com[†] joedson.santos@academico.ufpb.br[‡]

Abstract

Pietsch's Factorization Theorem is one of the main results of the theory of absolutely summing operators. This work aims to study some classes of linear and multilinear operators, in a certain abstract context, that satisfy a Pietsch-type Factorization theorem.

1 Introduction

The theory of absolutely summing linear operators plays a prominent role in Functional Analysis. A continuous linear operator between Banach spaces $T : X \to Y$ is *absolutely p-summing*, $1 \leq p < \infty$, if there is a constant $C > 0$ such that, for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, we have

$$\left(\sum_{j=1}^{n} \|T(x_j)\|^p\right)^{1/p} \leq C \sup_{\varphi \in B_{X^*}} \left(\sum_{j=1}^{n} |\varphi(x_j)|^p\right)^{1/p},$$

where B_{X^*} denotes the closed unit ball of the topological dual X^* of X.

If X is a Banach space, then the canonical map $I : X \to C(B_{X^*})$ given by

$$I(x)(\varphi) = \varphi(x), \text{ for all } x \in X \text{ and } \varphi \in B_{X^*},$$

is a linear isometry. In addition, if μ is a regular Borel probability measure on B_{X^*}, then the canonical inclusion map $J_p : C(B_{X^*}) \to L_p(\mu)$ is absolutely p-summing.

Theorem 1.1 (Pietsch's Factorization)**.** *An operator $T : X \to Y$ is absolutely p-summing if and only if there are a regular Borel probability measure μ in B_{X^*}, a closed subspace $X_p \subseteq L_p(\mu)$, and a linear continuous operator $\widehat{T} : X_p \to Y$ such that $J_p(I(X)) \subseteq X_p$ and*

$$T = \widehat{T} \circ J_p \circ I.$$

Due to the importance of this theorem and their applications, it is natural to investigate whether the above theorem holds in generalizations of the class of absolutely summing linear operators. However, this task is quite challenging as many tools used in linear theory do not work in more general contexts.

Thus, the main purpose here is to study some classes of linear and multilinear operators, introduced by Achour et al. in [1], that satisfy a Pietsch-type Factorization theorem.

2 Main Results

A map between vector spaces $\Psi : X \to Y$ is *absolutely homogeneous* if $\Psi(\lambda x) = |\lambda|\Psi(x)$, for all $x \in X$ and $\lambda \in \mathbb{K}$. If X and Y are normed spaces, then $\Psi : X \to Y$ is *bounded* if there is a constant $K_\Psi > 0$ such that $\|\Psi(x)\| \leq K_\Psi \|x\|$, for all $x \in X$.

Let $m \in \mathbb{N}$, and $X_1, \ldots, X_m$ and Y Banach spaces. The canonical map $I_m : X_1 \times \cdots \times X_m \to C(B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m)^*})$ given by

$$I_m(x)(\varphi) = \varphi(x^1 \otimes \cdots \otimes x^m), \text{ for all } x = (x^1, \ldots, x^m) \in X_1 \times \cdots \times X_m \text{ and } \varphi \in B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m)^*},$$

is m-linear and continuous. Denoting by $I_m(x^1, \ldots, x^m) = \langle x^1 \otimes \cdots \otimes x^m, \cdot \rangle$ and $\widehat{V_m} = \operatorname{span}\{I_m(X_1 \times \cdots \times X_m)\}$, the expression

$$\mathbf{x} = \sum_{k=1}^{d} \langle x^{1,k} \otimes \cdots \otimes x^{m,k}, \cdot \rangle.$$

provides the usual representation for an element $\mathbf{x} \in \widehat{V_m}$.

Let $\Phi : \widehat{V_m} \to C(B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m)^*})$ an absolutely homogeneous map, Z a Banach space and $J : \operatorname{span}\{\Phi(\widehat{V_m})\} \to Z$ a linear continuous operator. The expression

$$\rho_{\Phi,Z}(\mathbf{x}) = \inf \left\{ \sum_{i=1}^{r} \left\| J \circ \Phi \left(\sum_{k=1}^{d} \langle x_i^{1,k} \otimes \cdots \otimes x_i^{m,k}, \cdot \rangle \right) \right\|_Z : \mathbf{x} = \sum_{i=1}^{r} \sum_{k=1}^{d} \langle x_i^{1,k} \otimes \cdots \otimes x_i^{m,k}, \cdot \rangle \right\}$$

defines a seminorm in $\widehat{V_m}$. Consider the subspace $N = \{\mathbf{x} \in \widehat{V_m} : \rho_{\Phi,Z}(\mathbf{x}) = 0\}$ of $\widehat{V_m}$. The completion of $\widehat{V_m}/N$, denoted by $(\widehat{V_m})_\Phi^Z$ and called *domination space* of $\widehat{V_m}$ defined by Φ and Z, is a Banach space with the norm

$$\|[\mathbf{x}]\|_{(\widehat{V_m})_\Phi^Z} = \rho_{\Phi,Z}(\mathbf{x}).$$

Furthermore, if Φ is a bounded, then the canonical linear map $I_{(\widehat{V_m})_\Phi^Z} : \widehat{V_m} \to (\widehat{V_m})_\Phi^Z$ given by

$$I_{(\widehat{V_m})_\Phi^Z}(\mathbf{x}) = [\mathbf{x}], \text{ for all } \mathbf{x} \in \widehat{V_m},$$

is continuous.

Let $\Phi : \widehat{V_m} \to C(B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m)^*})$ an absolutely homogeneous map. A m-linear continuous operator $T : X_1 \times \cdots \times X_m \to Y$ is *strongly Φ-abstract p-summing*, $1 \leq p < \infty$, if there is a constant $C > 0$ such that, for all $n, d \in \mathbb{N}$ and $x_j^{t,k} \in X_t$, $(t, j, k) \in \{1, \ldots, m\} \times \{1, \ldots, n\} \times \{1, \ldots, d\}$, we have

$$\left(\sum_{j=1}^{n} \left\| \sum_{k=1}^{d} T(x_j^{1,k}, \ldots, x_j^{m,k}) \right\|^p \right)^{1/p} \leq C \left\| \left(\sum_{j=1}^{n} \left| \Phi \left(\sum_{k=1}^{d} \langle x_j^{1,k} \otimes \cdots \otimes x_j^{m,k}, \cdot \rangle \right) \right|^p \right)^{1/p} \right\|_{C(B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m)^*})}. \tag{1}$$

Theorem 2.1. *If $\Phi : \widehat{V_m} \to C(B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m)^*})$ is a bounded absolutely homogeneous map, then an operator $T : X_1 \times \cdots \times X_m \to Y$ is strongly Φ-abstract p-summing if and only if there are a regular Borel probability measure μ in $B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m)^*}$, a domination space $(\widehat{V_m})_\Phi^{L_p(\mu)}$ defined by Φ and $L_p(\mu)$, and a linear continuous operator $\widehat{T} : (\widehat{V_m})_\Phi^{L_p(\mu)} \to Y$ such that*

$$T = \widehat{T} \circ I_{(\widehat{V_m})_\Phi^{L_p(\mu)}} \circ I_m.$$

Remark 2.1. *By appropriately choosing the parameters of Theorem 2.1, we can recover and prove Pietsch-type Factorization theorems for several classes of operators, such as absolutely p-summing linear operators, $(p;\sigma)$-absolutely continuous linear operators, strongly p-summing multilinear operators, and factorable strongly $(p;\sigma)$-summing multilinear operators.*

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A CONVENIENT APPLICATION OF THE MULTILINEAR THEORY TO MID SUMMING LINEAR OPERATORS

ANTONIO A. DO NASCIMENTO[1,†] & JAMILSON R. CAMPOS[2,‡]

Departmento de Matemática, UFPB, PB, Brasil[1], Departmento de Ciências Exatas, UFPB, PB, Brasil[2]

antonionascimentoufpb@gmail.com[†] jamilson@dcx.ufpb.br[‡]

Abstract

In this work we present a certain class of mid summing linear operators, more general than the one defined in [1], and a somewhat convenient relation of this class with classes of multiple summing operators. This kind of relationship between linear and non-linear theory, object of study in our dissertation, establishes coincidence and inclusion results for the presented class of operators.

1 Introduction

In the paper [3], A. Karn and D. Sinha introduce and study the concepts of p-limited sets, p-compactness and weak p-compactness. These concepts are then worked on in various aspects, such as their relationship with certain classes of operator and the Dunford-Pettis property. In particular, this study gives rise to the *space of mid p-summable sequences* and this space was subject to a complementary and more detailed investigation in [1]. In this work, the authors work from the point of view of the abstract theory of sequence classes, which generalizes classes of operators characterized by transformations of vector-valued sequences, and study classes of mid summing operators.

From the work [2], which generalizes the space of mid p-summable sequences, we will present a result that relates the linear theory of mid summing operators to the multilinear theory of absolutely summing operators. This result leads us to inclusion and coincidence results that are not present in previous works.

In what follows, the letters E, F will denote Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The symbol $E \overset{1}{\hookrightarrow} F$ denotes $E \subset F$ and $\|x\|_F \leq \|x\|_E$, for all $x \in E$. The ideal of the multiple $(q_1, \ldots, q_n; p_1, \ldots, p_n)$-summing n-linear operators will be denoted by $\Pi^n_{q_1,\ldots,q_n;p_1,\ldots,p_n}$. Other notations and symbols used here are well known or can be found in [1, 2].

2 Main Results

The space of *mid (q,p)-summable* sequences in E, denoted by $\ell^{\mathrm{mid}}_{q,p}(E)$ is the space formed by all sequences $(x_j)_{j=1}^\infty$ in E satisfying $((\varphi_n(x_j))_{n=1}^\infty)_{j=1}^\infty \in \ell_q(\ell_p)$, whenever $(\varphi_n)_{n=1}^\infty \in \ell_p^w(E')$. The expression

$$\|(x_j)_{j=1}^\infty\|_{q,p} = \sup_{(\varphi_n)_{n=1}^\infty \in B_{\ell_p^w(E')}} \left(\sum_{j=1}^\infty \left(\sum_{n=1}^\infty |\varphi_n(x_j)|^p \right)^{q/p} \right)^{1/q}$$

defines a complete norm in $\ell^{\mathrm{mid}}_{q,p}(E)$ and when $p = q$ we have $\ell^{\mathrm{mid}}_{p,p}(E) = \ell^{\mathrm{mid}}_p(E)$, space studied in [1].

It is not difficult to show that $\ell_q(E) \overset{1}{\hookrightarrow} \ell^{\mathrm{mid}}_{q,p}(E) \overset{1}{\hookrightarrow} \ell_q^w(E)$, with $1 \leq p, q < \infty$, and that $\ell^{\mathrm{mid}}_{q,p}(\cdot)$ is a sequence class.

If $x = (x_j)_{j=1}^\infty \in \ell_q^w(E)$, it follows that the operator $\psi_x : E' \to \ell_q$, given by $\psi_x(\varphi) = (\varphi(x_j))_{j=1}^\infty$, is linear and continuous, with $\|\psi_x\| = \|(x_j)_{j=1}^\infty\|_{w,q}$. Using Minkowski's inequality and the operator ψ_x, we obtain the following result.

Theorem 2.1. *Let $x = (x_j)_{j=1}^\infty \in \ell_q^w(E)$.*

a) *If $q \leq p$ and $x \in \ell_{q,p}^{mid}(E)$, then $\psi_x \in \Pi_p(E', \ell_q)$ and $\pi_p(\psi_x) \leq \|(x_j)_{j=1}^{\infty}\|_{q,p}$.*

b) *If $q \geq p$ and $\psi_x \in \Pi_p(E', \ell_q)$, then $x \in \ell_{q,p}^{mid}(E)$ and $\|(x_j)_{j=1}^{\infty}\|_{q,p} \leq \pi_p(\psi_x)$.*

Of course, the above sentences are equivalent if $p = q$.

From the theorem above it follows that $\ell_{q,p}^{\mathrm{mid}}(E) \overset{1}{\hookrightarrow} \ell_{r,s}^{\mathrm{mid}}(E)$, when $q \leq p \leq s \leq r$. In particular, $\ell_p^{\mathrm{mid}}(E) \overset{1}{\hookrightarrow} \ell_q^{\mathrm{mid}}(E)$, if $p \leq q$.

Definition 2.1. *Let $q \geq r$ and $T \in \mathcal{L}(E;F)$. We say that T is weakly mid $(q,p;r)$-summing if $(T(x_j))_{j=1}^{\infty} \in \ell_{q,p}^{mid}(F)$, whenever $(x_j)_{j=1}^{\infty} \in \ell_r^w(E)$.*

The space of all weakly mid $(q,p;r)$-summing operators from E to F will be denoted by $W_{q,p;r}^{\mathrm{mid}}(E,F)$. The case $p = q$, denoted $W_{p;r}^{\mathrm{mid}}(E,F)$, was defined and studied in [1] and the case $p = q = r$, denoted $W_p^{\mathrm{mid}}(E,F)$, was defined in [3].

A necessary and sufficient condition to $T \in W_{q,p;r}^{\mathrm{mid}}(E,F)$ is that there exists a constant $B > 0$ such that

$$\left(\sum_{j=1}^{\infty}\left(\sum_{n=1}^{\infty}|\varphi_n(T(x_j))|^p\right)^{q/p}\right)^{1/q} \leq B \cdot \left\|(x_j)_{j=1}^{\infty}\right\|_{w,r} \cdot \|(\varphi_n)_{n=1}^{\infty}\|_{w,p}\,, \tag{1}$$

whenever $(x_j)_{j=1}^{\infty} \in \ell_r^w(E)$ and $(\varphi_n)_{n=1}^{\infty} \in \ell_p^w(F')$. Defining the bilinear (continuous) form $\Phi_T : F' \times E \to \mathbb{K}$ by $\Phi_T(\varphi, x) = \varphi(T(x))$, you can see that

$$\left(\sum_{j=1}^{\infty}\left(\sum_{n=1}^{\infty}|\varphi_n(T(x_j))|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} = \left(\sum_{j=1}^{\infty}\left(\sum_{n=1}^{\infty}|\Phi_T(\varphi_n, x_j)|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}.$$

Using the expression above, the characterization in (1) and the definition of multiple summing operators (see [2, Definition 4.3]) we get the result below.

Theorem 2.2. *The operator $T \in \mathcal{L}(E;F)$ is weakly mid $(q,p;r)$-summing if and only if the bilinear operator Φ_T be multiple $(p,q;p,r)$-summing.*

Corollary 2.1. *For an admissible choice of parameters q, q_1, p, p_1, r e r_1:*

(a) *$W_{q,p;r}^{mid}(E;F) = \mathcal{L}(E;F)$, if $\Pi_{p,q;p,r}^2(F', E; \mathbb{K}) = \mathcal{L}(F', E; \mathbb{K})$.*

(b) *$W_{q,p;r}^{mid}(E;F) \subseteq W_{q_1,p_1;r_1}^{mid}(E;F)$, if $\Pi_{p,q;p,r}^2(F', E; \mathbb{K}) \subseteq \Pi_{p_1,q_1;p_1,r_1}^2(F', E; \mathbb{K})$.*

As applications of the results above, for $1 \leq p, q < 2$, we have: i) $W_p^{\mathrm{mid}}(E;F) \subset W_q^{\mathrm{mid}}(E;F)$ when $p \leq q$; ii) If E, F' have cotype 2, then $W_p^{\mathrm{mid}}(E;F) = W_q^{\mathrm{mid}}(E;F)$.

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WEIGHTED NORM INEQUALITIES FOR INTEGRAL TRANSFORMS OF FOURIER TYPE (F-TRANSFORMS)

BRUNA Z. CENTURION[1,†] & THAÍS JORDÃO[2,‡]

Instituto de Ciências Matemáticas e de Computação, USP, SP, Brasil[1]
Supported by FAPESP, grant #2021/13082-1,
Instituto de Ciências Matemáticas e de Computação, USP, SP, Brasil[2]

bruna.centurion@usp.br[†] tjordao@icmc.usp.br[‡]

Abstract

In classical Fourier analysis the decay of the Fourier transform is a fundamental theme. Norm inequalities for the Fourier transform are known useful tools for relating smoothness and the decay of the transform. In this work we explore weighted norm inequalities for integral transforms generated by continuous kernels satisfying a size condition. We present Pitt's inequality for the integral tranforms we consider and, in particular, we recover important and classical inequalities for the Fourier, Hankel and Jacobi transforms.

1 Introduction

The classical Pitt's inequality for the Fourier transforms states that for $1 < p \leq q < \infty$, it holds

$$\left(\int_{\mathbb{R}^n} |\widehat{f}(y)|^q |y|^{-q\gamma} dy\right)^{1/q} \lesssim \left(\int_{\mathbb{R}^n} |f(x)|^q |x|^{p\beta} dx\right)^{1/p},$$

if and only if

$$\max\{0, n(1/p + 1/q - 1)\} \leq \gamma < n/q \quad \text{and} \quad \beta - \gamma = n(1 - 1/p - 1/q).$$

Above $\widehat{f}$ represents the Fourier transform of f and the sign $a(s) \lesssim b(s)$ means $a(s) \leq cb(s)$, for some constant does not depending on s. We will employ the notation $a(s) \asymp b(s)$, for $a(s) \lesssim b(s)$ and $b(s) \lesssim a(s)$.

We will present Pitt's inequality for a general class of integral transforms. The results are weighted norm inequalities for the so called F-transform. The goal is to characterize the pairs of the weights u and v, that are non-negative and locally integrable functions, such that inequalities of type

$$||F(f)||_{q,u} \lesssim ||f||_{p,v}$$

holds, with the operator $F : L^p_v \longrightarrow L^q_u$ given by

$$F(f)(x) = \int_0^\infty f(y)\mathcal{K}(x,y)s(y)dy, \quad x > 0,$$

where s a continuous function satisfying a monotonicity condition, $\mathcal{K}$ is a continuous kernel, and for f locally integrable functions.

2 Main Results

In this section we present the definition of the integral transforms we will work with and we state Pitt's inequality for the general transform. We follow the path designed in [1]. The classical setting, i.e., weighted inequalities for the Fourier transform, can be seen in [2].

Let $s>0$ a non-decreasing continuous function that satisfies

$$s(2y)\lesssim s(y),\quad y>0,$$

and consider $\mathcal{K}$ a continuous kernel. The integral transform

$$Ff(x):=\int_0^\infty f(y)\mathcal{K}(x,y)s(y)dy,\quad x>0,\quad f\in L^1_{loc},$$

is a **F-transform** if there exists a non-decreasing function $w\geq 0$ with $w(x)s\left(1/x\right)\asymp 1$, for $x>0$, such that, for all $f\in L^2_s$, the Bessel's inequality

$$||Ff||_{2,w}\lesssim ||f||_{2,s}\quad \text{or}\quad ||w^{1/2}Ff||_2\lesssim ||s^{1/2}f||_2$$

holds, and the kernel $\mathcal{K}$ satisfies

$$|\mathcal{K}(x,y)|\lesssim \min\left\{1,[w(x)s(y)]^{-1/2}\right\},\quad x,y>0.$$

In the main theorem we employ the notation

$$P_xg=\int_0^x g(y)dy,\ \text{and}\ \ Q_xg=\int_x^\infty g(y)dy,$$

for g a non-negative function and $x>0$. We will also write f^* the non-increasing rearrangement of $f\in L^p(\mathbb{R}^n)$ as in ([3], p. 189).

Theorem 2.1. ***(A)*** *Let $1<p\leq q<\infty$, $1<a\leq 2$, $(p,q,a)\neq(2,2,2)$, u and v locally integrable functions such that $u^*,v_*^{1-p'}\in L^1_{loc}$,*

$$\left(P_{1/r}u^*\right)^{\frac{1}{q}}\left(P_rv_*^{1-p'}\right)^{\frac{1}{p'}}\lesssim 1,\quad \text{and}\quad \left[Q_{1/r}\left(x^{-\frac{q}{a'}}u^*\right)\right]^{\frac{1}{q}}\left[Q_r\left(y^{-\frac{p'}{a'}}v_*^{1-p'}\right)\right]^{\frac{1}{p'}}\lesssim 1,\quad r>0.$$

Then, the Pitt's inequality

$$||w^{\frac{1}{a'}}Ff||_{q,u}\lesssim ||s^{\frac{1}{a}}f||_{p,v},$$

holds for the wights s and w from the definition of the F-transform. Moreover, if $p=q=a=2$, u and v are locally integrable functions such that $u^*,v_*^{-1}\in L^1_{loc}$, and

$$\left(P_{1/r}u^*\right)\left(P_rv_*^{-1}\right)\lesssim 1,\quad r>0,$$

then Pitt's inequality

$$||w^{\frac{1}{2}}Ff||_{2,u}\lesssim ||s^{\frac{1}{2}}f||_{2,v},$$

holds, for the wights s and w from the definition of the F-transform.

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INTRODUCTION TO THE THEORY OF DISTRIBUTIONS

DANIEL G. SIMÃO[1,†] & PAULO A. LIBONI FILHO[1,‡]

Universidade Estadual de Londrina, UEL, PR, Brasil[1]

danielsimao8@gmail.com[†] liboni@uel.br[‡]

Abstract

Our main goal in this work is to present the concept and main properties of the Theory of Distribution, which is fundamental in the study of solutions for PDEs. Observe that, in classical calculus, the equations $\dfrac{\partial^2 u}{\partial x \partial y} = 0$ and $\dfrac{\partial^2 u}{\partial y \partial x} = 0$ do not have the same solution: the first equation is satisfied by $u(x, y) = |x|$ while $\dfrac{\partial u}{\partial x}$ is not defined for $x = 0$. In distributions, we complement the function space with new objects, so that the derivatives are always possible.

1 Introduction

The discussion of solutions for partial differential equations is facilitated by the Theory of Distributions, as it generalizes classical notions of Mathematical Analysis, in order to derive a larger class of functions. In this text, the space of distributions and their essential operations are defined. There is the remarkable fact that, if a function is differentiable in the classical sense, then the derivative in both senses coincides. To illustrate the use of the presented techniques, fundamental solutions are calculated for some very important partial differential equations: the Cauchy-Riemann operator in $\mathbb{C}$ and the Laplacian $\triangle$ in $\mathbb{R}^{\mathbb{N}}$.

2 Main Results

Definition 2.1. *For an open subset Ω of $\mathbb{R}^{\mathbb{N}}$, the space of distribuitions $\mathcal{D}'(\Omega)$ is defined to be the dual space of $\mathcal{D}(\Omega)$. That is, $\mathcal{D}'(\Omega)$ is the space of all continuous linear maps T from $\mathcal{D}(\Omega)$ to $\mathbb{C}$. Analogously, $\mathcal{E}'(\Omega)$ is defined to be the dual space of $\mathcal{E}(\Omega)$. The pairing between an element T of $\mathcal{D}'(\Omega)$ and an element ϕ of $\mathcal{D}(\Omega)$ will be denoted by $(T, \phi)_\Omega \in \mathbb{C}$.*

Example 2.1. *Let T be a locally integrable function on Ω. Then T can be viewed as an element of $\mathcal{D}'(\Omega)$ by defining*

$$(T, \phi)\Omega = \int x \in \Omega T(x)\phi(x)dx \quad \text{for} \quad \phi \in \mathcal{D}(\Omega).$$

Example 2.2. *For a point $p \in \mathbb{R}^{\mathbb{N}}$, the delta function at p, δ_p defines an $\mathcal{E}'(\mathbb{R}^{\mathbb{N}})$-distribution defined by*

$$(\delta_p, f)_{\mathbb{R}^{\mathbb{N}}} = f(p) \quad , \quad f \in \mathcal{E}(\mathbb{R}^{\mathbb{N}}).$$

Definition 2.2. *If T is a smooth function on Ω and $\phi \in \mathcal{D}(\Omega)$, then*

$$\begin{aligned} (D^\alpha T, \phi)\Omega = \int x \in \Omega (D^\alpha T)\phi dx &= (-1)^{|\alpha|} \int_{x \in \Omega} T D^\alpha \phi dx \\ &= (-1)^{|\alpha|} (T, D^\alpha \phi)_\Omega. \end{aligned}$$

Example 2.3. *For the distribution δ_p then*

$$(D^\alpha \delta_p, \phi)_{\mathbb{R}^{\mathbb{N}}} = (-1)^{|\alpha|} (D^\alpha \phi)(p) \quad \text{for} \quad \phi \in \mathcal{E}(\mathbb{R}^{\mathbb{N}}).$$

Definition 2.3. *If T belongs to $\mathcal{E}(\mathbb{R}^{\mathbb{N}})$ and ψ belongs to $\mathcal{D}(\mathbb{R}^{\mathbb{N}})$, then the convolution $T * \psi$ is the smooth function defined by*

$$(T * \psi)(x) = \int_{y \in \mathbb{R}^{\mathbb{N}}} T(y)\psi(x-y)dy.$$

As a distribution on $\mathbb{R}^{\mathbb{N}}$, we have

$$\begin{aligned}(T * \psi, \phi)\mathbb{R}^{\mathbb{N}} &= \int x \in \mathbb{R}^{\mathbb{N}} \left(\int_{y \in \mathbb{R}^{\mathbb{N}}} T(y)\psi(x-y)dy \right) \phi(x)dx, \quad \phi \in \mathcal{D}(\mathbb{R}^{\mathbb{N}}) \\ &= \int_{y \in \mathbb{R}^{\mathbb{N}}} \left(\int_{x \in \mathbb{R}^{\mathbb{N}}} \psi(x-y)\phi(x)dx \right) dy.\end{aligned}$$

Taking $\widehat{\psi}(t) = \psi(-t)$, then we have

$$\begin{aligned}(T * \psi, \phi)\mathbb{R}^{\mathbb{N}} &= \int y \in \mathbb{R}^{\mathbb{N}} T(y)(\widehat{\psi} * \phi(y) \\ &= (T, \widehat{\psi} * \phi)_{\mathbb{R}^{\mathbb{N}}}.\end{aligned}$$

Lemma 2.1. *For $T \in \mathcal{D}'(\mathbb{R}^{\mathbb{N}})$, then $\delta_0 * T = T$.*

Definition 2.4. *$T \in \mathcal{D}'(\mathbb{R}^{\mathbb{N}})$ is a fundamental solution for $P(D)$ if*

$$P(D)\{T\} = \delta_0.$$

Theorem 2.1. *Let $P(D)$ is a partial differential operator with constant coefficients. Supose T is a fundamental solution for $P(D)$. If $\phi \in \mathcal{D}(\mathbb{R}^{\mathbb{N}})$ then $u = T * \phi$ is a solution to the differential equation $P(D)\{u\} = \phi$.*

Theorem 2.2. *The distribution $T(z) = \dfrac{1}{\pi z}$ is a fundamental solution for the Cauchy-Riemann operator $\dfrac{\partial}{\partial \overline{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$ on $\mathbb{C}$.*

Theorem 2.3. *Let*

$$T(x) = \begin{cases} \dfrac{1}{2\pi} \log |x| & \text{if } N = 2 \\ \frac{1}{(2-N)\omega_{N-1}} |x|^{2-N} & \text{if } N \geq 3. \end{cases}$$

where $\omega_{N-1} = \dfrac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}$ is the volume of unit sphere in $\mathbb{R}^{\mathbb{N}}$. Then T is a fundamental solution for the Laplacian $\Delta = \sum_{j=1}^{N} \left(\dfrac{\partial^2}{\partial x_j^2} \right)$ on $\mathbb{R}^{\mathbb{N}}$.

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BASIC SEQUENCES CRITERIA ON TOPOLOGICAL VECTOR SPACES

EVANDIO DEMÉTRIO JÚNIOR[1,†] & NACIB GURGEL ALBUQUERQUE[1,‡]

Departamento de Matemática, UFPB, PB, Brasil[1]

evandiodemetriojunior@gmail.com[†] nacib.albuquerque@academico.ufpb.br[‡]

Abstract

This work presents a brief overview of basic sequences in topological vector spaces. While this theory in normed spaces is well established, studying these sequences in more general spaces is rare in the literature. Our main goal is to present the Banach-Grunblum-Nikolskii criteria for basic sequences on general environments that are not necessarily convex, such as F-spaces.

1 Introduction

Definition 1.1. *Let X be a topological vector space. A sequence $(x_n)_{n=1}^{\infty}$ is said to be a basis in X if for every $x \in X$ there exists a unique sequence of scalars $(a_n)_{n=1}^{\infty}$ such that*

$$x = \sum_{n=1}^{\infty} a_n x_n.$$

When the sequence $(x_n)_{n=1}^{\infty}$ forms a basis for the space $\overline{span\{x_n : n \in \mathbb{N}\}}$, we simply say that $(x_n)_{n=1}^{\infty}$ is a basic sequence.

Our main goal is to investigate when a sequence is basic. In particular, in the context of normed spaces the Banach-Grunblum-Nikolskii criterion is widely known and plays a fundamental role in this analysis. We will now state this criterion, which detailed proof can be found in [1].

Theorem 1.1. *A sequence $(x_n)_{n=1}^{\infty}$ of non-zero vectors in a Banach space X is basic if and only if, there exist a constant $K \geq 1$ such that for every sequence of scalars $(a_n)_{n=1}^{\infty}$,*

$$\left\| \sum_{i=1}^{m} a_i x_i \right\| \leq K \left\| \sum_{i=1}^{n} a_i x_i \right\|, \qquad \text{whenever } n \geq m.$$

Topological vector spaces with a local base formed by convex sets are usual in the literature. We will study environments in which the local convex property is not mandatory. For this, the spaces are equipped with a structure similar to the norm. Firstly, let us recall some basic facts.

Definition 1.2. *A topological vector space X is said to be locally convex when the origin admits a (topologial) basis of convex neighborhoods.*

The following result brings us a characterization of these spaces.

Theorem 1.2. *A topological vector space X is locally convex if and only if its topology is generated by a family of seminorms.*

Definition 1.3. *A barrel in locally convex space X is a subset convex, balanced, absorvent and closed. A locally convex space X is barreled if each barrel in X is a neighborhoods of the origin.*

In the non-locally convex settings, a F-norm plays the role of a norm, providing the topological properties of the space.

Definition 1.4. *Let X be a vector space. A F-norm on X is a map $\|\cdot\| : X \to \mathbb{R}$ that fulfills the following properties:*

1. $\|x\| \geq 0$ *for every* $x \in X$ *and* $\|x\| = 0$ *if and only if,* $x = 0$*;*
2. $\|\alpha x\| \leq \|x\|$ *for every* $\alpha \in \mathbb{K}$ *with* $|\alpha| \leq 1$ *e* $x \in X$*;*
3. $\lim_{n\to\infty} \left\|\frac{x}{n}\right\| = 0$ *for every* $x \in X$*;*
4. $\|x+y\| \leq \|x\| + \|y\|$ *for every* $x, y \in X$*.*

Each F-norm induces a metric on X given by $d(x,y) = \|x-y\|$. When (X,d) is complete, we say that $(X, \|\cdot\|)$ is an F-space.

2 Main Results

J.R. Retheford and C.W. McArthur presented in [2] the first characterization of basic sequences on locally convex spaces.

Theorem 2.1. *Let (X,τ) is barreled space and $(x_n)_{n=1}^{\infty}$ be a sequence in X such that $x_n \neq 0$ for all $n \in \mathbb{N}$ and $X = \overline{span\{x_n : n \in \mathbb{N}\}}$. Let Γ be the collection of all continuous seminorms that generate the topology τ. Then $(x_n)_{n=1}^{\infty}$ is a base in X if, and only if, for every $\rho \in \Gamma$ there are $\sigma \in \Gamma$ and $K = K(\rho) > 0$ such that*

$$\rho\left(\sum_{i=1}^{n} a_i x_i\right) \leq K\sigma\left(\sum_{i=1}^{m} a_i x_i\right),$$

for any $m \geq n \in \mathbb{N}$ and all scalars $a_1, a_2, \cdots, a_m$.

In the context of F-spaces, P.K. Kamthan and M. Gupta proved an analogous version of Theorema 1.1, which we present next.

Theorem 2.2. *(Banach-Grunblum-Nikolskii Criteria). A sequence $(x_n)_{n=1}^{\infty}$ of non-zero vectors of a F-space $(X, \|\cdot\|)$ is basic if and only if there is a constant $M \geq 1$ such that for every sequence of scalars $(a_n)_{n=1}^{\infty}$,*

$$\left\|\sum_{i=1}^{m} a_i x_i\right\| \leq M \left\|\sum_{i=1}^{n} a_i x_i\right\| \text{ whenever } n \geq m.$$

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UNIVERSALITY LIMIT FOR STAHL-TOTIK REGULAR MEASURES

HERMES ALVES NETO[1,†]

Instituto de Ciências Matemáticas e de Computação, USP, SP, Brasil[1].

hermesalvesneto@usp.br[†].

Abstract

The universality limit, proved by Doron Lubinsky in 2009, was previously known only for a few cases of measures, namely weighted Legendre measures. In this work we present the technique employed by Lubinsky in order to prove the universality limit for the Stahl-Totik regular measures. The proof is based on approximations for the reproducing kernels of Stahl-Totik regular measures in terms of the reproducing Kernels of weighted Legendre measures.

1 Introduction

The universality limit explored in this work raises in the theory of random matrix with the model given by the Gaussian Unitary Ensemble (GUE). The GUE is an important tool in random matrix theory and closely related to the orthogonal polynomials theory (see [3, 5]). In this model, reproducing kernels of orthogonal polynomials can be seen as a stathistical model for the so-called determinantal point process and, the probability distribution can be described in terms of the reproducing kernel associated to Hermite polynomials.

Let μ be a finite Borel measure on the interval $[-1,1]$. There exists a sequence of polynomials $\{p_n\}$ on $[-1,1]$, each one with leading coefficient $\gamma_n > 0$ and, satisfying the following orthonormality relation

$$\int_{-1}^{1} p_n p_m d\mu = \delta_{mn},$$

where δ_{mn} represents the Kronecker delta. A positive finite Borel measure μ is *Stahl-Totik regular* (see [2]) if $\gamma_n^{1/n} \to 2$, as $n \to \infty$. The regularity described here can be obtained by the following criteria. If the Radon-Nikodym derivative of a measure supported in $[-1,1]$ with respect to the Lebesgue measure is positive almost everywhere on its support, then this measure is Stahl-Totik regular.

We consider a Stahl-Totik regular measure μ and ω its Radon-Nikodym derivative with respect to Lebesgue measure. The sequence of the normalized reproducing kernel associated to the measure μ is defined as follows

$$\tilde{K}_n(x,y) = \omega(x)^{1/2}\omega(y)^{1/2}\sum_{k=0}^{n-1} p_k(x)p_k(y), \quad x,y \in [-1,1], \quad n \in \mathbb{N}.$$

The Gaussian Unitary Ensemble (see [4]) is defined by the probability distribution

$$p_n(M)dM = c \cdot \mathrm{e}^{-\sum_{i=1}^n x_i^2} \prod_{1 \le i < j \le n}^{n} (x_i - x_j)^2 dx_1 \cdots dx_n,$$

where M is a Hermitian matrix of order $n \in \mathbb{N}$ and c is a normalizing constant such that $\int p_n(M)dM = 1$. For $k \in \{1,\ldots,n\}$, the k-th correlation function (see [4]) for the GUE is given in terms of the normalized reproducing kernels associated to the weighted measure ce^{-x^2}, with $R_{n,k}(x_1,\ldots,x_k) = \det[\tilde{K}_n(x_i,x_j)]_{i,j=1}^k$, for $x_1,\ldots,x_k \in \mathbb{R}$. This equality means that the GUE is a *Determinantal Point Process*. In Random Matrix theory, the limit

$$\lim_{n\to\infty} R_{n,k}\left(x + \frac{x_1}{\tilde{K}_n(x,x)},\ldots,x + \frac{x_k}{\tilde{K}_n(x,x)}\right)/\tilde{K}_n(x,x) = \det\left[\frac{\sin[\pi(x_i - x_j)]}{\pi(x_i - x_j)}\right]_{i,j=1}^k,$$

with $x, x_1, \ldots, x_k$ in compact subsets of $\mathbb{R}$, is called universality limit and, from the case $k = 2$, it is possible to recover it to any $k \in \{1, \ldots, n\}$, $n \in \mathbb{N}$. Our goal is to show that for x in compact subsets of $[-1, 1]$ and a, b in compact subsets of $\mathbb{R}$, then the limit above exists for Stahl-Totik regular measures.

2 Main Results

We present the following universality limit

$$\lim_{n\to\infty} \tilde{K}_n\left(x + \frac{a}{\tilde{K}_n(x,x)}, x + \frac{b}{\tilde{K}_n(x,x)}\right)/\tilde{K}_n(x,x) = \frac{\sin[\pi(a-b)]}{\pi(a-b)}, \tag{1}$$

for Stahl-Totik regular measures, with x in compact subsets of $[-1, 1]$ and a, b in compact subsets of $\mathbb{R}$. The proof is based on application of the *Lubisnky's inequality*, that provides an upper bound for the difference of two reproducing kernels associated with finite Borel measures on $[-1, 1]$ in terms of their Christoffel Functions.

For $n \in \mathbb{N}$, the *n-th Christoffel function* associated to finite Borel measure μ defined on $[-1, 1]$ is given by $\lambda_n(x) = 1/K_n(x, x)$.

Lemma 2.1 (Lubinsky's Inequality, [1]). *Let μ and μ^* be measures on $[-1, 1]$ such that $\mu \leq \mu^*$ in $[-1, 1]$. Then we have that,*

$$\frac{|K_n(x,y) - K_n^*(x,y)|}{K_n(x,x)} \leq \left(\frac{\lambda_n(x)}{\lambda_n(y)}\right)^{\frac{1}{2}} \left[1 - \frac{\lambda_n(x)}{\lambda_n^*(x)}\right]^{\frac{1}{2}}, \quad \forall x, y \in [-1, 1], \quad n \in \mathbb{N}.$$

Theorem 2.1 ([1]). *Let μ be a Stahl-Totik regular measure. If $J \subset (-1, 1)$ is a compact subset where μ is absolutely continuous in an open set containing J and its Radon-Nykodim deivative ω is positive and continuous in J, then universality limit (1) holds.*

The Christoffel Functions have the asymptotics well known, then the proof follows by arguments of localization and smoothing, i.e., by approximations of the Stahl-Totik regular measure in a neighborhood of a point by the Lebesgue measure.

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DISTRIBUTIONAL CHAOS FOR LINEAR OPERATORS ON FRÉCHET SPACES

JOÃO VICTOR ARAÚJO PINTO[1,†] & VINÍCIUS V. FÁVARO[2,‡]

Faculdade de Matemática, UFU, MG, Brasil[1]

joao.pinto@ufu.br[†] vvfavaro@ufu.br[‡]

Abstract

In this work we will study distributional chaos for linear operators on Fréchet spaces. We will present a characterization of distributional chaos using a distributional chaos criterion and also in terms of the existence of distributionally irregular vectors. This work is based on results of [1].

1 Introduction

From now on X denotes a Fréchet space and $(\|\cdot\|_n)_{n\in\mathbb{N}}$ denotes a sequence of seminorms in X that defines the metric of X. For any set A, we denote by $card A$ the cardinality of A.

Given a continuous map $f : X \to X$, for each pair $x, y \in X$ and each $n \in \mathbb{N}$ the *distributional function* $F^n_{xy} : \mathbb{R}^+ \to [0,1]$ is defined by

$$F^n_{xy}(\varepsilon) = \tfrac{1}{n} card \left\{0 \leq i \leq n-1 : d(f^i(x), f^i(y)) < \varepsilon\right\}.$$

We also define $F^*_{xy}(\varepsilon) := \limsup_n F^n_{xy}(\varepsilon)$ and $F_{xy}(\varepsilon) := \liminf_n F^n_{xy}(\varepsilon)$.

Given $A \subset \mathbb{N}$, its *upper* and *lower densities* are, respectively, defined by

$$\overline{dens}(A) = \limsup_n \tfrac{card(A\cap[0,n])}{n} \quad \text{and} \quad \underline{dens}(A) = \liminf_n \tfrac{card(A\cap[0,n])}{n}.$$

Definition 1.1. *A continuous map $f : X \to X$ is said to be distributionally chaotic if there exist an uncountable set $\Gamma \subset X$ and $\varepsilon > 0$ such that for everey $\tau > 0$ and each pair of distinct points $x, y \in \Gamma$, we have that $F_{xy}(\varepsilon) = 0$ and $F^*_{xy}(\tau) = 1$. In this case, the set Γ is called a distributionally ε-scrambled set and the pair (x, y) is called a distributionally chaotic pair.*

Definition 1.2. *Let $T : X \to X$ and $x \in X$. We say that x is a distributionally irregular vector for T if there are $m \in \mathbb{N}$, $A, B \subset \mathbb{N}$ with $\overline{dens}(A) = \overline{dens}(B) = 1$ such that*

$$\lim_{n\in A} T^n x = 0 \quad \textit{and} \quad \lim_{n\in B} \|T^n x\|_m > 0.$$

Definition 1.3. *Let $T : X \to X$ be a continuous linear operator and $x \in X$. The orbit of x (that is $\{x, T(x), T^2(x), \ldots\}$) is said distributionally near to 0 if there exists $A \subset \mathbb{N}$ with $\overline{dens}(A) = 1$ such that $\lim_{n\in A} T^n x = 0$. We say that x has a distributionally unbounded orbit if there exists $B \subset \mathbb{N}$ with $\overline{dens}(B) = 1$ and $m \in \mathbb{N}$ such that $\lim_{n\in B} \|Tx\|_m = \infty$.*

Definition 1.4. *Let $T : X \to X$ be a continuous linear operator. We say that T satisfies the Distributional Chaos Criterion (DCC) if there exists sequences $(x_k), (y_k) \in X$ such that*
a) *There exist $A \subset \mathbb{N}$ with $\overline{dens}(A) = 1$ such that $\lim_{n\in A} T^n x_k = 0$ for all k.*
b) *$y_k \in \overline{span\{x_n : n \in \mathbb{N}\}}$, $\lim_k y_k = 0$ and there exist $\varepsilon > 0$ and an increasing sequence (N_k) in $\mathbb{N}$ such that*

$$card\left\{1 \leq j \leq N_k : d(T^j x_k, 0) > \varepsilon\right\} \geq N_k(1 - k^{-1})$$

for all k.

2 Main Result

The main result of this work is a characterization of distributional chaos and it was proved in [1, Theorem 12]. First we enunciate a lemma which is a simple consequence of the definition of distributionally chaotic pair.

Lemma 2.1. *Let $T : X \to X$ be a continuous linear operator, and $x, y \in X$ with $x \neq y$. Then the pair (x, y) is distributionally chaotic if only if, there exists $\varepsilon > 0$ such that, for all $\tau > 0$,*

$$\underline{dens}\,\{n \in \mathbb{N} : d(T^n x, T^n y) < \varepsilon\} = 0 \quad and \quad \overline{dens}\,\{n \in \mathbb{N} : d(T^n x, T^n y) < \tau\} = 1.$$

Theorem 2.1. *If $T : X \to X$ is a continuous linear operator, then the following assertions are equivalent:*
i) *T satisfies DCC;*
ii) *T has a distributionally irregular vector;*
iii) *T is distributionally chaotic;*
iv) *T admits a distributionally chaotic pair.*

Sketch of the proof. (i) $\Rightarrow$ (ii): Consider the set $X_0 := \{x \in X : \lim_{n\in A} T^n x = 0\}$. Obviously X_0 is a subspace of X, $T(X_0) \subset X_0$ and $T(\overline{X_0}) \subset \overline{X_0}$. By hypothesis $x_k \in X_0$ and $y_k \in \overline{X_0}$ for all k. Using [1, Propositions 8 and 9] we get that the set of all vectors $x \in \overline{X_0}$ with distributionally unbounded orbit and with orbits distributionally near to 0 is residual in $\overline{X_0}$. Thus the set of all distributionally irregular vectors is residual in $\overline{X_0}$ and so there exists a distributionally irregular vector.
(ii) $\Rightarrow$ (iii): Let $u \in X$ be a distributionally irregular vector. It is not difficult to prove that the set $\{\lambda u : \lambda \in \mathbb{K}\}$ is an uncountable distributionally ε-scrambled set for a certain $\varepsilon > 0$.
(iii) $\Rightarrow$ (iv): Trivial.
(iv) $\Rightarrow$ (i): Consider a distributionally chaotic pair $(x, y) \in X \times X$ and put $u := x - y$. By Lemma 2.1 there exists $\varepsilon > 0$ such that

$$\overline{dens}\left\{j \in \mathbb{N} : d(T^j u, 0) > \varepsilon\right\} = 1 \quad \text{and} \quad \overline{dens}\left\{j \in \mathbb{N} : d(T^j u, 0) < \delta\right\} = 1, \tag{1}$$

for each $\delta > 0$. Hence there exists an increasing sequence (n_k) in $\mathbb{N}$ such that

$$card\left\{1 \leq j \leq n_k : d(T^j u, 0) < k^{-1}\right\} \geq n_k(1 - k^{-1}). \tag{2}$$

Consider $A_k := \left\{1 \leq j \leq n_k : d(T^j u, 0) < k^{-1}\right\}$ and $A := \bigcup_{k=1}^{\infty} A_k$. By (2), $\overline{dens}(A) = 1$ and $\lim_{n\in A} T^n u = 0$. Defining $x_k := T^k u$, we have $\lim_{n\in A} T^n x_k = 0$, for all $k \in \mathbb{N}$. Now choose s_k such that $\|T^{s_k} u\|_k < k^{-1}$ and put $y_k := T^{s_k} u$. Then $y_k \to 0$ and by (1), we have

$$\overline{dens}\left\{j \in \mathbb{N} : d(T^j y_k, 0) > \varepsilon\right\} = 1,$$

for all k. Therefore there exists an increasing sequence (N_k) in $\mathbb{N}$ such that

$$card\left\{j \in \mathbb{N} : d(T^j y_k, 0) > \varepsilon\right\} \geq N_k(1 - k^{-1}),$$

for all k. Thus T satisfies DCC. $\square$

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WEAKLY NULL SEQUENCES IN $L_1[0,1]$

JUAN DAVID RUBIO TABARES[1,†] & GERALDO BOTELHO[2,‡]

Faculdade de Matemática, UFU, MG, Brasil[1]

juan.tabares@ufu.br[†] botelho@ufu.br[‡]

Abstract

In this work, using only measure theoretic arguments, we give sufficient conditions for a sequence in $L_1[0,1]$ to be weakly null. We obtain the fact that the Rademacher sequence is weakly null in $L_1[0,1]$ as a particular case. Additional examples are provided.

1 Introduction

Let $r_n\colon [0,1] \longrightarrow \mathbb{R}$ be the n-th Rademacher function, that is,

$$r_n(t) = sgn(\sin(2^n \pi t)) \text{ for every } t \in [0,1].$$

The fact that the Rademacher sequence $(r_n)_{n=1}^{\infty}$ is weakly null in $L_1[0,1]$ is extremely useful in several areas of mathematical analysis and in probability theory. For example, it is used to show that $L_1[0,1]$ fails the Schur property, that is, it contains weakly null non-norm null sequences. This fact is crucial in the theory of Banach lattices, because it shows that there are Banach lattices with the positive Schur property (meaning that positive weakly null sequences are norm null) which fail the Schur property (see, e.g., [2]).

On the one hand, it is obvious that $(r_n)_{n=1}^{\infty}$ is non-norm null in $L_1[0,1]$, but the fact that it is weakly null is far from obvious. The usual proofs of this fact use powerful tools from Banach space theory, including the Khintchine inequality (see [3]).

In the main result of this work we give a purely measure theoretic proof of the fact that the Rademacher sequence is weakly null in $L_1[0,1]$. Actually, we give sufficient conditions for a sequence to be weakly null in $L_1[0,1]$, and we recover the case of the Rademacher sequence as a particular case. An additional example of a sequence that can be proved to be weakly null in $L_1[0,1]$ with the help of our main result is given.

2 Main Results

All integrals are taken in the sense of Lebesgue with respect to the Lebesgue measure m on $[0,1]$.

Definition 2.1. *Let $f_n\colon [0,1] \longrightarrow \mathbb{R}, n \in \mathbb{N}$, be a sequence of Lebesgue integrable functions. We say that the sequence $(f_n)_{n=1}^{\infty}$ has null integral of order even if, for every $k \in \mathbb{N}$,*

$$\int_{\frac{L}{2^k}}^{\frac{H}{2^k}} f_n(t)dt = 0$$

for every $n \geq k$ and all $L, H \in \left\{0, 1, \ldots, 2^k\right\}$ such that $H \leq L$ and $L - H$ is even.

Theorem 2.1. *Every bounded sequence in $L_\infty[0,1]$ having null integral of order even is weakly null in $L_1[0,1]$.*

Proof. We just give a sketch of the proof. Let $(f_n)_{n=1}^{\infty}$ be a bounded sequence in $L_\infty[0,1]$, we say $|f_n(t)| < M$ for some $M > 0$ and for every $n \in \mathbb{N}$ and almost every $t \in [0,1]$. Given a functional $\varphi \in L_1[0,1]^*$, by the Riesz Representation Theorem [1, Teorema 4.1.2], there exists $g \in L_\infty[0,1]$ such that

$$\varphi(f) = \int_0^1 g(t)f(t)dt \ \text{ for every } f \in L_1[0,1].$$

Using that the simple functions are dense in $L_1[0,1]$ (see [4]), for every $\varepsilon > 0$ there is a simple function ψ such that, for every $n \in \mathbb{N}$,

$$\left|\int_0^1 g(t)f_n(t)dt\right| = \left|\int_0^1 g(t)f_n(t) - \psi(t)f_n(t) + \psi(t)f_n(t)dt\right| \le \frac{\varepsilon}{2} + \left|\int_0^1 \psi(t)f_n(t)dt\right|.$$

Writing the canonical representation of ψ as $\psi = \sum\limits_{k=1}^{m} a_k\chi_{E_k}$, and using that the measure m if finite we get

$$\left|\int_0^1 \psi(t)f_n(t)dt\right| \le \sum_{k=1}^{m}\left|\int_0^1 a_k\chi_{E_k}(t)f_n(t)dt\right| = \frac{\varepsilon}{4} + \sum_{k=1}^{m}\left|\sum_{h=1}^{m_k} a_k\int_0^1 \chi_{J_h^k}(t)f_n(t)dt\right|,$$

where the sets J_n^k are such that $m\left(E_k\Delta\left(\bigcup\limits_{h=1}^{m_k} J_h^k\right)\right) < \dfrac{\varepsilon}{M2^{k+2}|a_k|}$ (by Δ we mean the symmetric difference). Finally, using that the sequence $(f_n)_{n=1}^{\infty}$ has null integral of order even, it follows that

$$\left|\int_0^1 g(t)f_n(t)dt\right| < \varepsilon \text{ whenever } n \ge N, \ \text{hence} \ \lim_{n\mapsto\infty}\left|\int_0^1 g(t)f_n(t)dt\right| = 0.$$

This proves that $f_n \xrightarrow{\omega} 0$ in $L_1[0,1]$. ∎

Example 2.1. *It is clear that the Rademacher sequence $(r_n)_{n=1}^{\infty}$ fulfills the conditions of Theorem 2.1, so it is weakly null in $L_1[0,1]$.*

Many other sequences can be proved to be weakly null in $L_1[0,1]$ by Theorem 2.1. We just give one more concrete example.

Example 2.2. *Let $a \in \mathbb{R}$ be given. For every $n \in \mathbb{N}$, define $f_n\colon [0,1] \longrightarrow \mathbb{R}$ by*

$$f_n(x) = \begin{cases} -2^{n+1}ax + a\,(2k-1) & \text{if} \quad \frac{k-1}{2^n} \le x < \frac{k}{2^n} \text{ and } k \in \{1,\ldots,2^n-1\} \\ \\ -2^{n+1}ax + a\,(2k-1) & \text{if} \quad \frac{k-1}{2^n} \le x \le \frac{k}{2^n} \text{ and } k = 2^n. \end{cases}$$

The sequence $(f_n)_{n=1}^{\infty}$ satisfies the conditions of Theorem 2.1, so it is weakly null in $L_1[0,1]$.

Remark 2.1. *As far as we know, this method of getting weakly null sequences in $L_1[0,1]$ has not appeared in the literature before.*

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ERROR ANALYSIS OF THE SOLUTIONS OF A NONLINEARPARABOLIC PROBLEM WITH TERMS CONCENTRATING IN AN OSCILLATORY NEIGHBORHOOD OF THE BOUNDARY

DANIEL MORALES[1,†] & GLEICIANE S. ARAGÃO[2,‡]

Universidade Federal de São Paulo, ICAQF-UNIFESP, Diadema, SP, Brazil[1],
Universidade de São Paulo, IME-USP, SP, Brazil[2]

gleiciane.aragao@unifesp.br[†] morales.daniel@ime.usp.br[‡]

Abstract

In this work, we analyze the numerical behavior of the solutions of a nonlinear parabolic problem with homogeneous Neumann boundary conditions, when some nonlinear reaction term is concentrated in a neighborhood of the boundary of a domain in $\mathbb{R}^2$ and this neighborhood shrinks to the boundary as the parameter ϵ goes to zero.

Our main objective is to analyze how the numerical solution of a parabolic problem with nonlinear Neumann boundary conditions is aproximated by the family of numerical solutions of the concentrated problem using the finite element method and we evaluate the error made in this approximation as ϵ goes to zero. Numerical results associated with the dynamics of these concentrated problems will be presented as a great novelty.

1 Introduction

Let $\Omega = \left\{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 < 1\right\}$ and $g_\epsilon(s) = g\left(s, \frac{s}{\epsilon}\right) = 2 + \cos\left(\frac{s}{\epsilon}\right)$, for $0 < \epsilon \leq 1$ and $s \in [0, 2\pi]$, that is, the oscillatory function g_ϵ presents a purely periodic behavior. Consider the following parametrization

$$\omega_\epsilon = \left\{\xi \in \mathbb{R}^2 : \ \xi = ((1-t)\cos(s), (1-t)\sin(s)), \quad s \in [0, 2\pi] \quad \text{and} \quad t \in [0, \epsilon g_\epsilon(s))\right\}, \quad 0 < \epsilon \leq 1.$$

Based in [2] and using the concentration tecnique developed in [3], we want to analyze, as ϵ goes to zero, the numerical behavior of the following concentrated nonlinear parabolic problem with homogeneous Neumann boundary conditions

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t} - \Delta u_\epsilon + u_\epsilon = \dfrac{1}{\epsilon}\chi_{\omega_\epsilon} u_\epsilon (1 - u_\epsilon), & \text{in } (0, \infty) \times \Omega \\ \dfrac{\partial u_\epsilon}{\partial \vec{N}} = 0, & \text{on } (0, \infty) \times \partial\Omega \\ u_\epsilon(0) = \sin(x^2 + y^2), \end{cases} \tag{1}$$

where χ_{ω_ϵ} is the characteristic function of the set ω_ϵ, $\vec{N}$ is the unit outward normal vector to $\partial\Omega$ and we refer to the term $\frac{1}{\epsilon}\chi_{\omega_\epsilon} u_\epsilon(1 - u_\epsilon)$ as the nonlinear reaction concentrating on the region ω_ϵ.

More precisely, we analyze how the solutions of (1) approximate to the solution of the following parabolic problem with nonlinear Neumann boundary conditions

$$\begin{cases} \dfrac{\partial u_0}{\partial t} - \Delta u_0 + u_0 = 0, & \text{in } (0, \infty) \times \Omega \\ \dfrac{\partial u_0}{\partial \vec{N}} = \mu u_0 (1 - u_0), & \text{on } (0, \infty) \times \partial\Omega \\ u_0(0) = \sin(x^2 + y^2), \end{cases} \tag{2}$$

and we evaluate the error made in this approximation for sufficiently small ϵ. The boundary coefficient $\mu \in L^\infty(\partial\Omega)$ in (2) is given by

$$\mu = \mu(s) = \frac{1}{2\pi} \int_0^{2\pi} g(s, \tau)\, d\tau, \quad \forall s \in (0, 2\pi).$$

2 Main Results

Initially, in [1] we establish an abstract form for the problems (1) and (2) and we prove that they have a unique global solution on $H^1(\Omega)$.

Using an appropiate finite difference scheme, we can obtain the numerical solutions of the problems (1) and (2) and then taking small arbitrarely values for ϵ, we can observe that the solutions of the problem (1) converge in $H^1(\Omega)$ to the solution of the problem (2), as ϵ goes to zero, in a numerical sense.

When ϵ is sufficiently small, the error made in the approximation remains constant around 0.37. Taking a tolerance of 10^{-1} for the error, our stopping criterion is satisfied.

We can see this graphically by considering the differents heat maps generated for the numerical solution of the problem (1). For example, the Figure 1 illustrates the heat maps for ϵ equals to 0.1, 0.01, 0.001 and 0.0001, respectively.

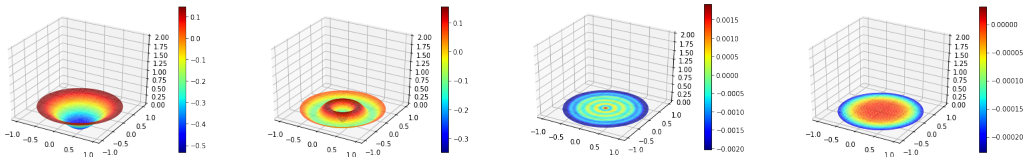

Figure 1: Evolution of the heat map for the numerical solution of the problem (1).

We compare this heat maps with the heat map for the numerical solution of the limit problem (2) given in the Figure 2.

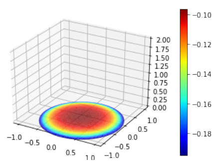

Figure 2: Heat map for the numerical solution of the limit problem (2).

As far as we know, these simulations are the first of their kind and constitutes a numerical verification of the results obtained in work [2], thanks to the work [3], where the concentration technique was developed.

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CORRELATION BETWEEN COVID-19 CASES AND TEMPERATURE IN SÃO PAULO CITY WITH WAVELET ANALYSIS

AMARI GOULART[1], LUCIANO A. MAGRINI.[2], MARTA C. GADOTTI[1] & MARIANA P. M. A. BARONI[1]

Sciences and Mathematics Department, Federal Institute of Education, Science and Technology (IFSP), Campus São Paulo, SP, Brazil[1],

Department of Mathematics, University of São Paulo State "Júlio de Mesquita Filho" (UNESP), Campus Rio Claro, SP, Brazil[2]

magrini@ifsp.edu.br[†] mc.gadotti@unesp.br[‡] mariana.baroni@ifsp.edu.br[§] amari.go@ifsp.edu.br[§§]

Abstract

In this work, we investigated the existence of a correlation between the average temperature and the daily cases of COVID-19 recorded for São Paulo city in the triennial 2020-2022 using wavelet analysis.

1 Introduction

Nonlinear and non-stationary phenomena present several difficulties in being characterized and for which the wavelet analysis has been successfully used in your investigation process [4]. Using the cross wavelet transform and wavelet coherence we investigated the possible correlations between air temperature and COVID-19 cases in São Paulo city considering the 2020-2022 years.

2 Cross Wavelet Transform and Wavelet Coherence

Let be $f(t)$ e $g(t)$ signals and $\psi(t)$ a wavelet function. The cross wavelet transform of $f(t)$ e $g(t)$ is defined as:

$$\mathcal{W}_{\psi}^{f,g}(a,\tau) = \int_{\mathbb{R}} f(t)\psi^{*}\left(\frac{t-\tau}{a}\right)\, dt \int_{\mathbb{R}} g^{*}\left(\frac{t-\tau}{a}\right)\, dt \tag{1}$$

where * indicate complex conjugation, $a > 0$ and $\tau \in \mathbb{R}$ varies continuously. Regardless of the magnitude of $\mathcal{W}_{\psi}^{f,g}(a,\tau)$ the Equation (1) shows the phase difference between $f(t)$ and $g(t)$ for $t = \tau$, as in [1]. The wavelet coherence (WC) is given by

$$\mathcal{R}^2(a,\tau) = \frac{|a^{-1}\mathcal{W}_{\psi}^{f,g}(a,\tau)|}{|a^{-1}\mathcal{W}_{\psi}^{f}(a,\tau)||a^{-1}\mathcal{W}_{\psi}^{g}(a,\tau)|}, \tag{2}$$

whose values vary in the interval $[0,1]$. Values biggest 0.5 indicates that the $f(t)$ e g are correlated [1, 3]. A summary of the theory and tips for the application of these tools is given in [5].

3 Data

We analyzed the air temperature and COVID-19 cases in São Paulo city between 2020/02/29 and 2022/12/31. For details about the official resources used see the file "readme.txt" in https://github.com/magriniluciano1983/enama2023. A graphical representation of these data can be seen in panels A and B of Figure (1).

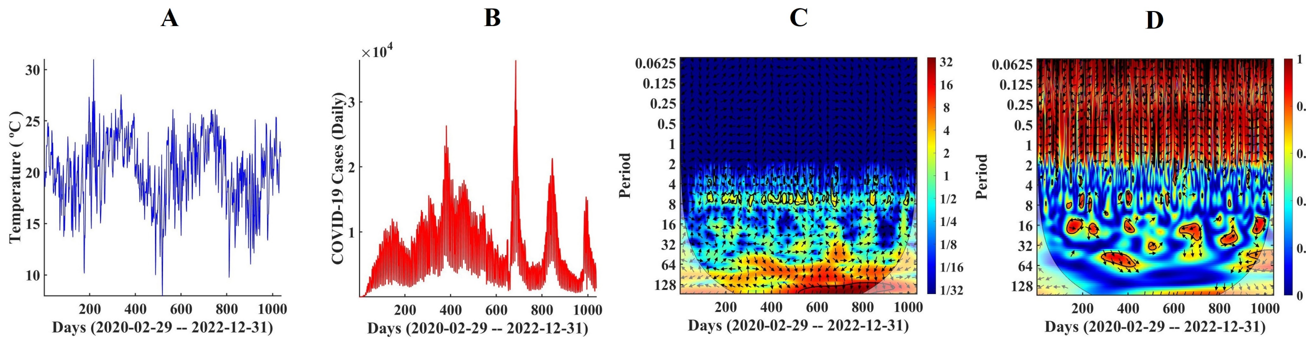

Figure 1: Panels A, B, C, and D: Air Temperature, COVID-19 Cases, CWT, and WC, respectively.

4 Numerical Analysis and Results

The CWT e a WC to data is represented in panels C and D of Figure (1). The CWT analysis shows that for the longest periods (and thus for low-frequency content) the COVID-19 cases are strongly correlated with daily temperature. In particular, for the period equal to 8 this correlation is present for all time analyzed. The analysis shows that the correlation is present for the smallest periods (and thus for high-frequency content). We can conclude, via wavelet analysis that the average temperature and daily cases of COVID-19 are correlated with whose investigation is in progress.

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A DISCRETIZATION OF ψ-RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE.

M. AURORA P. PULIDO[1,†], J. VANTERLER C. SOUSA[2,‡] & E. CAPELAS DE OLIVEIRA[3,§]

IMECC, UNICAMP, SP, Brasil, Research grant 40784/2020-9, CNPq[1],
DEMATI, UEMA, MA, Brasil[2], IMECC, UNICAMP, SP, Brasil[3]

m211681@dac.unicamp.br[†] vanterler@ime.unicamp.br[‡] capelas@unicamp.br[§]

Abstract

In this work, we present one discretization of ψ-Riemann-Liouville fractional derivative and show a one-dimensional application in order to assess the discretization.
Keywords. BE ψ-Riemann-Liouville Approximation, ψ-Riemann-Liouville fractional derivative, Backward Euler.

1 Introduction

Fractional calculus appeared at about the same time as the classical calculus of Newton and Leibniz, however, at that time it had little attraction for researchers. It was only in 1974, after the first conference dedicated to fractional calculus, that it began to gain prominence in the scientific community. In recent years, fractional calculus has been consolidated and highlighted as a tool to solve and describe complex phenomena. Several definitions of fractional derivatives have been introduced over the years, the best known being those of Riemann-Liouville and Caputo. On the other hand, the numerical solution of fractional differential equations (FDEs) is a topic of great interest to many researchers in the area, thus motivated by the ideas of the work [3], where the approximation $L1-2$ ψ-Caputo is discussed and, from the work [1], where the Riemann-Liouville integral is approximated, we develop an approximation for the Riemann-Liouville fractional derivative of one function with respect to another. In this work, a BE ψ-Riemann-Liouville approximation is presented.

2 Definitions and important considerations

In this section, two definitions are displayed: ψ-Riemann-Liouville fractional derivative and ψ-Riemann-Liouville approximation.

Definition 2.1. [3] *Let $\alpha > 0$, $n \in \mathbb{N}$, $I = [a,b]$ be the interval $(-\infty \leq a < t < b \leq \infty)$, $\psi(t)$ an increasing function and $\psi'(t) \neq 0$, for all $t \in I$. The Riemann-Liouville fractional derivative of a function u with respect to ψ of order α is given by*

$$\mathbb{D}_a^{\alpha;\psi} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_a^t \psi'(\xi)(\psi(t) - \psi(\xi))^{n-\alpha-1} u(\xi) d\xi, \tag{1}$$

where $n = [\alpha] + 1$, with $[\alpha]$ being the integer part of α.

On the other hand, let's consider a uniform mesh with points t_0, t_1, $\cdots$, t_N of an interval $[a,b]$, where Δt will be the amplitude of the step, $t_k = k\Delta t$ for $0 < k \leq N$ and u a continuous function on $[a,b]$.

The approximation presented below, arises as a result of using the backward Euler method to approximate the derivative that appears in (1) with $0 < \alpha < 1$, as discussed in [3] and defining the function $u(t)$, as being $u(t) = u(t_j)$, for all $t \in [t_j, t_{j+1})$, following the idea presented by [1] in the Riemann-Liouville integral approximation.

Definition 2.2. *Let $0 < \alpha < 1$, $I = [a, b]$, $u \in C(I, \mathbb{R})$ and $\psi(t)$ be an increasing function such that $\psi'(t) \neq 0$, for all $t \in I$. Given a uniform mesh of points $t_0, t_1, \cdots, t_N$ of the interval I, such that $t_0 = a$, $t_k = k\Delta t$ for $0 < k \leq N$, where Δt is the step width. The* BE *ψ-Riemann-Liouville approximation to u at the point t_k of order α, is given by*

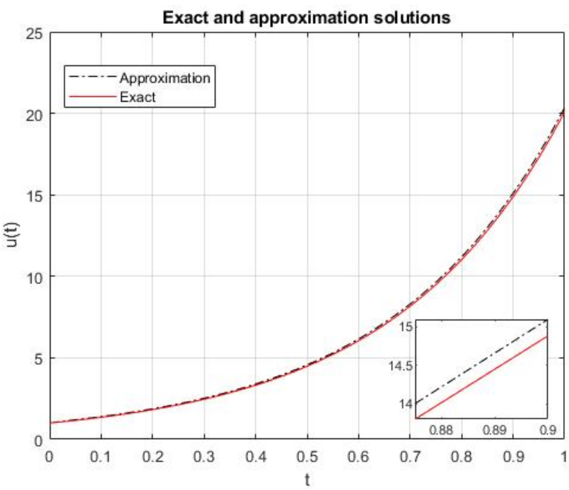
$$ {}^{RB}\mathbf{D}_{t_0}^{\alpha;\psi} u(t)\Big|_{t=t_k} = \frac{1}{\Gamma(2-\alpha)} \frac{1}{\psi'(t_k)\Delta t} \left\{ \sum_{j=1}^{k-1} u(t_j) \left[a_{k,j}^{\alpha;\psi} - a_{k-1,j}^{\alpha;\psi} \right] + u(t_k) a_{k,k}^{\alpha;\psi} \right\}, $$

where $a_{k,j}^{\alpha;\psi} = (\psi(t_k) - \psi(t_{j-1}))^{1-\alpha} - (\psi(t_k) - \psi(t_j))^{1-\alpha}$ and $\Gamma(\cdot)$ is a gamma function.

3 Application

We want to find $u(t)$ using the BE ψ-Riemann-Liouville approximation and knowing that

$$ \begin{cases} {}^{RL}\mathbb{D}_0^{\alpha;t^\rho} u(t) = \displaystyle\sum_{s=0}^{\infty} \frac{3^s}{s!} \frac{\Gamma((s+\rho)/\rho)}{\Gamma((s+\rho-\alpha\rho)/\rho)} t^{s-\alpha\rho}, \quad t \in (0,1], \alpha \in (0,1) \\ u(0) = 1. \end{cases} $$



$\rho = 0.2$	$\alpha = 0.7$	
Δt	Error $\lVert\cdot\rVert_\infty$	Rate
0.10000	1.37709e+00	1.19806
0.05000	6.01217e-01	1.19567
0.02500	2.63564e-01	1.18973
0.01250	1.16481e-01	1.17806
0.00625	5.19651e-02	1.16448
0.00313	2.31716e-02	1.16519
0.00156	9.88182e-03	1.22951
0.00078	3.38096e-03	1.54735

Figure 1: Numerical results comparing the exact and the approximate solution with $\alpha = 0.7$ and $\rho = 0.2$.

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MODELING CANCER CELL MIGRATION: TEMPORAL EVOLUTION OF NON-CONSERVED FIELD VARIABLES

RÉGIS S. DE QUADROS[1], JOÃO G. P. BANDEIRA.[2], GUSTAVO B. KURZ[1] & DANIELA BUSKE[1]

Postgraduate Program in Mathematical Modeling, UFPel, RS, Brazil[1]
Control and Automation Engineering, UFPel, RS, Brazil[2]

quadros99@gmail.com[†] joaogabrielpbandeira@gmail.com[‡] gustavobk@gmail.com[§] danielabuske@gmail.com[§§]

Abstract

The main objective of this work is to investigate the temporal evolution of non-conserved field variables through the application of the Allen-Cahn equation. The equation serves as the foundation for several phase-field models utilized in cell migration studies, particularly in the context of tumor cells and cancer metastasis. The model describes cells as 2D soft bodies, considering mechanical and biological factors to simulate cell movement.

1 Introduction

Mathematical models have been widely employed to study cell migration, especially in the context of tumor cells, as they play a significant role in cancer development and metastasis [2, 3]. These models incorporate various biological mechanisms and mechanical considerations to simulate cell movement. By employing efficient algorithms, researchers have been able to explore cell movement and interaction, contributing to a better understanding of cellular behavior in diverse environments [1, 2, 3].

The main objective of this work is to observe the temporal evolution of non-conserved field variables using the Allen-Cahn equation, which serves as the foundation for many phase-field models in the literature. The focus is on understanding cell migration, a crucial process in various biological phenomena, including cancer metastasis, which constitutes the key purpose of this study.

2 Main Results

In the model, each cell is described by an order parameter φ, which individually characterizes them in the phase field model, with a value of 1 inside the cell and 0 outside. They are treated as a 2D soft body, whose equilibrium shape minimizes the following free energy:

$$F_0 = \sum_n \left[\gamma_n \int_V \left[(\nabla \varphi_n)^2 + \frac{30}{\lambda^2} \varphi_n^2 (1 - \varphi_n)^2 \right] dV + \frac{\mu_n}{\pi R^2} \left(\pi R^2 - \int_V (\varphi_n^2)\, dV \right)^2 \right], \tag{1}$$

where R is the cell radius, λ represents the width of the cell boundary, μ_n is considered a parameter that determines the energetic cost associated with changes in the cell's area while keeping its volume approximately constant, and γ_n is a parameter that controls the elasticity of the cells, as described in [1].

The free energy F_0 represents the cells individually, while the total energy, taking into account the interactions between them, is given by $F = F_o + F_{int}$, with F_{int} being defined, in [1], as

$$F_{\text{int}} = \frac{30\kappa}{\lambda^2} \int_V \left(\sum_{n, m \neq n} \varphi_n^2 \varphi_m^2 \right) dV, \tag{2}$$

where κ is the coefficient of gradient energy. Furthermore, the temporal evolution of each cell is described as

$$\frac{\partial \varphi_n}{\partial t} + v_n \cdot \nabla \varphi_n = -\frac{1}{2}\frac{\delta F}{\delta \varphi_n}, \tag{3}$$

with the term v_n being defined as the time-dependent velocity of the cell, divided into two parts: $v_{n,I} + v_{n,A}$, as said in [1]. The term $v_{n,A}$ represents the active part of the velocity, i.e., the self-propulsion of the cell. In the model, it is considered to have a constant magnitude. On the other hand, the velocity $v_{n,I}$ is determined by the forces arising from the interaction with other cells and is defined as

$$v_{n,I} = \frac{60\kappa}{\xi\lambda^2}\int_V \left(\varphi_n\left(\nabla\varphi_n\right)\sum_{m\neq n}\varphi_m^2\right), \tag{4}$$

where ξ represents the friction between the cells and the liquid environment around, as seem in [1]. Finally, by solving the functional derivative in equation (3), we obtain the equation that describes the temporal evolution of the cells,

$$\frac{\partial \varphi_n}{\partial t} = \gamma_n \nabla^2 \varphi_n - \frac{30}{\lambda^2}\left[\gamma_n\varphi_n(1-\varphi_n)\left(1-2\varphi_n\right) + 2\kappa\sum_{m\neq n}\varphi_n\varphi_m^2\right] - \frac{2\mu}{\pi R^2}\varphi_n\left[\int_V \left(\varphi_n^2\right)dV - \pi R^2\right] - v_n \cdot \nabla\varphi_n. \tag{5}$$

It is shown in Figure 1 the time evolution of the multicellular system containing normal and five soft (cancer) cells, in three different times.

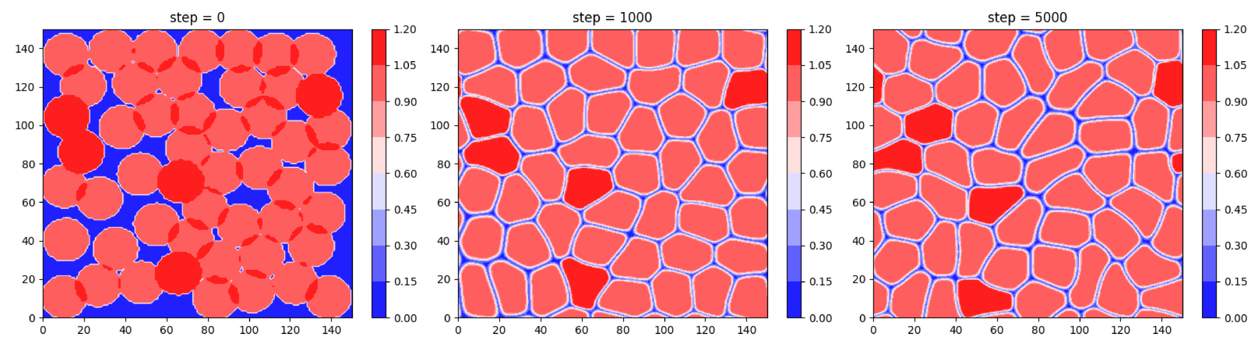

Figure 1: Temporal evolution of cell migration in three different times

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RESOLUTION OF THE LINEAR SYSTEM GENERATED BY APPLYING THE 3D-GILTT METHOD IN THE TRANSIENT THREE-DIMENSIONAL ADVECTION-DIFFUSION EQUATION

VILIAM C. DA SILVEIRA[1,†], GUSTAVO B. KURZ[1,‡], DANIELA BUSKE[1,§],
RÉGIS S. DE QUADROS.[1,§§] & GUILHERME J. WEYMAR[1,§§§]

Postgraduate Program in Mathematical Modeling, UFPel, RS, Brazil[1]

viliamcardoso2@gmail.com[†] gustavobrk@gmail.com[‡] danielabuske@gmail.com[§] quadros99@gmail.com[§§]
guilhermejahnecke@gmail.com[§§§]

Abstract

The aim of this work is to evaluate the computational time to simulate the pollutants dispersion considering different methodologies to solve the linear system that is obtained by applying the 3D-GILTT method in the three-dimensional advection-diffusion equation. To validate the model, unstable tank experiment data were considered. The results show that the Gauss-Seidel method has the shortest computational time to simulate the pollutant dispersion. As expected, similar results are obtained to the pollutant concentration, regardless of the methodology used to solve the linear system.

1 Introduction

The aim of this work is to evaluate the computational time to simulate the pollutants dispersion, considering different methods to solve the linear system that is generated by the application of the 3D-GILTT method in the transient three-dimensional advection-diffusion equation [1] given by

$$\frac{\partial \overline{c}}{\partial t} + \overline{u}\frac{\partial \overline{c}}{\partial x} + \overline{v}\frac{\partial \overline{c}}{\partial y} + \overline{w}\frac{\partial \overline{c}}{\partial z} = -\frac{\partial \overline{u'c'}}{\partial x} - \frac{\partial \overline{v'c'}}{\partial y} - \frac{\partial \overline{w'c'}}{\partial z} \tag{1}$$

where $\overline{c}$ is the average concentration of a passive contaminant and $\overline{u}$, $\overline{v}$ and $\overline{w}$ are the cartesian components of the mean wind (m/s) and $\overline{u'c'}$, $\overline{v'c'}$ e $\overline{w'c'}$ represent, respectively, the contaminant turbulent flow (g/sm^2) in the longitudinal, lateral and vertical directions. Therefore, after applying the 3D-GILTT method, the following equation can be written in matrix notation $Y''(x) + F.Y'(x) + G.Y(x) = 0$, where $Y(x)$ is the column vector whose components are $\{\overline{c_{n,i}}(x,r)\}$. The F and G matrixes are given by $F = B^{-1}.R$ and $G = B^{-1}.S$, respectively [2]. The GILTT technique combines series expansion with integration. In the expansion, is used a trigonometric base determined with the help of an auxiliary Sturm-Liouville problem. The ordinary differential equations resulting system is analytically solved using the Laplace transform and diagonalization.

Applying an order reduction in the equation $Z'(x) + H.Z(x) = 0$, we can write $Z(x) = X \,.\begin{bmatrix} e^{-d_1 x} & 0 & \dots & 0 \\ 0 & e^{-d_2 x} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{-d_n x}\end{bmatrix} . X^{-1} . Z(0)$, where H is the block matrix $H = \begin{bmatrix} 0 & -I \\ G & F \end{bmatrix}$, with X being the eigenvector matrix and d_n the eigenvalues and $Z(0)$ is the initial condition. Defining, $\xi = X^{-1} . Z(0)$, the following linear system must be solved $X\,\xi = Z(0)$. In order to verify the best methodology to be used, were tested different techniques to solving linear systems (LU decomposition method, iterative Jacobi method, Gauss-Seidel method and successive over-relaxation) [3] to reduce errors and computational time.

In this work, the advection-diffusion equation is considered in its most complete form, considering a three-dimensional model and the non-local closure term of the turbulence. To carry out the simulations, the ubuntu linux operating system was used on a notebook with core i5. The model was written in python language. The turbulent eddy diffusivities were parameterized as

$$K_\alpha = \frac{0,583 w_* z_i c_i \psi^{2/3} (z/z_i)^{4/3} X^* [0,55 (z/z_i)^{2/3} + 1,03 c_i^{1/2} \psi^{1/3} (f_m^*)_i^{2/3} X^*]}{[0,55 (z/z_i)^{2/3} (f_m^*)_i^{1/3} + 2,06 c_i^{1/2} \psi^{1/3} (f_m^*)_i X^*]^2} \tag{2}$$

where α refers to the directions x, y and z [2]. The classic tank experiment data [4] were used to simulate the pollutants dispersion. The parameters used are: Monin-Obukhov length $L = -10\,m$; convective velocity scale $w_* = 2\,m/s$; source intensity $Q = 10\,m/s$; source height $H_s = 300\,m$; boundary layer height $z = 1150\,m$; wind speed $V = 2.6\,m/s$; dimensionless distance $X^* = 0.5$ and the observed dimensionless pollutant concentration is 4.90.

2 Main Results

Table 1: Computational time to simulate the pollutants dispersion

Method	**Time**	**Concentration at ground level**
LU decomposition	15m44.006s	6.210295391310229
Jacobi	16m1.388s	6.210295391310232
Gauss-Seidel	15m41.430s	6.210295391310232
Successive over-relaxation	16m20.570s	6.210299612494771

As we can see from the table 1, the shortest computational time to simulate the pollutants dispersion is obtained with the iterative Gauss-Seidel method. Successive over-relaxation is the method that presents the highest computational time. As expected, the Jacobi method has a higher computational time when compared with the Gauss-Seidel method. The LU decomposition and Gauss-Seidel method presents similar simulation time. Regardless of the methodology used to solve the linear system, the pollutants concentration presents similar results and the user can decide which method to use.

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EXISTENCE OF POSITIVE SOLUTION FOR A SECOND-ORDER NONLINEAR PROBLEM WITH MIXED CONDITIONS

ADRIANO PEIXOTO[1,†]

Instituto de Matemática e Estatística, Universidade de São Paulo, USP, Brasil[1].

alcp@ime.usp.br[†]

Abstract

In this work, we present the existence of a positive solution to a second-order nonlinear problem with mixed boundary conditions. The proofs of the main results are based on the Mawhin's coincidence degree.

1 Introduction

Assuming that $f : [0,T] \times [0,+\infty[\times \mathbb{R} \to \mathbb{R}$ is a L^p-Carathéodory function, in this work, we prove the existence of a positive solution to the second-order nonlinear problem with boundary conditions

$$\begin{cases} u'' + f(t,u,u') = 0, \ 0 < t < T \\ \mathcal{B}(u) = (0,0), \end{cases} \tag{1}$$

where the linear operator $\mathcal{B} : C^1([0,T],\mathbb{R}) \to \mathbb{R}^2$ represents the boundary conditions, which can be

$$\mathcal{B}(u) \in= \{(u'(T) - u(0), u'(0) - u(0)), (u(T) - u(0), u'(0)), (u'(T) - u'(0), u(0))\} \tag{2}$$

A *solution* to problem (1) is a function $u : [0,T] \to \mathbb{R}$, of class C^1 such that $u'(t)$ is absolutely continuous and $u(t)$ satisfies (1) for almost every $t \in [0,T]$. We are interested in positive solutions of (1), i.e., solutions u such that $u(t) > 0$ for all $t \in [0,T]$. However, when the problem is studied with the boundary condition $\mathcal{B}(u) = (u'(T) - u'(0), u(0))$, we already know in advance that u vanishes at $t = 0$. In this case, the sought-after solution will be such that $u(t) > 0$ for all $t \in]0,T]$. In the problema (1), $f : [0,T] \times [0,+\infty[\times \mathbb{R} \to \mathbb{R}$ is an L^p-Carathéodory function, for some $1 \le p \le \infty$, satisfying the following conditions:

(f_1) $f(t,0,\xi) = 0$, for almost every $t \in [0,T]$ and for every $\xi \in \mathbb{R}$;

(f_2) there exists a nonnegative function $k \in L^1[0,T]$ and a constant $\rho > 0$ such that $|f(t,s,\xi)| \le k(t)(|s| + |\xi|)$, for almost every $t \in [0,T]$, for every $0 \le s \le \rho$, and $|\xi| \le \rho$;

(f_3) In addition to the above assumptions, we will also suppose that $f(t,s,\xi)$ satisfies a kind of Bernstein-Nagumo condition in order to have $|u'(t)|$ bounded whenever $u(t)$ is bounded.

For each $\eta > 0$, there exists a continuous function

$$\phi = \phi_\eta : [0,+\infty[\to [0,+\infty[, \text{ with } \int^\infty \frac{\xi^{\frac{p-1}{p}}}{\phi(\xi)} d\xi = \infty,$$

and a function $\psi = \psi_\eta \in L^p([0,T],[0,+\infty[)$ such that

$$|f(t,s,\xi)| \le \psi(t)\phi(|\xi|), \text{ for almost every } t \in [0,T], \forall\, s \in [0,\eta], \forall\, \xi \in \mathbb{R}.$$

For technical reasons, when dealing with Nagumo functions $\phi(\xi)$ as above, we always assume that

$$\liminf_{\xi \to +\infty} \phi(\xi) > 0.$$

This avoids the possibility of pathological examples as can be seen in [1, p. 46-47] and does not affect our application.

2 Main Results

Theorem 2.1. *Assume (f_1), (f_2), and (f_3), and suppose that there exist two constants $r, R > 0$, with $r \neq R$, such that the following hypotheses are true:*

(H_r) *The condition*

$$\int_0^T f(t, a + at, a)dt < 0, \text{ para } a = \frac{r}{1+T}$$

is satisfied. Moreover, any solution $u(t)$ of the problem

$$\begin{cases} u'' + \vartheta f(t, u, u') = 0, \quad 0 < t < T \\ \mathcal{B}(u) = (0, 0), \end{cases} \tag{1}$$

for $0 < \vartheta \leq 1$, such that $u(t) > 0$ in $[0, T]$, satisfies $|u|_\infty \neq r$.

(H_R) *There exists a non-negative function $v \in L^p([0, T], \mathbb{R})$ with $v \not\equiv 0$ and a constant $\alpha_0 > 0$, such that every solution $u(t) \geq 0$ of the problem*

$$\begin{cases} u'' + f(t, u, u') + \alpha v(t) = 0, \quad 0 < t < T \\ \mathcal{B}(u) = (0, 0), \end{cases} \tag{2}$$

for $\alpha \in [0, \alpha_0]$, satisfies $|u|_\infty \neq R$. Moreover, there are no solutions $u(t)$ of (2) for $\alpha = \alpha_0$ with $0 \leq u(t) \leq R$, for every $t \in [0, T]$.

Then the problem (1) has at least one positive solution $u(t)$ with

$$\min\{r, R\} < \max_{t \in [0,T]} u(t) < \max\{r, R\}.$$

The proof is given by a topological approach based on the Mawhin's coincidence degree introduced in [2, 3]. Furthermore, to ensure that the found solution is positive, we employ a maximum principle.

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PERSISTENCY OF ITÔ-HENSTOCK SDES WITH LÉVY NOISE ON HILBERT SPACES

ANTONIO VELOSO[1], RODOLFO COLLEGARI[2] & MARCIA FEDERSON[1]

Instituto de Ciências Matemáticas e de Computação, USP, SP, Brasil[1], Faculdade de Matemática, UFU, MG, Brasil[2]

antoniomavs@usp.br† collegari@gmail.com‡ federson@icmc.usp.br§

Abstract

We establish necessary conditions to ensure persistence of Itô-Henstock SDEs with Lévy noise on Hilbert spaces.

1 Introduction

It is our intention to present a new result concerning persistence of Ito-Henstock stochastic differential equations with Lévy noise on Hilbert spaces. Consider V a Hilbert space and $(\Omega, \mathcal{F}, \{F_t\}_{t\in I}, \mathbb{P})$ a filtering probability space, for $I \subset \mathbb{R}^+$ an unbounded set. We are going to investigate the following Ito Hentosck SDE with Lévy Noise

$$dX_t = f(X_t, t)dt + \sigma(X_t, t)dW_t + \int_{\mathbb{Y}} \gamma(t, y)N_{dt}(dy), \;\; t \in I \subset \mathbb{R}^+, \tag{1}$$

where $\{X_t : t \in I\}$ is a $\{F_t\}$-adapted process, where $X_t \in \mathcal{L}^p(\Omega, V)$ for $t \in I$, $f \in \mathcal{G}(\mathcal{L}^p(\Omega, V) \times I, h, Id)$, $\sigma \in \mathcal{G}(\mathcal{L}^p(\Omega, V) \times I, h, W)$, $\{W_t : t \in I\}$ is a Q-Wiener process on V, $N_{dt}(dy)$ is a real-valued Poisson counting measure with characteristic measure ν on a measurable set $\mathbb{Y} \subset \mathbb{R}^+$ with $\nu(\mathbb{Y}) < \infty$ and $\gamma : I \times \mathbb{Y} \to (-1, \infty)$.

The study of SDEs with Lévy noise is of big relevance in the population dynamics field since it can model more properly natural disasters, such as hurricanes, earthquakes, epidemics, ocean red tide, and more (see [1], [3]). Then, to bring this type of noise to the Ito-Henstock SDEs is an important step to develop the theory of population model systems, that is because instead of asking for continuity and boundedness of f and σ and boundedness of the Lévy noise we simply ask for the Lipschitz condition on the expectation of each function and still guarantee global solution and condition to persistence. To be clear, the term persistence in population dynamics theory can be roughly summarized as "the population may not be extinct in the future". Our main goal is to prove that under a certain condition, the solution of (1) is persistent. Let us give the proper definitions and provide the necessary tools to state the result.

Definition 1.1 (Class of $\mathcal{G}$ [2])**.** *Let $h : I \subset \mathbb{R} \to \mathbb{R}$ be a nondecreasing function, $\{W_t : t \in I\}$ be a Q-Wiener process and $T : \mathcal{L}^p(\Omega, V) \times I \to \mathcal{L}^p(\Omega, V)$ be a functional. We say that $T \in \mathcal{G}(\mathcal{L}^p(\Omega, V)) \times I, h, W)$ if, for every $\{F_t\}$-adapted process $\{X_t : t \in I\}$ on a filtering probability space with $X_t \in L^p(\Omega, V)$, for $t \in I$, the Itô-Henstock integral $\int_I T(Z_s, s)dW_s$ exists and, for $[s_1, s_2] \subset I$ we have*

- $$\mathbb{E}\left[\left\| \int_{s_1}^{s_2} T(Z_s, s)dW_s \right\|_V^p\right] \leq |h(s_2) - h(s_1)|;$$
- *for $\{Y_t : t \in I\}$ is also $\{F_t\}$-adapted on the filtering probability space*
$$\mathbb{E}\left[\left\| \int_{s_1}^{s_2} [T(Z_s, s) - T(Y_s, s)]dW_s \right\|_V^p\right] \leq \|Z - Y\|_{\mathcal{L}^p}^p |h(s_2) - h(s_1)|.$$

Definition 1.2 (Persistence)**.** *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtering probability space and $\{X_t : t \in I \subset \mathbb{R}^+\}$, where I is unbounded, be a $\{F_t\}$-adapted process. We say that X_t is*

- *weakly persistent if*
$$\limsup_{t\to\infty}\{\|X_t\|_V\} \neq 0, \quad (a.s);$$
- *strongly persistent if*
$$\liminf_{t\to\infty}\{\|X_t\|_V\} \neq 0, \quad (a.s);$$
- *persistent if it is strongly and weakly persistent and the limits coincide.*

For simplicity, let X and Y be Hilbert spaces, and $f\colon \mathcal{L}^p(\Omega, Y) \to \mathcal{L}^p(\Omega, Y)$ be a Itô-Henstock integrable function with respect to $W = \{W_t\colon t \in I \subset \mathbb{R}\}$ a Q-Wiener process, we define the following notation

$$\langle f \rangle_y^W = \frac{1}{t}\int_y^t f(X_s)dW_s, \quad t \in I, \quad X_t \in \mathcal{L}^p(\Omega, Y).$$

2 Main Result

Theorem 2.1. *[4] Let $f, g \in \mathcal{G}(L^p(\Omega, V) \times [t_0, \infty)], h, Id)$, where $g\colon [t_0,\infty) \times \mathcal{L}^p(\Omega, V) \to \mathcal{L}^p(\Omega, V)$ is a function given by*

$$g(t, Z_t) = Z_t \cdot \int_{\mathbb{Y}} \gamma(t, y) N_{dt}(dy), \quad y \in \mathbb{Y},$$

also $\sigma \in \mathcal{G}(L^p(\Omega, V) \times [t_0, \infty)], h, W)$, $\{X_t\colon [t_0,\infty)\}$ be a global solution of the stochastic differential equation (1) with $X_{t_0} = \tilde{X}$ and $X_t \in \mathcal{L}^p(\Omega, V)$, $t \in [t_0, \infty)$. Then, if

$$\lim_{t\to\infty}(\langle f+g \rangle_{t_0}^{Id} + \langle \sigma \rangle_{t_0}^{W}) \neq 0 \ \ (a.s) \Longrightarrow X_t \ \text{ is persistent.}$$

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STABILITY AND EIGENVALUE BOUNDS FOR MICROPOLAR SHEAR FLOWS

PABLO BRAZ E SILVA[1,†] & JACKELLYNY D. DO N. CARVALHO[1,‡]

Universidade Federal de Pernambuco, PE, Brasil[1]

pablo.braz@ufpe.br[†] jackellyny.dassy@ufpe.br[‡]

Abstract

We prove eigenvalue bounds for two-dimensional linearized disturbances of parallel flows of micropolar fluids, deriving the Orr-Sommerfeld equations and providing a sufficient condition for linear stability of such flows. We also derive wave speed bounds.

1 Introduction

In this short note, we study two dimensional perturbations of parallel flows of micropolar, also known as asymmetric, fluids. To this end, we first derive the Orr-Sommerfeld equations for this kind of fluids and, through a variational method, prove some bounds for their eigenvalues. We obtain bounds for the imaginary part of the eigenvalue ensuring stabilty in some regions of the parameters. We also prove some wave speed bounds for perturbations of the base flow. Our results here are the generalization to micropolar fluids of the results by Joseph [1] for the classical Navier-Stokes case.

First we construct, by application of elementary isoperimetric inequalities following the same procedure done by Joseph [1], some estimates to the real and imaginary parts of the eigenvalue

$$C = C_r + iC_i$$

of the Orr-Sommerfeld problem for micropolar fluids

$$i\alpha[(U-C)(D^2-\alpha^2)-U'']\varphi = \Big(\frac{1}{R_\mu}+\frac{1}{2R_k}\Big)(D^2-\alpha^2)^2\varphi + \frac{R_0}{R_k}(D^2-\alpha^2)\omega; \tag{1}$$

$$i\alpha[(U-C)\omega - W'\varphi] = \Big[\frac{1}{R_\gamma}(D^2-\alpha^2) - \frac{2R_0}{R_\nu}\Big]\omega - \frac{1}{R_\nu}(D^2-\alpha^2)\varphi \tag{2}$$

where $D = \frac{d}{dy}$, with boundary conditions

$$\varphi(0) = \varphi(1) = \varphi'(0) = \varphi'(1) = \omega(0) = \omega(1) = 0. \tag{3}$$

2 Main Results

Theorem 2.1. *Let* $R_1 = \min\left\{\frac{1}{R_\mu}, \frac{1}{2R_k}, \frac{R_0}{R_\nu}\right\}$ *and* $R_2 = \max\left\{\frac{R_0}{2R_k}, \frac{1}{2R_\nu}\right\}$. *If* $R_1 > R_2$, *then*

$$C_i \leq \frac{q_1+q_2}{2\alpha} - \frac{\pi^2+\alpha^2}{\alpha R},$$

where $\frac{1}{R} = \min\left\{R_1 - R_2, \frac{1}{R_\mu}+\frac{1}{2R_k}, \frac{1}{R_\gamma}\right\}$. *Moreover, no amplified disturbances* ($C_i > 0$) *of* (1), (2) *and* (3) *exist if*

$$2\alpha R q_1 < f(\alpha) := max\{M_1, M_2\}; \qquad 2\alpha R q_2 < g(\alpha) := max\{N_1, N_2\}, \tag{1}$$

where

$$\begin{cases} M_1 = (4.73)^2\pi + 2^{\frac{3}{2}}\alpha^3, \\ M_2 = (4.73)^2\pi + 2\alpha^2\pi; \end{cases} \qquad \begin{cases} N_1 = 2(4.73)^2\pi + 2^{\frac{3}{2}}\alpha\pi, \\ N_2 = 2(4.73)^2\pi + 2\alpha^3. \end{cases} \tag{2}$$

Theorem 2.2. *Let $C(\alpha, R)$ be any eigenvalue of (1), (2) and (3). Then, the following inequalities hold:*

a)
$$U_{min} - \frac{W'_{max}}{2\alpha} < C_r < U_{max} + \frac{U''_{max}}{2(\pi^2+\alpha^2)} + \frac{W'_{max}}{2\alpha} \quad (U''_{min} \geq 0 \ \ and \ \ W'_{min} \geq 0)$$

b)
$$U_{min} - \frac{W'_{max}}{\alpha} + \frac{W'_{min}}{2\alpha} < C_r < U_{max} + \frac{U''_{max}}{2(\pi^2+\alpha^2)} - \frac{W'_{min}}{\alpha} + \frac{W'_{max}}{2\alpha} \quad (U''_{min} \geq 0 \ \ and \ \ W'_{min} \leq 0 \leq W'_{max})$$

c)
$$U_{min} + \frac{W'_{min}}{2\alpha} < C_r < U_{max} + \frac{U''_{max}}{2(\pi^2+\alpha^2)} - \frac{W'_{min}}{2\alpha} \quad (U''_{min} \geq 0 \ \ and \ \ W'_{max} \leq 0)$$

d)
$$U_{min} + \frac{U''_{min}}{2\alpha^2} - \frac{W'_{max}}{2\alpha} < C_r < U_{max} + \frac{U''_{max}}{2(\pi^2+\alpha^2)} + \frac{W'_{max}}{2\alpha} \quad (U''_{min} \leq 0 \leq U''_{max} \ \ and \ \ W'_{min} \geq 0)$$

e)
$$U_{min} + \frac{U''_{min}}{2\alpha^2} - \frac{W'_{max}}{\alpha} + \frac{W'_{min}}{2\alpha} < C_r < U_{max} + \frac{U''_{max}}{2(\pi^2+\alpha^2)} - \frac{W'_{min}}{\alpha} + \frac{W'_{max}}{2\alpha} \quad (U''_{min} \leq 0 \leq U''_{max} \ \ and \ \ W'_{min} \leq 0 \leq W'_{max})$$

f)
$$U_{min} + \frac{U''_{min}}{2\alpha^2} + \frac{W'_{min}}{2\alpha} < C_r < U_{max} + \frac{U''_{max}}{2\alpha^2} - \frac{W'_{min}}{2\alpha} \quad (U''_{min} \leq 0 \leq U''_{max} \ \ and \ \ W'_{max} \leq 0)$$

g)
$$U_{min} + \frac{U''_{min}}{2\alpha^2} - \frac{W'_{max}}{2\alpha} < C_r < U_{max} + \frac{W'_{max}}{2\alpha} \quad (U''_{max} \leq 0 \ \ and \ \ W'_{min} \geq 0)$$

h)
$$U_{min} + \frac{U''_{min}}{2\alpha^2} - \frac{W'_{max}}{\alpha} + \frac{W'_{min}}{2\alpha} < C_r < U_{max} - \frac{W'_{min}}{\alpha} + \frac{W'_{max}}{2\alpha} \quad (U''_{max} \leq 0 \ \ and \ \ W'_{min} \leq 0 \leq W'_{max})$$

i)
$$U_{min} + \frac{U''_{min}}{2\alpha^2} + \frac{W'_{min}}{2\alpha} < C_r < U_{max} - \frac{W'_{min}}{2\alpha} \quad (U''_{max} \leq 0 \ \ and \ \ W'_{max} \leq 0).$$

Here, $U_{max}, U_{min}, U''_{max}, U''_{min}, W'_{max}$ and W'_{min} are maximum and minimum values on the range of $U(y), U''(y)$ and $W'(y)$ for $y \in [0,1]$.

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PARTIAL DIFFERENTIAL EQUATIONS AND (CO)HOMOLGY THEORY

ARTUR JORGE MARINHO[1,†]

Instituto de Matemática e Estatística, UFG, GO, Brasil[1].

arturxjorge@discente.ufg.br[†]

Abstract

We study here one paper of Kanishka Perera, namely *An abstract critical point theorem with applications to elliptic problems with combined nonlinearities* [1]. It is proved in the paper an abstract critical point theorem based on cohomological index theory that produces pairs of nontrivial critical points with nontrivial higher critical groups. This theorem yields pairs of nontrivial solutions that are neither local minimizers nor of mountain pass type for problems with combined nonlinearities. Applications are given to subcritical and critical p-Laplacian problems, Kirchhoff type nonlocal problems, and critical fractional p-Laplacian problems.

1 Introduction

The purpose of this work is to prove an abstract critical point theorem that can be used to obtain pairs of nontrivial solutions of problems of the type

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + \mu f(x,u) + |u|^{q-2}u \quad \text{in} \quad \Omega \\ \qquad u = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, $p > 1$, $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian of u, $p < q \leq p^* = Np/(N-p)$ if $p < N$ and $p < q < \infty$ if $p \geq N$, $\lambda, \mu > 0$ are parameters, and f is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying

$$f(x,t) = |t|^{\sigma-2}t + o(|t|^{\sigma-1}) \quad \text{as} \quad t \to 0, \text{ uniformly a.e. in} \quad \Omega \tag{2}$$

for some $1 < \delta < p$, the sign condition

$$F(x,t) = \int_0^t f(x,s)ds > 0 \quad \text{for a.a.} \quad x \in \Omega \quad \text{and all} \quad t \in \mathbb{R}\setminus\{0\} \tag{3}$$

and the growth condition

$$|f(x,t)| \leq a\left(|a|^{r-1} + 1\right) \quad \text{for a.a.} \quad x \in \Omega \quad \text{and all} \quad t \in \mathbb{R} \tag{4}$$

Let us denote

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega |u|^p dx - \mu\int_\Omega F(x,u)dx - \frac{1}{q}\int_\Omega |u|^q dx, \quad u \in W_0^{1,p}(\Omega)$$

the energy functional associated to 1.
Now let us recall the definition of the $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [2].

Definition 1.1. *Let W be a Banach space and let $\mathcal{A}$ denote the class of symmetric subsets of $W\backslash\{0\}$. For $A \in \mathcal{A}$, let $\overline{A} = A/\mathbb{Z}_2$ be te quotient space of A with each u and $-u$ identified, let $f : \overline{A} \to \mathbb{R}P^\infty$ be the classifying map of $\overline{A}$, and let $f^* : H^*(\mathbb{R}P^\infty) \to H^*(\overline{A})$ be the induced homomorphism of the Alexsander-Spanier cohomology rings. The cohomological index of A is defined by*

$$i(A) = \begin{cases} 0 & \text{if} \quad A = \emptyset \\ \sup\left\{m \geq 1 : f^*(\omega^{m-1}) \neq 0\right\} & \text{if} \quad A \neq \emptyset \end{cases} \tag{5}$$

where $\omega \in H^1(\mathbb{R}P^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}P^\infty) = \mathbb{Z}_2[\omega]$.

2 Main Results

Theorem 2.1. *Let E be a C^1-functional on W and let $\mathcal{M}$ be a bounded symmetric subset of $W\backslash\{0\}$ radially homeommorphic to the unit sphere $S = \{u \in W : \|u\| = 1\}$ and A_0, B_0 closed subsets of $\mathcal{M}$ such that*

$$i(A_0) = i(\mathcal{M}\backslash B_0) = k < \infty$$

Assume that there exist $\omega_0 \in \mathcal{M}\backslash A_0$, $0 \leq r < \rho < R$ and $a < b$ such that, setting

$$\begin{aligned} A_1 &= \{\pi_{\mathcal{M}}((1-s)v + sw_0) : v \in A_0, 0 \leq s \leq 1\} \\ A^* &= \{tu : u \in A_1, r \leq t \leq R\} \\ B^* &= \{tw : w \in B_0, \leq t \leq \rho\} \\ A &= \{ru : u \in A_1\} \cup \{tv : v \in A_0, r \leq t \leq R\} \cup \{Ru : u \in A_1\} \\ B &= \{\rho w : w \in B_0\} \end{aligned}$$

where $\pi_{\mathcal{M}} : S \to \mathcal{M}$ is the radial homeomorphism of $\mathcal{M}$ with S, we have

$$a < \inf_{B^*} E, \quad \sup_A E < \inf_B E, \quad \sup_{A^*} E < b \tag{1}$$

If E satisfies the $(PS)_c$ condition for all $c \in (a, b)$, then E has a pair of critical points u_1, u_2 with

$$\inf_{B^*} E \leq E(u_1) \leq \sup_A E, \quad \inf_B E \leq E(u_2) \leq \sup_{A^*} E.$$

If, in addition, E has only a finite number of critical points with the corresponding critical values in (a, b), then u_1 and u_2 can be chosen to satisfy

$$C^k(E, u_1) \neq 0 \quad C^{k+1}(E, u_2) \neq 0$$

where $C^*(f, p)$ denotes the critical groups of f with respect to the critical point p.

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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR KIRCHHOFF PROBLEMS VIA NONLINEAR RAYLEIGH QUOTIENT IN $\mathbb{R}^N$

EDUARDO D. LIMA[1], EDCARLOS D. SILVA[1] & JOSÉ C. OLIVEIRA JUNIOR[2]

Instituto de Matemática e Estatística, UFG, GO, Brasil[1], Colegiado de Matemática, UFNT, TO, Brasil[2]

eduardo.lima@discente.ufg.br[†] edcarlos@ufg.br[‡] jc.oliveira@uft.edu.br[§]

Abstract

In this work we consider existence of solutions for the following nonlocal elliptic problem:

$$\begin{cases} -m\left(\|\nabla u\|_2^2\right)\Delta u + V(x)u = \lambda a(x)|u|^{q-2}u - \theta b(x)|u|^{p-2}u \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \qquad (\mathcal{P}_{\lambda,\theta})$$

where $N \geq 3$, $\lambda, \theta > 0$, $2 < 2(\sigma+1) < q < p < 2^* = 2N/(N-2)$, $\sigma \in (0, 2/(N-2))$ and $a, b \in L^\infty(\mathbb{R}^N)$ with $a(x), b(x) > 0$ almost everywhere in $\mathbb{R}^N$. This type of problem contains the function $m : \mathbb{R}^+ \to \mathbb{R}^+$ known as the Kirchhoff function given by $m(t) = \alpha_1 + \alpha_2 t^\sigma$ with $\alpha_1, \alpha_2 > 0$ and $t \in \mathbb{R}^+$. Under our assumptions the potential $V : \mathbb{R}^N \to \mathbb{R}$ and the nonlinearities can be sign changing functions.

1 Introduction

The main objective in the present work is to investigate existence and multiplicity of solutions to Kirchhoff elliptic problem in the whole space $\mathbb{R}^N$ given in $(\mathcal{P}_{\lambda,\theta})$. Recall that the conditions about the functions a and b implies that $\mathcal{L}_\lambda(x,t) = \lambda a(x)|t|^{q-2}t - \theta b(x)|t|^{p-2}t$ is a sing changing function. An immediate consequence is that the above problem has at least one ground state solution and at least one bound state solution whenever $\lambda \in (\lambda^*, +\infty)$ for some suitable $\lambda^* > 0$. The main idea is to use the minimization method in the Nehari manifold together with the nonlinear Rayleigh quotient. Throughout this work we assume the following assumptions:

(m_1) The function m is defined as $m(t) = \alpha_1 + \alpha_2 t^\sigma$ with $t \in \mathbb{R}^+$ and $\alpha_1, \alpha_2 > 0$;

(a_1) $2 < 2(\sigma+1) < q < p < 2^* = 2N/(N-2)$ where $0 < \sigma < 2/(N-2)$ and $N \geq 3$;

(a_2) The functions $a, b : \mathbb{R}^N \to \mathbb{R}$ satisfy $a, b \in L^\infty(\mathbb{R}^N)$ with $a(x), b(x) > 0$ almost everywhere in $\mathbb{R}^N$;

(v_1) $V \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ and there exists a constant $V_0 > 0$ such that $V(x) \geq -V_0$ for all $x \in \mathbb{R}^N$;

(v_2) It holds that

$$d := \inf_{u \in X, \|u\|_{L^2(\mathbb{R}^N)}=1} \int_{\mathbb{R}^N} \left[\alpha_1 |\nabla u|^2 + V(x)u^2\right] dx > 0;$$

(v_3) For each $M > 0$, it follows that $\left|\left\{x \in \mathbb{R}^N : V(x) \leq M\right\}\right| < +\infty$.

It is important to mention that the working space is defined by

$$X := \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < +\infty\right\},$$

endowed with the inner product and norm given by

$$\langle u, \varphi \rangle = \int_{\mathbb{R}^N} \left[\alpha_1 \nabla u \nabla \varphi + V(x)u\varphi\right] dx \text{ and } \|u\| = \left(\int_{\mathbb{R}^N} \left[\alpha_1 |\nabla u|^2 + V(x)u^2\right] dx\right)^{\frac{1}{2}} \text{ for all } \varphi \in X.$$

Now, define the energy functional $J : X \to \mathbb{R}$ associated to problem $(\mathcal{P}_{\lambda,\theta})$ by

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{\alpha_2}{2(\sigma+1)}\|\nabla u\|_2^{2(\sigma+1)} - \frac{\lambda}{q}\int_{\mathbb{R}^N} a(x)|u|^q dx + \frac{\theta}{p}\int_{\mathbb{R}^N} b(x)|u|^p dx \text{ for all } u \in X.$$

According to our assumptions, we have that a function $u \in X$ is a critical point for the functional J if, and only if, u is a weak solution for problem $(\mathcal{P}_{\lambda,\theta})$. Furthermore, we define a set called *Nehari manifold* given by $\mathcal{N} = \{u \in X\backslash\{0\} : J'(u)u = 0\}$. As a product, due to the works [4] and [5], we consider the nonlinear Rayleigh quotients $R_n, R_e : X\backslash\{0\} \to \mathbb{R}$ associated with the parameters $\lambda, \theta > 0$ in the following way:

$$R_n(u) = \frac{\|u\|^2 + \alpha_2\|\nabla u\|_2^{2(\sigma+1)} + \theta\|u\|_{p,b}^p}{\|u\|_{q,a}^q} \text{ and } R_e(u) = \frac{\frac{1}{2}\|u\|^2 + \frac{\alpha_2}{2(\sigma+1)}\|\nabla u\|_2^{2(\sigma+1)} + \frac{\theta}{p}\|u\|_{p,b}^p}{\frac{1}{q}\|u\|_{q,a}^q}, \ u \in X\backslash\{0\}.$$

To simplify the notation, we define the extremal values as follows:

$$\lambda^* = \inf_{u\in X\backslash\{0\}} \inf_{t>0} R_n(tu) \text{ and } \lambda_* = \inf_{u\in X\backslash\{0\}} \inf_{t>0} R_e(tu).$$

2 Main Results

Theorem 2.1. *Suppose (m_1), (a_1)-(a_2), (v_1)-(v_3) and $c_{\mathcal{N}^-} < c_{\mathcal{N}^0}$ where $c_{\mathcal{N}^-} := \inf\{J(w) : w \in \mathcal{N}^-\}$ and $c_{\mathcal{N}^0} := \inf\{J(w) : w \in \mathcal{N}^0\}$. Then for each $\lambda \in (\lambda^*, +\infty)$ and $\theta > 0$, there exists $\theta_2 > 0$ such way that problem $(\mathcal{P}_{\lambda,\theta})$ admits at least a bound state solution $u \in X\backslash\{0\}$ satisfying $u \in \mathcal{N}^-$ whenever $0 < \theta < \theta_2$. Furthermore, we obtain that $0 < \lambda^* < \lambda_* < +\infty$.*

Theorem 2.2. *Suppose (m_1), (a_1)-(a_2), (v_1)-(v_3) and $c_{\mathcal{N}^+} < c_{\mathcal{N}^0}$ where $c_{\mathcal{N}^+} := \inf\{J(w) : w \in \mathcal{N}^+\}$ and $c_{\mathcal{N}^0} := \inf\{J(w) : w \in \mathcal{N}^0\}$. Then for each $\lambda \in (\lambda^*, +\infty)$ and $\theta > 0$, problem $(\mathcal{P}_{\lambda,\theta})$ admits at least a ground state solution $v \in X\backslash\{0\}$ satisfying $v \in \mathcal{N}^+$.*

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MORSE THEORY AND APPLICATIONS TO A CLASS OF SEMILINEAR ELLIPTIC PROBLEMS

HUGO HENRYQUE C. E SILVA[1,†] & JOSÉ CARLOS DE ALBUQUERQUE M. JÚNIOR[1,‡]

Departamento de matemática, UFPE, PE, Brasil[1]

hugo.henryque@ufpe.br[†] josecarlos.melojunior@ufpe.br[‡]

Abstract

In this work we will list some results of the Morse theory for functionals $I \in \mathcal{C}^2$ defined on a Hilbert space H. In certain cases, such results, when combined with deformation theorems, allow us to describe critical groups of certain critical points and, therefore, the acquisition of critical point theorems, which guarantee under which conditions I admits one or more non-trivial critical points.

As an application, we will study the existence and multiplicity of solutions for the following class of problems

$$\begin{cases} -\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $f \in \mathcal{C}^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$. To do so, we will use variational calculus tools and Morse Theory applied to the functional I, associated with the above problem, defined in the Sobolev space $H_0^1(\Omega)$.

1 Introdução

In this work, we will use methods from Algebraic Topology and Analysis with the goal of applying the tools of Morse Theory introduced by Gromoll-Meyer [2] to ensure the existence and multiplicity of non-trivial critical points for a class of functions $I \in \mathcal{C}^2(H, \mathbb{R})$ defined on a Hilbert space H. As an application, we obtain the existence and multiplicity of solutions for a class of elliptic problems involving the Laplacian operator.

The main motivation for using the aforementioned tools lies in the fact that when analyzing at critical values, the topology of the level sets $I^a = \{x \in H \,| I(x) \leq a\}$ undergoes changes. Thus, in finite-dimensional spaces, it is possible to study the topology of these spaces without resorting to techniques from Algebraic Topology. However, when dealing with infinite-dimensional spaces, we lose the geometric and topological intuition about them. Hence, we turn to homology groups in neighborhoods of critical points, known as critical groups, to establish the structure of these abelian groups and their modifications in the presence of critical points, thereby ensuring the existence or non-existence of new critical points.

In this sense, following the work of Silva [3], we will present three abstract results that ensure the existence and multiplicity of critical points for a class of functionals satisfying the Cerami (Ce) condition and:

(I_0) The origin is an isolated critical point, $I(0) = 0$, $I'(0) = 0$ and there exists $i \in \{0, 1, \cdots\}$, such that

$$C_q(I, 0) \cong \begin{cases} \mathbb{Z}, & q = i, \\ 0, & q \neq i. \end{cases}$$

(I_1) There exists $d_1 < 0$ such that $I'(u)(u) < 0$ for every $u \in I^{d_1}$.

(I_2) The set $S^- = \{u \in \partial B(0,1) \mid I(tu) \to -\infty \text{ as } t \to +\infty\}$ is a non-empty subset of $\partial B(0,1)$ homotopy equivalent to a point.

(I_3) There exists $d_2 < 0$ such that $I(tu) \geq d_2$ for all $t \geq 0$ and $u \in \partial B(0,1) \setminus S^-$.

(I_4) There exist $u_0 \in H$ and $\alpha, \rho > 0$ such that

$$I(u) \geq I(u_0) + \alpha, \text{ for every } u \in \partial B(0,\rho)$$

(I_5) There exists $e \notin B(u_0,\rho)$ such that $I(e) < I(u_0) + \alpha$.

(I_6) I'' is a Fredholm operator and $\dim[\mathrm{Ker}(I''(u))] \leq 1$, for every u critical point of I such that the Morse index of u is zero.

As an application, we will study the existence and multiplicity of solutions for the following class of problems

$$\begin{cases} -\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^N$, $f \in \mathcal{C}^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and satisfies some necessary assumptions.

2 Main Results

More precisely, in this work, we will present the following theorems from Silva [3], which use the techniques of critical group description to establish the existence and multiplicity of critical points for a class of functionals satisfying (Ce) and $(I_0) - (I_6)$.

The first result guarantees the existence of at least one non-trivial critical point as follows.

Theorem 2.1. *Suppose $I \in \mathcal{C}^1(H,\mathbb{R})$ and satisfies (Ce) and $(I_0) - (I_4)$. Then, I possesses at least one nonzero critical point in H.*

The next result establishes the existence of at least four critical points for this class of functionals.

Theorem 2.2. *Suppose $I \in \mathcal{C}^2(H,\mathbb{R})$ is such that $I''(u)$ is a Fredholm operator, for every u critical point of I. Assume I satisfies (I_0), with $i \neq 1$, $(I_1) - (I_3), (I_6)$ and (Ce). Then, I possesses at least four critical points in H provided that it has a local minimum $u_0 \neq 0$.*

Finally, as an application of the above theorems, under certain conditions on f, we will establish the existence and multiplicity of solutions for the problem (1). To achieve this, we will ensure that the functional I associated with this problem, defined in the Sobolev space $H_0^1(\Omega)$, satisfies conditions (Ce) and $(I_0) - (I_6)$. For this purpose, we will employ techniques from variational calculus and establish the critical groups of I based on the relationship between f and the eigenvalues of the Laplacian.

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CHARACTERIZATION OF THE MAXIMUM PRINCIPLE FOR LINEAR SECOND ORDER ELLIPTIC OPERATORS WITH NON-LOCAL TERM

ISMAEL OLIVEIRA[1,†] & WILLIAN CINTRA[1,‡]

Departamento de Matemática, UnB, DF, Brasil[1]

mael.anjo1412@gmail.com[†] willian@unb.br[‡]

Abstract

In this work we establishes a characterization of the maximum principle for a class of second order uniformly elliptic operators with a non-local term and mixed boundary conditions. The results will be presented in Sobolev spaces contexts. As a consequence we obtain several monotonicity properties of the principal eigenvalue of this non-local operator, as well as some existence and non-existence results for certain elliptic equations.

1 Introduction

The main goal of this work is to present a characterization of the maximum principle for a class of second order uniformly elliptic operators with a non-local term and under mixed boundary conditions.

$$\mathcal{L}_{\mathcal{I}} := -div(A\nabla\cdot) + \langle \vec{b}, \nabla\cdot\rangle + c - \int_{\Omega} K(x,y)\cdot dy,$$

$$\mathfrak{B}\psi := \begin{cases} \psi, \text{ on } \Gamma_0, \\ \frac{\partial\psi}{\partial\nu} + \beta\psi, \text{ on } \Gamma_1, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a regular boundary domains such that $\partial\Omega$ consists of two disjoint open and closed subsets, Γ_0 and Γ_1, $K \in L^\infty(\Omega\times\Omega)$ and $K \geq 0$. Moreover, $A = (a_{ij})$, $\vec{b} = (b_1, ..., b_N)$ with $a_{ij} \in W^{1,\infty}(\Omega), b_j, c \in L^\infty(\Omega), i, j = 1, 2, ..., N$.

The following definitions will play a key role for our purposes: a function $h \in W^{2,p}(\Omega)$, $p > N$, h is said to be a supersolution of $(\mathcal{L}_{\mathcal{I}}, \mathfrak{B}, \Omega)$ if

$$\begin{cases} \mathcal{L}_{\mathcal{I}} h \geq 0 \text{ in } \Omega, \\ \mathfrak{B}h \geq 0 \text{ on } \partial\Omega. \end{cases} \tag{1}$$

If, in addition, some of these inequalities is strict on a measurable set of positive measure, we say that h is a strict supersolution of $(\mathcal{L}_{\mathcal{I}}, \mathfrak{B}, \Omega)$.

A principal eigenvalue of

$$\begin{cases} \mathcal{L}_{\mathcal{I}} u = \sigma u \text{ in } \Omega, \\ \mathfrak{B}u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2}$$

is an eigenvalue whose associated eigenfunction does not change sign.

We emphasize that we proved the existence and uniqueness of the principal eigenvalue using the Krein-Rutman Theorem and it will be denoted by $\sigma_1(\mathcal{L}_{\mathcal{I}}, \mathfrak{B}, \Omega)$.

2 Main Results

A crucial step to obtain the main result is to establish the following proposition which provides the behavior of supersolutions of $(\mathcal{L}_{\mathcal{I}}, \mathfrak{B}, \Omega)$.

Proposition 2.1. *Suppose that* $(\mathcal{L}_{\mathcal{I}}, \mathfrak{B}, \Omega)$ *has a positive supersolution* $h \in W^{2,p}(\Omega), p > N$, *such that* $h(x) > 0$, *for all* $x \in \overline{\Omega}$. *Then any supersolution* $u \in W^{2,p}(\Omega)$ *of* $(\mathcal{L}_{\mathcal{I}}, \mathfrak{B}, \Omega)$ *must satisfy some the following alternatives:*

A1. $u = 0$ *in* Ω.

A2. u *is strongly positive, that is,* $u(x) > 0, \quad \forall x \in \Omega \cup \Gamma_1$ *and*

$$\frac{\partial u}{\partial \nu}(x) < 0, \forall x \in u^{-1}(0) \cap \Gamma_0.$$

A3. There exists a constant $m < 0$ *such that* $u = mh$ *in* $\overline{\Omega}$. *In such case* $\Gamma_0 = \emptyset$ *and*

$$u(x) < 0, \quad \forall x \in \overline{\Omega}.$$

We emphasize that this result is a version for $\mathcal{L}_{\mathcal{I}}$ of [2, Theorem 7.1.7].

The result that we present below is a generalization of [1, Lemma 2.4], where we are considering here a more general class of operators, as well as boundary conditions.

Theorem 2.1. *The following assertions are equivalent:*

1. *The principal eigenvalue associate with* (3) *that we will denote by* $\sigma[\mathcal{L}_{\mathcal{I}}, \mathfrak{B}, \Omega]$, *is positive;*
2. $(\mathcal{L}_{\mathcal{I}}, \mathfrak{B}, \Omega)$ *has a strict supersolution* $h \in W^{2,p}(\Omega)$, $p > N$*;*
3. *The tern* $(\mathcal{L}_{\mathcal{I}}, \mathfrak{B}, \Omega)$ *satisfies the strong maximum principle, i.e, if any supersolution* $h \in W^{2,p}(\Omega) \setminus \{0\}$, $p > N$, *of* $(\mathcal{L}_{\mathcal{I}}, \mathfrak{B}, \Omega)$ *is strongly positive;*
4. *The tern* $(\mathcal{L}_{\mathcal{I}}, \mathfrak{B}, \Omega)$ *satisfies the maximum principle, i.e, if any supersolution* $h \in W^{2,p}(\Omega)$, $p > N$, *of* $(\mathcal{L}_{\mathcal{I}}, \mathfrak{B}, \Omega)$ *satisfies* $h \geq 0$*;*
5. *The resolvent operator of the linear problem*

$$\begin{cases} \mathcal{L}_{\mathcal{I}} u = f \ \text{ in } \Omega, \\ \mathfrak{B} u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{3}$$

is strongly positive.

It should be noted that the hypothesis $K \geq 0$ is important for us to apply Krein-Rutman Theorem, since without it the resolvent operator may no longer be strongly positive.

This result provides us with important applications in the study of elliptic PDE's with non-local term, such as monotony of the principal eigenvalue with respect to several parameters, existence and non-existence of solutions and even the punctual characterization of the principal eigenvalue.

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NORMALIZED SOLUTION FOR SCHRÖDINGER EQUATIONS WITH L^P-SUBCRITICAL AND CRITICAL GROWTH IN $\mathbb{R}^N$.

JEFFERSON L. A. OLIVEIRA[1], EDCARLOS D. SILVA[1] & CLAUDINEY GOULART[2]

Instituto de Matemática e Estatística, UFG, GO, Brasil[1], Instituto de Ciências Exatas e Tecnológicas, UFJ, GO, Brasil[2]

jefferson_luis@discente.ufg.br[†] edcarlos@ufg.br[‡] claudiney@ufj.edu.br[§]

Abstract

It is established existence of solution with prescribed L^p norm for the following nonlocal elliptic problem:

$$\begin{cases} (-\Delta)^s_p u \ + \ V(x)|u|^{p-2}u \ = \lambda |u|^{p-2}u + \beta\,|u|^{q-2}\,u \text{ in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^p dx = m^p, \ u \in W^{s,p}(\mathbb{R}^N). \end{cases}$$

where $s \in (0,1), sp < N, \lambda > 0, \beta > 0$ and $q \in (p, \overline{p}_s]$ where $\overline{p}_s = p + sp^2/N$. The main feature here is to consider the cases L^p-subcritical and L^p-critical. Furthermore, we consider a huge class of potentials V taking into account periodic potentials, asymptotically periodic potentials and coercive potentials. More precisely, we ensure the existence of a normalized solution for the periodic and asymptotically periodic potential V in the L^p-subcritical for each $\beta > 0$. Furthermore, for the L^p critical case, we also prove an existence result for each $\beta > 0$ small enough.

1 Introduction

In this work we consider a class of problems that has been widely studied in recent years by several authors. As motivation, we consider the existence of normalized solution for the following nonlocal elliptic problem:

$$\begin{cases} (-\Delta)^s_p u \ + \ V(x)|u|^{p-2}u \ = \lambda |u|^{p-2}u + \beta\,|u|^{q-2}\,u \text{ in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^p dx = m^p, \ u \in W^{s,p}(\mathbb{R}^N), \end{cases} \tag{P$_m$}$$

where the parameter λ is given by the Lagrange Multiplier Theorem, $N > ps, s \in (0,1), \beta > 0$ and $1 < p < q \leq \overline{p} = p + sp^2/N$. Now, we define the space X for the Problem (P$_m$) as follows:

$$X = \left\{ u \in W^{s,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^p dx < \infty \right\},$$

Here, we remember that $W^{s,p}(\mathbb{R}^N)$ is the fractional Sobolev space. At this stage, we consider the sphere of radius m in space $L^p(\mathbb{R}^N)$ as follows:

$$S_m = \left\{ u \in X : \int_{\mathbb{R}^N} |u|^p dx = m^p \right\}.$$

Our main objective is to find the existence of minimizers for the functional $J : X \to \mathbb{R}$ restricted to the set S_m, that is, we need to ensure the existence of $u \in S_m$ that satisfies

$$\gamma_m = \inf\{J(w) : w \in S_m\} = J(u),$$

where the energy functional J is given by

$$J(u) = \frac{1}{p}[u]^p + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \frac{\beta}{q}\int_{\mathbb{R}^N} |u|^q dx.$$

In order to state our hypotheses we consider the important set $\mathcal{F}$ introduced in [2] which represents a class of functions $f \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that for all $\varepsilon > 0$ the Lebesgue measure of the set $\{x \in \mathbb{R}^N : |f(x)| \geq \varepsilon\}$ is finite. Under these conditions, we shall consider the following hypotheses:

(V_1) The potential $V \in L^\infty(\mathbb{R}^N)$ is $1-$ periodic and $V(x) \geq 0$, for all $x \in \mathbb{R}^N$.

(V_2) The potential $V \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is asymptotically periodic, i.e., there exists a potential $V_\theta \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, 1-periodic, with $V_\theta(x) \geq V(x) \geq 0$, $V \not\equiv V_\theta$ such that $V - V_\theta \in \mathcal{F}$.

In the present work, the continuous embedding from the Sobolev spaces into the Lebesgue spaces does not work directly. In fact, by using the fact that V can be zero in some subsets $\Omega \subset \mathbb{R}^N$, the standard continuous embedding is not verified. Hence, we need to apply a hypothesis introduced by Sirakov [3]. Namely, we consider the following statement:

(V_3) $\sigma = \inf_{u \in X} \left\{ \|u\|^p : \int_{\mathbb{R}^N} |u|^p dx = 1 \right\} > 0.$

For the case $q = p + sp^2/N$ we assume the following auxiliary assumption:

(V_4) There holds $\mu\left(\{x \in \mathbb{R}^N : V(x) \leq M\}\right) < \infty$ for each $M > 0$.

2 Main Results

Theorem 2.1 (L^p-subcritical case, periodic potential)**.** *Suppose that $q \in (p, p + sp^2/N)$ and (V_1), (V_3), $\beta > 0$. Then, for every $m > 0$, there exists $\delta = \delta(m) > 0$ such that if $\|V\|_\infty < \delta$, we obtain that the Problem (P_m) has at least one solution $u \in S_m$ satisfying $J(u) = \gamma_m < 0$.*

Now, by using hypothesis (V_2), we can consider the existence of local minimizers $u \in S_m$ for asymptotically periodic potentials. More precisely, we are able to consider the following result;

Theorem 2.2 (L^p-subcritical case, asymptotically periodic potential)**.** *Suppose $q \in (p, p + sp^2/N)$, $\beta > 0$, (V_2) and (V_3). Then, for every $m > 0$ there exists $\delta = \delta(m) > 0$ such that if $\|V_\theta\|_\infty < \delta$, the Problem (P_m) has at least one weak solution $u \in S_m$ such that $J(u) = \gamma_m < 0$.*

Theorem 2.3 (L^p-critical case)**.** *Suppose $q = p + sp^2/N$. Also assume that (V_3) and (V_4) are satisfied. Then, there exists $\beta_0 > 0$ such that the Problem (P_m) has at least one solution $u \in S_m$, for every $\beta \in (0, \beta_0)$ where $\beta_0 = \beta_0(N, s, p, m) > 0$.*

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GROUND STATE SOLUTION FOR A LINEARLY COUPLED STEIN-WEISS SYSTEM IN $\mathbb{R}^2$

J.L. SANTOS[1], J.C. DE ALBUQUERQUE.[1], P. K. MISHRA[2] & V. M. TRIPATHI[2]

Universidade Federal de Pernambuco, UFPE, PE, Brasil[1],
Indian Institute of Technology Bhilai, Raipur, Chhattisgarh, India[2].

luando.brito@ufpe.br† josecarlos.melojunior@ufpe.br‡ pawan@iitbhilai.ac.in§ vinayakm@iitbhilai.ac.in§§

Abstract

In this work, we study the following class of linearly coupled system involving Stein-Weiss type convolution

$$\begin{cases} -\Delta u + u = \dfrac{1}{|x|^\beta}\left(\displaystyle\int_{\mathbb{R}^2} \frac{F(u(y))}{|y-x|^\mu |y|^\beta}\,\mathrm{d}y\right) f(u(x)) + \lambda v & \text{in } \mathbb{R}^2, \\ -\Delta v + v = \dfrac{1}{|x|^\beta}\left(\displaystyle\int_{\mathbb{R}^2} \frac{G(v(y))}{|y-x|^\mu |y|^\beta}\,\mathrm{d}y\right) g(v(x)) + \lambda u & \text{in } \mathbb{R}^2, \end{cases} \tag{S_λ}$$

where $0 < \mu < 2$, $\beta \geq 0$, $0 < 2\beta + \mu < 2$ and $F(s), G(s)$ are the primitives of $f(s)$ and $g(s)$ respectively. The coupling parameter $\lambda \in (0,1)$. Assuming that the nonlinearities $f(s)$ and $g(s)$ have critical exponential growth in the sense of Trudinger-Moser inequality, we study the existence of ground state solution of the above system. Moreover, regularity results and asymptotic behaviour of solutions complement the study of the work.

1 Introduction

Recently, inspired by the physical appeal and the interesting mathematical point of view, elliptic problems involving Stein-Weiss type nonlinearities have been studied, we refer the readers to [3, 5]. There are a few works considering Stein-Weiss term and a nonlinearity with critical exponential growth, see for example [1, 2, 6]. In [6], authors have considered logarithmic potential combined with Stein-Weiss critical and subcritical nonlinearity. In [2] authors used the Mountain-Pass Theorem combined with shifted sequences jointly with Lions' vanishing-nonvanishing arguments which may not be applicable in the lack of periodicity of the nonlocal term. A similar observation is noted in [3], pp 2190.

The main purpose of the present work is to investigate the existence of ground state solution for the linearly coupled system (S_λ) involving doubly critical nonlinearity in $\mathbb{R}^2$ and our contributions are the following:
(i) In the above context, the result of our work completes the picture of [4] in two dimensional case. We complement and extend some works which consider the Choquard type nonlinearity;
(ii) The existence result obtained in [1] and (S_λ) is compatible in the light of study of asymptotic behaviour of solutions of (S_λ) with respect to $\lambda \to 0$ as done in the work;
(iii) The work presents an alternative approach to the standard arguments based on Lions' vanishing-nonvanishing and shifted sequences which is not applicable in the presence of $\beta \neq 0$.

2 Main Results

Inspired by the Trudinger - Moser type inequality, we say that a function $h : \mathbb{R} \to \mathbb{R}$ has α_0- critical exponential growth at $+\infty$ if there exists $\alpha_0 > 0$ such that $\lim_{s\to+\infty} h(s)/e^{\alpha s^2} = 0$, if $\alpha > \alpha_0$ and $\lim_{s\to+\infty} h(s)/e^{\alpha s^2} = +\infty$ if $\alpha < \alpha_0$. We suppose that the nonlinearity f and g has α_0- critical exponential growth at $+\infty$ and the following hypotheses:
(f_1) $f, g \in C(\mathbb{R})$, $f(s) = g(s) = 0$ for all $s \leq 0$ and $f(s) = g(s) = o(s^{\frac{2-2\beta-\mu}{2}})$;

(f_2) $\liminf_{|s|\to\infty} \frac{f(s)}{e^{\alpha_0 s^2}} = \liminf_{|s|\to\infty} \frac{g(s)}{e^{\alpha_0 s^2}} = \xi > \sqrt{\frac{(2-\mu)(3-\mu)(4-\mu)(4-2\beta-\mu)}{(\alpha_0/2)\rho^{4-2\beta-\mu}}}$, where ρ satisfies $\frac{4-2\beta-\mu}{4}\rho^2 < 1$;
(f_3) there exists $\theta > 2$ such that $0 < \theta F(s) \leq 2f(s)s$ and $0 < \theta G(s) \leq 2g(s)s$ for all $s > 0$;
(f_4) there exist $M_0 > 0$ and $m_0 \in (0,1]$ such that $0 < s^{m_0}F(s) \leq M_0 f(s)$ and $0 < s^{m_0}G(s) \leq M_0 g(s)$, for all $s \geq s_0$;
(f_5) the functions $s \mapsto f(s)$ and $s \mapsto g(s)$ are increasing for $s \geq 0$.

Theorem 2.1 (Existence)**.** *Suppose that f and g satisfy assumptions $(f_1)-(f_5)$ and $\lambda \in (0,1)$. Then, system (S_λ) has a positive ground state solution (u_λ, v_λ).*

Theorem 2.2 (Regularity)**.** *Let $(u_\lambda, v_\lambda) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ is the positive ground state solution of System (S_λ) obtained in Theorem 2.1. Then $(u_\lambda, v_\lambda) \in L^\infty(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$. In addition, the solution obtained is $C^{1,\gamma}(\mathbb{R}^2) \times C^{1,\gamma}(\mathbb{R}^2)$ for some $\gamma \in (0,1)$.*

Theorem 2.3 (Asymptotic behavior)**.** *Suppose that assumptions $(f_1) - (f_5)$ are satisfied by f and g. Let $(\lambda_n) \subset (0,1)$ be such that $\lambda_n \to 0^+$ as $n \to \infty$ and for each $n \in \mathbb{N}$, $(u_{\lambda_n}, v_{\lambda_n}) \in H^1_{rad}(\mathbb{R}^2) \times H^1_{rad}(\mathbb{R}^2)$ is the positive ground state solution of system (S_{λ_n}) obtained in Theorem 2.1. Then, up to a subsequence, $(u_{\lambda_n}, v_{\lambda_n}) \to (u_0, v_0)$ in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ as $n \to +\infty$, where either $v_0 = 0$ and $u_0 > 0$ or $u_0 = 0$ and $v_0 > 0$ is a ground state solution to the equation, respectively,*

$$-\Delta u + u = \frac{1}{|x|^\beta}\left(\int_{\mathbb{R}^2} \frac{F(u(y))}{|y-x|^\mu |y|^\beta}\,\mathrm{d}y\right) f(u(x)) \quad \textit{and} \quad -\Delta v + v = \frac{1}{|x|^\beta}\left(\int_{\mathbb{R}^2} \frac{G(v(y))}{|y-x|^\mu |y|^\beta}\,\mathrm{d}y\right) g(v(x)).$$

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ON FRACTIONAL MUSIELAK-SOBOLEV SPACES AND APPLICATIONS TO NONLOCAL PROBLEMS

L.R.S. DE ASSIS[1], J.C. DE ALBUQUERQUE.[1], M.L.M. CARVALHO[2] & A. SALORT[3]

Universidade Federal de Pernambuco, UFPE, PE, Brasil[1], Universidade Federal de Goiás, UFG, GO, Brasil[2], Universidad de Buenos Aires, Buenos Aires, Argentina[3].

lazaro.assis@ufpe.br[†] josecarlos.melojunior@ufpe.br[‡] marcos_leandro_carvalho@ufg.br[§] asalort@dm.uba.ar[§§]

Abstract

In this work, we establish some abstract results on the perspective of the fractional Musielak-Sobolev spaces, such as: uniform convexity, Radon-Riesz property with respect to the modular function, (S_+)-property and other monotonicity results. Moreover, we apply the theory developed to study the existence of solutions to the following class of nonlocal problems

$$\begin{cases} (-\Delta)^s_{\Phi_{x,y}} u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{$\mathcal{P}_\Phi$}$$

where $N \geq 2$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function not necessarily satisfying the Ambrosetti-Rabinowitz condition.

1 Introduction

Recently, Azroul et al. [1, 2] have considered a new class of fractional problems driven by nonlocal integro-differential operator of elliptic type defined as follows

$$(-\Delta)^s_{\Phi_{x,y}} u(x) := 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \varphi\left(x, y, \frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{u(x)-u(y)}{|x-y|^s} \frac{dy}{|x-y|^{N+s}}, \quad x \in \mathbb{R}^N, \tag{1}$$

where $s \in (0,1)$ and $\Phi_{x,y}(t) := \Phi(x,y,t) = \int_0^{|t|} \tau\varphi(x,y,\tau)\,\mathrm{d}\tau$ is a generalized N-function that satisfies some suitable assumptions. Due to the nonlocality of the operator (1), they introduced the new fractional Musielak-Sobolev space in a domain $\Omega \subset \mathbb{R}^N$ defined as

$$W^{s,\Phi_{x,y}}(\Omega) := \left\{ u \in L_{\widehat{\Phi}_x}(\Omega) \colon J_{s,\Phi}(u) < \infty \right\}, \quad \text{where} \quad L_{\widehat{\Phi}_x}(\Omega) := \left\{ u \colon \Omega \to \mathbb{R} \text{ measurable} : J_{\widehat{\Phi}}(u) < \infty \right\},$$

$\widehat{\Phi}(x,t) := \Phi(x,x,t)$ and the modular functions $J_{\widehat{\Phi}}$ and $J_{s,\Phi}$ are determined in the following form:

$$J_{s,\Phi}(u) = \int_\Omega \int_\Omega \Phi\left(x, y, \frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{dxdy}{|x-y|^N}, \qquad J_{\widehat{\Phi}}(u) = \int_\Omega \widehat{\Phi}(x, |u(x)|)\,dx.$$

Furthermore, they considered the following work space

$$W_0^{s,\Phi_{x,y}}(\Omega) = \{u \in W^{s,\Phi_{x,y}}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}.$$

The fractional Musielak-Sobolev space extends many other known concepts in the literature, for instance, the fractional Sobolev with variable exponents space [5] and fractional Orlicz-Sobolev space [4]. In [1], the authors proved that $W^{s,\Phi_{x,y}}(\Omega)$ is a separable and reflexive Banach space when endowed with the Luxemburg norms. Moreover, if Φ satisfies the Δ_2-condition and $t \mapsto \Phi_{x,y}(\sqrt{t})$ is convex for all $(x,y) \in \Omega \times \Omega$, then $W^{s,\Phi_{x,y}}(\Omega)$ is a uniformly convex space. This convexity assumption has also been used to obtain the (S_+)-property for a wide class of operators associated to the fractional Orlicz-Sobolev space, see [3].

Motivated by the above discussion, our main goal in this work is extend and complement the previous results on the perspective of the new class of fractional Musielak-Sobolev space $W^{s,\Phi_{x,y}}(\Omega)$ and the nonlocal operator (1) assuming only that Φ and its conjugate function satisfy the Δ_2 condition.

2 Main Results

Throughout this work, we assume that $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with Lipschitz boundary and $\varphi : \Omega \times \Omega \times (0, +\infty) \to [0, +\infty)$ is a Carathéodory function satisfying the following assumptions:

(φ_1) $\lim_{t\to 0} t\varphi_{x,y}(t) = 0$ and $\lim_{t\to +\infty} t\varphi_{x,y}(t) = +\infty$, where $\varphi_{x,y}(t) := \varphi(x, y, t)$ for all $(x, y) \in \Omega \times \Omega$;

(φ_2) $t \mapsto t\varphi_{x,y}(t)$ is nondecreasing on $(0, +\infty)$;

(φ_3) there exist $1 < \ell \leq m < +\infty$ such that $\ell \leq \frac{t^2\varphi_{x,y}(t)}{\Phi_{x,y}(t)} \leq m$ for all $(x, y) \in \Omega \times \Omega$ and $t > 0$;

The assumptions $(\varphi_1)-(\varphi_3)$ imply that Φ and its conjugate are generalized N-functions and satisfy Δ_2 condition.

Theorem 2.1. *The space $W^{s,\Phi_{x,y}}(\Omega)$ is a uniformly convex. Furthermore, if $u_n \rightharpoonup u$ in $W_0^{s,\Phi_{x,y}}(\Omega)$ and $J_{s,\Phi}(u_n) \to J_{s,\Phi}(u)$, then $u_n \to u$ in $W_0^{s,\Phi_{x,y}}(\Omega)$.*

The next results establish some monotonicity properties of the operator $J'_{s,\Phi} : W_0^{s,\Phi_{x,y}}(\Omega) \to \left(W_0^{s,\Phi_{x,y}}(\Omega)\right)^*$.

Proposition 2.1. *The operator $J'_{s,\Phi}$ satisfies the following properties:*

(i) $J'_{s,\Phi}$ is bounded, coercive and monotone;

(ii) $J'_{s,\Phi}$ is pseudomonotone.

Theorem 2.2. *The operator $J'_{s,\Phi}$ satisfies the (S_+)-property, that is, given $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{s,\Phi_{x,y}}(\Omega)$ satisfying $u_n \rightharpoonup u$ weakly in $W_0^{s,\Phi_{x,y}}(\Omega)$ and $\limsup_{n\to\infty}\langle J'_{s,\Phi}(u_n), u_n - u\rangle \leq 0$, there holds $u_n \to u$ strongly in $W_0^{s,\Phi_{x,y}}(\Omega)$.*

Proposition 2.2. *Assume that (φ_1), (φ_3) hold and $t \mapsto t\varphi_{x,y}(t)$ is strictly increasing. Then, $J'_{s,\Phi}$ is a homeomorphism strictly monotone.*

In second part this work, we apply the theory developed to obtain the existence of weak solutions to the class of nonlocal problem $(\mathcal{P}_\Phi)$ where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying suitable growth assumptions.

In order to prove our main existence result, we need to assume that φ is symmetric with respect to x, y and that there exist constants $C_1, C_2 > 0$ such that $C_1 \leq \Phi_{x,y}(1) \leq C_2,$ for all $(x, y) \in \Omega \times \Omega$. The first assumption is used to define the notion of weak solution and the second one plays a key role in the continuous embedding theorem and Poincaré inequality for $W_0^{s,\Phi_{x,y}}(\Omega)$.

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SEMILINEAR ELLIPTIC PROBLEMS VIA THE NONLINEAR RAYLEIGH QUOTIENT WITH TWO NONLOCAL NONLINEARITIES

MARLOS R. ROCHA[1], EDCARLOS D. SILVA[1] & JEFFERSON S. SILVA[1]

IME , UFG, GO, Brasil [1]

edcarlos@ufg.br † marlosrodrigues14@gmail.com ‡ jeffersonsantos.mat@gmail.com §

Abstract

It is establish existence and multiplicity of solutions for nonlocal elliptic problems where the nonlinearity is driven by two convolutions terms. More specifically, we shall consider the following Choquard type problem:

$$\begin{cases} -\Delta u + V(x)u &= \mu(I_{\alpha_1} * |u|^q)|u|^{q-2}u - \lambda(I_{\alpha_2} * |u|^p)|u|^{p-2}u \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $p > q, \lambda, \mu > 0$, $\alpha_1 \leq \alpha_2$; $\alpha_1, \alpha_2 \in (0, N), N \geq 3$; $p \in (2_{\alpha_2}, 2^*_{\alpha_2})$; $q \in (2_{\alpha_1}, 2^*_{\alpha_1})$, $2_{\alpha_j} = (N + \alpha_j)/N$ and $2^*_{\alpha_j} = (N + \alpha_j)/(N-2), j = 1, 2$.

1 Introduction

In the present work we shall consider the Choquard problem with two convolutions terms defined in the whole space. Namely, we shall consider the following nonlocal elliptic problem:

$$\begin{cases} -\Delta u + V(x)u = \mu(I_{\alpha_1} * |u|^q)|u|^{q-2}u - \lambda(I_{\alpha_2} * |u|^p)|u|^{p-2}u, \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1}$$

where $p > q$ and $\lambda > 0, \mu > 0, \alpha_1 \leq \alpha_2$; $\alpha_1, \alpha_2 \in (0, N), N \geq 3$; $p \in \left(2_{\alpha_2}, 2^*_{\alpha_2}\right)$; $q \in \left(2_{\alpha_1}, 2^*_{\alpha_1}\right)$. Throughout this work we write $2_{\alpha_j} = (N + \alpha_j)/N$ and $2^*_{\alpha_j} = (N + \alpha_j)/(N - 2)$ for $j = 1, 2$. The potential $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function. Later on, we shall consider some hypotheses for the potential V. Recall that the Riesz potential can be written as follows:

$$I_\alpha(x) = \frac{A_\alpha(N)}{|x|^{N-\alpha}}, x \in \mathbb{R}^N \quad \text{and} \quad A_\alpha(N) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}} 2^\alpha}$$

where $\alpha \in (0, N)$ and Γ denotes the Gamma function, see [1]. Throughout this work we assume the following assumptions:

(h_1) The potential $V : \mathbb{R}^N \to \mathbb{R}$ is continuous function and there exists a constant $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^N$.

(h_2) For each $M > 0$ it holds that $|\{x \in \mathbb{R}^N : V(x) \leq M\}| < +\infty$.

(h_3) There exists $r \in [2, 2^*)$ such that $d_1 > 0$ where $d_1 = \inf_{u \in X \setminus \{0\}} \left\{ \dfrac{\|u\|_r^{2\frac{(p-q)}{p-1}} B(u)^{\frac{q-1}{p-1}}}{A(u)} \right\}$ where

$$B(u) = \int_{\mathbb{R}^N} (I_{\alpha_2} * |u|^p)|u|^p dx, \ A(u) = \int_{\mathbb{R}^N} (I_{\alpha_1} * |u|^q)|u|^q dx, \tag{2}$$

2 Main Results

Theorem 2.1. *Suppose* $(h_1)-(h_3)$ *holds. Assume also that* $c_{\mathcal{N}^+_{\lambda,\mu}} < c_{\mathcal{N}^0_{\lambda,\mu}}$ *holds. Then there exists at least one weak solution* $v_{\lambda,\mu} \in \mathcal{N}^+_{\lambda,\mu}$ *for the problem* (1). *Furthermore,* $v_{\lambda,\mu}$ *is a ground state solution such that*

$$c_{\mathcal{N}^+_{\lambda,\mu}} = J_{\lambda,\mu}(v_{\lambda,\mu}) = \inf_{w\in\mathcal{N}^+_{\lambda,\mu}} J(w).$$

Moreover, we obtain that $J_{\lambda,\mu}(v_{\lambda,\mu}) > 0$ *whenever* $\mu \in (\mu_n, \mu_e)$. *Similarly,* $J_{\lambda,\mu}(v_{\lambda,\mu}) = 0$ *for* $\mu = \mu_e$ *and* $J_{\lambda,\mu}(v_{\lambda,\mu}) < 0$ *for each* $\mu > \mu_e$.

Corollary 2.1. *Suppose* $(h_1)-(h_3)$ *holds. Assume that at least one of the following items is satisfied:*

i) $\lambda > 0$, $\mu \geq \mu_e$;

ii) $\lambda \in (0, \lambda^*)$, $\mu \in (\mu_n, \mu_e)$;

iii) $\lambda > 0$, $\mu \in (\mu_e - \epsilon, \mu_e)$;

where $\lambda^* > 0$ *is fixed and* $\epsilon > 0$ *is small enough. Then the problem* (1) *admits at least one weak solution* $v_{\lambda,\mu} \in \mathcal{N}^+_{\lambda,\mu}$ *which is a ground state solution. Furthermore, we obtain that*

$$c_{\mathcal{N}^+_{\lambda,\mu}} = J_{\lambda,\mu}(v_{\lambda,\mu}) = \inf_{w\in\mathcal{N}^+_{\lambda,\mu}} J_{\lambda,\mu}(w).$$

Theorem 2.2. *Suppose* $(h_1)-(h_3)$ *holds. Assume also that* $c_{\mathcal{N}^-_{\lambda,\mu}} < c_{\mathcal{N}^0_{\lambda,\mu}}$ *holds. Then there exists at least one weak solution* $u_{\lambda,\mu} \in \mathcal{N}^-_{\lambda,\mu}$ *for the problem* (1) *such that*

$$c_{\mathcal{N}^-_{\lambda,\mu}} = J_{\lambda,\mu}(u_{\lambda,\mu}) = \inf_{w\in\mathcal{N}^-_{\lambda,\mu}} J_{\lambda,\mu}(w).$$

Corollary 2.2. *Suppose* $(h_1)-(h_3)$ *and* $\mu > \mu_n$ *holds. Then there exists* $\lambda_* > 0$ *such that for every* $\lambda \in (0, \lambda_*)$ *the problem* (1) *admits at least one weak solution* $u_{\lambda,\mu} \in \mathcal{N}^-_{\lambda,\mu}$. *Furthermore, we also have*

$$c_{\mathcal{N}^-_{\lambda,\mu}} = J_{\lambda,\mu}(u_{\lambda,\mu}) = \inf_{w\in\mathcal{N}^-_{\lambda,\mu}} J_{\lambda,\mu}(w).$$

As a consequence, by using Theorems 2.1 and 2.2, we obtain the following result:

Corollary 2.3. *Suppose* $(h_1)-(h_3)$ *holds. Assume also that* $\mu > \mu_n$ *and* $\lambda \in (0, \min(\lambda_*, \lambda^*))$ *hold. Then the problem* (1) *admits at least two weak solutions* $v_{\lambda,\mu} \in \mathcal{N}^+_{\lambda,\mu}$ *and* $u_{\lambda,\mu} \in \mathcal{N}^-_{\lambda,\mu}$ *such that*

$$c_{\mathcal{N}^+_{\lambda,\mu}} = J_{\lambda,\mu}(v_{\lambda,\mu}), c_{\mathcal{N}^-_{\lambda,\mu}} = J_{\lambda,\mu}(u_{\lambda,\mu}). \tag{3}$$

Theorem 2.3. *Suppose* $(h_1)-(h_3)$ *holds. Assume also that* $\mu < \mu_n$ *and* $\lambda > 0$. *Then the Problem* (1) *does not admit any nontrivial solution.*

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HAMILTONIAN SYSTEMS IN THE PLANE INVOLVING VANISHING POTENTIALS

MARTA MENEZES[1,†]

PPGMAT-UFPB, PB, Brasil[1].

mnm@academico.ufpb.br[†]

Abstract

In this work, we establish the existence of nontrivial solutions for the following class of Hamiltonian systems:

$$\begin{cases} -\Delta u + V(x)u = Q(x)g(x,v), & x \in \mathbb{R}^2, \\ -\Delta v + V(x)v = Q(x)f(x,u), & x \in \mathbb{R}^2, \end{cases} \tag{1}$$

where V and Q decay to zero at infinity as $(1+|x|^{\alpha})^{-1}$ with $\alpha \in (0,2)$, and $(1+|x|^{\beta})^{-1}$ with $\beta \in [2,+\infty)$, respectively. The nonlinear terms $f(x,s)$ and $g(x,s)$ have exponential subcritical or critical growth. We show an alternative proof of a weighted Trudinger-Moser-type inequality and combine with a Galerkin approximation method and a linking theorem.

1 Introduction

We shall consider the following assumptions:

(V) $V \in C(\mathbb{R}^2)$, there exists $\alpha, a > 0$ such that $\dfrac{a}{1+|x|^{\alpha}} \leq V(x)$, and $V(x) \sim |x|^{-\alpha}$ as $|x| \to \infty$;

(Q) $Q \in C(\mathbb{R}^2)$, there exists $\beta, b > 0$ such that $0 < Q(x) \leq \dfrac{b}{1+|x|^{\beta}}$, and $Q(x) \sim |x|^{-\beta}$ as $|x| \to \infty$;

(h_0) $f, g : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ are continuous;

(h_1) $f(x,s) = o(|s|)$ and $g(x,s) = o(|s|)$ at the origin and uniformly on $x \in \mathbb{R}^2$;

(h_2) for any bounded interval $J \subset \mathbb{R}^2$, there exists $C > 0$ such that $|f(x,s)| \leq C$ and $|g(x,s)| \leq C$ for all $(x,s) \in \mathbb{R}^2 \times J$;

(h_3) there exists $\theta > 0$ such that $\begin{cases} 0 < \theta F(x,s) := \theta \displaystyle\int_0^s f(x,t)tdt \leq sf(x,s) \\ 0 < \theta G(x,s) := \theta \displaystyle\int_0^s g(x,t)tdt \leq sg(x,s) \end{cases}$ for all $(x,s) \in \mathbb{R}^2 \times (0,\infty)$;

(h_4) there exists constants $M_0 > 0$ and $s_1 > 0$ such that

$$\begin{cases} 0 < \theta F(x,s) \leq M_0 f(x,s) \\ 0 < \theta G(x,s) \leq M_0 g(x,s) \end{cases} \text{for all } (x,s) \in \mathbb{R}^2 \times [s_1,\infty).$$

We say $f(x,s)$ has *critical growth* if there exists *critical exponent* $\gamma_0 > 0$ such that,

$$\lim_{s\to+\infty} \frac{f(x,s)}{e^{\gamma_1 s^2}} = \begin{cases} 0, & \text{for all } \gamma_1 > \gamma_0 \\ +\infty, & \text{for all } \gamma_1 < \gamma_0 \end{cases}, \text{ uniformly in } x,$$

and has *subcritical growth* if such limit is zero for all $\gamma_1 > 0$, uniformly in x.

We denote by $L^p_w(\mathbb{R}^2)$ the weighted L^p-space consisting of all measurable functions $u : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $\int_{\mathbb{R}^2} w(x)|u|^p dx < \infty$, and introduce the weighted Sobolev space

$$H^1_V(\mathbb{R}^2) := \left\{ u \in L^2_V(\mathbb{R}^2 : |\nabla u| \in L^2(\mathbb{R}^2) \right\},$$

with norm $\|u\|^2 := \|\nabla u\|_2^2 + \int_{\mathbb{R}}^2 V(x)u^2$.

We denote the product space $E = H^1_V \times H^1_V$ equipped with the inner product

$$\langle (u,v), (\varphi, \psi) \rangle_E = \int_{\mathbb{R}^2} [\nabla u \nabla v + V(x)u\varphi + \nabla u \nabla \psi + V(x)v\psi] dx,$$

for all $(u,v), (\varphi,\psi) \in E$, to which corresponds to the norm $\|(u,v)\|_E = \langle (u,v),(u,v)\rangle_E^{1/2}$.

2 Main Results

First, we prove an Trudinger-Moser-type inequality with an alternative proof to the ones presented in [3, 4].

Theorem 2.1. *Suppose that (V) and (Q), with $\alpha \in (0,2)$ and $\beta \in [2,+\infty)$, are satisfied. For any $\gamma > 0$ and $u \in E$, we have $Q(\cdot)(e^{\gamma u^2} - 1) \in L^1(\mathbb{R}^2)$. Moreover, for any $0 < \gamma < 4\pi$,*

$$\sup_{u \in E, \|u\|_E \leq 1} \int_{\mathbb{R}^2} Q(x)(e^{\gamma u^2} - 1) dx < \infty.$$

Moreover, we can prove that $H^1_V(\mathbb{R}^2) \hookrightarrow L^p_Q(\mathbb{R}^2)$, for any $p \in [2,\infty)$. In particular, since $\beta > \alpha$, such embedding is compact. Equipped with this and inspired by [1, 2], our main results are:

Theorem 2.2 (Subcritical case)**.** *Suppose $f(x,s)$ has exponential subcritical growth or critical growth, $g(x,s)$ has exponential subcritical growth, (V), (Q) and $(h_0)-(h_3)$ are satisfied. Then (1) possesses a nontrivial weak solution $(u,v) \in E$.*

For the next result, we assume that there exists $\gamma_0 > 0$ such that the functions f, g satisfy

$$\lim_{|s|\to\infty} \inf \frac{sf(s)}{e^{\gamma_0 s^2}}, \quad \lim_{|s|\to\infty} \inf \frac{sg(s)}{e^{\gamma_0 s^2}} = \beta_0 > \mathcal{M},$$

where $\mathcal{M} =: \inf_{r>0} \dfrac{4e^{1/2r^2 V_{\max,r}}}{\gamma_0 r^2 Q_{\min,r}}$, $V_{\max,r} := \max_{|x|\leq r} V(x) > 0$ and $Q_{\min,r} := \min_{|x|\leq r} Q(x) > 0$.

Theorem 2.3 (Critical case)**.** *Suppose $f(x,s)$ and $g(x,s)$ has exponential critical growth, (V), (Q) and $(h_0)-(h_4)$ are satisfied. Then (1) possesses a nontrivial weak solution $(u,v) \in E$.*

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EXTREMAL SHAPE FOR EIGENVALUES OF THE GRUSHIN LAPLACIAN

NAÍSA CAMILA GARCIA[1,†] & MARCUS ANTONIO MENDONÇA MARROCOS[2,‡]

Universidade Federal do ABC, UFABC, SP, Brasil[1], Universidade Federal do Amazonas, UFAM, AM, Brasil[2]

naisa.camila@ufabc.edu.br[†]

Abstract

In this talk, we will address the eigenvalue problem of the Grushin Laplacian with Dirichlet and Neumann boundary conditions. We consider isovolumetric bounded domains on product manifolds of the form $\mathcal{M} = \mathbb{R}^k \times N$, and analyze the shape of the domain that maximize the principal eigenvalue of the Grushin Laplacian with Neumann boundary condition.

1 Introduction

Let $\mathcal{M}$ be a product manifold of the form $\mathcal{M} := \mathbb{R}^k \times N$, where (N, g_N) is a closed Riemannian manifold and $\mathcal{M}$ is endowed with the product Riemannian metric. The Grushin Laplacian operator acts in $u \in C^\infty(\mathcal{M})$ by:

$$\Delta_G u = \Delta^{\mathbb{R}^k} u + \|x\|^{2s}_{\mathbb{R}^k} \Delta^N u$$

where $s \in \mathbb{R}$, $\Delta^{\mathbb{R}^k}$ and Δ^N respectively denote the Laplacian Beltrami in $\mathbb{R}^k$ and in N.

For a bounded domain (open and connected set) $\Omega \subset \mathcal{M}$, with C^2 boundary, we consider the eigenvalue problems for the Laplacian Grushin operator with Neumann or Dirichlet boundary conditions:

$$\begin{cases} -\Delta_G \ u &= \lambda u \quad em \ \Omega \\ \mathcal{B}_\alpha(u) &= 0 \quad sobre \ \ \partial\Omega \end{cases} \tag{1}$$

$\mathcal{B}_\alpha(u) = \alpha\langle \nabla_G u, \nu\rangle + (1-\alpha)u$, for $\alpha \in \{0, 1\}$. In other words, when $\alpha = 0$ we are considering the Dirichlet boundary condition and when $\alpha = 1$ we are considering the Neumann boundary condition, $u : \Omega \subset \mathcal{M} \to \mathbb{R}$, ν is the unit normal vector exterior to $\partial\Omega$ and $\nabla_G u$ is called Grushin gradient of u given by: $\nabla_G u = (\nabla_x u, \|x\|^{2s}_{\mathbb{R}^k} \nabla_y u)$, where ∇_x and ∇_y denote the gradients of $\mathbb{R}^k$ and N respectively.

The Grushin Laplacian is not uniformly elliptic since it degenerates to $\Delta^{\mathbb{R}^k}$ on points of the fiber $\{0\} \times N$ however, some classic results from the theory of elliptic operators remain valid for the Grushin operator, such as Sobolev inequality, Poincaré inequality, and the existence of weak solutions.

Our goal is to examine isovolumetric bounded domains that maximize the principal eigenvalue of the Grushin Laplacian with Neumann boundary conditions. To achieve this, we will first present some useful in order to ensure the tools and conditions that ensure the existence the maximizing domain.

2 Preliminary

2.1 Weighted Sobolev Spaces

Let Ω be a domain in $\mathcal{M}$. We denote by $W^{1,2}_G$ the space of real-valued functions in $L^2(\Omega)$ such that $\frac{\partial u}{\partial x_i} \in L^2(\Omega)$ for all $i = 1, \cdots, k$ and $\|x\|^s \frac{\partial u}{\partial y_j} \in L^2(\Omega)$, endowed with norm $\|u\|_{W^{1,2}_G(\Omega)} := \left(\int_\Omega u^2 + \int_\Omega |\nabla_G \ u|^2\right)^{\frac{1}{2}}$ and we denote by $W^{1,2}_{G,0}(\Omega)$ the closure of $C^\infty_c(\Omega)$ in $W^{1,2}_G(\Omega)$.

2.2 Spectral Theorem

Definition 2.1. *We say that a function* $u \in W^{1,2}_{G,0}(\Omega)\left(W^{1,2}_{G}(\Omega)\right)$ *is a weak solution for the problem (1), when it satisfies*

$$\int_\Omega \nabla_G u \cdot \nabla_G \varphi = \lambda \int_\Omega u\varphi \quad \forall \varphi \in W^{1,2}_{G,0}(\Omega)\left(W^{1,2}_{G}(\Omega)\right)$$

Theorem 2.1. *Let* $\Omega \subset \mathcal{M}$ *be a domain (of class* C^∞*). Then the eigenvalue problem (1) has an infinity countable number of eigenvalues* $0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$ *with* $\lim_{n\to\infty} \lambda_m = \infty$ *and eigenfunctions* $u_i \in W^{1,2}_{G,0}(\Omega)\left(W^{1,2}_{G}(\Omega)\right)$ *orthonormal in* $L^2(\Omega)$ *that satisfy* $B_\alpha(u) = 0$ *on* $\partial\Omega$.

3 Main Results

Let Ω_r be a cylindrical domain $\Omega_r := \{(x, y) \in \mathcal{M}; x \in \mathbb{R}^k, \|x\| \le r, y \in N\} \subset \mathcal{M}$, $r > 0$ and consider the eigenvalue problem of the Laplacian Grushin (1) on Ω_r.

In what follows we will only consider the Neumann boundary condition, that is, we consider $\alpha = 1$ in (1).

The following proposition is the first step in the proof of Theorem 3.1.

Proposition 3.1. *Let* $r > 0$*. If* $\Omega \subset \mathcal{M}$ *is a bounded open set, satisfies* $|\Omega| = |\Omega_r|$*, then* $\lambda_2(\Omega) \le \mu^r = \mu_2(B_r)$*, where* $\mu_2(B_r)$ *is the first nonzero eigenvalue of the Laplacian on the ball.*

Theorem 3.1. *Let* $|\Omega_r| = v$*. Then we have* $\lambda_2(\Omega) \le \lambda_2(\Omega_r)$ *for every bounded open set* $\Omega \subset \mathcal{M}$ *with* $|\Omega| = v$.

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INFINITELY MANY SOLUTIONS FOR AN ELLIPTIC EQUATION WITH ARBITRARY GROWTH IN THE UPPER-HALF SPACE

RODOLFO F OLIVEIRA[1,†] & MARCELO F. FURTADO[1,‡]

Departamento de Matemática, UnB, DF, Brasil[1]

rodolfo@mat.unb.br[†] mfurtado@unb.br[‡]

Abstract

In this work, we study an elliptic problem in $\mathbb{R}^N_+$ with non-linearities with arbitrary growth at infinity. We obtain infinitely many solutions by supposing that this nonlinearity is concave-convex and odd near the origin.

1 Introduction

In this paper, we consider the following equation

$$\begin{cases} -\Delta u - \dfrac{1}{2}\left(x \cdot \nabla u\right) = 0, & \text{in} \quad \mathbb{R}^N_+ \\ \dfrac{\partial u}{\partial \eta} = \mu a(x')|u|^{q-2}u + f(u), & \text{on} \quad \mathbb{R}^{N-1} \end{cases}, \tag{1}$$

where f is odd and superlinear near to 0. This type of problem arises when we look for self-similar solutions for the heat equation [2]. We obrain infinitely many solutions for the equation.

2 Main Results

We suppose that a and f verify the follwing:

(f_1) $f \in C(\mathbb{R};\mathbb{R})$,

(f_2) There exists $p \in (2, 2_*)$ such that $\dfrac{f(s)}{s^{p-1}} \to 0$ if $s \to 0$,

(a_1) $a \in L^{\sigma_q}_K(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$, where

$$\left(\frac{p}{q}\right)' < \sigma_q \leq \left(\frac{2}{q}\right)', \tag{1}$$

(a_2) $\Omega_a^+ = \{x \in \mathbb{R}^{N-1} : a(x) > 0\} = \mathbb{R}^{N-1}$.

With the assumptions above, we shall to prove the following result

Theorem 2.1. *There exists $C = C(N,p,q) > 0$ such that, if f is odd in $[-C, C]$, then (1) has infinitely many solutions for $\mu > 0$ small.*

Proof (Sketch) We follow similar arguments of [1]. By setting $F(s) = \int_0^s f(t)dt$, we first notice that the formal functional associated to the problem (1), namely

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx - \frac{\mu}{q}\int_{\mathbb{R}^{N-1}} K(x')a(x')|u|^q dx' - \int_{\mathbb{R}^{N-1}} K(x')F(u)dx', \tag{2}$$

is not be well defined, since f has arbitrary growth at infinity. So, we follow [1] and define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(s) = \begin{cases} f(s), |s| \leq A \\ \dfrac{f(A)}{A^{p-1}}|s|^{p-2}s, |s| > A, \end{cases} \tag{3}$$

where $A > 0$ is free for now.

Once know that f is odd in $[-A, A]$, we have that g is an odd function in the entire real line. Moreover, there exists $c_g > 0$ such that $|g(s)| \leq c_g|s|^{p-1}$ for every $s \in \mathbb{R}$. Let $G(s) = \int_0^s g(t)dt$ and define the class C^1 functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx - \frac{\mu}{q}\int_{\mathbb{R}^{N-1}} K(x')a(x')|u|^q dx' - \int_{\mathbb{R}^{N-1}} K(x')G(u)dx'. \tag{4}$$

Our goal is to prove the existence of infinitely many solutions for the problem (1) replacing f by g. After that, we show that these solutions belongs to $L^\infty(\mathbb{R}^{N-1})$ and satisfies $|u|_\infty \leq C$, and therefore they are solutions of the original problem. By Hölder's inequality and making use of the Sobolev embeddings for our working space $D^{1,2}_K(\mathbb{R}^N_+)$, we have positive constants C_1, C_2 in a such way that

$$I(u) \geq \frac{1}{2}\|u\|^2 - \frac{C_1\mu}{q}|a|_{\sigma_q}\|u\|^q - C_2\|u\|^p := h(\|u\|),$$

By straightforward computation, for $\mu > 0$ small enough, h has only two positive roots, namely R_1 and R_2, with $R_1 < R_2$. We now consider a cutoff function $\phi \in C_c^\infty([0, +\infty))$ such that $\phi \equiv 1$ in $[0, R_1]$ and $\phi \equiv 0$ on $[R_2, +\infty)$. With this, we can define the coercive and C^1 functional given by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx - \frac{\mu}{q}\int_{\mathbb{R}^{N-1}} K(x')a(x')|u|^q dx' - \phi(\|u\|)\int_{\mathbb{R}^{N-1}} K(x')G(u)dx'. \tag{5}$$

Since Φ is coercive and even, we can apply genus theory to obtain infinitely many critical points of Φ with negative energy. Furthermore, we prove that: If $\Phi(u) < 0$ then $\|u\| < R_1$ and $I(v) = \Phi(v)$ for a small neighborhood of u. Moreover, Φ satisfies a local $(PS)_c$-condition for $c < 0$. Thus, these points are critical points of I as well. For the final step we adapt the Moser's Iteration method and we find a constant $C = C(N, p, q) > 0$, uniformly in A, which bound from above the L^∞ norm of any these critical points. So, taking $A = C$ in (3) and repeating all previous steps we conclude the proof. □

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ON A ELLIPTIC PROBLEM INVOLVING THE $P(X)-$LAPLACIAN AND SOBOLEV-HARDY CRITICAL EXPONENTS

RONALDO L. DA SILVA[1,†] & AUGUSTO C. R. COSTA[2,‡]

ICEN - PDM, UFPA, PA, Brasil[1], ICEN - FACMAT, UFPA, PA, Brasil[2]

ronaldo.silva@icen.ufpa.br[†] aug@ufpa.br[‡]

Abstract

This work presents a result of the existence of solutions for the class of $p(x)-$Laplacian problems

$$\begin{cases} -div\ (|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u &= \displaystyle\sum_{i=1}^{k} h_i(x)\frac{|u|^{p^*_{s_i}(x)-2}u}{|x|^{s_i(x)}} + f(x,u), \quad in\ \Omega, \\ u &= 0 \quad on\ \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $0 \in \Omega$; $p(x), s_1(x), \cdots, s_k(x)$ are Lipschitz continuous and radially symmetric in $\overline{\Omega}$ such that $1 < p^- \leq p(x) \leq p^+ < N$; $0 \leq s_i(x) \ll p(x), \quad \forall i \in \{1,\cdots,k\}$ and $|\{x \in \Omega\ /\ s_i(x) = s_j(x),\ \forall i \neq j\}| = 0$; $p^*_{s_i}(x) = \frac{p(x)\cdot(N-s_i(x))}{N-p(x)}$ and the functions $h_i \quad (i = 1,\cdots,k)$ and f are functions whose properties will be given later. We obtain this result via the Lions' concentration-compactness principle for critical Sobolev-Hardy exponent in $W_0^{1,p(x)}(\Omega)$ due to Yu , Fu and Li [3] and the Fountain Theorem in [1].

1 Introduction

Throughout this work we assume the following: The functions $h_i : \overline{\Omega} \to \mathbb{R}$ are continuous and satisfy

$$h_i(x) = h_i(|x|) > 0,\ \forall x \in \overline{\Omega} - \{0\} \quad and \quad h_i(0) = 0, \tag{2}$$

$$\lim_{x\to 0} h_i(x)\cdot\frac{1}{|x|^{s_i(x)}} = +\infty, \quad \forall i \in \{1,\cdots,k\}. \tag{3}$$

The required conditions for the function $f : \overline{\Omega}\times\mathbb{R} \longrightarrow \mathbb{R}$ are below

(f_1) f satisfies the Caratheodory condition;

(f_2) There are constants c_1, c_2 such that $|f(x,t)| \leq c_1 + c_2|t|^{q(x)-1}$, where $q : \overline{\Omega} \longrightarrow \mathbb{R}$ is a measurable Lebesgue function such that $p(x) \ll q(x) \ll p^*_{s_i}(x);\ \forall i \in \{1,\cdots,k\}$, for all $x \in \overline{\Omega}$;

(f_3) $f(x,t) = f(|x|,t),\ \forall (x,t) \in \Omega\times\mathbb{R}$;

(f_4) $f(x,t) = -f(x,-t),\ \forall (x,t) \in \Omega\times\mathbb{R}$.

2 Main Results

Theorem 2.1. *Assume $\Omega \subset \mathbb{R}^N$ bounded, $0 \in \Omega$, and conditions (2)−(3) and $(f_1)-(f_4)$ hold, then problem (1) has a sequence $(u_n) \subset W_0^{1,p(x)}(\Omega)$ of solutions such that, for its energy functional $J : W_0^{1,p(x)}(\Omega) \longrightarrow \mathbb{R}$, $J(u_n) \to +\infty$, as $n \to +\infty$.*

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SPECTRAL THEORY FOR THE $\mathcal{L}_{\mathcal{T}}$ GRUSHIN OPERATOR

WANESSA F. TAVARES[1,†] & MARCUS A. M. MARROCOS[2,‡]

CMCC, UFABC, SP, Brasil[1], Departamento de Matemática, UFAM, AM, Brasil[2]

wanessa.ferreira@ufabc.edu.br[†] marcusmarrocos@ufam.edu.br[‡]

Abstract

This work aims to investigate the behavior of the eigenvalues of a family of degenerate elliptic operators defined on a Riemannian manifold. We are interested in the family of operators parameterized by the Riemannian metric on the manifold.

1 Introduction

The spectral theory for elliptic operators has several branches and applications. The most common of these operators is known as the Laplace-Beltrami operator which acts on smooth functions on Riemannian manifolds and is defined as the divergence of the gradient of the function or, equivalently, the trace of the Hessian of the function.

The Grushin Laplacian operator that acts on functions $C^\infty(\mathbb{R}^k \times \mathbb{R}^h))$ is defined by

$$\Delta_G u(x,y) := \Delta^{\mathbb{R}^k} u + |x|^{2s}_{\mathbb{R}^k} \Delta^{\mathbb{R}^h} u,$$

where $s > 0$, $\Delta^{\mathbb{R}^k}$ and $\Delta^{\mathbb{R}^h}$ denote, respectively, the Laplacian Beltrami in $\mathbb{R}^k$ and in $\mathbb{R}^h$. Note that Δ_G is not uniformly elliptic throughout $\mathbb{R}^k \times \mathbb{R}^h$, because it degenerates for $\Delta^{\mathbb{R}^k}$ in points on the axis y. Despite this, some classic results from the theory of elliptic operators remain valid for the Grushin operator, as such the Sobolev's inequality and Poincaré's inequality, existence of weak solutions and Hölder's regularity, Maximum principles, Hardy's inequalities and an Rellich-Kondrachov theorem analogue.

In [2] it is studied the spectral problem for the Grushin Laplacian under to Dirichlet boundary conditions on a bounded open subset of $\mathbb{R}^n$. It was proved that the symmetric functions of a given subset of eigenvalues really depend real analytically on the perturbations of the domain and a Hadamard type formula was given for the Grushin eigenvalues. As an application, the critical domains for the symmetric functions under isovolumetric and isoperimetric perturbations were characterized in terms of overdetermined problems. In the present work, we are interested in the Spectral Theory for the differential operators that generalize the Grushin Laplacian considering both Neumann and Dirichlet boundary conditions.

We propose to investigate whether the results of Lamberti [2] remain valid in more general situations, considering wider class of degenerate elliptic operators and in Riemannian Manifolds and not just in $\mathbb{R}^n$.

Let (M, g) be an n-dimensional compact Riemannian manifold with boundary ∂M. Consider a symmetric positive semidefinite $(0, 2)$-tensor T, and a smooth function $\eta : M \to \mathbb{R}$. The Grushin-type Laplacian operator is defined by

$$\mathcal{L}_{T,\eta} f = div_\eta(T\nabla f) = div_g(T\nabla f) - g(\nabla\eta, T\nabla f)$$

Note that $\mathcal{L}_{T,\eta}$ is a degenerate operator since T is positive semidefinite. Note that the operator also smoothly depends on the metric g and η. Here, div stands for the divergence of smooth vector fields and ∇ for the gradient of smooth functions.

The existence of eigenvalues and eigenfunctions for the operator $\mathcal{L}_{T,\eta}$ with Dirichlet or Neumann condition defined in domains Ω of $\mathbb{R}^n$ depends on suitable assumptions over T and Ω, see [3]. There is a sequence of eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \longrightarrow +\infty$ and $\phi_i : M \to \mathbb{R}$ associated eigenfunctions, for the eigenvalues problem given by

$$\begin{cases} -\mathcal{L}_{T,\eta}\phi_i & = \lambda_\eta \phi_i \quad em \ M \\ \mathcal{B}_\alpha(u) & = 0 \quad sobre \ \ \partial M \end{cases}$$

where $\mathcal{B}_\alpha(u) = \alpha\langle \nabla u, T\nu\rangle + (1-\alpha)u$, for $\alpha \in \{0,1\}$, in other words, when $\alpha = 0$ the Dirichlet boundary condition is met, and when $\alpha = 1$, it satisfies the Neumann boundary condition.

As the operator $\mathcal{L}_{T,\eta}$ clearly depends on the metric g let's consider the family of operators $\mathcal{L}_{gT,\eta}$ parameterized by g. It is known in the literature that the eigenvalues of elliptic operators depend continuously on g. We are interested in checking the regularity of the dependence on the eigenvalues of $\mathcal{L}_{gT,\eta}$ degenerate with respect to the metric g. In general, we cannot expect the $\lambda_n : M \to \mathbb{R}$ that associates each metric g to the nth eigenvalue of $\mathcal{L}_{T,\eta}$ to be differentiable.

Kato's choice theorem tells us that in an analytic family of operators it is possible to parameterize the eigenvalues analytically without necessarily putting them in ascending order. For our problem we will consider an analytic family of metrics parameterized by a real parameter t.

2 Main Results

Let $\mathcal{M}$ be the set of all smooth Riemannian metrics on M.

Definition 2.1. *The application $\Lambda_{F,\tau} : \mathcal{M} \longrightarrow \mathbb{R}$ is defined as the elementary symmetric functions of the eigenvalues, and are given by*

$$\Lambda_{F,\tau}[g] := \sum_{\substack{j_1,\ldots,j_\tau \in F \\ j_1 < \cdots < j_\tau}} \lambda_{j_1}[g]\cdots\lambda_{j_\tau}[g] \quad \forall\ g \in \mathcal{M}, F \subset \mathbb{N}, \tau \in \{1,\cdots,|F|\}$$

Definition 2.2. *$\bar{g}$ is said a critical metric of $\Lambda_{F,\tau}$ under the volume constraint $g \in V[v]$ if it satisfies*

$$d_{g=\bar{g}}\Lambda_{F,\tau}[H] + \bar{c}d_{g=\bar{g}}\mathcal{V}[H] = 0 \ \ \forall\ H \in S^2(\mathcal{M}), \textit{all } c \in \mathbb{R} \textit{ where } H-(0,2)-\textit{tensor.}$$

Let $V[v] := \{g \in \mathcal{M}\ /Vol_g = v\}$. Note that T' is the directional derivative of $\mathcal{F}$ in the direction of H,

$$d\mathcal{F}_{\bar{g}}(H) = \frac{d}{dt}\bigg|_{t=0}\mathcal{F}(g(t)) = \frac{d}{dt}\bigg|_{t=0} T_g = T'$$

Theorem 2.1. *If $\bar{g}$ is a critical metric for $\Lambda_{F,\tau}$ under the volume constraint $g \in V[v]$, then there exists a constant $c \in \mathbb{R}$ such that*

$$\sum_{i\in F} div_\eta(T\nabla\phi_i^2)\bar{g} + 2\nabla\phi_i \otimes T\nabla\phi_i - d\mathcal{F}^*_{\bar{g}}(\nabla\phi_i \otimes \nabla\phi_i) = c\bar{g}$$

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ASYMPTOTIC BEHAVIOR OF THE ELASTIC WAVE EQUATION WITH LOCALIZED KELVIN-VOIGT DAMPING AND MEMORY EFFECTS

ANTONIO G. LEITE[1,†], MARIA ASTUDILLO ROJAS[1,‡] & HIGIDIO P. OQUENDO[1,§]

Departamento de Matemática, UFPR, PR, Brasil[1]

antonioguimaraes@ufpr.br[†] maria.astudillo@ufpr.br[‡] higidio@ufpr.br[§]

Abstract

We study the asymptotic behavior of the wave equation with localized Kelvin-Voigt damping and memory effects. We show that there is no exponential stability of the solution when the purely elastic part is connected only with the Kelvin-Voigt damping. In this case we show that the solutions decay polynomially with the rate $1/t^2$.

1 Introduction

In this work, we consider an elastic string with local viscoelastic damping of Kelvin-Voigt type and past history damping. The mathematical model is the following partial differential equation:

$$u''(x,t) - \Big[\beta u_x(x,t) - a(x)\int_0^\infty g(s)u_x(x,t-s)ds + b(x)u_x'(x,t)\Big]_x = 0, \quad \text{in} \quad (0,L)\times(0,\infty) \tag{1}$$

with initial and boundary conditions

$$u(x,0) = u_0(x); \quad u'(x,0) = u_1(x), \quad u(x,-s) = \phi_0(s), \quad \text{in} \quad (0,L)\times(0,\infty),$$
$$u(0,t) = u(L,t) = 0, \quad \text{in} \quad (0,\infty).$$

Here we consider the coefficients

$$a(x) = a_0\chi_{(a_1,a_2)(x)} \quad \text{and} \quad b(x) = b_0\chi_{(b_1,b_2)(x)},$$

where (a_1,a_2) and (b_1,b_2) are subintervals of $(0,L)$. The e coefficients β, a_0 and b_0 are positive and the kernel of the memory satisfies:

$$\begin{cases} g \in L^1([0,\infty)) \cap C^1([0,\infty)) \text{ is a positive function such that} \\ g_0 := g(0) > 0,\ \tilde{g} := \int_0^\infty g(s)ds,\ \tilde{a}(x) := \beta - a(x)\tilde{g} > 0, \text{ and} \\ g'(s) \le -c_0 g(s), \text{ for some } c_0 > 0, \forall s \ge 0. \end{cases} \tag{2}$$

As in Dafermos [1], we introduce the following change of variable:

$$\eta(x,s,t) := u(x,t) - u(x,t-s), \quad (x,s,t) \in (0,L)\times(0,\infty)\times(0,\infty).$$

Then, the equation (1) becomes

$$u_{tt}(x,t) - [\tilde{a}(x)u_x(x,t) + a(x)\int_0^\infty g(s)\eta_x(x,s)ds + b(x)u_{xt}(x,t)]_x = 0,\ (x,t) \in (0,L)\times(0,\infty), \tag{3}$$
$$\eta_t(x,s,t) + \eta_s(x,s,t) - u_t(x,t) = 0, \quad (x,s,t) \in (0,L)\times(0,\infty)\times(0,\infty). \tag{4}$$

satisfying the initial and boundary conditions

$$u(\cdot,-s)=\phi_0(s),\quad u(\cdot,0)=u_0(\cdot),\quad u_t(\cdot,0)=u_1(\cdot),\quad \eta(\cdot,s,0)=\eta_0(\cdot,s):=u_0(\cdot)-\phi_0(s),\ \text{in } (0,L)\times(0,\infty);$$
$$u(0,t)=u(L,t)=0,\ \text{in } (0,\infty),\quad \eta(\cdot,0,t)=0,\ \text{in } (0,L)\times(0,\infty),\quad \eta(0,s,t)=\eta(L,s,t)=0,\ \text{in } (0,\infty)\times(0,\infty).$$

In this work we use the following phase space $\mathcal{H} := H_0^1(0,L)\times L^2(0,L)$ where $\mathcal{G}_g$ is the weighted space $L_g^2((0,\infty);H_0^1(0,L))$ with the inner product $(\eta^1,\eta^2)_{\mathcal{G}_g} := \int_0^L\int_0^\infty g(s)\eta_x^1\overline{\eta_x^2}dsdx,\ \forall\, \eta^1,\eta^2\in\mathcal{G}_g$. The Hilbert space $\mathcal{H}$ is equipped with the inner product defined by

$$(U_1,U_2)_{\mathcal{H}}=\int_0^L v^1\overline{v^2}dx+\int_0^L \tilde{a}(\cdot)u_x^1\overline{u_x^2}dx+\int_0^L\int_0^\infty a(\cdot)g(s)\eta_x^1(\cdot,s)\overline{\eta_x^2}(\cdot,s)dsdx,$$

for $U_1=(u^1,v^1,\eta^1(\cdot,s))$, $U_2=(u^2,v^2,\eta^2(\cdot,s))$ in $\mathcal{H}$. We define the unbounded linear operator $\mathcal{A}: D(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$ defined by

$$\mathcal{A}(u,v,\eta(\cdot,s))=\left(v,(\tilde{a}(\cdot)u_x+a(\cdot)\int_0^\infty g(s)\eta_x(\cdot,s)ds+b(\cdot)v_x)_x,v-\eta_s(\cdot,s)\right)^\top,\quad \forall\, U=(u,v,\eta(\cdot,s))^\top\in D(\mathcal{A})$$

with domain

$$D(\mathcal{A})=\left\{\begin{array}{l} U=(u,v,\eta(\cdot,s))^\top\in\mathcal{H}\mid v\in H_0^1(0,L),\eta_s(\cdot,s)\in\mathcal{G}_g,\eta(\cdot,0)=0 \text{ in } (0,L)\\ (\tilde{a}(\cdot)u_x+a(\cdot)\int_0^\infty g(s)\eta_x(\cdot,s)ds+b(\cdot)v_x)_x\in L^2(0,L)\end{array}\right\}.$$

2 Main Results

Our main result is as follows:

Theorem 2.1. *Under the hypotheses (2). If the purely elastic part is connected only with the Kelvin-Voigt damping then the system (3)-(4) is not exponentially stable. Moreover, the system decays polynomially with the rate* t^{-2}*, that is*

$$\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}}\le Ct^{-2}\|U_0\|_{D(\mathcal{A})},\quad \forall\, U_0\in D(\mathcal{A}),\quad t\ge 1,$$

where the constant $C>0$ *is independent of* U_0*.*

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EFFECT OF SOIL RETARDATION FACTOR ON A POLLUTANT TRANSPORT MODEL

DANIELA BUSKE[1,†], JOSIANE KONRADT[1,‡], GUILHERME J. WEYMAR[1,§], IGOR C. FURTADO.[2,§§] & RÉGIS S. QUADROS[1,§§§]

Postgraduate Program in Mathematical Modeling, UFPel, RS, Brazil[1],
Department of Mathematics, IFSul, RS, Brazil[2].,

josianekonradt@gmail.com[†] danielabuske@gmail.com[‡] guilhermejahnecke@gmail.com[§]
igorjara@gmail.com[§§] quadros99@gmail.com[§§§]

Abstract

The paper presents a study of a two-dimensional model of pollutant dispersion in landfills, considering the governing equation in dimensionless form. The adopted model is solved using the Generalized Integral Laplace Transform Technique (GILTT), and the obtained solution is analytical. The results are presented, highlighting the influence of the soil retardation factor parameter, demonstrating that the proposed method is effective in reproducing the physical characteristics of the problem.

1 Introduction

The Eq. (1), written in dimensionless form, found in the work of Albuquerque [1], describes the transport of pollutants in a saturated porous medium, occurring underneath a landfill site where urban solid waste (USW) is disposed:

$$R\frac{\partial C}{\partial \tau} = L^*\frac{\partial^2 C}{\partial X^2} + \frac{\partial^2 C}{\partial Y^2} - Pe\frac{\partial C}{\partial Y}, \tag{1}$$

where R represents the soil retardation factor, C is the concentration of the contaminant in the liquid phase, τ denotes time, L^* is the aspect ratio of the problem between X and Y dimensions $\left(L^* = (\frac{L_1}{L_2})^2\right)$ and Pe is the Peclet number.

The initial condition of the problem is given by:

$$C(X, Y, 0) = C_0, \tag{2}$$

where C_0 is the initial concentration of the contaminant in the USW storage cell.

The boundary conditions in dimensionless form and in the X direction are given by:

$$\frac{\partial C}{\partial X}(0, Y, \tau) = 0, \qquad \frac{\partial C}{\partial X}(1, Y, \tau) = 0, \tag{3}$$

where null flux conditions are used at the boundaries of the domain in X.

The boundary conditions in the Y direction are given by:

$$C(X, 0, \tau) = 1, \tag{4}$$

which represents the interface condition corresponding to continuous and uniform leachate leakage in a USW cell, and:

$$\frac{\partial C}{\partial Y}(X, 1, \tau) + BiC(X, 1, \tau) = 0, \tag{5}$$

where Bi is the Biot number and this condition represents the convective flow located at the contact zone between the soil and the groundwater table.

Using the Generalized Integral Laplace Transform Technique (GILTT) to obtain the analytical solution of the two-dimensional mass transport model in a saturated porous medium, we obtain:

$$C(X,Y,\tau) = \sum_{n=0}^{N} \varphi_n(X) \left[\sum_{k=0}^{K} \psi_k(Y) \widetilde{C}_k(\tau) \right] + C_E(Y), \tag{6}$$

where $\varphi_n(X) = \cos\left(\frac{\lambda_n}{\sqrt{L^*}} X\right)$, $\psi_k(Y) = \text{sen}(\beta_k Y)$, $\widetilde{C}_k(\tau)$ is defined by $Z(\tau) = X \cdot G(\tau) \cdot X^{-1} \cdot Z(0)$, which is part of a series in terms of eigenfunctions, and $C_E(Y)$ is the solution for the steady-state problem, given by $C_E(Y) = \frac{e^{Pe}(Pe+Bi)-Bie^{PeY}}{e^{Pe}(Pe+Bi)-Bi}$.

2 Main Results

After obtaining the solution of the two-dimensional pollutant transport model in the porous medium, the results obtained through the online software Google Colaboratory in the Python language will be presented and analyzed. The influence of the parameter R on the concentration field will be observed, where the retardation factor is a measure of the soil's capacity to retain contaminants and slow down their movement.

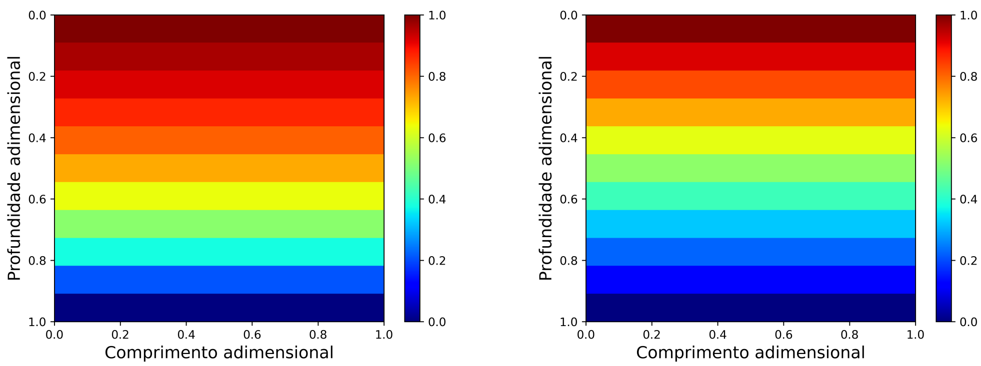

Figure 1: On the left side, $Pe = 2$, $R = 1$, and $\tau = 0.50$ were considered. On the right side, $Pe = 2$, $R = 5$, and $\tau = 0.50$ were considered.

Considering the same values for the parameters Pe and τ in both cases, it can be observed that the parameter R causes a change in the results of the pollutant concentration field. For $R = 1$, it can be seen that the pollutant reaches about 50% of the soil with elevated pollutant levels, approximately 0.7. By increasing the parameter to $R = 5$, it can be observed that the concentration levels are different; in this case, the pollutant reaches about 30% of the soil with a level of 0.7. In other words, from the figure, it is evident that as the value of R increases, the dimensionless concentration will be lower, meaning that the concentration of the contaminant is retained by the porosity of the soil. Therefore, the soil retardation factor parameter is an essential consideration for understanding and managing soil and groundwater contamination, aiming to protect human health and the environment.

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EXACT BOUDARY CONTROLABILLITY FOR THE KORTEWEG-DE VRIES ON BOUDED DOMAINS

ERIC A. S. SANTOS[1,†]

Instituto de Matemática, UFAL, AL, Brasil[1]

eric.santos@im.ufal.br[†]

Abstract

This work is developed in a dissertation where studying controlabillity for the linearized Korteweg-de Vries equation, and the nonlinear Korteweg-de Vries equation in bounded domains. The aim of this work is to prove the controllability of these systems through the Hilbert's Uniqueness Principle.

1 Introduction

Given two states, one initial and one final, is it possible to move the system of control from the initial state to the final state? We consider nonlinear partial differential equation of KdV as a system control.

The well-posedness nonlinear equation of KdV is given by

$$\begin{cases} y_t + y_x + yy_x + y_{xxx} = 0, \;\; t \in (0,T), \;\; x \in (0,L), \\ y(t,0) = y(t,L) = 0, \;\; y_x(t,L) = u(t), \;\; t \in (0,T), \\ y(0,x) = y^0(x), \;\; x \in (0,L). \end{cases} \tag{1}$$

The solution equation model solitary waves. The linearized equation of KdV is obtained by the operator

$$Af = -f_x - f_{xxx}, \quad \forall x \in D(A) := \left\{ f \in H^3(0,L); f(0) = f(L) = f_x(L) = 0 \right\}.$$

That is closed, and dense in $L^2(0,L)$. Note that both A and A^* are dissipative which proves existence and uniqueness (see [2]) of solution of (1).

Definition 1.1. (Controllability) *Let $T > 0$. The control system of the KdV equation is controlable in time T if and only if for every $y^0 \in L^2(0,L)$ and avery $y^1 \in L^2(0,L)$, there exist a $u \in L^2(0,T)$ such that the solution y of Cauchy problem (1) satisfies $y(T,.) = y^1$.*

2 Main Results

It is a curious fact that the linearized control system at (1) has the folowing condition.

Theorem 2.1. *Let $T > 0$. Let the set*

$$\mathcal{N} := \left\{ 2\pi \sqrt{\frac{j^2 + l^2 + jl}{3}}; j, l \in \mathbb{N} - \{0\} \right\} \tag{1}$$

The control sistem is controlable in time T iff $L \notin \mathcal{N}$.

Now we can reconfigure the controlabillity problem with duality between controllability and observability. So let $T > 0$. Let us define a linear map $F_T : L^2(0,T) \longrightarrow L^2(0,L)$ as following. Let $u \in L^2(0,T)$. Let $y \in C^0([0,T]; L^2(0,L))$ be the wake solution of the Cauchy problem (1) with $y^0 := 0$. Then $F_T(u) := y(T,.)$.

Proposition 2.1. *The control system of KdV equation is controllable in time T if and only if F_T is onto.*

Proposition 2.2. (Observability Inequalities) *Let H_1 and H_2 be two Hilbert spaces. Let F be a linear continuous map from H_1 into H_2. Then F is onto if and only if there exists $c > 0$ such that*

$$\|F^*(x_2)\|_{H_1} \geq c\|x_2\|_{H_2}, \forall x2 \in H_2.$$

The proofs of (1) is completed by $\mathbb{C}$-linear map $A : N_{T'} \longrightarrow N_{T'}$ using the lemma

Lemma 2.1. *Let $T > 0$ and $y^0 \in L^2(0, L)$ such that $y_x(., 0) = 0$ for all $y := S(t)y^0$ at semigroup form. If $L \in \mathcal{N}$ then $y^0 = 0$.*

Theorem 2.2. *Let $T > 0$ and $L > 0$. Then there exist $r_0 > 0$ such that for all $y^0, y^T \in L^2(0, L)$, with $\|y^0\| < r_0$, $\|y^T\| < r_0$, there exist a solution $y \in C([0, T]; L^2(0, L)) \cap L^2(0, T, H^1(0, L)) \cap W^{1,1}(0, T, H^{-2}(0, L))$*

$$\begin{aligned} y_t = (y_x + yy_x + y_{xxx}) \in D'(0, T, H^{-2}(0, L)); \\ y(\cdot, 0) = 0 \in L^2(0, T) \end{aligned} \tag{2}$$

satisfazendo $y(\cdot, 0) = y^0$, e $y(T, \cdot) = y^T$

By Hilbert's uniqueness principle there existe a linear map $\Pi : y^T \in L^2(0, L) \longrightarrow u_x(\cdot, L) \in L^2(0, T)$. Let F a nonlinear map definide by

$$F : L^2(0, T, H^1(0, L)) \longrightarrow B$$

$$y \longmapsto F(y) := S(\cdot)y^0 + \Psi_1 \circ \Gamma(y^T - S(T)y^0 + \Psi_2(yy_x)(T, \cdot)) + \Psi_2(-yy_x). \tag{3}$$

Using the Banach's fixed point contraction we prove that there exist a control u such thath the nonlinear system control is locally controllable.

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A LYAPUNOV FUNCTION FOR FULLY NONLINEAR DEGENERATE EQUATIONS

ESTER BEATRIZ[1,†] & PHILLIPO LAPPICY[2,‡]

Instituto de Ciências Matemáticas e de Computação, USP, SP, Brasil[1], Universidade Complutense de Madrid, Espanha[2]

esterbiaeb@usp.br[†] phillipolappicy@gmail.com[‡]

Abstract

Energy (or Lyapunov) functions are used to prove stability of equilibria, or to indicate a gradient-like structure of a dynamical system. Matano constructed a Lyapunov function for quasilinear non-degenerate parabolic equations [1]. We modify Matano's method to construct an energy formula for fully nonlinear degenerate parabolic equations.

1 Introduction

Lyapunov functions occur in physical systems that naturally lose energy, for example, certain dissipative dynamical systems. Mathematically, these functions are used to prove the stability of equilibrium points.
For certain PDEs, we know the existence of Lyapunov functions, and in particular cases, we know its specific formula. This is the case of the ρ-Laplacian equation,

$$u_t = (p-1)|u_x|^{p-2}u_{xx} + f(u), \tag{1}$$

where $p \in \mathbb{N}$ e $p > 2$. It energy is given by:

$$E = \int_0^1 \frac{|u_x|^p}{p} - \int_0^u f(u_1)du_1 \; dx. \tag{2}$$

In this setting, this present work consists of showing a method to construct a Lyapunov function for fully nonlinear degenerate parabolic equations:

$$f(x, u, u_x, u_{xx}, u_t) = f(x, u, p, q, r) = 0, \tag{3}$$

where $f \in C^2$, with $f_q \cdot f_r \leqslant 0$, $f_r \neq 0$ and $f_q(x, u, p, 0, 0) \not\equiv 0$.

2 Main Results

Given an differential equation as in (3), a Lyapunov function is a non-negative map E, such that:

$$\frac{dE}{dt}(u(x,t)) \leq 0, \tag{4}$$

along solutions $u(x,t)$ of (3). We follow the method proposed by H. Matano [1] for quasilinear equations, and by B. Fiedler and P. Lappicy [2] for fully nonlinear equations. First we rewrite (3) as

$$f_q(x, u, p, 0, 0)u_{xx} = F^0(x, u, p) + F^1(x, u, p, q, r). \tag{5}$$

We want to find a function $L(x, u, p)$ such that

$$E = \int_0^1 L(x, u, p) \; dx. \tag{6}$$

It is found through a function $g(x,u,p)$, according to

$$L_{pp} = f_q(x,u,p,0,0) \cdot exp(g(x,u,p)). \tag{7}$$

We prove the existence of $g(x,u,p)$ by the characteristic method, so we guarantee that $L(x,u,p)$ exists, and we recover it integrating twice the equation (7). Then we can obtain the desired:

$$\frac{dE}{dt} = -\int_0^1 exp(g)F^1 u_t \; dx \leqslant 0. \tag{8}$$

Therefore, if the characteristic equations have a solution g, we obtain a Lyapunov function for (3). This method allow us to calculate explicitly the Lyapunov function for some generalizations of the equation (3), for example:

$$u_t = (\rho-1)|u_x|^{\rho-2}u_{xx} + u_x^n, \tag{9}$$

with $n \in \mathbb{R}$. Using our method, we obtain the energy function:

$$E = k_* \int_0^1 \frac{(\rho-1)|u_x|^{\rho-n}}{(\rho-n)(\rho-(n+1))} - u \; dx, \tag{10}$$

for $n \neq \rho, \rho-1$ and $k_* \in \mathbb{R}$ a constant.
By modifying the method introduced by H. Matano, we expand the scope of equations that possess a Lyapunov function. We mention that our method doesn't work for all degenerate equation, since it depends on the existence of solutions of the ODE system obtained by the characteristic method, and furthermore, the equations need to comply with the restrictions. This doesn't occur, for example, for the Trudinger's equations:

$$(u^\alpha)_t = u_{xx}, \tag{11}$$

where $\alpha > 0$, since $f_r = 0$.

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SUSPENSION BRIDGE WITH INTERNAL DAMPING

LEANDRO ARAÚJO[1], CARLOS RAPOSO.[2], JOILSON RIBEIRO[2] & ARTHUR CUNHA[2]

Department of Exact and Technological Sciences, UESB, BA, Brasil[1], Departament of Mathematics, UFBA, BA, Brazil[2]

leandro.araujo@uesb.edu.br[†] carlos.raposo@ufba.br[‡] joilsonor@ufba.br[§] cunha.arthurc@gmail.com[§§]

Abstract

This manuscript deals with a suspension bridge model with internal damping. We use semigroup theory. The existence of solution is proved by applying the Lumer-Phillips theorem. Exponential stability is obtained due to the analyticity of the semigroup associated with the energy space.

1 Introduction

In this paper we study the existence of solutions and analyticity for the initial boundary value problem of a suspension bridge with internal damping of the type

$$u_{tt} - \alpha u_{xx} - \lambda(\varphi - u) + \gamma_1 u_t = 0, \tag{1}$$

$$\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + \lambda(\varphi - u) + \gamma_2 \varphi_t = 0, \tag{2}$$

$$\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma_3 \psi_t = 0. \tag{3}$$

The equations above are considering that the deck has negligible transversal section dimensions compared to the length (span of the bridge), it is modeled in Timoshenko's theory as a one-dimensional extensible beam of length L, see [5]. As in [3], where we are denoting by $\varphi = \varphi(x,t)$ the displacement of the cross-section on the point $x \in (0, L)$, by $\psi = \psi(x,t)$ the rotation angle of the cross-section and the suspender cables are assumed to be linear elastic springs with standard stiffness $\lambda > 0$. The constant $\alpha > 0$ is the elastic modulus of the string (holding the main cable to the deck). The positive coefficients ρ_1 and ρ_2 are the mass density and the moment of mass inertia of the beam, respectively. Moreover, b represents the cross section's rigidity coefficient, and k represents the elasticity's shear modulus. Finally, the constants $\gamma_1, \gamma_2, \gamma_3 > 0$ are the coefficients of the damping force.

System (1)-(3) is subject to initial data and Dirichlet boundary conditions

$$\begin{cases} u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), \; x \in (0,L), \\ \varphi(x,0) = \varphi_0(x), \;\; \varphi_t(x,0) = \varphi_1(x), \; x \in (0,L), \\ \psi(x,0) = \psi_0(x), \;\; \psi_t(x,0) = \psi_1(x), \; x \in (0,L), \end{cases} \tag{4}$$

$$\begin{cases} u(0,t) = u(L,t) = 0, \; t \geq 0, \\ \varphi(0,t) = \varphi(L,t) = 0, \; t \geq 0, \\ \psi(0,t) = \psi(L,t) = 0, \; t \geq 0. \end{cases} \tag{5}$$

We introduce the Hibert Space

$$\mathcal{H} = H_0^1(0,L) \times L^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \times H_0^1(0,L) \times L^2(0,L)$$

endowed with the following inner product,

$$\langle U, \tilde{U} \rangle_{\mathcal{H}} = \int_0^L v\overline{\tilde{v}} dx + \alpha \int_0^L u_x \overline{\tilde{u}}_x dx + \rho_1 \int_0^L w\overline{\tilde{w}} dx + \rho_2 \int_0^L z\overline{\tilde{z}} dx + b \int_0^L \psi_x \overline{\tilde{\psi}}_x$$
$$+ \lambda \int_0^L (\varphi - u)(\overline{\tilde{\varphi}} - \overline{\tilde{u}}) dx + k \int_0^L (\varphi_x + \psi)(\overline{\tilde{\varphi}}_x + \overline{\tilde{\psi}}) dx,$$

being $U=(u,v,\varphi,w,\psi,z)^T$ and $\tilde{U}=(\tilde{u},\tilde{v},\tilde{\varphi},\tilde{w},\tilde{\psi},\tilde{z})^T$, with $u_t=v, \varphi_t=w$ and $\psi_t=z$. With this notation, we rewrite (1) - (3) as the following first-order Cauchy problem

$$\begin{cases} U_t-\mathcal{A}U=0, \\ U(0)=U_0, \end{cases} \tag{6}$$

where

$$\mathcal{A}\,:\,D(\mathcal{A})\subset\mathcal{H}\to\mathcal{H},\ \text{with } D(\mathcal{A})=[H_0^1(0,L)\cap H^2(0,L)\times H_0^1(0,L)]^3$$

is defined by (6).

For existence of solution, the main idea is to use the well-known Lummer-Phillips Theorem (see [2]). As $D(\mathcal{A})$ is dense in $\mathcal{H}$, to get that $\mathcal{A}$ is the infinitesimal generator of $S(t)=e^{\mathcal{A}t}$, a C_0-semigroup of contractions on $\mathcal{H}$, we prove that $\mathcal{A}$ is dissipative and that $0\in\rho(\mathcal{A})$ the resolvent set of $\mathcal{A}$. We prove the following theorem.

Theorem 1.1. *Let $U_0\in\mathcal{H}$, then there exists a unique weak solution U of problem (6) satisfying $U\in C^0([0,+\infty);\mathcal{H})$. Moreover, if $U_0\in D(\mathcal{A})$, then $U\in C^0([0,+\infty);D(\mathcal{A}))\cap C^1([0,+\infty);\mathcal{H})$.*

2 Analiticity (Main Results)

The main result is the analiticity of semigroup associated. We prove that $i\mathbb{R}\subset\rho(\mathcal{A})$ and that

$$\overline{\lim_{|\beta|\to\infty}}\ \|\beta(i\beta I-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}<\infty, \tag{7}$$

so, by theorem 1.3.3, p. 5 of [2], we get

Theorem 2.1. *The semigroup $S(t)=e^{\mathcal{A}t},\ t\geq 0$, generated by $\mathcal{A}$ is analytic.*

As directly consequence, by using Geahart-Prüss-Greiner, see theorem 1.11, p. 302 of [1], we deduce,

Corollary 2.1. *The C_0-semigroup of contractions $S(t)=e^{\mathcal{A}t}, t\geq 0$, generated by $\mathcal{A}$ is exponentially stable.*

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STRONG STABILIZATION FOR A TIMOSHENKO BEAM SYSTEM WITH INTERNAL FRACTIONAL DAMPING

RAFAEL O. DE JESUS[1,†], CARLOS A. RAPOSO[2,‡] & OCTAVIO V. VILLAGRAN[3,§]

Mathematics Collegiate, UPE, PE, Brasil[1], Departament of Mathematics, UFBA, BA, Brazil[2],
Department of Mathematics, University of Tarapacá, Arica, Chile[3]

rafael.oliveira@upe.br[†] carlos.raposo@ufba.br[‡] opverav@academicos.uta.cl[§]

Abstract

In this article, we consider the system with Timoshenko beam system with internal fractional damping in a bounded domain. We study the strong stabilization, and the well posed for a Timoshenko beam system with different dissipations of the type fractional integro-differential operators with weight exponential.

1 Introduction

We consider the system

$$\rho_1\phi_{tt}(x,t) - k\left(\phi_x(x,t)+\psi(x,t)\right)_x + a\partial_t^{\alpha,\eta}\phi(x,t) = 0 \quad \text{in} \quad (0,L)\times(0,+\infty) \tag{1}$$

$$\rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + k\left(\phi_x(x,t)+\psi(x,t)\right) + c\partial_t^{\beta,\zeta}\psi(x,t) = 0 \quad \text{in} \quad (0,L)\times(0,+\infty) \tag{2}$$

where $\eta,\zeta \geq 0$, $0<\alpha<1$, $0<\beta<1$ and ρ_1,ρ_2,k,a,b,c positive real constants.

The dampers used are of the type fractional integro-differential operators with exponential weight, i.e.

$$\partial_t^{\omega,\xi} f(t) = \frac{1}{\Gamma(1-\omega)}\int_0^t (t-s)^{-\omega}e^{-\xi(t-s)}f'(s)\;ds,$$

where $0<\omega<1$, $\xi\geq 0$ and $f\in W([0,L);X)$. For to prove the well posed we write the equations as augmented system: where $\gamma_1 = \frac{a\sin\alpha\pi}{\pi} = \frac{a}{\Gamma(\alpha)\Gamma(1-\alpha)}$, $p(y)=|y|^{\frac{2\alpha-1}{2}}$, $\gamma_2 = \frac{c\sin\beta\pi}{\pi} = \frac{c}{\Gamma(\beta)\Gamma(1-\beta)}$ and $q(y)=|y|^{\frac{2\beta-1}{2}}$.

We introduce the functions $u=\phi_t$, $v=\psi_t$, and transform the initial boundary value problem (**??**)-(**??**) into an abstract problem

$$\begin{cases} U_t = \mathcal{A}U;\; t>0, \\ U(0)=U_0. \end{cases} \tag{3}$$

on Hilbert space $\mathcal{H} = [H_0^1(0,L)]^2\times[L^2(0,L)]^2\times[L^2(\mathbb{R};L^2(0,L))]^2$, where $U_0 = (\phi^0,\psi^0,\phi^1,\psi^1,0,0)^T$ and

$$\mathcal{A}:\mathcal{D}(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$$

is the linear operator defined by

$$\mathcal{A}U = \begin{pmatrix} u \\ v \\ \frac{1}{\rho_1}\left[k\left(\phi_x+\psi\right)_x - \gamma_1\displaystyle\int_{\mathbb{R}} p(y)\varphi_1(y)\;dy\right] \\ \frac{1}{\rho_2}\left[b\psi_{xx} - k\left(\phi_x+\psi\right) - \gamma_2\displaystyle\int_{\mathbb{R}} q(y)\varphi_2(y)\;dy\right] \\ -\left(|y|^2+\eta\right)\varphi_1(y)+p(y)u \\ -\left(|y|^2+\zeta\right)\varphi_2(y)+q(y)v \end{pmatrix}. \tag{4}$$

2 Main Results

Theorem 2.1. *If $U_0 \in \mathcal{H}$, the Cauchy problem (3) admits a unique weak solution*

$$U \in C^0\left([0, +\infty); \mathcal{H}\right),$$

given by $U(t) = e^{t\mathcal{A}} U_0$.

If $U_0 \in \mathcal{D}(\mathcal{A})$, then the solution obtained is a strong solution with the following regularity

$$U \in C^0\left([0, +\infty); \mathcal{D}\left(\mathcal{A}\right)\right) \cap C^1\left([0, +\infty); \mathcal{H}\right).$$

Theorem 2.2. *The C_0-semigroup of contraction $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is strongly stable on $\mathcal{H}$, i.e.*

$$\lim_{t\to+\infty} \left\|e^{t\mathcal{A}U_0}\right\|_{\mathcal{H}} = 0;\ \forall\ U_0 \in \mathcal{H}.$$

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EXPONENTIAL STABILITY OF THE VON KÁRMÁN SYSTEM WITH INTERNAL DAMPING

ROSEANE MARTINS[1], CARLOS RAPOSO.[1], JOILSON RIBEIRO[1] & OCTAVIO VERA[2]

Departament of Mathematics, UFBA, BA, Brazil[1], Department of Mathematics, University of Tarapacá, Arica, Chile[2]

roseane.smartins@hotmail.com[†] jessicamanghi@gmail.com[‡] joilsonor@ufba.br[§] opverav@academicos.uta.cl[§§]

Abstract

This work deals with a von Kármán system with internal damping. For the solution's existence, we use nonlinear semigroup theory tools. We construct an evolution system by nonlinear Lipschitz perturbation of a semigroup of contractions. We apply the energy method for the asymptotic behavior, which uses suitable multipliers to construct a Lyapunov functional that leads to exponential decay.

1 Introduction

In this paper, we study the existence of solution and asymptotic behavior for the initial boundary value problem of the von Kármán beam system of the type

$$\begin{cases} \rho A w_{tt} - EA\left[\left(u_x + \frac{1}{2}w_x^2\right)w_x\right]_x + EIw_{xxxx} = 0 \quad \text{in} \quad (0,\, L)\times(0,\, T), \\ \rho A u_{tt} - EA\left[u_x + \frac{1}{2}w_x^2\right]_x = 0 \quad \text{in} \quad (0,\, L)\times(0,\, T). \end{cases} \tag{1}$$

where $w(x,\, t)$ is the transverse displacement of a generic point, $u(x,\, t)$ the longitudinal displacement, $(0,\, L)$ is the segment occupied by the beam, and T is a given positive time. The physical parameters represent the properties of the material being E the Young's modulus, A the cross-sectional area of the beam, L the beam length, ρA the weight per unit length and EI the beam stiffness or flexural rigidity. The model (1) was proposed by J. E. Lagnese and J. L. Lions, see [3, 4].

Here we are interested in studying the existence of solution and asymptotic behavior, considering frictional damping, which is a natural problem, given by

$$\begin{cases} w_{tt} - b_1\left[\left(u_x + \frac{1}{2}w_x^2\right)w_x\right]_x + b_2 w_{xxxx} + a_1 w_t = 0 \quad \text{in} \quad (0,\, L)\times(0,\, T), \\ u_{tt} - b_1\left[u_x + \frac{1}{2}w_x^2\right]_x + a_2 u_t = 0 \quad \text{in} \quad (0,\, L)\times(0,\, T). \end{cases} \tag{2}$$

We consider the initial data and boundary conditions, respectively

$$\begin{cases} w(x,\, 0) = w_0(x),\ w_t(x,\, 0) = w_1(x), \\ u(x,\, 0) = u_0(x),\ u_t(x,\, 0) = u_1(x), \end{cases} \tag{3}$$

$$\begin{cases} u(0,\, t) = u(L,\, t) = 0, \\ w(0,\, t) = w(L,\, t) = 0, \\ w_x(0,\, t) = w_x(L,\, t) = 0. \end{cases} \tag{4}$$

Now, we introduce the Hilbert space

$$\mathcal{H} = H_0^2(0,\, L)\times L^2(0,\, L)\times H_0^1(0,\, L)\times L^2(0,\, L),$$

equipped with the inner product given by

$$\langle U,\, \widetilde{U}\rangle_{\mathcal{H}} = b_2\int_0^L w_{xx}\overline{\widetilde{w}}_{xx}dx + \int_0^L \varphi\overline{\widetilde{\varphi}}dx + b_1\int_0^L u_x\overline{\widetilde{u}}_x dx + \int_0^L \psi\overline{\widetilde{\psi}}dx, \tag{5}$$

where $U = (w,\, \varphi,\, u,\, \psi)^T$, $\widetilde{U} = (\widetilde{w},\, \widetilde{\varphi},\, \widetilde{u},\, \widetilde{\psi})^T$, we introduce the functions $\varphi = w_t$ and $\psi = u_t$. We now wish to transform the initial boundary value problem (2)-(4) to an abstract problem in the Hilbert space $\mathcal{H}$. Rewrite the system (2)-(4) as the following initial value problem

$$\begin{cases} U_t = \mathcal{A}U + \mathcal{F}(U), \\ U(0) = (w_0,\, \varphi_0,\, u_0,\, \psi_0)^T, \quad ,\forall\, t > 0, \end{cases} \tag{6}$$

The domain of operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$D(\mathcal{A}) = H^4(0,\, L) \cap H_0^2(0,\, L) \times H_0^2(0,\, L) \times H^2(0,\, L) \cap H_0^1(0,\, L) \times H_0^1(0,\, L).$$

The main idea is to consider the nonlinear evolution system (6) as a locally Lipschitz perturbation $\mathcal{F}$ of a linear contraction semigroup $S(t) = e^{\mathcal{A}t}$ on $\mathcal{H}$. Since the nonlinear term $\mathcal{F}$ is locally Lipschitz, then abstract results (see [2], Chap. 6 and [5],Theorem 7.1) on the generation of nonlinear semigroups apply in order to conclude the existence of a nonlinear semigroup on $\mathcal{H}$. Nonlinear semigroup theory also implies that for initial data taken from the domain of the generator, the corresponding solutions are continuous in time with the values in $\overline{D(\mathcal{A})}$. For an outline of the proof, see [[6], Appendix]. Thus strong solutions possesses the property $\in C([0,\, T), \mathcal{H})$.

To get $S(t) = e^{\mathcal{A}t}$ on $\mathcal{H}$, we will use the well known the Lumer-Phillips theorem (see [2]) and $\mathcal{F}$ is locally Lipschitz we adapt the idea as in [1], Lemma 3.

Our solution existence result is given by

Theorem 1.1. *If $U_0 \in \mathcal{H}$, then problem (6) has a unique mild solution $U(t) = e^{\mathcal{A}t}U_0 + \int_0^t e^{\mathcal{A}(t-s)}\mathcal{F}(U(s))ds$, $U \in C([0,\, \infty) : \mathcal{H})$, with $U(0) = U_0$. Moreover, if $U_0 \in D(\mathcal{A})$ the mild solution is a strong solution globally defined.*

2 Asymptotic behaviour (Main result)

Theorem 2.1. *Let $(w,\, u)$ be a solution of (2) where the initial data are given in $D(\mathcal{A})$. Then, the energy $\mathcal{E}(t)$ satisfies* $\mathcal{E}(t) \leq C\mathcal{E}(0)e^{-\alpha t}, \quad \alpha, C > 0, \quad for\ all\ t > 0.$

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