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CONVERGENCE RESULTS FOR THE PRIMAL HYBRID METHOD WITH SERENDIPITY BASED SPACES ON QUADRILATERALS

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Abstract

In this work, we discuss some of the main results of [1], where the authors have proven sufficient, and in some cases also necessary, conditions to obtain optimal convergence rates for the Primal Hybrid Method (PHM) on meshes of general convex quadrilaterals. These results were obtained by combining the original analysis of [2] for the PHM on affine meshes with approximation results for finite element spaces on general quadrilateral meshes discussed in [3]. Numerical results are presented to illustrate the application of the theoretical findings when Serendipity based approximation spaces are used.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded and convex domain with a Lipschitz continuous boundary $\partial\Omega$. The Darcy problem consists of finding the pressure $p \in H^1(\Omega)$ in the flow of an incompressible fluid in a rigid saturated porous medium such that

$$-\operatorname{div}(\mathcal{K}\nabla p) = f \qquad \text{in} \quad \Omega \tag{1a}$$

$$p = g \qquad \text{on} \quad \partial\Omega, \tag{1b}$$

where $f \in L^2(\Omega)$ is a given source/sink function and $g \in H^{1/2}(\partial\Omega)$ describes the Dirichlet boundary conditions on $\partial\Omega$.

The symmetric and uniformly positive definite tensor $\mathcal{K} = \mathcal{K}(x)$ represents the permeability of the porous matrix divided by the fluid viscosity [1].

Finite element methods to solve problem (1a) involve the construction of finite-dimensional spaces X_h, where the approximate pressure p_h is sought. To construct such spaces, we first need a mesh for Ω, i.e., a subdivision $\mathcal{T}_h$ of Ω into non-overlapping sub-domains K, called elements. Each element $K \in \mathcal{T}_h$ is described as the image of an isomorphism F_K acting over a reference element $\hat{K}$. The usual construction of X_h begins by defining a finite-dimensional subspace $\hat{X} \subset H^1(\hat{K})$, which is then mapped to local spaces $X_K \subset H^1(K)$ through the isomorphisms F_K. The spaces X_K are then tied together to construct the global approximation space X_h.

For the case of meshes of parallelograms, $\hat{K}$ is usually the unitary square $[0,1] \times [0,1]$, and in this case, the isomorphisms F_K are bilinear and affine transformations. If we enable general convex elements in our meshes, the transformations F_K will remain bilinear but no longer affine. In [3], the authors analyze the approximation properties of finite element spaces generated through non-affine bilinear transformations. Those properties are used in [1] to expand the analysis of the PHM, developed for affine meshes in [2], to general quadrilateral meshes. Here we use the theory of [1] to discuss the use of Serendipity based spaces in the PHM.

2 The Primal Hybrid Method

For $r \geq 2$, let $S_r(\hat{K})$ denote the Serendipity spaces in two dimensions as defined in [4], and $S_r^+(\hat{K})$ the space spanned by the polynomials of $S_r(\hat{K})$ plus a function v_0, as defined in [2]. Denote by $E_m(\partial\hat{K})$ the space of all functions defined over $\partial\hat{K}$ whose restrictions to any edge are polynomials of degree less or equal $m \geq 0$.

In addition to the approximation space X_h, the PHM also needs an auxiliary space M_h. When Serendipity spaces are used, those spaces are defined as

$$X_h^r = \{v \in L^2(\Omega) \,:\, \forall\, K \in \mathcal{T}_h\,,\, v|_K \in X_K\}, \tag{2}$$

$$M_h^m = \left\{ \mu \in \prod_{K\in\mathcal{T}_h} \Lambda_K : \mu|_{\partial K_1} + \mu|_{\partial K_2} = 0 \text{ on } K_1 \cap K_2\,,\ \text{ for every pair of adjacent elements } K_1, K_2 \in \mathcal{T}_h \right\}, \tag{3}$$

with X_K and Λ_K being local spaces given by

$$X_K = \{v \in H^1(K) \,:\, v = \hat{v} \circ F_K^{-1}\,,\, \hat{v} \in S_r^+(\hat{K})\} \quad \text{ and } \quad \Lambda_K = \{\mu \in L^2(\partial K) \,:\, \mu = \hat{\mu} \circ F_K^{-1}\,,\, \hat{\mu} \in E_m(\partial\hat{K})\}. \tag{4}$$

The PHM with Serendipity based spaces is then defined as: Find the pair $(p_h, \lambda_h) \in X_h \times M_h$ such that

$$\sum_{K\in\mathcal{T}_h} \int_K (\mathcal{K}\nabla p_h)\cdot\nabla v \,\mathrm{dx} + \sum_{K\in\mathcal{T}_h} \int_{\partial K} v\lambda_h \,\mathrm{ds} = \int_\Omega f v \,\mathrm{dx} \qquad \forall\, v \in X_h^r \tag{5a}$$

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} p_h\mu \,\mathrm{ds} = \sum_{e\in\varepsilon_{\partial\Omega}} \int_e g\mu \,\mathrm{ds} \quad \forall\, \mu \in M_h^m, \tag{5b}$$

where $\varepsilon_{\partial\Omega}$ denotes the set of all edges e on the boundary of Ω. It follows from the analysis of [2] that, if $r \geq m+1$, the problem (5) always admit a unique solution (p_h, λ_h).

3 Convergence results for Serendipity based spaces

Now, we present our main result, concerning the convergence of the variable p_h on affine and non-affine quadrilateral meshes. This result, summarized in Theorem 6, is a direct application of the analysis developed in [1].

Theorem 3.1. *Consider $\mathcal{T}_h$ a regular quadrilateral mesh for Ω according to [3]. Let X_h^r and M_h^m be the spaces constructed in Section 2 satisfying $r \geq m+1$, and p_h be the solution for the discrete problem 5. Assuming that the exact solution p is regular enough, there is a constant C independent of the mesh parameter h such that*

$$\|p - p_h\|_{0,\Omega} \leq C h^s |p|_{r+1,\Omega}, \tag{6}$$

where $s = \min\{r, m+1\} + 1$ for affine meshes and $s = \min\{\lfloor r/2 \rfloor, m+1\} + 1$ for non-affine ones.

Theorem 3.1 indicates that the Serendipity based PHM achieves optimal convergence orders on affine meshes, but only sub-optimal orders on non-affine ones. These theoretical estimates are confirmed by numerical experiments.

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A SIMPLIFIED METHOD FOR THE RESPONSE ANALYSIS OF STOCHASTIC VIBRATION SYSTEMSWITH NONLINEAR ADJUSTABLE STIFFNESS

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Abstract

In many practical situations, stochastic vibration systems subject to random excitations should be integrated over very long time intervals. In this circumstance conventional methods usually show exploding behavior or their implementation is computationally demanding. In this work we propose and analyze a simplified weak simulation method for the integration of nonlinear vibration systems with adjustable stiffness under random excitations. We prove that it reproduces the same statistical properties that characterize the exact solution to the linearized equation associated to the underlying system. Computational simulations are carried out to illustrate its practical performance.

1 Introduction

In many situations of direct engineering interest, structures are subject to random excitations with a mixture of different intensities of external noisy forces (e.g., structures under earthquake, wind or sea wave excitations). This randomness has an important impact on the vibratory behavior of mechanical and structural systems [2], [4], [5], [6]. To study these systems deterministic models are not adequate. Thus, physics-based modeling through stochastic differential equations is one of the approaches commonly used for the proper mathematical description of the phenomena. A relevant mathematical model for studying nonlinear random vibration problems is the nonlinear stochastic systems [3].

$$\begin{aligned} u''(t) + \eta u'(t) + f(u(t)) + \alpha Y(t) &= c\xi(t), \\ Y'(t) + \beta Y(t) &= u'(t), \end{aligned} \tag{1}$$

where $u(t)$ represents the displacement response, f is a nonlinear stiffness function, α, β, η are parameters, vector $c \in \mathbb{R}^{1\times m}$ is the amplitude of the random forcing, and $\xi(t) = (\xi^1(t), \ldots, \xi^m(t))^\intercal$ is a vector of zero mean Gaussian white noise excitations with $\mathbb{E}(\xi^i(t_1)\xi^j(t_2)) = Q_{ij}\delta(t_1 - t_2)$, where $Q = [Q_{ij}]$ is the $(m \times m)$-matrix of excitation intensities and $\delta(\cdot)$ is the Dira's delta function. Several approximate techniques have been systematically developed over the years to evaluate the stochastic responses of (1). However, there are limitations in the practical applications of these techniques, especially for the response analysis over long time intervals. Hence, the search for alternative efficient simulation methods is currently of great interest.

In this work we propose a numerical simulation method for (1). We devise the integrator regarding its efficiency and capability to preserve meaningful statistical features of the system. We carry out the construction based on the weak approximation approach which gives us much more freedom as to the generation of the necessary random variables. With respect to its long-term behavior, it is notable that for any value of the step-size of integration, the approximate solution obtained by the proposed method shares the same statistical properties that characterize the exact solution to the linearized system corresponding to the underlying system (1). As revealed by our studies, the method is valuable for evaluating late-time statistics of nonlinear vibration systems.

2 The proposed simplified weak method and main results

Let $Z(t) = (u(t), Y(t), u'(t))^{\intercal}$, then starting from $Z_0 = Z(0)$, we construct the approximations $\{Z_n\}$ to $Z(t_n)$ $(n = 1, \ldots, N)$ as follows. For each time interval $[t_n\ t_{n+1}]$ we rewrite (1) in the semilinear form

$$\left[dZ(t) = AZ(t)dt + g(Z(t))dt + \sum_{j=1}^{m} b_j dW_t^j, \right]$$

where W_t^j are uncorrelated standard Wiener processes and b_j are those values corresponding to the excitation intensities matrix Q of the white noise process $\xi(t)$. The proposed method is

$$Z_{n+1} = e^{Ah} Z_n + \frac{h}{2}(g(Z_n) + g(e^{Ah} Z_n + U) + AU) + \sqrt{h} \sum_{j=1}^{m} b_j \zeta_n^j, \quad U = g(Z_n)h + \sqrt{h} \sum_{j=1}^{m} b_j \zeta_n^j,$$

where ζ_n^j is a three-point distributed random variable [1] satisfying

$$\mathbb{P}(\zeta_n^j = \sqrt{3}) = \frac{1}{6}; \quad \mathbb{P}(\zeta_n^j = -\sqrt{3}) = \frac{1}{6}; \quad \mathbb{P}(\zeta_n^j = 0) = \frac{2}{3}.$$

We have the following results concerning the rate of weak convergence and the long time behavior of the method:

Theorem 2.1. *The weak order of accuracy of the method is 2. That is,*

$$\max_{1 \le n \le N} |\mathbb{E}(\phi(Z(t_n))) - \mathbb{E}((\phi(Z_n))| = \mathcal{O}(h^2)$$

where $\phi : \mathbb{R}^3 \to \mathbb{R} \in C_P^k(\mathbb{R}^3, \mathbb{R})$

Theorem 2.2. *For any step-size h, the numerical realization given by the proposed integrator replicates the long time properties that characterize the exact solution to the linearized equation corresponding to the underlying continuous system (1). That is, for any step-size h there exist a unique matrix D_h such that the numerical sample trajectory $\{Z_n\}_{n=0,\ldots}$ satisfies*

$$\lim_{n \to \infty} \mathbb{E}((Z_n) = \mathbf{0}; \quad \lim_{n \to \infty} Cov(Z_n) = D_h$$

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ANALYSIS NUMERICAL OF A STABILIZED HYBRID FINITE ELEMENT METHODS FOR THE HELMHOLTZ PROBLEM

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Abstract

In this work, error estimates are shown for the stabilized hybrid finite element methods studied in [6] for the Helmholtz equation with Robin's Boundary Condition. These methods use Lagrange multipliers associated with the pressure trace and they are introduced to weakly enforce continuity on the finite element interfaces. The choice of the Lagrange multiplier in this way allows the use of the static condensation technique, which consists of solving local problems in each element, resulting in a global system that involves only the degrees of freedom associated with the multipliers of Lagrange. Assuming that the exact value of the Lagrange multiplier is known, we present a numerical analysis of the problems local, obtaining the sub-optimal convergence rate in the energy norm and optimal in the L^2 norm. Next, we determine the estimate of the global error in the L^2 norm.

1 Introduction

We will study the Helmholtz equation

$$-\Delta p - \kappa^2 p = f, \quad \text{em} \quad \Omega \tag{1}$$

$$-\nabla p \cdot \mathbf{n} + \mathrm{i}\kappa p = g, \quad \text{em} \quad \partial\Omega \tag{2}$$

To derive the method, we first define a Helmholtz equation (1) at each K mesh element $\mathcal{T}_h$ with the interface conditions defined on the interior edges: $[|p|]|_e = 0$ and $[|\nabla p|]|_e = 0, \quad \forall e \in \mathcal{E}_h^0$, and the Robin condition defined on the edges of the boundary. We obtain the weak formulation of the local problem and introduce the multiplier de Lagrange (λ_h) defined as the trace of the pressure. We include a symmetrization term and a stabilization term on ∂K and formulate the following local problems about each K element:
Local Problems: For each $K \in \mathcal{T}_h$, find $p_h \in \mathbb{S}_l(K)\,[= \mathbb{P}_l(K) \text{ or } \mathbb{Q}_l(K)]$, such that

$$\begin{aligned}(\nabla p_h, \nabla \bar{q}_h)_K - \kappa^2 (p_h, \bar{q}_h)_K - \langle \nabla p_h \cdot \mathbf{n}, \bar{q}_h\rangle_{\partial K} - \langle \nabla \bar{q}_h \cdot \mathbf{n}, p_h - \lambda_h\rangle_{\partial K} \\ +\beta \langle p_h - \lambda_h, \bar{q}_h\rangle_{\partial K} = (f, \bar{q}_h)_K, \quad \forall \bar{q}_h \in \mathbb{S}_l(K).\end{aligned} \tag{3}$$

To complete the system above, we add a global equation relative to the multiplier and Robin's condition:
Global Problem: Find $\lambda_h \in \mathcal{M}_h^s = \left\{\lambda \in M : \lambda|_e = \mathbb{P}_s(e), \forall e \in \mathcal{E}_h^0\right\}$, such that

$$\langle \nabla p_h \cdot \mathbf{n}, \bar{\mu}_h\rangle_{\partial \mathcal{T}_h} - \mathrm{i}\kappa \langle \lambda_h, \bar{\mu}_h\rangle_{\partial\Omega} + \beta \langle \lambda_h - p_h, \bar{\mu}_h\rangle_{\partial \mathcal{T}_h} = -\langle \mathbf{g}, \bar{\mu}_h\rangle_{\partial\Omega}, \quad \forall \bar{\mu}_h \in \mathbb{S}_l(K) \tag{4}$$

In the equation (4) the first term weakly imposes the continuity of the normal flow component and the third term concerns the stabilization of the Lagrange multiplier related to the pressure trace.

2 Main Results

For the numerical analysis we first use the local projection of the primal variable, defined inside each element, to the exact multiplier λ. To formally introduce the problem associated with λ, we define the operators $\mathcal{A}_K(P_h, q_h)$, $\mathcal{B}_K(\lambda, q_h)$ and the linear functional $\mathcal{F}_K(q_h)$ from (3) such that

$$\begin{aligned} \mathcal{A}_K(P_h, q_h) &:= (\nabla P_h, \nabla \bar{q}_h)_K - \kappa^2 (P_h, \bar{q}_h)_K - \langle \nabla P_h \cdot \mathbf{n}, \bar{q}_h \rangle_{\partial K} - \langle \nabla \bar{q}_h \cdot \mathbf{n}, P_h \rangle_{\partial K} + \beta \langle P_h, \bar{q}_h \rangle_{\partial K} \quad (1) \\ \mathcal{G}_K(q_h) &:= \mathcal{F}_K(q_h) - \mathcal{B}_K(\lambda, q_h) = (f, \bar{q}_h)_K - \langle \nabla \bar{q}_h \cdot \mathbf{n}, \lambda \rangle_{\partial K} + \beta \langle \lambda, \bar{q}_h \rangle_{\partial K} \quad (2) \end{aligned}$$

Thus, the problem corresponding to the exact value of the multiplier is given by:

For given λ, find $P_h \in \mathbb{S}_l(K)$ such that

$$\mathcal{A}_K(P_h, q_h) = \mathcal{G}_K(q_h), \quad \forall q_h \in \mathbb{S}_l(K). \quad (3)$$

Theorem 2.1. [1] *If $P_h \in \mathbb{S}_l(K)$ is the approximate solution of the problem (3) obtained by the* **LDGl** *method when the Lagrange multiplier λ is given and the exact solution p of the problem (1)-(2) belongs to $H^{l+1}(K)$ then we have the errors in the energy norm and in the norm L^2:*

$$|||p - P_h|||_{LDGl} \leq C(K, \alpha_{lc}, \alpha_{le}) h^l |p|_{H^{l+1}(K)} \quad \text{and} \quad \|p - P_h\|_K \leq \widehat{C}(K, \alpha_{lc}, \alpha_{le}) \kappa h^{l+1} |p|_{H^{l+1}(K)}, \quad (4)$$

respectively, where $C(K, \alpha_{lc}, \alpha_{le})$ and $\widehat{C}(K, \alpha_{lc}, \alpha_{le})$ are constants that depend of the finite element K, α_{lc} and α_{le}.

Theorem 2.2. [2] *If $[p_h, \lambda_h] \in \mathcal{W}_h^l \times \mathcal{M}_h^s$ is the approximate solution of the problem (3)-(4), $P_h \in \mathbb{S}_l(K)$ is the approximate solution of the problem (3) obtained by* **LDGl** *method when the Lagrange multiplier λ is given and the exact solution p of the problem (1)-(2) belongs to $H^{l+1}(\mathcal{T}_h)$ then we have the following estimate:*

$$\|p - p_h\|_{\mathcal{T}_h} \leq \sqrt{2M_1\rho^*}\kappa h^{l+1} |p|_{H^{l+1}(\mathcal{T}_h)} + \sqrt{2hM_2\rho^*}\delta \|\lambda - \lambda_h\|_{\partial\mathcal{T}_h}$$

where $\mathcal{W}_h^l = \left\{p \in L^2(\mathcal{T}_h) : p|_K \in \mathbb{S}_l(K), \forall K \in \mathcal{T}_h\right\}$, $M_1 = \max\left\{\widehat{C}^2(K, \alpha_{lc}, \alpha_{le}), K \in \mathcal{T}_h\right\}$, $M_2 = \max\left\{\dfrac{C_1(K)}{\alpha_{le}^2}, K \in \mathcal{T}_h\right\}$ and ρ^* is the number of finite elements K.

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DYSON'S SPLIT ACTION FORMULA FOR TRANSPORT OPERATORS

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Abstract

Dyson's formula is presented. Its proof stems from the Lax-Richtmyer stability of convergent discretizations, which is the "principle of uniform boundedness" of Numerical Analysis. Among its applications is a method for the dynamical stabilization (i.e. initial value sensitivity reduction) of a non-hydrostatic cloud-resolving atmospheric model.

Lemma: (Dyson's formula for matrices) Let $\mathcal{M}(k)$ be the set of real square matrices of order k. If $\mathrm{A}, \mathrm{B} \in \mathcal{M}(k)$ and $x_0 \in \mathbb{R}^k$, then $e^{t(\mathrm{A}+\mathrm{B})}x_0 = e^{t\mathrm{A}}x_0 + \int_0^t e^{(t-s)(\mathrm{A}+\mathrm{B})}\mathrm{B}e^{s\mathrm{A}}x_0\, ds$.

proof: Let $x(t) = e^{t(\mathrm{A}+\mathrm{B})}x_0 - e^{t\mathrm{A}}x_0$. Taking the derivative of $x(t)$, one obtains the initial value problem

$$x'(t) = (\mathrm{A}+\mathrm{B})x(t) + \mathrm{B}e^{t\mathrm{A}}x_0, \quad t \in \mathbb{R}, \tag{1}$$

$$x(0) = 0 \tag{2}$$

Eq. 3 is a non-homogeneous linear ordinary differential equation with constant coefficient. Duhamel's formula for the solution of problem 3-2 is $x(t) = \int_0^t e^{(t-s)(\mathrm{A}+\mathrm{B})}\mathrm{B}e^{s\mathrm{A}}x_0\, ds$. Thus $e^{t(\mathrm{A}+\mathrm{B})}x_0 - e^{t\mathrm{A}}x_0 = \int_0^t e^{(t-s)(\mathrm{A}+\mathrm{B})}\mathrm{B}e^{s\mathrm{A}}x_0\, ds$. □

Theorem: (Dyson's formula for transport operators) If

1. Ω is an bounded open set in $\mathbb{R}^n$ with volume V;
2. $a, b : \Omega \to \mathbb{R}^n$ are $\{\mathrm{C}_b^1(\Omega)\}^n$ vector fields with flow functions defined for all $t \geq 0$;
3. $\mathrm{S}(t) : \mathrm{C}_b^1(\Omega) \to \mathrm{C}_b^1(\Omega)$, $t \geq 0$, is the solution operator of $\partial_t u(t,x) = a(x) \cdot \nabla u(t,x)$ with $\mathrm{S}(0)u_0 = u_0$;
4. $\mathrm{T}(t) : \mathrm{C}_b^0(\Omega) \to \mathrm{C}_b^0(\Omega)$, $t \geq 0$, is the solution operator of $\partial_t w(t,x) = (a(x) + b(x)) \cdot \nabla w(t,x)$ with $\mathrm{T}(0)w_0 = w_0$;

then

$$\mathrm{T}(t)u_0 = \mathrm{S}(t)u_0 + \int_0^t \mathrm{T}(t-s)\,\mathrm{B}\,\mathrm{S}(s)u_0\, ds,$$

for all $u_0 \in \mathrm{C}_b^1(\Omega)$, where $\mathrm{C}_b^0(\Omega)$ is the space of bounded continuous functions on Ω and $\mathrm{C}_b^1(\Omega)$ is the space of bounded continuous functions with bounded continuous partial derivatives on Ω, and $\mathrm{B} = b(x) \cdot \nabla : \mathrm{C}_b^1(\Omega) \to \mathrm{C}_b^0(\Omega)$.

proof: Let $\mathrm{A}_k, \mathrm{B}_k : \mathbb{R}^{p(k)} \to \mathbb{R}^{p(k)}$ be convergent discretizations of A and B with uniform mesh width Δx, where $k = \mathrm{V}/(\Delta x)^n$ is the number of grid cells and $p(k)$ is the number of grid points. Let $\mathrm{N}/\Delta x$, $\mathrm{N} > 0$, be a bound to the infinite operator norm of B_k, so that $||\mathrm{B}_k\,[v]||_k \leq (\mathrm{N}/\Delta x)\,||[v]||_k$ for all $v \in \mathrm{C}_b^1(\Omega)$, where $[f]$ denotes a grid vector and $||[f]||_k = \max_{i=1,\ldots,p(k)} |[f]_i|$ for $[f] \in \mathbb{R}^{p(k)}$ (so that $||[f]||_k \to ||f||$ as $k \to +\infty$ upon $\Delta x \to 0$,

for any bounded $f : \Omega \to \mathbb{R}$, where $||\cdot||$ is the sup norm). Let also $\mathrm{T}_k(t), \mathrm{S}_k(t) : \mathbb{R}^{p(k)} \to \mathbb{R}^{p(k)}$ be convergent discretizations of $\mathrm{T}(t)$ and $\mathrm{S}(t)$ with uniform time step Δt and uniform mesh width Δx associated to A_k and B_k, which converge as $k \to +\infty$ upon $\Delta x \to 0$ and $\Delta t \to 0$ provided $\Delta t \leq \mathrm{M}\,\Delta x$ for some $\mathrm{M} > 0$. Then:
$||\mathrm{T}_k(t-s)\,\mathrm{B}_k\,\mathrm{S}_k(s)[u_0] - [\mathrm{T}(t-s)\,\mathrm{B}\,\mathrm{S}(s)u_0]||_k$
$= ||\mathrm{T}_k(t-s)\,\mathrm{B}_k\,\mathrm{S}_k(s)[u_0] \pm \mathrm{T}_k(t-s)\,[\mathrm{B}\,\mathrm{S}(s)u_0] \pm \mathrm{T}_k(t-s)\,\mathrm{B}_k\,[\mathrm{S}(s)u_0] - [\mathrm{T}(t-s)\,\mathrm{B}\,\mathrm{S}(s)u_0]||_k$
$= ||\mathrm{T}_k(t-s)\,\big(\mathrm{B}_k\,(\mathrm{S}_k(s)[u_0] - [\mathrm{S}(s)u_0]) + (\mathrm{B}_k[\mathrm{S}(s)u_0] - [\mathrm{BS}(s)u_0])\big) + \big(\mathrm{T}_k(t-s)\,[\mathrm{BS}(s)u_0] - [\mathrm{T}(t-s)\mathrm{BS}(s)u_0]\big)||_k$
$\leq |||\mathrm{T}_k(t-s)|||_k\,\big(||\mathrm{B}_k\,(\mathrm{S}_k(s)[u_0] - [\mathrm{S}(s)u_0])||_k + ||\mathrm{B}_k[v_0(s)] - [\mathrm{B}v_0(s)]||_k\big) + ||\mathrm{T}_k(t-s)[w_0(s)] - [\mathrm{T}(t-s)w_0(s)]||_k$,
where $v_0(s) = \mathrm{S}(s)u_0$, $w_0(s) = \mathrm{BS}(s)u_0$ and $|||\mathrm{T}_k(t)|||_k = \sup\{||\mathrm{T}_k(t)[w]||_k : [w] \in \mathbb{R}^{p(k)}, ||[w]||_k = 1\}$. Since, according to Lax's Equivalence Theorem [1], $\mathrm{T}_k(t)$ is Lax-Richtmyer stable provided $\Delta t \leq \mathrm{M}\,\Delta x$, i.e. the set $\{|||\mathrm{T}_k(t)|||_k : k \in \mathbb{N}^*, \Delta x = \sqrt[n]{\mathrm{V}/k}, \Delta t \leq \mathrm{M}\,\Delta x\}$ is bounded, and since $||\mathrm{B}_k\,v_k(s)||_k \leq (\mathrm{N}/\Delta x)\,||v_k(s)||_k \to 0$ if $||v_k(s)||_k = \emptyset(\Delta x)$, which is the case for $v_k(s) = \mathrm{S}_k(s)[u_0] - [\mathrm{S}(s)u_0]$ if $\mathrm{S}_k(s)$ is first order accurate in space, one concludes that $\mathrm{T}_k(t-s)\,\mathrm{B}_k\,\mathrm{S}_k(s)[u_0] \to \mathrm{T}(t-s)\,\mathrm{B}\,\mathrm{S}(s)u_0$ as $k \to +\infty$ upon $\Delta x \to 0$ and $\Delta t \to 0$, provided $\Delta t \leq \mathrm{M}\,\Delta x$. Hence $\mathrm{T}_k(t)[u_0] - \mathrm{S}_k(t)[u_0] - \mathrm{R}_m(t)[u_0] \to \mathrm{T}(t)u_0 - \mathrm{S}(t)u_0 - \int_0^t \mathrm{T}(t-s)\,\mathrm{B}\,\mathrm{S}(s)u_0\,ds$ as $k, m \to +\infty$ upon $\Delta x \to 0$ and $\Delta t \to 0$, provided $\Delta t \leq \mathrm{M}\,\Delta x$, where $\mathrm{R}_m[u_0] = \sum_{i=0}^m (\mathrm{T}_k(t-s_i)\,\mathrm{B}_k\,\mathrm{S}_k(s_i)[u_0])\Delta t$, $m = t/\Delta t$, $s_i = s_{i-1} + \Delta t$, $s_0 = 0$. But $\mathrm{T}_k(t)[u_0] - \mathrm{S}_k(t)[u_0] - \mathrm{R}_m(t)[u_0] \to e^{t(\mathrm{A}_k+\mathrm{B}_k)}[u_0] - e^{t\mathrm{A}_k}[u_0] + \int_0^t e^{(t-s)(\mathrm{A}_k+\mathrm{B}_k)}\mathrm{B}_k e^{s\mathrm{A}_k}[u_0]\,ds$ as $m \to +\infty$ upon $\Delta t \to 0$ with k and Δx fixed, and this limit value vanishes according to the above Lemma. Therefore $\mathrm{T}(t)u_0 - \mathrm{S}(t)u_0 - \int_0^t \mathrm{T}(t-s)\,\mathrm{B}\,\mathrm{S}(s)u_0\,ds = 0$. □

Among the applications of Dyson's formula is a method, named IBM, for the dynamical stabilization (i.e. initial value sensitivity reduction) of a non-hydrostatic cloud-resolving atmospheric model, named CRM, both presented in [2]. Figure (1) shows some prediction error profiles obtained with the CRM under three different prediction schemes. The error profiles obtained with the IBM method are stable.

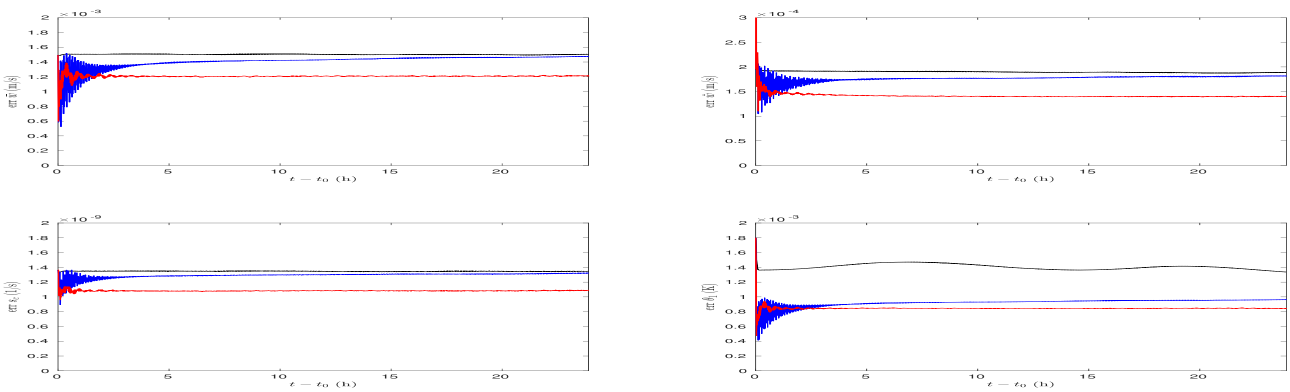

Figure 1: Prediction error profiles obtained without model initialization (black), with model initialization (blue) and with model initialization & use of the IBM method (red) for the grid vertical velocity (frame (1,1)), subgrid vertical velocity (frame (1,2)), condensation rate (frame (2,1)) and temperature (frame (2,2)). The atmospheric truth was generated by the CRM.

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ON LATTICE-ALMOST COPIES OF $C_0(\Gamma)$ AND $L_1(\Gamma)$ IN BANACH LATTICES

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Abstract

For a set Γ, we denote by $l_\infty(\Gamma)$ the Banach space of all bounded families $(a_\gamma)_{\gamma\in\Gamma}$ endowed with the sup norm. Let $c_0(\Gamma)$ be the closed subspace of $l_\infty(\Gamma)$ such that for each $\varepsilon > 0$ the set $\{\gamma \in \Gamma : |a_\gamma| \geq \varepsilon\}$ is finite. Also let $l_1(\Gamma)$ be the Banach space of all summable families $(a_\gamma)_{\gamma\in\Gamma}$ endowed with the usual l_1-norm. When Γ is countable, we denote $c_0(\Gamma)$ and $l_1(\Gamma)$ by c_0 and l_1 respectively. In this talk we show the results obtained in [4] about the corresponding lattice version of James distortion theorems for $c_0(\Gamma)$ and $l_1(\Gamma)$: if a Banach lattice contains lattice copy of $c_0(\Gamma)$ ($l_1(\Gamma)$ respectively), then it contains lattice-almost isometric copies of $c_0(\Gamma)$ ($l_1(\Gamma)$ respectively).

1 Introduction

Let X and Y be Banach spaces. We say that Y contains a copy of X if there is an isomorphism from X into Y. We also say that X and Y are almost-isometric if for each $\varepsilon > 0$ there exists a linear isomorphism $T_\varepsilon : X \to Y$ such that $\|T_\varepsilon\|\|T_\varepsilon^{-1}\| \leq 1 + \varepsilon$. Finally, we say Y contains almost-isometric copies of X, if for each $\varepsilon > 0$ there exists a subspace Z_ε of Y which is almost-isometric to X.

A classical result of James establishes that if a Banach space X contains a copy of c_0 (respectively l_1), then X contains almost isometric copies of c_0 (respectively l_1). This is so called James distortion theorem for c_0 and l_1 (see [3, Lemma 2.1 and 2.2]). Now, in real Banach lattices we say that Y contains a lattice copy of X if X is Banach-lattice isomorphic to a sub-lattice of Y; we also say that Y contains lattice-almost isometric copies of X if for each $\varepsilon > 0$ there exist a sub-lattice Z_ε of Y and a Banach-lattice isomorphism $T_\varepsilon : X \to Y$ such that $\|T_\varepsilon\|\|T_\varepsilon^{-1}\| \leq 1 + \varepsilon$.

Meanwhile, Chen in [1, Theorem 2] and [2, Theorem 1] proved that if a Banach lattice contains a lattice copy of c_0 (respectively l_1), then it contains lattice-almost copies of c_0 (respectively l_1). The aim of the talk is to prove a version for $c_0(\Gamma)$ (respectively $l_1(\Gamma)$).

To give a proof, is necessary to fix some notations: If A is a set, and τ is an ordinal, then $|A|$ and $[A]^{<\tau}$ denote, respectively, the cardinality of A and the family of all subsets of A with cardinality less than τ. We denote by ω the first infinite ordinal. If X is a Banach lattice, B_X, X^+ and B_X^+ denote respectively the closed united ball of X, the positive cone of X and the set $B_X \cap X^+$.

2 Main Results

Theorem 2.1 (Main theorem)**.** *Let X be a Banach lattice. If X contains a lattice copy of* $c_0(\Gamma)$ *(respectively* $l_1(\Gamma)$*), then X contains lattice-almost isometric copies of* $c_0(\Gamma)$ *(respectively* $l_1(\Gamma)$*).*

To give a proof for this result, we need some lemmas:

Lemma 2.1. *Let X be a Banach lattice. Then X contains a lattice copy of* $c_0(\Gamma)$ *iff there exists a disjoint family* $(x_\gamma)_{\gamma\in\Gamma}$ *in X^+ satisfying* $\inf\{||x|| : \gamma \in \Gamma\} > 0$ *and* $\sup\left\{\left|\left|\sum_{\gamma\in\Gamma} x_\gamma\right|\right| : F \in [\Gamma]^{<\omega}\right\} < \infty$.

Lemma 2.2. *Let X be a Banach lattice containing a lattice copy of $c_0(\Gamma)$. Then for each $0<\delta<1$, there exists a disjoint family $(u_\gamma)_{\gamma\in\Gamma}$ in B_X^+ such that for each $F\in[\Gamma]^{<\omega}$ and any set of scalars $\{a_\gamma:\gamma\in F\}$ we have*

$$(1-\delta)\max_{\gamma\in F}|a_\gamma|\leq\left\|\sum_{\gamma\in F}a_\gamma u_\gamma\right\|\leq\max_{\gamma\in F}|a_\gamma|.$$

Lemma 2.3. *Let X be a Banach lattice. Then X contains a lattice copy of $l_1(\Gamma)$ iff there exists a disjoint family $(x_\gamma)_{\gamma\in\Gamma}$ in X^+ and two positive constants m,M such that*

$$m\sum_{\gamma\in F}|a_\gamma|\leq\left\|\sum_{\gamma\in F}a_\gamma x_\gamma\right\|\leq M\sum_{\gamma\in F}|a_\gamma|$$

for all $F\in[\Gamma]^{<\omega}$ and every family of scalars $\{a_\gamma:\gamma\in F\}$.

Lemma 2.4. *Let X be a Banach lattice containing a lattice copy of $l_1(\Gamma)$. Then for each $\delta>0$, there exists a disjoint family $(u_\gamma)_{\gamma\in\Gamma}$ in B_X^+ such that for each $F\in[\Gamma]^{<\omega}$ and any set of scalars $\{a_\gamma:\gamma\in F\}$ we have*

$$(1-\delta)\max_{\gamma\in F}|a_\gamma|\leq\left\|\sum_{\gamma\in F}a_\gamma u_\gamma\right\|\leq\max_{\gamma\in F}|a_\gamma|.$$

Corollary 2.1 (Main theorem)**.** *Let X be a Banach space containing a copy of $c_0(\Gamma)$. Then X contains almost isometric copies of $c_0(\Gamma)$.*

Proof. Let $\varepsilon>0$ be given and choose $0<\delta<1$ such that $\frac{1}{1+\varepsilon}<1-\delta$. By Lemma 2.2, there exists $(u_\gamma)_{\gamma\in\Gamma}$ in B_X^+ such that for each $F\in[\Gamma]^{<\omega}$ and any set of scalars $\{a_\gamma:\gamma\in F\}$ we have

$$(1-\delta)\max_{\gamma\in F}|a_\gamma|\leq\left\|\sum_{\gamma\in F}a_\gamma u_\gamma\right\|\leq\max_{\gamma\in F}|a_\gamma|.$$

Let $T_\varepsilon\colon c_0(\Gamma)\to X$ be defined as $T_\varepsilon((x_\gamma)_{\gamma\in\Gamma})=\sum_{\gamma\in\Gamma}x_\gamma u_\gamma$, for $(x_\gamma)_{\gamma\in\Gamma}\in c_0(\Gamma)$. Then T_ε is a Banach-lattice isomorphism from $c_0(\Gamma)$ to X with $\|T_\varepsilon\|\|T_\varepsilon^{-1}\|\leq 1+\varepsilon$. ☐

Corollary 2.2 (Main theorem)**.** *Let X be a Banach space containing a copy of $l_1(\Gamma)$. Then X contains almost isometric copies of $l_1(\Gamma)$.*

Proof. The proof follows the same lines of the proof of Corollary 2.1, with $l_1(\Gamma)$ instead of $c_0(\Gamma)$. ☐

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A COMPOSITION OPERATOR APPROACH TO THE INVARIANT SUBSPACE PROBLEM

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Abstract

Let H be a complex Hilbert space. An operator $U \in B(H)$ is called universal for H (in the sense of Rota) if for every operator $T \in B(H)$ there exists a scalar $\alpha \in \mathbb{C}$, an invariant subspace M of U and an isomorphism $S : M \to H$ such that $\alpha T = S \circ U_{|M} \circ S^{-1}$. The concept of universal operator provides an alternative approach to the Invariant Subspace Problem (ISP). In this work we use recent results due to Carmo and Noor [1] and study the composition operator $C_{\phi_a} \in B(H^2)$ with symbol given by $\phi_a = az + 1 - a$ for $0 < a < 1$. The main objective is to characterize the minimal invariant subspaces of C_{ϕ_a} and make progress towards the ISP. Here, we shall discuss some recent results and advances in this direction. This work is in collaboration with Waleed Noor and João R. Carmo.

1 Introduction

The Invariant Subspace Problem (ISP) is one of the most important open problems in operator theory: given a complex, infinite dimensional and separable Hilbert space H, does every bounded linear operator have a non-trivial invariant subspace? By a non-trivial invariant subspace of T we mean a closed subspace $\{0\} \subset N \subset H$ such that $T(N) \subseteq N$.

There exists a lot of approaches and remarkable results about this question; one of these approaches is based on the notion of Universal Operators developed by Rota in [2]. Given a universal operator U, it is possible to prove that the ISP is true if, and only if, every minimal invariant subspace of U has dimension 1. Recently, Carmo and Noor [1] found a new class of universal composition operators on the Hardy-Hilbert space.

The Hardy-Hilbert space of the disk is denoted by $H^2(\mathbb{D})$ or simply H^2 and defined as:

$$H^2(\mathbb{D}) := \{f \in Hol(\mathbb{D}) \ | \ f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{with} \ \sum_{n=0}^{\infty} |a_n|^2 < \infty\}.$$

It is not difficult to see that H^2 becomes a separable and infinite dimensional Hilbert space when we consider the inner product given by $\langle f, g\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$. A special class of bounded linear operators in H^2 are the composition operators: given a holomorphic function $\phi : \mathbb{D} \to \mathbb{D}$, the composition operator with symbol ϕ is defined in H^2 and given by $C_\phi(f) = f \circ \phi$ for every $f \in H^2$. In this direction a special type of symbols play a central role: the hyperbolic maps. A linear fractional map $\phi : \mathbb{D} \to \mathbb{D}$ is called hyperbolic if it has two distinct fixed points outside $\mathbb{D}$.

2 Main Results

Our starting point is the following theorem proved in [1]:

Theorem 2.1. *If ϕ is a linear fractional self-map of $\mathbb{D}$ then $C_\phi - \lambda$ is universal in $H^2(\mathbb{D})$ for some $\lambda \in \mathbb{C}$ if, and only if, ϕ is hyperbolic.*

Our approach to ISP consists in understanding the minimal invariant subspaces of C_{ϕ_a} where ϕ_a $(0 < a < 1)$ is the hyperbolic map given by $\phi_a(z) = az + 1 - a$. More clearly, we have the following equivalence:

Corollary 2.1. *Let $0 < a < 1$ and consider ϕ_a as defined above. Then the ISP has a positive solution if, and only if, every minimal invariant subspace of C_{ϕ_a} has dimension 1.*

If M is a minimal invariant subspace of the operator C_{ϕ_a} we see that

$$M = K_f := \overline{span\{f, C_{\phi_a} f, C_{\phi_a}^2 f, \dots\}}$$

for every $f \in M$ with $f \neq 0$, so we can study these subspaces in relation to the function-theoretic properties of f. In [1] Carmo and Noor initiated this approach and obtained many results about these minimal invariant subspaces. Recently, we achieved a partial solution supposing some additional properties of f. Given $f \in H^2$ we consider three cases.

1. $\lim_{n\to\infty} f(1 - a^n) = L \neq 0$
2. $\lim_{n\to\infty} f(1 - a^n) = 0$
3. $\lim_{n\to\infty} f(1 - a^n)$ does not exists.

Our main Theorem ensures that with an additional hypothesis, the cases 1. and 3. are well understood.

Theorem 2.2. *Let $f \in H^2$ such that $f' \in H^2$. Suppose that $\lim_{n\to\infty} f(1 - a^n)$ exists and isn't 0 or $\lim_{n\to\infty} f(1 - a^n)$ doesn't exists. If K_f is minimal then K_f has dimension 1.*

Supposing that $f, f' \in H^2$, the only open case for understand K_f is the case 2; in fact if we suppose that f is analytic at 1 or if $f(1 - a^n)$ goes to 0 slowly enough then the conclusion of Theorem 2.2 also holds.

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DENSITY AND ORTHOGONALITY IN H^2 AND ZETA ZEROS

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Abstract

Báez-Duarte's criterion asserts that the Riemann hypothesis (RH) is equivalent to the density of the linear span of a particular sequence in $L^2([0,1])$. This work deals with a unitarily equivalent version of this result in the Hardy space H^2 of the unit disk. It poses density and orthogonality questions arising naturally from this criterion, leading to weak versions of RH. Furthermore, a sufficient conditions for obtaining zero free half-planes for the zeta function is given. This is a joint work with Waleed Noor (IMECC/Unicamp), A. Ghosh and K. Kremnitzer (Oxford University, UK).

1 Introduction

Let ζ be defined initially as $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for complex s such that $\operatorname{Re} s > 1$. It can be analytically extended to the whole plane, except for a simple pole at $s = 1$. Consider the half-planes $\Omega_r = \{s \in \mathbb{C} : \operatorname{Re} s > r\}$. An old problem, still open, is to determine whether ζ does not vanish in Ω_r for some $1/2 \leq r < 1$. The case $r = 1/2$ corresponds to the Riemann hypothesis. Nyman [5] obtained a criterion in terms of density and approximation in $L^2([0,1])$, generalized later by Beurling [3] to the L^p context and refined – in the L^2 case – by Báez-Duarte [1] (see [2]). In [4], Báez-Duarte criterion is brought to H^2 by means of a unitary isomorphism. Given $p > 0$, H^p is the Hardy space of exponent p in the unit disc $\mathbb{D}$:

$$H^p = \left\{ f : \mathbb{D} \to \mathbb{C} \ : \ f \text{ is holomorphic and } \|f\|_p := \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \right)^{1/p} < \infty \right\}.$$

Each H^p is an F-space not locally convex if $p < 1$, and a Banach space if $p \geq 1$. In particular, H^2 is a separable Hilbert space.

Theorem 1.1 (4, Theorem 8)**.** *Let $\mathcal{N}$ be the linear span of $\{h_k\}_{k=2}^{\infty}$, where*

$$h_k(z) = \frac{1}{1-z} \log\left(\frac{1 + z + \cdots + z^{k-1}}{k} \right), \qquad z \in \mathbb{D}, \ k \geq 2\,.$$

The Riemann hypothesis is true if and only if $\mathcal{N}$ is dense in H^2.

This work concerns with two problems directly related to the criterion above.

- *The density problem*: find topologies in H^2, weaker than the one generated by the norm, with respect to which $\mathcal{N}$ is dense.
- *The orthogonality problem*: find classes of functions $V \subset H^2$, as large as possible, satisfying $\mathcal{N}^{\perp} \cap V = \{0\}$.

Partial answers to these questions provide weak versions of RH, i.e., assertions implied by RH but proven unconditionally.

2 Main results

Proposition 2.1. *For any $0 < p < 1$, $\mathcal{N}$ is dense in H^p.*

Proof There are two main steps in this proof: *(i)* The linear span of $\{g_k\}_{k=2}^{\infty}$ is dense in H^r for any $r > 0$, where $g_k(z) = (1-z)h_k(z)$. *(ii)* Given $p \in (0,1)$, there exists $r > 0$ large enough such that the operator of multiplication by $1/(1-z)$ is continuous from H^r to H^p. ■

Given $s \in \mathbb{C}$, consider the linear functional defined initially in the space of analytic polynomials as

$$\Lambda^{(s)}(1) = -\frac{1}{s} \quad \text{and} \quad \Lambda^{(s)}(z^k) = -\frac{1}{s}\left[(k+1)^{1-s} - k^{-s}\right],\ k \geq 2\,.$$

Proposition 2.2. *Given* $p \in (0,2]$ *and* $s \in \Omega_{1/p}$, $\Lambda^{(s)}$ *is bounded on* H^p *and satisfies* $\Lambda^{(s)}(h_k) = -\frac{\zeta(s)}{s}(k^{-s} - k^{-1})$.

Corollary 2.1. *If* $\mathcal{N}$ *is dense in* H^p *then* ζ *does not have zeros in* $\Omega_{1/p}$.

Proof Under the hypothesis above, the constant 1 is the H^p-limit of a sequence $\{f_n\} \subset \mathcal{N}$. If $\zeta(s) = 0$ then $\Lambda^{(s)}(f_n) = 0$ for all n, contradicting the fact that $\Lambda^{(s)}(1) \neq 0$. ■

Therefore, if one proves the density of $\mathcal{N}$ in H^p for any $p > 1$, a consequence we will be a previously unknown zeta zero free half-plane, which would be a huge achievement from the arithmetical viewpoint. The proven result for $p < 1$ furnishes the already known half-plane Ω_1.

Proposition 2.3. *Given* $\alpha > 1/2$, $\operatorname{span}\{(1-z)^\alpha h_k \,:\, k \geq 2\}$ *is dense in* H^2.

Proposition 2.4. *Given* $\alpha \in (1/2, 1]$*, the equality* $\mathcal{N}^\perp \cap V = \{0\}$ *holds for*

$$V = (1-z)^\alpha H^2 + \mathbb{C}\,.$$

Proof In [6], the space V above is identified as a de Branges-Rovnyak space, which in its turn is proven in [7] to be the domain of the adjoint of the (unbounded) multiplication operator of symbol $(1-z)^{-\alpha}$. Finally, it is an elementary verification that the intersection of this domain with $\mathcal{N}^\perp$ is trivial using Proposition 2.3.

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RIGIDITY OF COMBINATORIAL BANACH SPACES

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Abstract

Combinatorial spaces are Banach spaces whose norm is induced by families of finite subsets of some index set. Inspired by the classical Banach-Stone theorem, we investigate the impact of the properties of the family to the collection of isometries of the corresponding Banach space. We characterize the isometries as signed permutations of the canonical basis and try to determine which permutations induce isometries.

1 Introduction

Rigid objects in mathematics are objects with few automorphisms. In the context of Banach spaces, there are classical results on rigidity, where the automorphisms are linear isometries. Given a Banach space X, let $Iso(X)$ be the collection of all linear isometric bijections $T : X \to X$.

If K is a compact space, let $C(K)$ be the Banach space of all scalar valued continuous functions defined on K, with the supremum norm. Given a homeomorphism $\varphi : K \to K$ and $g \in C(K)$ such that $|g(x)| = 1$ for every $x \in K$, the map $T_{\varphi,g} : C(K) \to C(K)$ given by $T_{\varphi,g}(f) = g \cdot (f \circ \varphi)$ is an isometry. Hence,

$$Iso(C(K)) \supseteq \{T_{\varphi,g} : \varphi : K \to K \text{ homeomorphism } g \in C(K) \text{ such that } |g(x)| = 1\}.$$

Theorem 1.1 (Banach-Stone)**.** *Given compact Hausdorff spaces K and L, $T : C(K) \to C(L)$ is an isometry iff there are a homeomorphism $\varphi : L \to K$ and $g \in C(L)$ such that $|g(y)| = 1$ for every $y \in L$ and $T = T_{\varphi,g}$. Therefore,*

$$Iso(C(K)) = \{T_{\varphi,g} : \varphi : K \to K \textit{ homeomorphism } g \in C(K) \textit{ such that } |g(x)| = 1\}.$$

Turning to the classical sequence spaces c_0 or ℓ_p's, we have that given a bijection $\pi : \mathbb{N} \to \mathbb{N}$ and a sequence $\bar{\theta} = (\theta_n)_n$ such that $|\theta_n| = 1$ for every $n \in \mathbb{N}$, the map $T_{\pi,\bar{\theta}} : X \to X$ given by $T_{\pi,\bar{\theta}}(e_n) = \theta_n e_{\pi(n)}$ is an isometry, if $X = c_0$ or $x = \ell_p$ for some $1 \leq p < \infty$. Hence,

$$Iso(X) \supseteq \{T_{\pi,\bar{\theta}} : \pi : \mathbb{N} \to \mathbb{N} \text{ bijection, } \bar{\theta} = (\theta_n)_n \text{ such that } |\theta_n| = 1\}.$$

Theorem 1.2 (folklore)**.** *For $X = c_0$ or $X = \ell_p$, $1 \leq p < \infty$, $p \neq 2$, given an isometry $T : X \to X$, there are a bijection $\pi : \mathbb{N} \to \mathbb{N}$ and a sequence $(\theta_n)_n$ such that $|\theta_n| = 1$ for every $n \in \mathbb{N}$ and $T(e_n) = \theta_n e_{\pi(n)}$ for every $n \in \mathbb{N}$. Therefore,*

$$Iso(X) = \{T_{\pi,\bar{\theta}} : \pi : \mathbb{N} \to \mathbb{N} \textit{ bijection, } \bar{\theta} = (\theta_n)_n \textit{ such that } |\theta_n| = 1\}.$$

Similar results also exist for the Tsirelson space or for nonseparable versions of c_0 and ℓ_p. In this work, we discuss results in the same direction for the so called combinatorial Banach spaces.

2 Main Results

Given a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ containing all singletons, we can define a norm on $c_{00} = \{(\lambda_n)_n : \{n \in \mathbb{N} : \lambda_n \neq 0\}$ is finite$\}$ as follows:

$$\|(\lambda_n)_n\|_{\mathcal{F}} = \sup\{\sum_{n\in s} |\lambda_n| : s \in \mathcal{F}\}.$$

The completion of c_{00} is called the combinatorial space induced by $\mathcal{F}$ and is denoted by $X_{\mathcal{F}}$. Notice that c_0 and ℓ_1 can be seen as the combinatorial spaces of the families of all singletons and that of all finite subsets of $\mathbb{N}$, respectively.

Without loss of generality, the family $\mathcal{F}$ can be assumed to be closed under subsets and $\mathcal{F}$ can be embedded in the Cantor space, by identifying each element to its characteristic function. Combinatorial and topological properties of $\mathcal{F}$ imply geometrical properties of $X_{\mathcal{F}}$. For example, if $\mathcal{F}$ is compact, the Schauder basis $(e_n)_n$ is shrinking.

In [1] investigate the rigidity properties of $X_{\mathcal{F}}$ and prove the following result:

Theorem 2.1. *For every regular families $\mathcal{F}$ and $\mathcal{G}$, given an isometry $T : X_{\mathcal{F}} \to X_{\mathcal{G}}$, there are a bijection $\pi : \mathbb{N} \to \mathbb{N}$ and a sequence $(\theta_n)_n$ such that $|\theta_n| = 1$ for every $n \in \mathbb{N}$ and $T(e_n) = \theta_n e_{\pi(n)}$. Therefore,*

$$Iso(X_{\mathcal{F}}) \subseteq \{T_{\pi,\bar{\theta}} : \pi : \mathbb{N} \to \mathbb{N} \text{ bijection, } \bar{\theta} = (\theta_n)_n \text{ such that } |\theta_n| = 1\}.$$

Our main purpose is to present the techniques involved in the proof, as well as generalizations of this result published in [2].

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INHOMOGENEOUS CANCELLATION CONDITIONS AND CALDERÓN-ZYGMUND TYPE OPERATORS ON $H^P(\mathbb{R}^N)$

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Abstract

In this talk we present a new approach to atoms and molecules on Goldberg's local Hardy spaces $h^p(\mathbb{R}^n)$, $0 < p \leq 1$, assuming an appropriate cancellation condition. As applications, we prove improved continuity results for inhomogeneous Calderón-Zygmund operators on these spaces. Joint work with Galia Dafni and Chun Ho Lau (Concordia University).

1 Introduction

It is well known that the homogeneous Hardy spaces $H^p(\mathbb{R}^n)$, defined for $0 < p < \infty$ represents, in certain aspects, a good substitute for $L^p(\mathbb{R}^n)$ when $0 < p \leq 1$. However, this breaks down in certain aspects; for instance, $H^p(\mathbb{R}^n)$ are not closed under multiplication by test functions, since this may destroy the global vanishing moment conditions; they do not contain $\mathcal{S}(\mathbb{R}^n)$, the Schwartz space; they are not well defined in manifolds and in general pseudodifferential operators are not bounded on $H^p(\mathbb{R}^n)$. For this reason, Goldberg in [3] introduced the local (or inhomogeneous) version of Hardy spaces, which he called *local Hardy spaces* and denoted by $h^p(\mathbb{R}^n)$. These spaces contains $H^p(\mathbb{R}^n)$, are equal to $L^p(\mathbb{R}^n)$ when $p > 1$, and satisfies the desired properties, in particular if $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $f \in h^p(\mathbb{R}^n)$ then $\varphi f \in h^p(\mathbb{R}^n)$.

From the comparison between $H^p(\mathbb{R}^n)$ and $h^p(\mathbb{R}^n)$, a natural atomic decomposition for $h^p(\mathbb{R}^n)$ arises, which allows to write any tempered distribution $f \in h^p(\mathbb{R}^n)$ as an infinite linear combination $\sum_j \lambda_j a_j$ of atoms a_j, with $\|f\|_{h^p} \approx \inf \left(\sum |\lambda_j|^p\right)^{1/p}$ over all such decompositions. An atom is a function supported in a ball which satisfies a size condition relative to that ball, and vanishing moment conditions when the atom is supported in small balls, that is

$$\int a(x)x^\alpha dx = 0, \ \text{ for all } \ |\alpha| \leq N_p := n(1/p - 1) \ \text{ if } \ \text{supp}\,(a) \subset B(x_0, r) \ \text{ with } \ r < 1.$$

Several properties and applications of $h^p(\mathbb{R}^n)$ follow from this important decomposition. For instance, if $T : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is a linear and continuous operator, then its extension and continuity on $h^p(\mathbb{R}^n)$ can be established by just verifying that $\|Ta_j\|_{H^p} \leq C$ uniformly. For the homogeneous case $H^p(\mathbb{R}^n)$, we can also derive a more general decomposition in terms of molecules, usually denoted by M, in which we do not require compact support.

In contrast to $H^p(\mathbb{R}^n)$, the molecular theory of $h^p(\mathbb{R}^n)$ for $0 < p \leq 1$ is still not completely established. Komori [4], defined molecules for $n/(n+1) < p < 1$, assuming a uniform control of the zeroth moment of the molecule, that is

$$\left| \int M(x)dx \right| \leq C.$$

This estimate holds trivially for the case $p = 1$ and is therefore not sufficient to characterize $h^1(\mathbb{R}^n)$.

2 Main Results

We established in [1] a new atomic and molecular characterization of $h^p(\mathbb{R}^n)$, for all $0 < p \leq 1$, and apply it to show improved continuity results for inhomogeneous Calderón-Zygmund operators, that includes pseudodifferential operators in the class $OpS^0_{1,0}(\mathbb{R}^n)$, on $h^p(\mathbb{R}^n)$. The key is to introduce inhomogeneous cancellation conditions on both the operators and the atoms and molecules. In particular, we extend Komori's molecular approach to $p \leq n/(n+1)$ and $p = 1$, by giving different cancellation properties when $p = n/(n+k)$ for $k \in \mathbb{Z}^+$, compared to $n/(n+k+1) < p < n/(n+k)$. In fact, we call M an approximate molecule if it satisfies the standard size conditions, relative to the ball $B(x_0, r) \subset \mathbb{R}^n$, and the following cancellation condition

$$\left| \int_{\mathbb{R}^n} M(x)(x - x_0)^\alpha dx \right| \leq C, \text{ if } |\alpha| < n(1/p - 1) \tag{1}$$

and

$$\left| \int_{\mathbb{R}^n} M(x)(x - x_0)^\alpha dx \right| \leq \left[\log\left(1 + \frac{C}{r}\right)\right]^{-1/p} \text{ if } |\alpha| = N_p = n(1/p - 1). \tag{2}$$

An atomic decomposition of $h^p(\mathbb{R}^n)$ in terms of approximate atoms with analogous cancellation conditions (1) and (2) were also established.

As an application of the approximate atomic and molecular decomposition developed, we showed the following improved continuity result for inhomogeneous Calderón-Zygmund operators, extending results of [2].

Theorem 2.1. *Let $0 < p \leq 1$ and T be an (δ, μ) inhomogeneous Calderón-Zygmund with $\delta, \mu > 0$. Then T can be extended to a bounded operator from $h^p(\mathbb{R}^n)$ to itself provided that $\min\{\mu, \delta\} > n(1/p - 1)$ and there exists $C > 0$ such that for any ball $B(x_0, r) \subset \mathbb{R}^n$ with $r < 1$ and $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leq N_p$,*

$$f = T^*[(\cdot - x_0)^\alpha] \quad \textit{satisfies} \quad \left(\frac{1}{|B|} \int_B |f(y) - P_B^{N_p}(f)(y)|^2 dy \right)^{1/2} \leq C\, \Psi_{p,\alpha}(r), \tag{3}$$

where $P_B^{N_p}(f)$ is the polynomial of degree $\leq N_p$ that has the same moments as f over B up to order N_p, and

$$\Psi_{p,\alpha}(t) := \begin{cases} t^{n(1/p-1)} & \text{if } |\alpha| < n(1/p-1), \\ t^{n(1/p-1)} \left[\log\left(1 + \frac{C}{r}\right)\right]^{-1/p} & \text{if } |\alpha| = n(1/p-1) = N_p. \end{cases}$$

□

One can also replace (3) by the stronger condition that $f \in \dot{\Lambda}_{n(1/p-1)}(\mathbb{R}^n)$ if $|\alpha| < n(1/p-1)$ and $f \in L^{2,\Psi_{p,\alpha}}_{N_p}(\mathbb{R}^n)$, the generalized Campanato space, if $n(1/p-1) \in \mathbb{Z}$ and $|\alpha| = n(1/p-1)$.

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COHERENCE OF IDEALS OF GENERALIZED SUMMING MULTILINEAR OPERATORS BY BLOCKS

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Abstract

We prove downward coherence and coincidence results for the ideals of generalized summing multilinear operators by blocks introduced in [2].

1 Introduction

Coherence and coincidence results are classical topics in the study ideals of multilinear operators between Banach spaces (multi-ideals). Coherence is the issue that conncects multi-ideals with polynomial ideals and to holomorphy types (see, e.g., [3]). In this work we address these issues to the multi-ideals of $(B_{\mathcal{I}}; X_1, \ldots, X_d; Y_1, \ldots, Y_k)$-summing multilinear operators, introduced in [2], which we describe next.

Henceforth, d and k are natural numbers with $d \geq 2$ and $1 \leq k \leq d$, and $E_1, \ldots, E_d, F$ are Banach spaces. The symbol $[\overset{r}{\ldots}]$ means that the r-th coordinate has been omitted. Given a continuous d-linear operator $A\colon E_1 \times \cdots \times E_d \longrightarrow$, for $r = 1, \ldots, d$ and $x^r \in E_r$, the map $A_{x^r}\colon E_1 \times [\overset{r}{\ldots}] \times E_d \longrightarrow F$, $A_{x^r}(x^1, [\overset{r}{\ldots}], x^d) = A(x^1, \ldots, x^d)$, is a continuous $(d-1)$-linear operator and $\|A_{x^r}\| \leq \|A\| \cdot \|x^r\|$.

By X we mean a sequence class according to [1], that is, to each Banach space E corresponds a Banach space $X(E)$ of E-valued sequences such that $c_{00}(E) \subset X(E) \overset{1}{\hookrightarrow} \ell_\infty(E)$ and $\|e_j\|_{X(\mathbb{K})} = 1$ for every $j \in \mathbb{N}$, where the e_j-s are the canonical vectors of scalar-valued sequence spaces. From now on, $X_1, \ldots, X_d, Y_1, \ldots, Y_k$ stand for sequence classes.

By $\mathcal{I} = \{I_1, \ldots, I_k\}$ we denote a partition of $\{1, \ldots, d\}$ formed by pairwise disjoint subsets $\{1, \ldots, d\}$ whose union is $\{1, \ldots, d\}$. By $x * e_j$ we mean the d-tuple $(0, \ldots, 0, x, 0, \ldots, 0)$, where x appears at the j-th coordinate, either x belonging to a Banach space or being a natural number.

Definition 1.1. Fixed a partition $\mathcal{I} = \{I_1, \ldots, I_k\}$ and d sequences of natural numbers $(j_n^r)_{n=1}^\infty$, $r = 1, \ldots, d$, such that the correspondence $(n_1, \ldots, n_k) \in \mathbb{N}^k \longmapsto \sum_{s=1}^k \sum_{r \in I_s} j_{n_s}^r * e_r$ is injective, the *block of* $\mathbb{N}^d$ *associated to the partition* $\mathcal{I}$ *and to the sequences* $(j_n^r)_{n=1}^\infty$, $r = 1, \ldots, d$, is the set

$$B_{\mathcal{I}} = \left\{ \sum_{s=1}^k \sum_{r \in I_s} j_{n_s}^r * e_r \in \mathbb{N}^d : n_1, \ldots, n_k \in \mathbb{N} \right\}.$$

When the partition is the trivial one, that is, $\mathcal{I}_t = \{\{1, 2, , \ldots, d\}\}$, we say that this is isotropic case. Since $\mathbb{N}$ is countable, there are sequences $(j_n^r)_{n=1}^\infty$, $r = 1, \ldots, d$ such that $B_{\mathcal{I}_t} = \mathbb{N}^d$. All the other cases are called anisotropic. At the opposite side, we have $\mathbb{N}^d = \{(n_1, \ldots, n_d) : n_i \in \mathbb{N}\} = B_{\mathcal{I}_d}$, where $\mathcal{I}_d = \{\{r\} : r = 1, \ldots, d\}$ is the discrete partition and $(j_n^r)_{n=1}^\infty = (n)_{n=1}^\infty$, $r = 1, \ldots, d$. In order to get one degree of multilinearity lower, it is necessary to obtain of block of $\mathbb{N}^{d-1}$ by fixing one coordinate of the block $\mathbb{N}^d$. This is why we are going to consider the block $\mathbb{N}^d$ associated to the partitions above, where we can assure that $\mathbb{N}^{d-1}$ is a block associated to the partitions by omitting one coordinate. Of course, the space $Y_1(Y_2(F))$ if formed by all $Y_2(F)$-valued sequences belonging to Y_1. Repeating the process we can consider the space $Y_1(\cdots Y_k(F) \cdots)$.

Definition 1.2. A d-linear operator $A\colon E_1\times\cdots\times E_d\longrightarrow F$ is *partially* $(B_{\mathcal{I}};X_1,\ldots,X_d;Y_1,\ldots,Y_k)$*-summing* if

$$\left(\cdots\left(A\left(\sum_{s=1}^{k}\sum_{r\in I_s}x^r_{j^r_{n_s}}*e_r\right)\right)_{n_k=1}^{\infty}\cdots\right)_{n_1=1}^{\infty}\in Y_1(\cdots Y_k(F)\cdots)$$

whenever $(x_j^r)_{j=1}^\infty\in X_r(E_r)$, $r=1,\ldots,d$. The space of such operators is denoted by $\mathcal{L}^{B_{\mathcal{I}}}_{X_1,\ldots,X_d;Y_1,\ldots,Y_k}(E_1,\ldots,E_d;F)$.

2 Main Results

A sequence class X is *0-invariant* if, regardless of the Banach space E and the sequence $(x_j)_{j=1}^\infty$ in E, it holds $(x_j)_{j=1}^\infty\in X(E)\iff(x_j^0)_{j=1}^\infty\in X(E)$ and $\|(x_j)_{j=1}^\infty\|_{X(E)}=\|(x_j^0)_{j=1}^\infty\|_{X(E)}$, where $(x_j^0)_{j=1}^\infty$ is the zero-free version of $(x_j)_{j=1}^\infty$, meaning that x_j^0 is the j-th nonzero coordinate of $(x_j)_{j=1}^\infty$ if it exists, or zero otherwise.

Proposition 2.1. *Let $1\le m\le d$ and $a^m\in E_m$. If Y is 0-invariant and $A\in\mathcal{L}^{\mathbb{N}^d}_{X_1,\ldots,X_d;Y}(E_1,\ldots,E_d;F)$, then $A_{a^m}\in\mathcal{L}^{\mathbb{N}^{d-1}}_{X_1,[\overset{m}{\cdots}],X_d;Y}(E_1,[\overset{m}{\cdots}],E_d;F)$ e $\|A_{a^m}\|_{\mathbb{N}^{d-1};X_1,[\overset{m}{\cdots}],X_d;Y}\le\|A\|_{\mathbb{N}^d;X_1,\ldots,X_d;Y}\cdot\|a^m\|$.*

Corollary 2.1. *If Y is 0-invariant and $\mathcal{L}^{\mathbb{N}^d}_{X_1,\ldots,X_d;Y}(E_1,\ldots,E_d;F)=\mathcal{L}(E_1,\ldots,E_d;F)$, then $\mathcal{L}^{\mathbb{N}^{d-1}}_{X_1,[\overset{m}{\cdots}],X_d;Y}(E_1,[\overset{m}{\cdots}],E_d;F)=\mathcal{L}(E_1,[\overset{m}{\cdots}],E_d;F)$ para todo $m=1,\ldots,d$.*

Corollary 2.2. *Let $X_1,\ldots,X_d,Y$ be sequence classes and suppose that Y is 0-invariant. If $\mathcal{L}(E_1,\ldots,E_d;F)=\mathcal{L}^{\mathbb{N}^d}_{X_1,\ldots,X_d;Y}(E_1,\ldots,E_d;F)$, then $\mathcal{L}(E_r;F)=\mathcal{L}_{X_r;Y}(E_r;F)$ for every $r=1,\ldots,d$.*

Corollary 2.3. *[4, Corollay 3.4] Let $q,p_1,\ldots,p_d\ge 1$ be given and let $E_1,\ldots,E_d,F$ be Banach spaces such that $\mathcal{L}(E_1,\ldots,E_d;F)=\mathcal{L}^{\mathbb{N}^d}_{\ell^w_{p_1}(\cdot),\ldots,\ell^w_{p_d}(\cdot);\ell_q(\cdot)}(E_1,\ldots,E_d;F)$. Then $\Pi_{q;p_r}(E_r;F)=\mathcal{L}(E_r;F)$ for all $r=1,\ldots,d$.*

Proposition 2.2. *If Y is a linearly stable sequence class, $a\in E_1$ and $A\in\mathcal{L}^{\mathbb{N}^d}_{X_1,\ldots,X_d;{}^dY}(E_1,\ldots,E_d;F)$, then $A_a\in\mathcal{L}^{\mathbb{N}^{d-1}}_{X_2,\ldots,X_d;{}^{d-1}Y}(E_2,\ldots,E_d;F)$ and $\|A_a\|_{\mathbb{N}^{d-1};X_2,\ldots,X_d;{}^{d-1}Y}\le\|A\|_{\mathbb{N}^d;X_1,\ldots,X_d;{}^dY}\cdot\|a\|$.*

Corollary 2.4. *Let X and Y be sequence classes with Y linearly stable, $a\in E$ and $m\in\{1,\ldots,d\}$. If $A\in\mathcal{L}^{\mathbb{N}^d}_{{}^dX;{}^dY}({}^dE;F)$ symmetric, then $A_a^m\in\mathcal{L}^{\mathbb{N}^{d-1}}_{{}^{d-1}X;{}^{d-1}Y}({}^{d-1}E;F)$ and $\|A_a^m\|_{\mathbb{N}^{d-1};{}^{d-1}X;{}^{d-1}Y}\le\|A\|_{\mathbb{N}^d;{}^dX;{}^dY}\cdot\|a\|$.*

Proposition 2.3. *Let $X_1,\ldots,X_d,Y$ be sequence classes with Y linearly stable and let $E_1,\ldots,E_d,F$ be Banach spaces. If $\mathcal{L}(E_1,\ldots,E_d;F)=\mathcal{L}^{\mathbb{N}^d}_{X_1,\ldots,X_d;{}^dY}(E_1,\ldots,E_d;F)$, then $\mathcal{L}(E_2,\ldots,E_d;F)=\mathcal{L}^{\mathbb{N}^{d-1}}_{X_2,\ldots,X_d;{}^{d-1}Y}(E_2,\ldots,E_d;F)$.*

Corollary 2.5. *Let X,Y be sequence classes with Y linearly stable and let E,F be Banach spaces. If $\mathcal{L}({}^dE;F)=\mathcal{L}^{\mathbb{N}^d}_{{}^dX;{}^dY}({}^dE;F)$ for some $d\in\mathbb{N}$, then $\mathcal{L}(E;F)=\mathcal{L}_{X;Y}(E;F)$.*

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$G-$TOPOLOGICAL SPACES: GENERALIZING THE CONCEPT OF $G-$SPACE

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Abstract

In this work, we present a generalization of the concept of $G-$space, introduced in [1], to the case in which G is a topological group and, then, we show how to adapt results from [1] on $G-$spaces to the more general context of $G-$topological spaces. This research will serve as a basis for the elaboration of a PhD thesis.

1 Introduction

This work has two main goals: firstly, we present a generalization of the concept of $G-$space, introduced in [1], to the case in which G is a topological group; then, we show how to adapt results from Castillo and Ferenczi on $G-$spaces to the more general context of $G-$topological spaces. From this point onwards, we are supposing, implicitly, that G is a topological group, that τ_G is the topology of G, and that $\mathcal{V}$ is the collection of the neighborhoods of the identity element of G.

2 $G-$topological spaces

We start this section recalling two preliminary definitions. Then, we introduce the concept of $G-$topological space.

Definition 2.1. *A **bounded linear left action** of a group H (with identity element e_H) on a normed space X is a map u from H into the set $\mathcal{B}(X)$ of the continuous linear maps from X into X such that: i) $u(e_H) = \mathrm{id}_X$; ii) for every $(g,h) \in H \times H$, $u(g \cdot h) = u(g) \circ u(h)$; and iii) $||u|| := \sup\left\{||u(g)||_{\mathcal{B}(X)} : g \in H\right\} < +\infty$.*

Definition 2.2. *If H is a group, then a $H-$**space** (respectively, a $H-$**Banach space**) is, by definition, a triple (H, X, u), where X is a normed space (respectively, a Banach space), and u is a bounded linear left action of H on X.*

Definition 2.3. *We say that a $G-$space (G, X, u) is a $G-$**topological space** if the action u is, moreover, $\left(\tau_G, \tau^{SOT}_{\mathcal{B}(X)}\right)-$continuous (where $\tau^{SOT}_{\mathcal{B}(X)}$ is the strong operator topology on $\mathcal{B}(X)$). If (G, X, u) is a $G-$topological space, and if X is a Banach space, then we call (G, X, u) a $G-$**topological Banach space**.*

Remark 2.1. *Note that, if τ_G is the discrete topology, then every $G-$space is also a $G-$topological space. Therefore, we say that the concept of $G-$topological space in fact generalizes the notion of $G-$space.*

Example 2.1. *Let us consider the topological group $(H, \cdot, \tau_H)$, where $H := \{-1, 1\}^{\mathbb{N}}$, $\cdot$ is the usual product, and τ_H is the product topology (with each copy of $\{-1, 1\}$ being equipped with the discrete topology). Under these conditions, it is not difficult to show that, if $p \in [1, +\infty[$, and $u \colon H \to \mathcal{B}(\ell^p)$ is such that, for each $(\varepsilon_n)_{n\in\mathbb{N}} \in H$ and each $(x_n)_{n\in\mathbb{N}} \in \ell^p$, $\Big[u\big((\varepsilon_n)_{n\in\mathbb{N}}\big)\Big]\big((x_n)_{n\in\mathbb{N}}\big) = (\varepsilon_n \cdot x_n)_{n\in\mathbb{N}}$, then (H, ℓ^p, u) is a $H-$topological Banach space.*

Next, we recall the notion of $G-$equivariant map (as presented in [1]).

Definition 2.4. *Given $G-$spaces (G, X, u) and (G, Y, v), we say that a map $T \colon X \to Y$ is $G-$**equivariant** if , for every $g \in G$ and every $x \in X$, $T\Big(\big[u(g)\big](x)\Big) = \big[v(g)\big]\big(T(x)\big)$. If, in addition, T is also linear and continuous, we say that T is a $G-$**operator**.*

The category of $G-$spaces is the category that has $G-$spaces as objects and $G-$operators as morphisms. The category of $G-$topological spaces, in turn, is a full subcategory of the category of $G-$spaces.

3 Topological amenability, $G-$operators and $G-$centralizers

As mentioned in the introduction, one of the objectives of this work is to adapt results from [1] on $G-$spaces to the more general context of topological $G-$spaces. In this section, we state, as an example, a topological version of a particular case of Proposition 3.10 of [1]. To do so, however, we firstly recall preliminary concepts and definitions.

Definition 3.1. *Given a normed space X, we say that a map $f\colon G \to X$ is $(\mathcal{U}_R, ||\cdot||_X)-$**uniformly continuous** if, for every $\varepsilon > 0$, there exists a $V \in \mathcal{V}$ such that, for any $(g,h) \in G \times G$,*

$$h \cdot g^{-1} \in V \Rightarrow ||f(g) - f(h)||_X < \varepsilon. \tag{1}$$

*Furthermore, we say that a map $\sigma\colon G \to \mathcal{B}(X)$ is $(\mathcal{U}_R, \mathcal{B}(X)_{SOT})-$**uniformly continuous** if, for all $n \in \mathbb{N}$, all $(x_1, \ldots, x_n) \in X^n$ and all $\varepsilon > 0$, there exists a $V \in \mathcal{V}$ such that, for any $(g,h) \in G \times G$,*

$$h \cdot g^{-1} \in V \Rightarrow \forall\, j \in \{1, \ldots, n\},\ \big|\big|\big[\sigma(g)\big](x_j) - \big[\sigma(h)\big](x_j)\big|\big|_X < \varepsilon. \tag{2}$$

We will denote by $UCB_R(G)$ the set of $(\mathcal{U}_R, |\cdot|)-$uniformly continuous and bounded maps from G into $\mathbb{K}$ (where, as usual, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). The map from G into $\mathbb{K}$ which takes the value 1 at all points of G, in turn, will be denoted by 1_G.

Definition 3.2. *We say that a $G-$space (respectively, a $G-$Banach space) is a $G-$**topological space** (respectively, a $G-$**topological Banach space**) if u is, moreover, $\big(\tau_G, \tau^{SOT}_{\mathcal{B}(X)}\big)-$continuous (where $\tau^{SOT}_{\mathcal{B}(X)}$ is the strong operator topology on $\mathcal{B}(X)$).*

Definition 3.3. *Let $\star\colon G \times UCB_R(G) \to UCB_R(G)$ be the left action (in the usual sense) that, to every pair $(g, \varphi) \in G \times UCB_R(G)$, associates the function φ_g which maps each x in G to $\varphi(x \cdot g)$. We say that G is **topologically amenable** if there exists a $(G, \star)-$invariant mean on $UCB_R(G)$ (that is, if there exists a mean M on $UCB_R(G)$ such that $M(\varphi_g) = M(\varphi)$ for any $g \in G$ and any $\varphi \in UCB_R(G)$).*

We are now almost ready to state a topological version of a particular case of proposition 3.10 of [1]. Before that, however, we advise readers who are not familiar with the concepts of quasilinear map, trivial quasilinear map, $G-$centralizer and $G-$ultrasummand to look for these definitions in [1]. With that said, we can finally state the promised result.

Proposition 3.1. *Let (G, Y, v) be a $G-$Banach space, and let (G, X, u) be a $G-$ultrasummand. Suppose further that G is topologically amenable, that v is $(\mathcal{U}_R, \mathcal{B}(Y)_{SOT})-$uniformly continuous, and that u is $(\tau_G, \tau_{||\cdot||_{\mathcal{B}(X)}})-$continuous (where $\tau_{||\cdot||_{\mathcal{B}(X)}}$ is the topology induced on $\mathcal{B}(X)$ by $||\cdot||_{\mathcal{B}(X)}$). Then, given a $G-$centralizer $\Omega\colon \Delta \subseteq Y \to X$ such that, for each $y \in \Delta$, the restriction of Ω to the set $\big\{\big[v(g)\big](y) : g \in G\big\}$ is uniformly continuous, there exists a $G-$equivariant map $\omega\colon \Delta \to X$ such that $\Omega - \omega$ is a quasilinear bounded map. Furthermore, if Ω is also trivial, then ω can be chosen to be linear.*

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LATTICEABILITY IN SPACES OF BOUNDED SEQUENCES

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Abstract

For a Banach lattice E, we prove that some subsets of the Banach lattice $\ell_\infty(E)$ formed by sequences with distinguished properties contain a closed infinite dimensional sublattice.

1 Introduction

A subset A of a topological vector space E is *spaceable* (see [1]) if there exists a closed infinite dimensional subspace of E all of whose elements but the origin belong to A. Recently, Oikhberg [7] coined the following terms:

Definition 1.1. A subset A of a Banach lattice is *latticeable* (*completely latticeable*) if there exists a (closed) infinite dimensional sublattice of E all of whose elements but the origin belong to A (see also [8]).

A number of results on latticeability in spaces of vector-valued sequences appear in [2]. To state some of these results we need a few definitions.

Definition 1.2. (i) A Banach lattice E has the *positive Schur property* if positive – or, equivalently, disjoint – weakly null sequences in E are norm null.
(ii) A sequence $(x_j)_{j=1}^\infty$ in a Banach lattice E is *regular-polynomially null* if $P(x_j) \longrightarrow 0$ for every scalar-valued regular homogeneous polynomial P on E. A Banach lattice E is *positively polynomially Schur* if positive regular-polynomially null E-valued sequences are norm null.
(iii) A sequence $(x_j)_{j=1}^\infty$ in a Banach space E is *polynomially null* if $P(x_j) \longrightarrow 0$ for every scalar-valued continuous homogeneous polynomial P on E. A Banach space E is *polynomially Schur* if every polynomially null E-valued sequence is norm null.

The literature on the positive Schur property is vast. The class of Banach lattices in (ii) was introduced in [3]. The class of Banach spaces in (iii) was introduced by Carne, Cole and Gamelin [4] and have been developed by several authors.

2 Main Results

Recall that $\ell_\infty(E)$, the space of bounded E-valued sequences with the sup norm, is a Banach lattice with the coordinatewise order whenever E is a Banach lattice, and that $c_0^w(E)$, the space of weakly null E-valued sequences, is a closed subspace of $\ell_\infty(E)$ whenever E is a Banach space.

The next result was inspired by Jiménez-Rodríguez [6].

Theorem 2.1. (a) *Let E be a non-polynomially Schur Banach space. Then the set of E-valued non-norm null polynomially null sequences is spaceable in $c_0^w(E)$.*
(b) *Let E be a Banach lattice failing the positive Schur property. Then the set of E-valued disjoint non-norm null weakly null sequences is completely latticeable in $\ell_\infty(E)$.*
(c) *Let E be a non-positively polynomially Schur Banach lattice. Then the set of E-valued disjoint non-norm null regular-polynomially null sequences is completely latticeable in $\ell_\infty(E)$.*

In general, the closed sublattices of $\ell_\infty(E)$ obtained in the proofs of (b) and (c) above are not ideals in $\ell_\infty(E)$.

Remark 2.1. (i) We cannot use $c_0^w(E)$ instead of $\ell_\infty(E)$ in Theorem 2.1(b) and (c) because $c_0^w(E)$ is not always a sublattice of $\ell_\infty(E)$. For instance, for $1 \leq p < \infty$, $c_0^w(L_p[0,1])$ is not a Riesz space due to the fact that the lattice operations in $L_p[0,1]$ are not weakly sequentially continuous. But the proof makes clear that the sublattices of $\ell_\infty(E)$ created in Theorem 2.1(b) and (c) are contained in $c_0^w(E)$. Sometimes $c_0^w(E)$ is a Banach lattice, for instante when E is either an AM-space or an atomic Banach lattice with order continuous norm. In these cases, $\ell_\infty(E)$ can be replaced with $c_0^w(E)$ in Theorem 2.1(b) and (c).
(ii) Castillo, García and Gonzalo in [5, Theorem 5.5] proved that the sum of two polynomially null sequences is not necessarily polynomially null. This is why we cannot pass to a space smaller than $c_0^w(E)$ in Theorem 2.1(a).
(iii) We have already explained why $c_0^w(E)$ cannot be used in general in Theorem 2.1(c). But one might wonder if we could have gone to a smaller space, formed by regular-polynomially null sequences. In order to see that we cannot, next we show that the counterxample given in [5, Theorem 5.5] is good enough to show that the sum of two regular-polynomially null sequences may fail to be regular-polynomially null.

Proposition 2.1. *The sum of two regular-polynomially null sequences in a Banach lattice is not necessarily regular-polynomially null.*

Now it is easy to see that, for every Banach space E, the set PN of polynomially null E-valued sequences is spaceable in $c_0^w(E)$: if E is not polynomially Schur, in Theorem 2.1(a) we proved that a set much smaller than PN is spaceable; if E is polynomially Schur, it is easy to check that $PN = c_0(E)$, the closed subspace of $c_0^w(E)$ formed by norm null sequences.

The lattice setting gives room to several questions that are senseless in the environment of Banach spaces. Our last result is typical:

Proposition 2.2. *For every infinite dimensional Banach lattice E, the set of E-valued disjoint norm null sequences is completely latticeable in $c_0(E)$.*

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ORDER CONTINUITY OF ARENS EXTENSIONS OF REGULAR MULTILINEAR OPERATORS

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Abstract

First we give a counterexample showing that recent results on separate order continuity of Arens extensions of multilinear operators cannot be improved to get separate order continuity on the product of the whole of the biduals. Then we establish conditions on the operators and/or on the underlying Riesz spaces/Banach lattices so that the separate order continuity holds on the product of the whole biduals. We also prove that all Arens extensions of any regular multilinear operator is order continuous in at least one variable.

1 Introduction

Bidual extensions of multilinear operators have been studied since Arens' seminal paper [1]. By $E^\sim$ we denote the order dual of a Riesz space E, hence $E^{\sim\sim} = (E^\sim)^\sim$. For a Banach lattice E, E^* denotes is topological dual, hence E^{**} stands for its bidual. A net $(x_\alpha)_{\alpha\in\Omega}$ in a Riesz space E is *order convergent* to $x \in E$, denoted $x_\alpha \xrightarrow{o} x$, if there are a net $(y_\alpha)_{\alpha\in\Omega}$ in E and $\alpha_0 \in \Omega$ such that $|x_\alpha - x| \leq y_\alpha \downarrow 0$ for every $\alpha \geq \alpha_0$. A linear operator $T\colon E \longrightarrow F$ is *order continuous* if $x_\alpha \xrightarrow{o} x$ in E implies $T(x_\alpha) \xrightarrow{o} x$ in F. The symbols $(E^\sim)_n^\sim$ and $(E^*)_n^*$ stand for the corresponding subspaces formed by the order continuous functionals. The results that motivated our research are the following:

- Buskes and Roberts (2019) [3, Theorem 3.4]: If $A\colon E_1 \times \cdots \times E_m \longrightarrow F$ is an m-linear operator of order bounded variation between Riesz spaces, then its Arens extension $A^{[m+1]*}\colon E_1^{\sim\sim} \times \cdots \times E_m^{\sim\sim} \longrightarrow F^{\sim\sim}$ is separately order continuous on $(E_1^\sim)_n^\sim \times \cdots \times (E_m^\sim)_n^\sim$.
- Boyd, Ryan and Snigireva (2021) [2, Theorem 1]: If $A\colon E_1 \times \cdots \times E_m \longrightarrow F$ is a regular m-linear operator between Banach lattices, with F Dedekind complete, then its Arens extension $A^{[m+1]*}\colon E_1^{**} \times \cdots \times E_m^{**} \longrightarrow F^{**}$ is separately order continuous on $(E_1^*)_n^* \times \cdots \times (E_m^*)_n^*$.

In this work we investigate the possibility (or not) to get, in the results above, separate order continuity on the product of the whole of the biduals.

2 Main Results

By $J_E\colon E \longrightarrow E^{\sim\sim}$ we denote the canonical operator ($J_E(x)(x'') = x''(x)$), which happens to be a Riesz homomorphism. Given Riesz spaces $E_1, \ldots, E_m, F$, the space of regular m-linear operators from $E_1 \times \cdots \times E_m$ to F is denoted by $\mathcal{L}_r(E_1, \ldots, E_m; F)$. When F is the scalar field we write $\mathcal{L}_r(E_1, \ldots, E_m)$. S_m stands for the set of permutations of $\{1, \ldots, m\}$.

Given a permutation $\rho \in S_m$ and a regular m-linear operator $A\colon E_1 \times \cdots \times E_m \longrightarrow F$, the Arens extension of A with respect to ρ is the operator

$$AR_m^\rho(A)\colon E_1^{\sim\sim} \times \cdots \times E_m^{\sim\sim} \longrightarrow F^{\sim\sim}$$

as defined in [4]. In this fashion, $AR_m^\rho(A)$ is a regular m-linear operator that extends A in the sense that $AR_m^\rho(A) \circ (J_{E_1}, \ldots, J_{E_m}) = J_F \circ A$ and $AR_m^\rho(A)$ is positive for positive A (see [4, Theorem 2.2]).

The extension $A^{[m+1]*}$ from [2,3] is recovered by considering the permutation $\theta(m) = 1, \theta(m-1) = 2, \ldots, \theta(2) = m-1, \theta(1) = m$, that is, $AR_m^\theta(A) = A^{*[m+1]}$. In particular, $AR_2^\theta(A) = A^{***}$ in the bilinear case $m = 2$.

The counterexample. Consider the positive bilinear form given by the duality $c_0^* = \ell_1$:

$$A\colon \ell_1 \times c_0 \longrightarrow \mathbb{R}\ ,\ A\left((x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty\right) = \sum_{n=1}^\infty x_n y_n.$$

With the help of some technical claims we prove that the Aron-Berner extension A^{***} is order continuous in the first variable but not in the second one, while the other Aron-Berner extension $AR_2^{id}(A)$ is separately order continuous. This shows that the two results stated in the Introduction cannot be improved, in general, to get separate order continuity on the product of the whole of the biduals.

The results we obtained where the order continuity on the whole of the biduals is achieved are the following.

Proposition 2.1. *All Arens extensions of a multilinear operator of finite type bettween Riesz spaces coincide, are of finite type and are separately order continuous.*

Theorem 2.1. *Let $E_1, \ldots, E_m, F$ be Riesz spaces, $\rho \in S_m$ and $A \in \mathcal{L}_r(E_1, \ldots, E_m; F)$.*

(a) *For all $j \in \{1, \ldots, m\}$, $x''_{\rho(i)} \in E^{\sim\sim}_{\rho(i)}, i = 1, \ldots, j-1$, and $x''_{\rho(i)} \in (E^\sim_{\rho(i)})^\sim_n, i = j+1, \ldots, m$, the operator $x''_{\rho(j)} \in E^{\sim\sim}_{\rho(j)} \mapsto AR_m^\rho(A)(x''_1, \ldots, x''_{\rho(j)}, \ldots, x''_m) \in F^{\sim\sim}$ is order continuous on $E^{\sim\sim}_{\rho(j)}$.*

(b) *$AR_m^\rho(A)$ is separately order continuous on $(E_1^\sim)^\sim_n \times \cdots \times (E_m^\sim)^\sim_n$.*

(c) *$AR_m^\rho(A)$ is order continuous in the $\rho(m)$-th variable on the whole of $E^{\sim\sim}_{\rho(m)}$.*

Theorem 2.2. *Let $m \geq 2$ and $E_1, \ldots, E_m$ be Banach lattices such that:*
(i) *For $j = 2, \ldots, m-1$, and $i = 1, \ldots, m-j$, every regular linear operator from E_j to E^*_{j+i} is weakly compact;*
(ii) *For all $k = 2, \ldots, m$, $x_1^{**} \in E_1^{**}$ and $T \in \mathcal{L}_r(E_1; E_k^*)$, the functional $T^{**}(x_1^{**})$ is order continuous on E_k^{***}.*

Then, for every Banach lattice F and any $A \in \mathcal{L}_r(E_1, \ldots, E_m; F)$, the Arens extension $A^{*[m+1]}$ is separately order continuous on $E_1^{**} \times \cdots \times E_m^{**}$.

Recall that a Banach space E is *Arens regular* if every bounded linear operator from E to E^* is weakly compact (see, e.g., [5]). The Banach lattices c_0, ℓ_∞ and $C(K)$, where K is a compact Hausdorff space, in particular AM-spaces with order unit, are Arens regular.

Corollary 2.1. *Let E be an Arens regular Banach lattice. Then, for every Banach lattice F, the Arens extension $A^{*[m+1]}$ of any regular m-linear operator $A\colon E^m \longrightarrow F$ is separately order continuous on $(E^{**})^m$.*

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A CHARACTERIZATION FOR COMPLEX SYMMETRIC TOEPLITZ OPERATOR ON THE HARDY SPACE OVER DISK

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Abstract

In these notes, we present a characterization for complex symmetric Toeplitz operator over the Hardy space of the disk. As a consequence, we get examples of complex symmetry of Toeplitz operators when the symbol is a rational function.

1 Introduction

A *conjugation* C on a separable complex Hilbert space $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ such that:

(a) C is *isometric*: $\langle Cf, Cg\rangle = \langle g, f\rangle$, $\forall f, g \in \mathcal{H}$.

(b) C is *involutive*: $C^2 = I$.

A bounded linear operator T on $\mathcal{H}$ is said to be *complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $CT = T^*C$. We will often say that T is C-symmetric. Equivalently, T is C-symmetric if there exists an orthonormal basis $\{e_n\}$ of $\mathcal{H}$ with respect to which T has a symmetric matrix representation.

The concept of complex symmetric operators on separable Hilbert spaces in a natural generalization of complex symmetric matrices and their general study was initiated by Garcia, Putinar and Wogen [2,3,4,5]. The class of complex symmetric operators includes other basic classes of operators such as normal, Hankel, compressed Toeplitz and some Volterra operators.

In these notes, we assume that $\mathbb{T}$ is the boundary of the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. Let $L^2(\mathbb{T})$ be the space of square integrable functions on $\mathbb{T}$ with the inner product defined by

$$\langle u, v\rangle = \int_{\mathbb{T}} u\overline{v}\,ds.$$

Let $H^2(\mathbb{T})$ denote the classical *Hardy space* associated to $\mathbb{D}$ which is the space of holomorphic functions on $\mathbb{D}$ with $L^2(\mathbb{T})$-boundary values in $\mathbb{T}$. Since the set of monomials $\left\{\frac{1}{\sqrt{2\pi}}z^n : n = 0, 1, 2, \dots\right\}$ is an orthonormal basis for $H^2(\mathbb{T})$, we have that $f \in H^2(\mathbb{T})$ if and only if

$$f(z) = \sum_{n=0}^{\infty} a_n \frac{1}{\sqrt{2\pi}} z^n \text{ where } \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Let $L^\infty(\mathbb{T})$ be the space of essentially bounded measurable functions on $\mathbb{T}$ and let φ be in $L^\infty(\mathbb{T})$. The *Toeplitz operator* $T_\varphi : H^2(\mathbb{T}) \to H^2(\mathbb{T})$, with symbol φ, is defined by

$$T_\varphi f = P(\varphi f),$$

for all $f \in H^2(\mathbb{T})$, where $P : L^2(\mathbb{T}) \to H^2(\mathbb{T})$ is the orthogonal projection.

The purpose of this lecture is to present a characterization for complex symmetric Toeplitz operators finding an orthonormal basis for $H^2(\mathbb{T})$ so that the Toeplitz operator T_φ has a symmetric matrix representation. As a consequence we provide an example of complex symmetric Toeplitz operator T_φ with non-trigonometric symbol.

2 Main Results

Let $p \in \mathbb{D}$. For each non-negative integer n, considering the rational function R_n given by

$$R_n = \sqrt{\frac{1-|p|^2}{2\pi}}\frac{(z-p)^n}{(1-\overline{p}z)^{n+1}},$$

we have that $\mathfrak{B} = \{R_n : n = 0, 1, 2, \dots\}$ is an orthonormal basis for $H^2(\mathbb{T})$ (see [1] for more details). Thus, we get the following:

Lemma 2.1. *Let $\mathfrak{J} : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ be defined by*

$$\mathfrak{J}\left(\sum_{n=0}^{\infty} a_n \frac{(z-p)^n}{(1-\overline{p}z)^{n+1}}\right) = \sum_{n=0}^{\infty} \overline{a_n} \frac{(z-p)^n}{(1-\overline{p}z)^{n+1}}. \tag{1}$$

Then $\mathfrak{J}$ is a conjugation on $H^2(\mathbb{T})$.

Below we present our main result:

Theorem 2.1. *Let $\varphi(z) = \sum_{n=-\infty}^{\infty} \widehat{\varphi}(n) z^n \in L^\infty(\mathbb{T})$ and $p \in \mathbb{D}$. The following statements are equivalent:*

(i) T_φ is $\mathfrak{J}$-symmetric.

(ii) For all non-negative integer k, holds

$$\widehat{\varphi}(k) = \widehat{\varphi}(-k) + \overline{p}\left\{\widehat{\varphi}(-k-1) - \widehat{\varphi}(k-1)\right\}.$$

(iii) $\varphi(z) = \varphi(\overline{z})\dfrac{(1+\overline{p\overline{z}})}{1+\overline{p}z}$.

It follows from the previous theorem that there is no complex symmetric Toeplitz operator T_φ with conjugation $\mathfrak{J}$ given by (1) with finite symbol φ.

Corollary 2.1. *Let $p \in \mathbb{D}$ nonzero. If $\varphi(z) = \sum_{n=-M}^{N} \widehat{\varphi}(n) z^n$ where $N \geq M > 0$ with nonzero $\widehat{\varphi}(-M), \widehat{\varphi}(N)$, then T_φ is not $\mathfrak{J}$-symmetric.*

As a particular case of Theorem 2.1, we get examples of Toeplitz operators $\mathfrak{J}$-symmetric with non-trigonometric symbols.

Corollary 2.2. *Let $p \in (-1, 1)$ a real number. If $\varphi(z) = \dfrac{1+p\overline{z}}{|1+p\overline{z}|}$, then T_φ is $\mathfrak{J}$-symmetric.*

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FIXED POINT THEOREMS FOR GENERALIZED FUNCTIONS

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Abstract

We prove a Fixed Point Theorem for internal sets of finite diameter. This generalizes the Fixed Point Theorems proved by Juriaans-Oliveira and Juriaans-Oliveira-Queiroz in two recent papers. It opens a wide range for applications in non-linear PDE's. In particular, it can be used to prove existence results for PDE's in the context of Hilbert $\widetilde{\mathbb{C}}-$modules.

1 Introduction

Schwartz's Theory is a linear theory and hence no products are allowed. Attempts to create a non-linear theory are due to J.F. Colombeau and E.E. Rosinger. In this paper we focus on Colombeau's proposal of a Theory of Generalized Functions. Colombeau's proposal still did not have a natural domain of the new functions and a derivations defined by variation of some variable of which these new functions depend. The remedy for this started in the seminal paper of Kunzinger-Obberggenberger which assigned to a Colombeau generalized function a natural domain, thus becoming a function whose values belong to a ring. Since latter is not a field, it was highly unlikely to be a good candidate to replace $\mathbb{R}$ or $\mathbb{C}$ as the basic underlying algebraic and topologic milieu. This ring, the ring of Colombeau generalized numbers $\overline{\mathbb{K}}$, had to be better understood if it were to become the basic underlying milieu of a Generalized Calculus. This meant that its algebraic and topological properties had to be studied and very well understood. One of the first results of algebraic nature in the Theory of Colombeau Generalized functions can be traced back to M. Kunzinger. Aragona-Juriaans started a systematic study of this ring thus bridging algebra, analysis and topology in the field. The Biagioni-Scarpalézos topology, the sharp topology, was revisited by Aragona-Fernandez-Juriaans to make it algebraically more suitable. Together with the crucial idea of point value introduced by Kuzinger-Obberguggenberger, this let to a proposal of a Generalized Differential Calculus by Aragona-Fernandez-Juriaans being the ring of Colombeau generalized numbers, $\overline{\mathbb{K}}$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, the basic underlying milieu. At first, it might look odd that a calculus developed in a non-arquimedian environment can be a perfect match with classical calculus. However, this turned out to be the case as proved in the Embedding Theorem of Aragona-Fernandez-Juriaans (see [3,4]). As already stated, the idea of generalized points goes back to the seminal paper of Kunzinger-Oberguggenberger. Their idea was generalized by Aragona-Fernadnes-Juriaans-Oberguggenberger and the notion of a membranes was introduced. Subsequently, the notion of membranes was generalized by Vernaeve-Oberguggenberger who introduced the notion of internal sets of a locally convex spaces. One of the first compactness type results proved in Generalized Calculus, and thus in the context of Colombeau Generalized Function, was proved by Aragona-Fernandez-Juriaans. Vernaeve-Oberguggenberger generalized this result, proving the *Saturation Principal.* This Principal is an important tool to prove existence theorems for PDE's involving non-linearities. Topological Fixed Point Theorems where recently proved in [1,2]. Here we prove a Fixed Point Theorem for internal sets which generalized these two results, thus becoming one more tool in the Generalized Calculus. The Fixed Point Theorem of the present paper can also be applied in the context of Hilbert $\widetilde{\mathbb{C}}-$modules.

2 A Fixed Point Theorem for Internal Sets of a Locally Convex Space

We refer the reader to [1,2] for the definition of an internal set of a locally convex space $\mathbb{E}$. In this section, all internal sets are defined on $\mathbb{E}$.

Definition 2.1. *Let M be an internal set and and $F : M \longrightarrow M$. We say that F is an internal Lipschitz function if $F(M)$ is an internal set and t here exists $\lambda \in \overline{\mathbb{R}}$ such that $|F(x) - F(y)| \leq \lambda |x - y|$, $\forall\ x, y \in M$. If $\lambda \in B_1(0)$ then F is said to be an internal contraction on M. In this case, the sequence $(\lambda^n)_{n\in\mathbb{N}}$ converges to zero in $\overline{\mathbb{K}}$.*

Let M be a sharply bounded internal set, i.e., there exists $r \in \mathbb{R}$ such that for $x, y \in M$ we have that $\|x - y\| \leq \alpha^r$, where α is the natural gauge of this milieu. In this case we say that M has *finite diameter*, denoted by $\mu(M) < \infty$. The Saturation Principal guarantees that given a sequence of internal sets with the finite intersection property such that any finite intersection of elements of this sequence has finite diameter then the intersection of the elements of this sequence is non-empty. In the context of membranes, it suffices to use a result of Aragona-Fernandez-Juriaans, since a membrane, by definition, is an internal set with finite diameter.

Theorem 2.1. *Let M be a sharply bounded internal set and $F : M \longrightarrow M$ an internal contraction. Then there exists a unique $x_0 \in M$ such that $F(x_0) = x_0$, i.e., F has a unique fixed point in M.*

Corollary 2.1. *Let (T_ε) be a net of Lipschitz functions defined on an internal set $M = (M_\varepsilon)$, $n = (n_\varepsilon)$ a hyper natural number and $T = (T_\varepsilon^{n_\varepsilon})$. If T is a an internal contraction then it has a fixed point in M.*

3 Supports of Functions and Points

Let $\Omega \subset \mathbb{K}$, $x \in \widetilde{\Omega}_c$ and $f \in \mathcal{G}(\Omega)$. In [1] we introduced the notion of the support of a point in $\overline{\mathbb{K}}^n$. The definition of the support, $supp(f)$, of f is well known. As proved in [1], we may write $x = \sum_e e \cdot x_e$ where the e's appearing in the summation form a complete set of orthogonal idempotents in the boolean algebra $\mathcal{B}(\overline{\mathbb{K}})$. Using this, we have that $f(x) = \sum_e e \cdot f(x_e)$.

Theorem 3.1. *Let $f \in \mathcal{G}(\Omega)$ and $x \in \widetilde{\Omega}_c$. If $supp(x) \cap supp(f) = \emptyset$ then $f(x) = 0$.*

Consider the classical definition of the notion of the support of the function f defined in $\widetilde{\Omega}_c$ and write this support as $Supp(f)$. If $f = \delta$ is the Dirac function, then $Supp(\delta) \subset \{x \in \widetilde{\mathbb{R}}_c\ :\ \exists\ e \in\ \mathcal{B}(\overline{\mathbb{R}})$ such that $e \cdot x \in \overline{\mathbb{K}}_0\}$. Hence, $Supp(\delta)$ is contained in the set consisting of those points whose interleaving contains at least one element of $\overline{\mathbb{K}}_0$.

Lemma 3.1. *Let $f \in \mathcal{G}(\Omega)$. Then $Supp(f) \subset \{x \in \widetilde{\mathbb{R}}_c\ :\ \exists\ e \in\ \mathcal{B}(\overline{\mathbb{R}})$ and $r > 0$, such that $e \cdot x \approx y \in supp(f)\}$.*

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HYPER-IDEALS OF MULTILINEAR OPERATORS GENERATED BY SEQUENCE CLASSES

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Abstract

We develop a new technique, based on the concept of sequence classes introduced in [2], to create new examples of hyper-ideals of multilinear operators between Banach spaces, as well as of polynomial hyper-ideals and polynomial two-sided ideals.

1 Introduction

Throughout the text n is a natural number, $E, E_1, \ldots, E_n, F, G, H$ are Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the Banach space of continuous n-linear operators from $E_1 \times \cdots \times E_n$ to F is denoted by $\mathcal{L}(E_1, \ldots, E_n; F)$.

A first attempt to treat ideals of multilinear operators through the transformation os vector-valued sequence classes was made in [2]. A more effective approach appeared in [2] using the concept of sequence classes.

A *sequence class* is a rule $E \mapsto X(E)$ that assigns to each Banach space E a Banach space $X(E)$ formed by E-valued sequences such that $c_{00}(E) \subseteq X(E) \overset{1}{\hookrightarrow} \ell_\infty(E)$ and $\|e_j\|_{X(\mathbb{K})} = 1$ for every $j \in \mathbb{N}$.

Sequence classes that are linearly stable and multilinearly stable are defined in [2].

Definition 1.1. Given $n \in \mathbb{N}$, sequence classes X, Y and Banach spaces $E_1, \ldots, E_n, F$, an n-linear operator $A \in \mathcal{L}(E_1, \ldots, E_n; F)$ is said to be $(X;Y)$-*summing* if $(A(x_j^1, \ldots, x_j^n))_{j=1}^\infty \in Y(F)$ whenever $(x_j^m)_{j=1}^\infty \in X(E_m)$, $m = 1, \ldots, n$. In this case, the induced n-linear operator

$$\widetilde{A} \colon X(E_1) \times \cdots \times X(E_n) \longrightarrow Y(F) \ , \ A\left((x_j^1)_{j=1}^\infty, \ldots, (x_j^n)_{j=1}^\infty\right) = (A(x_j^1, \ldots, x_j^n))_{j=1}^\infty,$$

is continuous and we define $\|A\|_{X_1,\ldots,X_n;Y} = \|\widetilde{A}\|$.

The space of all such operator is denoted by $\mathcal{L}_{X_1,\ldots,X_n;Y}(E_1, \ldots, E_n; F)$. If $X_1 = \cdots = X_n = X$ we simply write $\mathcal{L}_{X;Y}$.

Theorem 1.1. *[2] If $X_1, \ldots, X_n, Y$ are linearly stable sequence classes and $X(\mathbb{K}) \overset{1}{\hookrightarrow} Y(\mathbb{K})$, then $(\mathcal{L}_{X_1,\ldots,X_n;Y}, \|\cdot\|_{X_1,\ldots,X_n;Y})$ is a Banach ideal of multilinear operators.*

2 Main Results

The results we describe in this section will appear in [5].

A hyper-ideal of multilinear operators (see [3]) is an ideal of multilinear operators that is stable with respect to the composition with multilinear operators in the left-hand side. Our first result reads as follows.

Theorem 2.1. *Let X and Y be sequence classes with X multilinearly stable, Y linearly stable and $X(\mathbb{K})^n \hookrightarrow Y(\mathbb{K})$ for every $n \in \mathbb{N}$. Then $(\mathcal{L}_{X;Y}, \|\cdot\|_{X;Y})$ is a Banach hyper-ideal of multilinear operators.*

Classes of polynomials defined by the transformation of vector-valued sequences were first treated in [1]. By $\mathcal{P}(^nE;F)$ we denote the Banach space of continuous n-homogeneous polynomials from E to F. By $\widehat{A}$ we mean the n-homogeneous polynomial defined by the n-linear operator A and by $\check{P}$ the symmetric n-linear operator associated to the n-homogeneous polynomial P.

Theorem 2.2. *Let X and Y be sequence classes. The following are equivalent for a polynomial $P \in \mathcal{P}(^nE;F)$:*
(a) $\check{P} \in \mathcal{L}_{X;Y}(^nE;F)$.
(b) There exists $A \in \mathcal{L}_{X;Y}(^nE;F)$ such that $\widehat{A} = P$.
(c) $(P(x_j))_{j=1}^\infty \in Y(F)$ whenever $(x_j)_{j=1}^\infty \in X(E)$.
(d) The induced operator $\widetilde{P}\colon X(E) \longrightarrow Y(F)$, $\widetilde{P}((x_j)_{j=1}^\infty) = (P(x_j))_{j=1}^\infty$, is a well defined continuous n-homogeneous polynomial. If the conditions above hold, then

$$\|\widetilde{P}\| \leq \|\check{P}\|_{X;Y} = \inf\{\|A\|_{X;Y} : A \in \mathcal{L}_{X;Y} \text{ e } \widehat{A} = P\} \leq \frac{n^n}{n!}\|\widetilde{P}\|.$$

The theorem above leads us to define the class of $(X;Y)$-summing n-homogeneous polynomials.

Definition 2.1. Given sequence classes X and Y, we say that a polynomial $P \in \mathcal{P}(^nE;F)$ is $(X;Y)$-*summing*, in symbols $P \in \mathcal{P}_{X;Y}(^nE;F)$, if the equivalent conditions of the previous theorem hold for P. In this case we define $\|P\|_{X;Y;1} = \|\widetilde{P}\|$ and $\|P\|_{X;Y;2} = \|\check{P}\|_{X;Y}$.

Polynomial hyper-ideals and two-sided ideals were define in [4].

Theorem 2.3. *Let X and Y be sequence classes such that $X(\mathbb{K})^n \hookrightarrow Y(\mathbb{K})$ for every $n \in \mathbb{N}$. Then*
(a) If X is multilinearly stable and Y linearly stable, then $(\mathcal{P}_{X;Y}, \|\cdot\|_{X;Y;1})$ and $(\mathcal{P}_{X;Y}, \|\cdot\|_{X;Y;2})$ are $\left(\frac{n^n}{n!}\right)_{n=1}^\infty$-polynomial Banach hyper-ideals.
(b) If X and Y are multilinearly stable, then $(\mathcal{P}_{X;Y}, \|\cdot\|_{X;Y;2})$ is a $\left(\frac{n^n}{n!}, \frac{n^n}{n!}\right)_{n=1}^\infty$-polynomial Banach two-sided ideal.

Definition 2.2. A sequence class X is *polinomially stable* if, regardless of the $n \in \mathbb{N}$, the Banach spaces E and F and $P \in \mathcal{P}(^nE;F)$, $(P(x_j))_{j=1}^\infty \in X(F)$ whenever $(x_j)_{j=1}^\infty \in X(E)$ and, denoting by $\widetilde{P}\colon X(E) \longrightarrow X(F)$ the induced n-homogeneous polynomial, it holds $\|\widetilde{P}\| = \|P\|$.

Theorem 2.4. *Let X and Y sequence classes such that $X(\mathbb{K})^n \hookrightarrow Y(\mathbb{K})$ for every $n \in \mathbb{N}$. Then*
(a) If X is polinomially stable and Y linearly stable, then $(\mathcal{P}_{X;Y}, \|\cdot\|_{X;Y;1})$ is a polynomial Banach hyper-ideal.
(b) If X is polinomially stable and Y multilinearly stable, then $(\mathcal{P}_{X;Y}, \|\cdot\|_{X;Y;1})$ is a $\left(1, \frac{n^n}{n!}\right)_{n=1}^\infty$-polynomial Banach two-sided ideal.
(c) If X and Y are polinomially stable, then $(\mathcal{P}_{X;Y}, \|\cdot\|_{X;Y;1})$ is a Banach polynomial two-sided ideal.

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FUNCTION SPACES OF GENERALIZED SMOOTHNESS

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Abstract

The function spaces of generalized smoothness (generalized Besov/Lipschitz spaces) will be defined in terms of majorant functions and the fractional modulus of smoothness. The well definition of these spaces, embedding theorems between them and applications will be discussed.

1 Introduction

A majorant function is a nondecreasing measurable function $\varphi : (0,\infty) \longrightarrow \mathbb{R}_+$, such that $\varphi(t) \to 0$ as $t \to 0_+$, and satisfying

$$\int_0^t \frac{\varphi(u)}{u} du \lesssim \varphi(t) \quad \text{for all } t > 0. \tag{1}$$

The notation $A(t) \lesssim B(t)$ means that $A(t) \leq c\, B(t)$, for some constant $c > 0$, not depending upon t. We denote by $\mathcal{M}$ the collection of all majorant functions. For $\beta > 0$, we define the following

$$\Omega_\beta := \left\{ \varphi \in \mathcal{M} : \int_t^\infty \frac{\varphi(u)}{u^{\beta+1}} du \lesssim \frac{\varphi(t)}{t^\beta}, \quad t > 0 \right\}. \tag{2}$$

Several examples of functions in Ω_β, including the usual power function $\varphi(t) = t^\alpha$ (it belongs to Ω_β if and only if $0 < \alpha < \beta$) employed in the definition of the standard Besov/ Lipschitz spaces, can be considered in terms of the *regularly varying* concept of Karamata ([2]).

For $d \geq 1$ the Fourier transform $\widehat{f}$ of a function f, in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{i\xi\cdot x} dx, \quad \xi \in \mathbb{R}^d.$$

For $1 \leq p \leq \infty$ we write $L^p(\mathbb{R}^d) := (L^p(\mathbb{R}^d), \|\cdot\|_p)$ for the usual Banach spaces of p-integrable functions. The fractional moduli of smoothness, for $\gamma > 0$, is given ([3]) in terms of the fractional difference as follows

$$\omega_\gamma(f,t)_p := \sup_{|h|<t} \{\|\Delta_h^\gamma f\|_p\}, \quad t > 0, \quad 1 < p < \infty, \quad \text{with} \quad \Delta_h^\gamma f(\,\cdot\,) := \sum_{j=0}^{\infty} (-1)^j \binom{\gamma}{j} f(\,\cdot\, + jh), \quad f \in L^p(\mathbb{R}^d).$$

For $0 < q, \gamma < \infty$, the generalized Besov space is defined in terms of the following set of majorant functions,

$$\Omega_\gamma^q := \left\{ \varphi \in \Omega_\gamma : \int_0^1 \frac{1}{[\varphi(t^{-1})]^q} \frac{dt}{t} < \infty \right\}.$$

Definition 1.1. *For $0 < q < \infty$ and $\varphi \in \Omega_{2\beta}^q$ the generalized Besov space is*

$$B_{p,q}^\varphi(\mathbb{R}^d) = \left\{ f \in L^p(\mathbb{R}^d) : |f|_{B_{p,q}^\varphi} := \int_0^1 \left(\frac{\omega_\beta(f,t)_p}{\varphi(t)} \right)^q \frac{dt}{t} < \infty \right\}. \tag{3}$$

For $q = \infty$ and $\varphi \in \Omega_\gamma$,

$$B_{p,\infty}^\varphi(\mathbb{R}^d) := \left\{ f \in L^p(\mathbb{R}^d) : |f|_{B_{p,\infty}^\varphi} := \sup_{t>0} \left\{ \frac{\omega_\beta(f,t)_p}{\varphi(t)} \right\} < \infty \right\}.$$

As usual, if $q < \infty$ we endow $B_{p,q}^\varphi$ with the norm $\|\cdot\|_{B_{p,q}^\varphi} := \left(\|\cdot\|_p^q + |\cdot|_{B_{p,q}^\varphi} \right)^{1/q}$, otherwise $\|\cdot\|_{B_{p,\infty}^\varphi} := \|\cdot\|_p + |\cdot|_{B_{p,\infty}^\varphi}$. In particular, for $q = \infty$ these spaces are the generalized Lipschitz ones, denoted by $\text{Lip}(p, \beta, \varphi)$.

2 Main Results

The main results about embedding between the spaces of generalized smoothness are the following. The first one is about the well definition of the generalized Besov spaces, and in the sequence the natural relation of this space with the usual Besov space.

Theorem 2.1. *Let $1 \le p, q < \infty$ and $\varphi \in \Omega_\gamma$. The space $B^{\varphi}_{p,q}$ does not depend on γ.*

The standard Lipschitz and Besov spaces are recovered by definition above by taking $\varphi(t) = t^{\alpha} \in \Omega_\gamma$, for $0 < \alpha \le \gamma$. In this case, we write $\mathrm{Lip}(p, \alpha)$ and $\mathrm{B}^{\alpha}_{p,q}$, respectively.

Proposition 2.1. *For any $0 < \alpha \le \delta$, the following holds*

$$Lip(p, \alpha) \subset Lip(p, \delta, \varphi) \quad and \quad B^{\alpha}_{p,q} \subset B^{\delta,\varphi}_{p,q}, \quad for\ all\ \varphi \in \Omega_\alpha. \tag{1}$$

An interesting and non-trivial embedding is given by the next theorem.

Theorem 2.2. *Consider $1 < p < q < q_1/2 < \infty$, $\delta > 0$, and $1/q = 1/p - \delta/d$. For $\varphi \in \Omega_{\gamma+\delta}$, it holds*

$$B^{\varphi}_{p,q_2} \subset B^{\phi}_{q,q_1}, \tag{2}$$

for any $\phi \in \Omega_\gamma$ and $q_2 = q_1 q/(q_1 - q)$.

The application of the spaces of generalized smoothness in the study of the decay of the Fourier transform is the content of the next (Titchmarsh type) theorem ([1]).

Theorem 2.3. *Let $\{T_t\}_{t>0}$ be a β-admissible family of multipliers operators on $L^p(\mathbb{R}^d)$ and $\varphi \in \Omega_{2\beta}$.*
(A) Let $1 < p \le 2$ and $p \le q \le p'$. If $f \in Lip\,(p, \beta, \varphi)$, then

$$\left(\int_{t \le |\xi| \le 2t} \left[|\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\xi \right)^{1/q} = O\left(\varphi(t^{-1})\right), \quad as\ t \to \infty. \tag{3}$$

(B) Let $2 \le p < \infty$, $|\cdot|^{d(1-1/p-1/q)} \widehat{f}(\cdot) \in L^q$ and $p' \le q \le p$. If

$$\left(\int_{t \le |\xi| \le 2t} \left[|\xi|^{d(1-1/p-1/q)} |\widehat{f}(\xi)| \right]^q d\xi \right)^{1/q} = O\left(\varphi(t^{-1})\right), \quad as\ t \to \infty,$$

then $f \in Lip\,(p, \beta, \varphi)$.

The Titchmarsh type theorem above for the generalized Besov spaces and applications of these results can be found in ([1]), where a Riemann-Lebesgue type inequality is obtained as consequence.

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ON THE BISHOP-PHELPS-BOLLOBÁS PROPERTY FOR OPERATORS DEFINED ON C_0-SUMS OF EUCLIDEAN SPACES

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Abstract

In this talk we study the Bishop-Phelps-Bollobás theorem for operators defined on c_0-sums of euclidean spaces to uniformly convex spaces.

1 Introduction

Let X and Y be Banach spaces. The theory of norm-attaining operators is a recent field of research and started with the classical Bishop-Phelps Theorem, in 1961. The theorem states that $NA(X,\mathbb{K})$ is a dense set of X^*. In general, is not true that $\overline{NA(X,Y)} = L(X,Y)$, for all Banach spaces X and Y. In [1], the authors defined a new property for the pair of Banach spaces (X,Y), called Bishop-Phelps-BollobÃ¡s property for operators (BPBp in short). If (X,Y) satisfies the BPBp then $\overline{NA(X,Y)} = L(X,Y)$, but the converse is not true. Considering Y a uniformly convex Banach space, in [1] they proved that (ℓ_∞, Y) satisfies the BPBp for operators and left open the question if (c_0, Y) satisfies this property. In [2], they answered this question in a positive way.

We prove that $(c_0(\oplus_{n=1}^{\infty}\ell_2^n), Y)$ satisfies the BPBp for operators, whenever Y is a uniformly convex Banach space. Considering the real case, when Y is strictly convex, whe show that if $(c_0(\oplus_{n=1}^{\infty}\ell_2^n), Y)$ satisfies BPBp for operators then Y is uniformly convex. These results are part of the paper [4].

2 Main Results

Definition 2.1. *Let X and Y be Banach spaces. We say that the pair (X,Y) has the Bishop-Phelps-Bollobás property for operators (BPBp for operators, in short) if given $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ such that whenever $T \in S_{\mathcal{L}(X,Y)}$ and $x_0 \in S_X$ satisfy that $\|Tx_0\| > 1-\eta(\varepsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in S_{\mathcal{L}(X,Y)}$ satisfying the following conditions*

$$\|Su_0\| = 1, \quad \|u_0 - x_0\| < \varepsilon, \quad \text{and} \quad \|S - T\| < \varepsilon.$$

Theorem 2.1. *Let $c_0\left(\oplus_{k=1}^{\infty}\ell_2^k\right)$ a real Banach space and Y a strictly convex real Banach space. If $\left(c_0\left(\oplus_{k=1}^{\infty}\ell_2^k\right), Y\right)$ satisfies the BPBp, then Y is an uniformly convex Banach space.*

Idea of the proof Suppose that Y is not uniformly convex Banach space. Then there exist $\epsilon > 0$ and sequences $(y_k), (z_k) \subset S_Y$ such that

$$\lim_{k\to\infty}\left\|\frac{y_k + z_k}{2}\right\| = 1 \quad \text{and} \quad \|y_k - z_k\| > \epsilon,\ \forall k \in \mathbb{N}. \tag{1}$$

Then it is possible to define a sequence of bounded linear operator $T_i : c_0\left(\oplus_{k=1}^{\infty}\ell_2^k\right) \to Y$ such that $||T_{i_0}(e_1)|| > 1-\eta\left(\frac{\epsilon}{2}\right)$, for some $i_0 \in \mathbb{N}$. As the pair $\left(c_0\left(\oplus_{k=1}^{\infty}\ell_2^k\right), Y\right)$ satisfies the BPBp for operators, there exist an operator $R \in S_{\mathcal{L}\left(c_0\left(\oplus_{k=1}^{\infty}\ell_2^k\right),Y\right)}$ and a point $u \in S_{c_0\left(\oplus_{k=1}^{\infty}\ell_2^k\right)}$ such that

$$\|R(u)\| = 1,\ \|R - T_{i_0}\| < \frac{\epsilon}{2},\ \|u - e_1\| < 1.$$

Therefore, $||y_{i_0} - z_{i_0}|| < \epsilon$. It is a contradiction, so Y is a uniformly convex Banach space.

Proposition 2.1. *If Y is an uniformly convex Banach space, then the pair $\left(\ell_\infty\left(\bigoplus_{k=1}^{n}\ell_2^k\right), Y\right)$ satisfies the Bishop-Phelps-Bollobás for operators.*

Theorem 2.2. *If Y is an uniformly convex Banach space, then the pair $\left(c_0\left(\oplus_{k=1}^{\infty}\ell_2^k\right), Y\right)$ satisfies the Bishop-Phelps-Bollobás for operators.*

Idea of the proof Let $T \in S_{\mathcal{L}\left(c_0\left(\oplus_{k=1}^{\infty}\ell_2^k\right), Y\right)}$ and $x = \sum_{n=1}^{\infty}\sum_{k\in I(n)} x_k e_k \in S_{c_0\left(\oplus_{k=1}^{\infty}\ell_2^k\right)}$ such that $\|T(x)\| > 1 - \delta(\epsilon) + \gamma(\epsilon)$, where $\delta(\epsilon) > 0$ is the modulus of convexity of Y, $\gamma(\epsilon) > 0$ and $\lim_{\epsilon\to o}\gamma(\epsilon) = 0$. Let $J : \ell_\infty\left(\bigoplus_{k=1}^{n}\ell_2^k\right) \to c_0\left(\oplus_{k=1}^{\infty}\ell_2^k\right)$ be the map defined by

$$J(w) = \begin{cases} w_i, \text{ se } i \in A \\ 0, \text{ se } i \in \mathbb{N}\setminus A, \end{cases}$$

and $Q : \ell_\infty\left(\bigoplus_{k=1}^{n}\ell_2^k\right) \to Y$ be the bounded linear operator defined by $Q(w) = \frac{TP_AJ}{\|TP_AJ\|}(w)$. Then is easy to check that $\|Q\| = 1$ and almost attain its norm in some element $z = (z_i) \in \ell_\infty\left(\bigoplus_{k=1}^{n}\ell_2^k\right)$. By Proposition 2.1, Q and z can be approximated by a norm-attaining operator $\tilde{R}$ and a vector $\tilde{u}$, respectively. Finally, defining $R : c_0\left(\oplus_{k=1}^{\infty}\ell_2^k\right) \longrightarrow Y$ be the bounded linear operator given by

$$R(y) = \sum_{j=1}^{\infty}\sum_{i\in I(j)} y_i R(e_i),$$

where

$$R(e_i) = \begin{cases} \tilde{R}(f_i), \text{ if } i \in A \\ 0, \text{ if } i \in \mathbb{N}\setminus A, \end{cases}$$

and the vector $v = (v_i)_i \in c_0\left(\oplus_{k=1}^{\infty}\ell_2^k\right)$ defined by

$$v_i = \begin{cases} \tilde{u}_i, \ i \in A \\ x_i, \ i \in \mathbb{N}\setminus A, \end{cases}$$

then $||R(v)|| = 1$, $||R - T|| < \epsilon$ and $||v - x|| < \epsilon$.

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THE LINEARIZATION METHOD FOR IDEALS OF MULTIPOLYNOMIALS

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Abstract

We develop a method that generates an ideal of multipolynomials $\mathcal{MP}_{[\mathcal{I}]}$ starting with an operator ideal $\mathcal{I}$ that encompasses, as particular cases, the well studied ideals of multilinear operators and of polynomials generated by the linearization method. We prove that $\mathcal{MP}_{[\mathcal{I}]}$ inherits good properties from $\mathcal{I}$. Some of the results we prove are new even for multilinear operators and polynomials.

1 Introduction

The recent theory of multipolynomials, see [1,2,3,4,5,6,7], encompasses as particular cases the theories of multilinear operators, homogeneous polynomials and many other nonlinear operators between Banach spaces. Ideals of multipolynomials have already been treated in [1,2,3], but thus far the linearization method, which is a classical method to generate ideals of multilinear operators and ideals of homogeneous polynomials, has not been developed for multipolynomials. We developed the linearization method for multipolynomials. The main difficulty we faced was that, while the tools to study the linearization method in the multilinear and polynomials cases have been known for a long time, the basic results needed in the multipolynomial case are not available. In the construction of the basic tools and in the development of the method, the arguments from the multilinear and polynomial cases proved to be not good enough, new techniques are needed in this more general setting of multipolynomials. For the basic theory of multipolynomials we refer to [4].

2 Main Results

Let $E_1, \ldots, E_m, F$ be Banach spaces and $n_1, \ldots, n_m \in \mathbb{N}$. A map $P\colon E_1 \times \cdots \times E_m \longrightarrow F$ is a *continuous* $(n_1, \ldots, n_m)$*-homogeneous multipolynomial* if, for every $j \in \{1, \ldots, m\}$, P is a continuous n_j-homogeneous polynomial in the j-th variable, that is, for fixed $x_1 \in E_1, \ldots, x_{j-1} \in E_{j-1}, x_{j+1} \in E_{j+1}, \ldots, x_n \in E_m$, the map

$$P_{j;x_1,\overset{[j]}{\ldots},x_m}\colon E_j \longrightarrow F \ , \ P_{j;x_1,\overset{[j]}{\ldots},x_m}(x_j) = P(x_1, \ldots, x_m),$$

is a continuous n_j-homogeneous polynomial. Of course, the notation $x_1, \overset{[j]}{\ldots}, x_m$ means that the j-th term has been omitted. We can consider its associated symmetric n_j-linear operator $(P_{j;x_1,\overset{[j]}{\ldots},x_m})^\vee\colon E_j^{n_j} \longrightarrow F$.

Proposition 2.1. (a) *Let $P \in \mathcal{MP}(^{n_1}E_1, \ldots, ^{n_m}E_m; F)$, $j \in \{1, \ldots, n\}$ and $x_j \in E_j$ be given. Then*

$$T_j(P)(x_j)\colon E_1 \times \cdots \times E_m \longrightarrow F \ , \ T_j(P)(x_j)(y_1, \ldots, y_m) = (P_{j;y_1,\overset{[j]}{\ldots},y_m})^\vee(x_j, y_j^{n_j-1}),$$

is a continuous $(n_1, \ldots, n_j - 1, \ldots, n_m)$-homogeneous multipolynomial.
(b) *For $j = 1, \ldots, n$, the operator*

$$T_j\colon \mathcal{MP}(^{n_1}E_1, \ldots, ^{n_m}E_m; F) \longrightarrow \mathcal{L}(E_j; \mathcal{MP}(^{n_1}E_1, \ldots, ^{n_j-1}E_j, \ldots ^{n_m}E_m; F)),$$

where $T_j(P)$ is defined in (a)*, is an isomorphism into.*

Now we are ready to define the ideals of polynomials provided by the factorization method.

Definition 2.1. Let $\mathcal{I}$ be an operator ideal. A multipolynomial $P \in \mathcal{MP}({}^{n_1}E_1, \ldots, {}^{n_m}E_m; F)$ belongs to $\mathcal{MP}_{[\mathcal{I}]}({}^{n_1}E_1, \ldots, {}^{n_m}E_2; F)$ if, for every $j = 1, \ldots, m$,

$$T_j(P) \in \mathcal{I}(E_j; \mathcal{MP}({}^{n_1}E_1, \ldots, {}^{n_j-1}E_j, \ldots, {}^{n_m}E_m; F)).$$

Theorem 2.1. *For every operator ideal $\mathcal{I}$, $\mathcal{MP}_{[\mathcal{I}]}$ is an ideal of multipolynomials.*

To study the injectivity of $\mathcal{MP}_{[\mathcal{I}]}$ we need the following lemma.

Lemma 2.1. *Let $n_1, \ldots, n_m \in \mathbb{N}$, $j \in \{1, \ldots, m\}$, $u \in \mathcal{L}(E; F)$ and $\varphi \in E^*$ with $\|\varphi\| = 1$ be given. Then:*
(a) *The map $P^{j,u} \colon \mathbb{K}^{j-1} \times E \times \mathbb{K}^{m-j} \longrightarrow F$ given by*

$$P^{j,u}(\lambda_1, \ldots, \lambda_{j-1}, x, \lambda_{j+1}, \ldots, \lambda_m) = \lambda_1^{n_1} \overset{[j]}{\cdots} \lambda_m^{n_m} \varphi(x)^{n_j-1} u(x),$$

is an $(n_1, \ldots, n_m)$-homogeneous multipolynomial.
(b) *If $v \in \mathcal{L}(F; G)$, then $P^{j, v \circ u} = v \circ P^{j,u}$.*
(c) *The linear operator $L_F^j \colon F \longrightarrow \mathcal{MP}({}^{n_1}\mathbb{K}, \ldots, {}^{n_{j-1}}\mathbb{K}, {}^{n_j-1}E, {}^{n_{j+1}}\mathbb{K}, \ldots, {}^{n_m}\mathbb{K}; F)$,*

$$L_F^j(y)(\lambda_1, \ldots, \lambda_{n_{j-1}}, x, \lambda_{n_{j+1}}, \ldots, \lambda_m) = \lambda_1^{n_1} \overset{[j]}{\cdots} \lambda_{n_m}^{n_m} \cdot \varphi(x)^{n_j-1} y,$$

is a metric injection.
(d) *There exists a finite rank linear operator $W \colon E_1 \longrightarrow \mathcal{MP}({}^{n_1}\mathbb{K}, \ldots, {}^{n_{j-1}}\mathbb{K}, {}^{n_j-1}E, {}^{n_{j+1}}\mathbb{K}, \ldots, {}^{n_m}\mathbb{K}; F)$ such that $n_j T_j(P^{j,u}) = (L_F^j \circ u) + W$.*

Theorem 2.2. *The following statements are equivalent for an operator ideal $\mathcal{I}$:*
(a) *The operator ideal $\mathcal{I}$ is injective.*
(b) *The ideal of multipolinomials $\mathcal{MP}_{[\mathcal{I}]}$ is injective.*
(c) *There are $n_1, \ldots, n_m \in \mathbb{N}$ such that the ideal $(\mathcal{MP}_{[\mathcal{I}]})_{n_1, \ldots, n_n}$ of $(n_1, \ldots, n_m)$-homogeneous polynomials is injective.*

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FOURIER-JACOBI COEFFICIENTS

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Abstract

Positive definite functions on two-point homogeneous spaces were characterized by R. Gangolli some forty years ago and are very useful for solving scattered data interpolation problems on the spaces. Such characterization is related to the so called Fourier-Jacobi coefficients and can be found in [6]. This work provides relations between these coefficients

1 Introduction

Let $\mathbb{M}^d$ denote a d dimensional compact two-point homogeneous space. It is well known that spaces of this type belong to one of the following categories ([7]): the unit spheres S^d, $d = 1, 2, \ldots$, the real projective spaces $\mathbb{P}^d(\mathbb{R})$, $d = 2, 3, \ldots$, the complex projective spaces $\mathbb{P}^d(\mathbb{C})$, $d = 4, 6, \ldots$, the quaternionic projective spaces $\mathbb{P}^d(\mathbb{H})$, $d = 8, 12, \ldots$, and the Cayley projective plane $\mathbb{P}^d(Cay)$, $d = 16$. In general this classification is decisive in analysis of problemas involving the compact two-point homogeneous spaces, as can be seen in [2,3,5] and others mentioned there.

A zonal kernel K on $\mathbb{M}^d$ can be written in the form $K(x, y) = K_r^d(\cos|xy|/2)$, $x, y \in \mathbb{M}^d$, for some function $K_r^d : [-1, 1] \to \mathbb{R}$, the *radial* or *isotropic part* of K. A result due to Gangolli ([6]) established that a continuous zonal kernel K on $\mathbb{M}^d$ is positive definite if and only if

$$K_r^d(t) = \sum_{k=0}^{\infty} a_k^{\alpha,\beta} P_k^{\alpha,\beta}(t), \quad t \in [-1, 1], \tag{1}$$

in which $\sum_{k=0}^{\infty} a_k^{\alpha,\beta} P_k^{\alpha,\beta}(1) < \infty$ and $a_k^{\alpha,\beta} \in [0, \infty)$, $k \in \mathbb{Z}_+$. Here, $\alpha = (d-2)/2$ and $\beta = (d-2)/2, -1/2, 0, 1, 3$, depending on the respective category $\mathbb{M}^d$ belongs to, among the five we have mentioned in the beginning of this section. The symbol $P_k^{(d-2)/2,\beta}$ stands for the Jacobi polynomial of degree k associated with the pair (α, β). The coefficients $a_k^{\alpha,\beta}$ are given by

$$a_k^{\alpha,\beta} := \frac{\left[P_k^{\alpha,\beta}(1)\right]^2 (2k+\alpha+\beta+1)\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+1}\,\Gamma(k+\alpha+1)\Gamma(k+\beta+1)} \int_{-1}^{1} f(t) R_k^{(\alpha,\beta)}(t)(1-t)^{\alpha}(1+t)^{\beta} dt,$$

and they are called *Fourier-Jacobi coefficients.*

2 Main Results

The main results to be proved in this work are based on those presented in [4] and are described below.

Theorem 2.1 ([1]. *] Let K be a continuous, isotropic and positive definite kernel on $\mathbb{M}^d$, and $a_k^{\alpha,\beta}$ the Fourier-Jacobi coefficients presented in (1). Then*

$$a_k^{\alpha,\beta} = \sum_{j=0}^{\infty} \left(\prod_{l=1}^{j} \omega_{k+l-1}^{\alpha,\beta} \right) \gamma_{k+j}^{\alpha,\beta}\ a_{k+j}^{\alpha+1,\beta} = \sum_{j=0}^{\infty} \left(\prod_{l=1}^{j} \varphi_{k+l-1}^{\alpha,\beta} \right) \xi_{k+j}^{\alpha,\beta}\ a_{k+j}^{\alpha,\beta+1}$$

in which,

$$\begin{aligned}
\omega_k^{\alpha,\beta} &= \frac{(k+1)(n+\beta+1)(2k+\alpha+\beta+1)}{(k+\alpha+1)(k+\alpha+\beta+1)(2k+\alpha+\beta+3)} \\
\gamma_k^{\alpha,\beta} &= \frac{(\alpha+1)(2k+\alpha+\beta+1)}{(k+\alpha+1)(k+\alpha+\beta+1)} \\
\varphi_k^{\alpha,\beta} &= \frac{2k+\alpha+\beta+1}{k+\alpha+\beta+1} \\
\xi_k^{\alpha,\beta} &= \frac{(k+1)(2k+\alpha+\beta+1)}{(k+\alpha+\beta+1)(2k+\alpha+\beta+3)}.
\end{aligned}$$

We can obtain an application of previous result involving the positive definiteness and strictly positive definiteness of a kernel on a two-point homogeneos space $\mathbb{M}^d$.

Theorem 2.2 ([1]**.** *] Let $d, d' \geq 2$ be integers. If K is a positive definite kernel on a two-point homogeneous space $\mathbb{M}^{2d}$ and a strictly postive definite kernel on $\mathbb{M}^{2d'}$, such that $\mathbb{M}^{2d}$ and $\mathbb{M}^{2d'}$ belong to same category we have mentioned in the beginning of previous section, then K is a strictly postive definite kernel on $\mathbb{M}^{2d}$.*

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OPERATOR EXTENSION ON TOTALLY ORDERED COMPACTA

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Abstract

In this work we investigate versions of the classical theorem of Sobczyk and we discuss the problem of extending operators defined on unital Banach subalgebras of $C(K)$, when K is a compact ordered space and c_0 is replaced by its non-separable version, $c_0(I)$. This is an article based on a joint work with Daniel V. Tausk.

1 Introduction

It follows from the Stone-Weierstrass theorem that the unital Banach subalgebras of $C(K)$ are precisely the images of composition operators $\varphi^*(f) = f \circ \varphi$, induced by continuous surjective functions $\varphi : K \longrightarrow L$ between compact Hausdorff spaces. In [3] the problem of characterizing maps $\varphi : K \longrightarrow L$ such that $\varphi^*\big[C(L)\big]$ has the c_0-EP in $C(K)$ was studied in the context of totally ordered spaces K and L, under the assumption that φ is increasing, surjective and continuous. The case where φ is not an increasing function is far more complicated to analyse and was considered in [2], only with L countable.

2 Main Results

In this article a compact ordered space is a totally ordered set $(K, \leq)$ that is bounded and Dedekind-complete, $\mathrm{M}(K)$ denotes the space of signed σ-additive regular Borel measures in K and $\mathrm{NBV}(K)$ denotes the space of right-continuous functions $f : K \longrightarrow \mathbb{R}$ of bounded variation. We assume the Riesz representation theorems for compact ordered spaces, i.e. $C(K)^* \equiv \mathrm{M}(K) \equiv \mathrm{NBV}(K)$ and all of these spaces are endowed with the weak* topology induced from $C(K)^*$ via isometry. For an infinite set I (frequently assumed to be uncountable) we denote:

$$c_0(I) = \Big\{(x_i)_{i\in I} \in \mathbb{R}^I : x_i \xrightarrow{w^*} 0\Big\},$$

where $x_i \xrightarrow{w^*} 0$ means: $\forall \varepsilon > 0$, $\{i \in I : |x_i| \geq \varepsilon\}$ is finite. In this case we say that the family $(x_i)_{i\in I}$ weak*-null.

We denote by $\varphi : K \xrightarrow{c.i.s.} L$ a continuous increasing surjective function between ordered compact spaces and $Q(\varphi) = \{t \in L : |\varphi^{-1}(t)| > 1\}$. Because bounded operators $T : C(L) \longrightarrow c_0(I)$ are associated with a weak*-null family of $(\alpha_i)_{i\in I}$ in $C(L)^*$ we can analyse the problem of operator extension in terms of the coordinate functionals α_i. Because of the Riesz representation theorem and the measure push-forward $\varphi_* : \mathrm{M}(K) \longrightarrow \mathrm{M}(L)$ we define:

Definition 2.1. *A bounded operator $T : C(L) \longrightarrow c_0(I)$ extends through φ^* to $C(K)$ iff there exists a weak*-null family $(F_i)_{i\in I} \subseteq \mathrm{NBV}(K)$ such that $T \equiv (\varphi_*(F_i))_{i\in I}$.*

In [3] the authors characterize extension of $T : C(L) \longrightarrow c_0$, associated with a weak*-null family $(F_n)_{n\geq 1}$ in $\mathrm{NBV}(L)$, through $\varphi : K \xrightarrow{c.i.s.} L$ in therms of the set $\{t \in Q(\varphi) : F_n(t) \not\longrightarrow 0\}$ being countable. Unfortunately this characterization does not remains valid for $c_0(I)$-valued operators:

Proposition 2.1. *There exist K and L compact ordered spaces, $\varphi : K \xrightarrow{c.i.s.} L$ and a weak*-null family $(F_i)_{i\in I}$ in $C(L)$ that extends through φ^* with $\{t \in Q(\varphi) : F_n(t) \not\longrightarrow 0\}$ uncountable.*

The construction of K and L relies on the existence of compact ordered spaces K_0 and L_0, $\psi : K_0 \xrightarrow{c.i.s.} L_0$ and an operator $T : C(L) \longrightarrow c_0$ such that $\{t \in Q(\psi) : F_n(t) \not\longrightarrow 0\}$ is non-empty. Then one needs to guarantee that the lexicographic product of uncountable copies of these spaces have the desired properties. On the bright side, the characterization given in [3] is not entirely lost: the converse is always true. If we want to characterize operator extension in this more general context, we need to introduce different tools:

Definition 2.2. *A family of real valued functions $(F_i)_{i\in I}$ is of type $c_0\ell_1$ over a set Q if there exists a decomposition:*

$$F_i(t) = a_{i,t} + b_{i,t}, \qquad \forall t \in Q,$$

where the families of real numbers $(a_{i,t})_{i\in I, t\in Q}$ and $(b_{i,t})_{i\in I, t\in Q}$ satisfies the following:

- *(c_0 component) For each $t \in Q$, $\lim_{i\in I} a_{i,t} = 0$;*
- *(ℓ_1 component) $\sup_{i\in I} \sum_{t\in Q} |b_{i,t}| < +\infty$.*

In the above definition the set Q try to code a piece of L where one can define an extension of $T : C(L) \longrightarrow c_0(I)$ that is viewed as a weak*-null family $(F_i)_{i\in I}$ in $\mathrm{NBV}(L)$. Notice that $(F_i)_{i\in I}$ is always of type $c_0\ell_1$ over the set $\{t \in Q : F_i(t) \longrightarrow 0\}$, so it is somewhat natural that we have the following characterization:

Theorem 2.1. *Let $\varphi : K \xrightarrow{c.i.s.} L$ and a weak*-null family $(F_i)_{i\in I} \subseteq \mathrm{NBV}(L)$. Are equivalent:*

1. *$(F_i)_{i\in I}$ extends through φ^*;*
2. *$(F_i)_{i\in I}$ is of type $c_0\ell_1$ over $Q(\varphi)$.*

Moreover, if T extends through φ^*, there exists an extension $\widetilde{T}$ with $\|\widetilde{T}\| \leq 2\|T\|$.

For a set I, we say that a Banach space X has the $c_0(I)$-*extension property* ($c_0(I)$-EP) if every operator defined on a closed subspace $T : Y \longrightarrow c_0(I)$ admits an extension to X. It was already known that c_0-EP and $c_0(I)$-EP were not the same. In [1] the authors present an a WCG space $C(K)$ with non-complemented copy of $c_0(I)$, with $|I| \geq \beth_\omega$ and the space K in this example is an Eberlein compact. Because odered compact spaces are well behaved, it is possible to construct an example of $c_0(I)$ valued operator without extension with $|I| = \omega_1$, the construction of this operator is contained in [4] and it is carefully discussed in [5].

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THE COARSE P-LIMITED SETS IN BANACH SPACES

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Abstract

A wide new class of subsets of a Banach space X named coarse p-limited sets ($1 \leq p < \infty$) is introduced by considering weak* p-summable sequences in X' instead of weak* null sequences. We study its basic properties and compare it with the class of compact and weakly compact sets. Results concerning the relationship of coarse p-limited sets with operators are obtained. This is a joint work with Pablo Galindo.

1 Introduction

Recall that a subset A of a Banach space X is said to be *limited* if weak*-null sequences in X' converge uniformly to 0 on A. Or equivalently, if for every linear operator $T : X \to c_0$, $T(A)$ is a relatively compact set. Notions alike to limitedness have been considered in several contexts. For instance, the approximation properties, the $p-$*limited* ($1 \leq p < \infty$) sets were defined by Karn and Sinha [4] and then studied by Delgado and Piñeiro [2]. A subset A of X is *p-limited* ($1 \leq p < \infty$) if for every weak* p-summable sequence $(x'_n) \subset X'$, there is $a = (a_n) \in \ell_p$ such that $|x'_n(x)| < a_n$ for all $x \in A$ and $n \in \mathbb{N}$.

It is natural to wander if the p-limited sets may be characterized by bounded operators with range in ℓ_p. In particular, we proved the following result:

Proposition 1.1. *If A is a p-limited set, $1 \leq p < \infty$, then $T(A)$ is relatively compact in ℓ_p for all $T \in L(X; \ell_p)$.*

However, the converse of Proposition 1.1 is not true. Indeed, by Pitt's theorem, every bounded operator $T : c_0 \to \ell_p$ is compact, i.e. $T(B_{c_0})$ is a compact set in ℓ_p for all $T \in L(c_0; \ell_p)$. Nevertheless, B_{c_0} is not a p-limited set. This fact, allows us to introduce the following definition:

Definition 1.1. *Let $1 \leq p < \infty$. We say that a subset A of X is a coarse p-limited set if $T(A) \subset \ell_p$ is a relatively compact set for all $T \in L(X; \ell_p)$.*

It follows from Proposition 1.1 that every p-limited set is coarse p-limited. However, in every infinite dimensional Banach space X there are coarse p-limited sets that are not p-limited. This remarkable difference led us to choose the word coarse in our definition.

A bunch of examples of this new class of sets were given. In particular, we have the following scheme:

$$\text{limited or } p\text{-limited} \Rightarrow \text{coarse } p\text{-limited} \not\Rightarrow \text{limited nor } p\text{-limited.}$$

As we pointed out, the classes of coarse p-limited and p-limited cannot coincide in infinite dimensional Banach spaces. Nevertheless, in ℓ_p, for $1 \leq p < \infty$, the classes of coarse p-limited and limited coincide.

2 Main Results

Bourgain and Diestel showed that every limited set is conditionally weakly compact [1]. It is natural to wonder if every coarse p-limited set is conditionally weakly compact. In the next proposition, we give a positive answer in the case $2 \leq p < \infty$.

Proposition 2.1. *If $2 \leq p < \infty$ and if $A \subset X$ is a coarse p-limited set, then A is conditionally weakly compact.*

The above result fails for $p = 1$. For example, the unit ball of $C([0,1])$ is a coarse 1-limited set which is not conditionally weakly compact.

Remark 2.1. *In general, one cannot establish an inclusion relationship between the class of coarse p-limited sets and the class of coarse q-limited sets for $p \neq q$.*

By comparing the class of coarse p-limited sets with the classes of compact and weakly compact sets, we defined two new properties in the class of Banach spaces:

Definition 2.1. *We say that a Banach space X has the*

1. coarse p-DP* property *if every relatively weakly compact set is coarse p-limited.*
2. coarse p-Gelfand-Phillips property *if every coarse p-limited subset of X is relatively compact.*

We provided examples and compare these new properties with the already known properties DP* and Gelfand-Phillips in Banach spaces. Besides, we proved the following results:

Theorem 2.1. *A Banach space X has the coarse p-DP* property if and only if every bounded operator $T : X \to \ell_p$ is completely continuous.*

Theorem 2.2. *A Banach space X has the coarse p-DP* property if, and only if, every conditionally weakly compact subset of X is coarse p-limited.*

Theorem 2.3. *If a Banach space X has the coarse p-Gelfand-Phillips property, then every weakly null coarse p-limited sequence in X is norm null. The converse holds if $2 \leq p < \infty$.*

A class of associated operators is introduced in a natural way: a bounded linear operator $T : X \to Y$ is said to be *coarse p-limited* if $T(B_X)$ is a coarse p-limited set in Y. This class of operators allowed us to prove the following characterization:

Theorem 2.4. *For a Banach space X, the following are equivalent:*

1. *X has the coarse p-DP* property .*
2. *Every weakly compact operator $T : Z \to X$ is coarse p-limited for any Banach space Z.*
3. *Every weakly compact operator $T : \ell_1 \to X$ is coarse p-limited.*

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DAMAGE IDENTIFICATION IN KIRCHHOFF PLATE BASED ON THE TOPOLOGICAL DERIVATIVE METHOD

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Abstract

This paper presents the topological derivative formula for damage identification in Kirchhoff bending plate model.

1 Introduction

Let us consider a plate represented by a two-dimensional domain $D \subset \mathbb{R}^2$. We assume that the boundary of D, denoted by ∂D, is a curvilinear polygon of class $\mathcal{C}^{1,1}$. The damage is represented by a variation in the plate thickness, which is assumed to be given by a piecewise constant function of the form $h(x) = h_0$, if $x \in \Omega$ and $h(x) = h_1$, if $x \in \omega$ where $\Omega = D \setminus \omega$ and ω are used to represent the healthy and damaged regions of the plate, respectively. We want to minimize a shape functional measuring the misfit between the available data (measurement) and the solution computed from the model problem, with respect to the support ω of the damaged region. The shape functional $\mathcal{J}(u,\theta)$ is defined as

$$\mathcal{J}(u,\theta) = \sum_{i=1}^{N} \int_D (|u-u^*|^2 + \|\theta-\theta^*\|^2)\delta(x-x_i), \tag{1}$$

where $\delta(x,x_i)$ is the Dirac delta function with pole at $x_i \in D$, $u : D \to \mathbb{R}$ and $\theta = \nabla u : D \to \mathbb{R}^2$ are the transverse displacement and rotation vector field of the plate, respectively. The perturbed counterpart of the plate thickness h is denoted as $h_\varepsilon = \gamma_\varepsilon h$, where $\gamma = \gamma(x)$ is used to denote the contrast in the plate thickness, namely

$$\gamma_\varepsilon(x) := \begin{cases} 1, \text{ if } x \in D \setminus B_\varepsilon(\hat{x}), \\ \gamma, \text{ if } x \in B_\varepsilon(\hat{x}). \end{cases} \qquad \gamma(x) := \begin{cases} h_1/h_0, \text{ if } x \in \Omega, \\ h_0/h_1, \text{ if } x \in \omega. \end{cases} \tag{2}$$

According to (2), the perturbed counterpart of the shape functional is given by

$$\mathcal{J}(u_\varepsilon,\theta_\varepsilon) = \sum_{i=1}^{N} \int_D (|u_\varepsilon-u^*|^2 + \|\theta_\varepsilon-\theta^*\|^2)\delta(x-x_i), \tag{3}$$

for $x_i \notin B_\varepsilon(\hat{x})$, $i = 1, \cdots, N$, where $u_\varepsilon : D \to \mathbb{R}$ and $\theta_\varepsilon = \nabla u_\varepsilon : D \to \mathbb{R}^2$ are the transverse displacement and rotation vector field of the plate, respectively, associated with the perturbed counterpart of the model problem. The transverse displacement (or deflection) of the plate in the time harmonic regime written in the frequency domain is solution to the following variational problem:

$$u \in \mathcal{U} : \int_D h^3 \mathcal{M}(u) \cdot \nabla\nabla\eta - k^2 \int_D hu\eta = \int_D b\eta \quad \forall \eta \in \mathcal{U}, \tag{4}$$

where k is the wave number defined as $k^2 = \rho(2\pi f)^2$, with f the working frequency and ρ the material density, h is the plate thickness, b is the source-term, $u : D \to \mathbb{R}$ is the transverse displacement and $\theta = \nabla u$ is the rotation. In addition, $\mathcal{M}(u)$ the moment tensor, namely

$$\mathcal{M}(u) = \frac{E}{12(1-\nu^2)}[(1-\nu)\mathbb{I} + \nu \mathrm{I} \otimes \mathrm{I}]\nabla\nabla u, \tag{5}$$

with I and $\mathbb{I}$ used to denote the second and fourth order identity tensors, respectively, whereas E is the Young modulus and ν is the Poisson ratio. The set of kinematically admissible displacements $\mathcal{U} := \{\varphi \in H^2(D) : \varphi_{|\partial D} = 0\}$. According to (2), the perturbed counterpart of the variational problem (4) reads

$$u_\varepsilon \in \mathcal{U} : \int_D h_\varepsilon^3 \mathcal{M}(u_\varepsilon) \cdot \nabla\nabla\eta - k^2 \int_D h_\varepsilon u_\varepsilon \eta = \int_D b\eta \quad \forall \eta \in \mathcal{U}. \tag{6}$$

Finally, let us introduce the following fourth-order polarization tensor associated with the plate bending model

$$\mathbb{P}_\gamma = -\frac{1-\gamma^3}{1+\gamma^3\beta}\left((1+\beta)\mathbb{I} + \frac{1-\gamma^3}{2}\frac{\alpha-\beta}{1+\gamma^3\alpha}\mathrm{I}\otimes\mathrm{I}\right), \tag{7}$$

where the symbols $\mathbb{I}$ and I are used to denote the second and fourth order identity tensor, respectively.

2 Main Results

The existence of the topological derivative is proved in [1] and the theory of the method is presented in [2]. By setting the constants α and β in the definition of the polarization tensor (7) as follows

$$\alpha = \frac{1+\nu}{1-\nu} \quad \text{and} \quad \beta = \frac{1-\nu}{3+\nu}, \tag{8}$$

we can present the main result, namely:

Theorem 2.1. *The topological derivative of the tracking-type shape functional $\mathcal{J}(u,\theta)$ from (1), where $\theta = \nabla u$, with respect to the nucleation of a small damage represented by a piecewise constant variation in the plate thickness, is given by*

$$\mathcal{T}(x) = h^3\mathbb{P}_\gamma\mathcal{M}(u)\cdot\nabla\nabla v(x) + k^2(1-\gamma)huv(x), \quad \forall x \in D\setminus\{x_1,\cdots,x_N\}, \tag{9}$$

where u is solution to (4), k is the wave number, h is the plate thickness and γ its contrast. The polarization tensor $\mathbb{P}_\gamma$ is defined by (7) together with the coefficients α and β according to (8).

Proof See [1].

Corollary 2.1. *For $h = h_0$, the limit case $\gamma \to 0$ ($h_1 \to 0$) in (9) is well defined and given by*

$$\mathcal{T}(x) = h_0^3\mathbb{P}_0\mathcal{M}(u)\cdot\nabla\nabla v(x) + k^2 h_0 uv(x), \quad \forall x \in D\setminus\{x_1,\cdots,x_N\}, \tag{10}$$

where the polarization tensor $\mathbb{P}_0$ is written as

$$\mathbb{P}_0 = \frac{-1}{3+\nu}\left(4\mathbb{I} + \frac{1+3\nu}{1-\nu}\mathrm{I}\otimes\mathrm{I}\right). \tag{11}$$

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CONVERSE LYAPUNOV THEOREMS FOR REGULAR STABILITY FOR LINEAR GENERALIZED ODES

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Abstract

We establish converse Lyapunov theorems for backward solutions of linear generalized ODEs with weakly right-continuous operator.

1 Introduction

The aim of this presentation is to address a new kind of stability for linear generalized ODE which are known to encompass several types of integral and differential equations. We are going to investigate linear generalized ODEs of the form

$$\frac{dx}{d\tau} = D[A(t)x + g(t)], \tag{1}$$

where $A\colon (-\infty, t_0] \to L(X)$ is an operator taking values in the well-known space of linear bounded operators from a Banach space X to itself and $g\colon (-\infty, t_0] \to X$ is a regulated function (i.e. g admits lateral limits). When $g \equiv 0$, the linear generalized ODE (1) is called homogeneous, otherwise it is non-homogeneous.

We require that A is locally of bounded variation and weakly right-continuous and, contrary to what we encounter in the literature, this is enough to ensure that the homogeneous linear generalized ODE (1) has a global backward solution (see [3]). It is our will to investigate a kind of stability that evaluates the distance between any two solutions using the usual supremum norm. In order to give the reader a rough idea of the kind of stability we are dealing with, we point out that the usual (Lyapunov) uniform stability is equivalent to the notion of regular stability.

Our main goal is to prove that regular stability implies the existence of a Lyapunov functional satisfying very weak conditions. Some of the necessary tools to explore regular stability are described next.

Definition 1.1 (Lyapunov Functional)**.** *Let $B \subseteq \mathcal{O} \subseteq X$, where $0_X \in \mathcal{O}$, $t_0 \in \mathbb{R}$. We say that $V\colon (-\infty, t_0] \times B \to \mathbb{R}$ is a Lyapunov functional with respect to (1) if*

LF1 For every $x \in B$, $V(\cdot, x)$ is right-continuous on $(-\infty, t_0)$;

LF2 There exists a strictly increasing continuous function $b\colon \mathbb{R}^+ \to \mathbb{R}^+$, such that

$$b(0) = 0 \quad \text{and} \quad V(t,x) \geqslant b(\|x\|), \quad \text{for } (t,x) \in (-\infty, t_0] \times B;$$

LF3 For every backward solution $x\colon J \subset (-\infty, t_0] \to X$, for a given $t \in J \setminus \sup\{J\}$, it is true that

$$D^+V(t, x(t)) = \limsup_{\eta \to 0^+} \left[\frac{V(t+\eta, x(t+\eta)) - V(t, x(t))}{\eta} \right] \geqslant 0.$$

Definition 1.2 (Regular stability)**.** *Let $A\colon (-\infty, t_0] \to L(X)$ be weakly right-continuous, $g\colon (-\infty, t_0] \to X$ be a right-continuous function and $x \equiv 0$ the trivial solution of the linear generalized ODE (1). We say that x is*

- *regular stable, if for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that for* $\overline{x}\colon [\alpha, \beta] \subset (-\infty, t_0] \to X$ *is a right-continuous function which satisfies* $\|\overline{x}(\beta)\| < \delta$ *and*

$$\sup_{\zeta \in [\alpha,\beta]} \left\{ \left\| \overline{x}(\zeta) - \overline{x}(\beta) + \int_{\zeta}^{t_0} d[A(\tau)]\overline{x}(\tau) \right\| \right\} < \delta,$$

then

$$\|x(t)\| < \epsilon, \ \ t \in [\alpha, \beta];$$

- *regular stable with respect to perturbations, if for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that if* $\|\widetilde{x}(\beta)\| < \epsilon$*, and* $\sup_{\zeta \in [\alpha,\beta]}\{\|g(\zeta) - g(\beta)\|\} < \delta$*, then*

$$\|\widetilde{x}(t)\| < \epsilon, \ \ t \in [\alpha, \beta],$$

where $\widetilde{x}\colon [\alpha, \beta] \subset (-\infty, t_0] \to X$ *is a backward solution of the non-homogeneous linear generalized ODE (1).*

2 Main Results

Theorem 2.1. *Let* $A\colon (-\infty, t_0] \to L(X)$ *be weakly right-continuous,* $g\colon (-\infty, t_0] \to X$ *be right-continuous, and* $x \equiv 0$ *be the trivial solution of the linear generalized ODE (1). Then,* $x \equiv 0$ *is regularly stable if and only if it is regularly stable with respect to perturbations.*

Theorem 2.2. *Let* $A\colon (-\infty, t_0] \to L(X)$ *be weakly right-continuous and* $g\colon (-\infty, t_0] \to X$ *be a regulated function. If the trivial solution* $x \equiv 0$ *is regularly stable, then there exists a Lyapunov functional* $V\colon (-\infty, t_0] \times X \to \mathbb{R}$ *satisfying*

1. *for all* $t \in (-\infty, t_0]$*,* $V(t, 0) = 0$ *and there exists an increasing continuous function* $a\colon \mathbb{R}^+ \to \mathbb{R}^+$ *satisfying*

$$a(0) = 0 \quad \text{and} \quad a(\|x\|) \geqslant V(t, x), \quad \text{for } x \in X;$$

2. *the mapping* $t \mapsto V(t, x(t))$ *is non-decreasing along global backward solutions of linear generalized ODE (1).*

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EXISTENCE AND UNIQUENESS FOR SOLUTIONS OF LINEAR DYNAMIC EQUATIONS WITH PERRON Δ-INTEGRALS

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Abstract

Our goal is to investigate the existence and uniqueness of a solution for a homogeneous and a nonhomogeneous linear dynamic equations on time scales, whose integral forms contain Perron Δ-integrals defined in Banach spaces. Since we work in the framework of Perron Δ-integrals, we can handle functions not only having many discontinuities, but also being highly oscillating. Our results require weaker conditions than those in the literature.

1 Introduction

Calculus on time scales, introduced in 1988 by Stefan Hilger, allows us to describe continuous, discrete and hybrid systems which have several applications. One of the main concepts of the time scale theory is the delta derivative, which is a generalization of the classical time derivative in the continuous time and the finite forward difference in the discrete time. As a consequence, differential equations as well as difference equations are naturally accommodated in this theory.

The best known results on the existence and uniqueness of a solution for a nonhomogeneous linear dynamic equation of the form

$$x^{\Delta} = a(t)x + f(t) \tag{1}$$

and for its corresponding homogeneous equation

$$x^{\Delta} = a(t)x \tag{2}$$

on a time scale $\mathbb{T}$, take into account that a is a regressive and rd-continuous $n \times n$-matrix-valued function and $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Moreover, the integrals appearing in the solutions of the dynamic equations (1) and (2) are in the sense of the Riemann Δ-integral.

In the paper, we are interested in proving existence and uniqueness of a solution for linear homogeneous and nonhomogeneous dynamic equations on time scales, where their integral forms contain Banach spaces-valued Perron Δ-integrals.

We emphasize that Lemma 2.1 in the sequel shows that every rd-continuous functions a and f satisfy all the hypotheses of our results. Therefore, our results on the existence and uniqueness of a solution for a linear nonhomogeneous dynamic equation and for a linear homogeneous dynamic equation generalize the results from the classical theory of dynamic equations. Moreover, since we are considering Perron Δ-integrals instead of Riemann Δ-integrals, our integrands may be highly oscillating and have many discontinuities.

2 Main Results

Let $\mathbb{T}$ be a time scale. Given $t_0 \in \mathbb{T}$, we define $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}$ and we denote by $G_0(\mathbb{T}_0, X)$ the vector space of all regulated functions $x : \mathbb{T}_0 \to X$ such that

$$\|x\|_{\mathbb{T}_0} := \sup_{s \in \mathbb{T}_0} e^{-(s-t_0)} \|x(s)\| < \infty.$$

Consider the nonhomogeneous linear dynamic equation (1) and its corresponding homogeneous equation (2), where both functions $a : \mathbb{T} \to L(X)$ and $f : \mathbb{T} \to X$ satisfy the following conditions.

(A1) The Perron Δ-integrals

$$\int_{t_1}^{t_2} f(s)\Delta s \quad \text{and} \quad \int_{t_1}^{t_2} a(s)y(s)\Delta s$$

exist for all $t_1, t_2 \in \mathbb{T}_0$, whenever $y : \mathbb{T}_0 \to X$ is regulated.

(A2) There is a locally Perron Δ-integrable function $L : \mathbb{T}_0 \to \mathbb{R}$ such that

$$\left\| \int_{t_1}^{t_2} a(s)[z(s) - y(s)]\Delta s \right\| \leq \|z - y\|_{\mathbb{T}_0} \int_{t_1}^{t_2} L(s)\Delta s,$$

for all $z, y \in G_0(\mathbb{T}_0, X)$ and all $t_1, t_2 \in \mathbb{T}_0$.

(A3) There is a locally Perron Δ-integrable function $K : \mathbb{T}_0 \to \mathbb{R}$ such that

$$\left\| \int_{t_1}^{t_2} f(s)\Delta s \right\| \leq \int_{t_1}^{t_2} K(s)\Delta s,$$

for all $t_1, t_2 \in \mathbb{T}_0$.

Theorem 2.1. *Assume that $a : \mathbb{T} \to L(X)$ and $f : \mathbb{T} \to X$ satisfy conditions* (A1), (A2) *and* (A3). *Then, the dynamic equations (1) and (2) admit unique solutions.*

Lemma 2.1. *Let $\mathbb{T}$ be a time scale and $\mathbb{T}_0 = [t_0, +\infty) \cap \mathbb{T}$, with $t_0 \in \mathbb{T}$. If $\overline{f} : \mathbb{T} \to \mathbb{R}^n$ and $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ are rd-continuous, then conditions* (A1), (A2), (A3) *are satisfied.*

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TIME-SCALE ANALYSIS FOR A HOST-VECTOR TRANSMISSION MODEL INCLUDING SPATIAL DYNAMICS

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Abstract

Vector-borne diseases are a big cause of concern due to its high potential to invade new areas and habitats, which occurs mainly as a result of climate changes and human mobility. Mathematical models applied to describe infectious diseases like dengue, must couple the dynamics of hosts and vectors, whose parameters have different time scales, mosquitoes have a life cycle of days while the human life cycle is years. In this work, we consider a model described by a system of Ordinary Differential Equations (ODE). We show that taking into account the difference of time-scale between hosts and vectors, it is possible to reduce the order of the model so the mosquitoes equations do not appear explicitly in the system. Next, we set up the network dynamic, introducing a diffusion operator. We show a formal expansion that reflects the general ODE singular perturbation results. Finally, we estimate the parameters considering a simplified mobility network that represents the initial spread of dengue in Rio de Janeiro state.

1 Introduction

We consider a $SIRS_mI_m$ model, following the frequency-dependent structure of the well-known Ross-Macdonald models. The total host population N_h is divided into susceptible S, infected I and recovered R and it is coupled with the compartments of susceptible S_m and infected I_m mosquitoes with total population given by N_m. The interaction dynamics between the compartments is described through the ODE system:

$$\begin{aligned} dS/dt &= \mu_h(N_h - S) - \beta S I_m/N_m \\ dI/dt &= \beta S I_m/N_m - (\gamma + \mu_h)I \\ dR/dt &= \gamma I - \mu_h R \\ dS_m/dt &= \mu_m(N_m - S_m) - \Omega S_m I/N_h \\ dI_m/dt &= \Omega S_m I/N_h - \mu_m I_m \end{aligned} \tag{1}$$

First, this system is reduced by considering the populations remain constant. Then, to describe the time scale separation, we add the singular term $1/\varepsilon$ [1,2,3]. Defining $\mu_m = \overline{\mu_m}/\varepsilon$ and $\Omega := \overline{\Omega}/\varepsilon$, with $\overline{\mu_m}$ in the time scale of μ_h, and setting up $\overline{\mu_m} := \mu_h$, we obtain $\varepsilon = \mu_h/\mu_m$. The resulting system is given by:

$$\begin{aligned} dS/dt &= \mu_h(N_h - S) - \beta S I_m/N_m \\ dI/dt &= \beta S I_m/N_m - (\gamma + \mu_h)I \\ \varepsilon dI_m/dt &= \overline{\Omega}(N_m - I_m)I/N_h - \overline{\mu_m} I_m \end{aligned} \tag{2}$$

So, as in [1,3], we establish a system where the vector population dynamics is much faster than hosts one as $\varepsilon \approx 0$. At $\varepsilon = 0$, (convergence showed in [3]), $I_m(t)$ can be obtained as a function of $I(t)$ at any time t. This give us a new equivalent system with a nonlinear incidence rate without the mosquitoes equation. The limit system, named SI, characterizes the dynamics of a disease within a population. If the purpose is to describe its transmission dynamics more realistically, it is necessary to consider a mobility network that includes interaction between populations. The parameter d_{rs} corresponds to the mobility rate from the population r to s per unit time. The system is then given by:

$$\begin{aligned}\frac{dS_r}{dt} &= \mu_h(N_{hr} - S_r) - \frac{\beta_r\overline{\Omega_r}I_rS_r}{\overline{\Omega_r}I_r + \overline{\mu_{m_r}}N_{hr}} + \sum_{r\neq s}(d_{sr}S_s - d_{rs}S_r)\\ \frac{dI_r}{dt} &= \frac{\beta_r\overline{\Omega_r}I_rS_r}{\overline{\Omega_r}I_r + \overline{\mu_{m_r}}N_{hr}} - (\gamma+\mu_h)I_r + \sum_{r\neq s}(d_{sr}I_s - d_{rs}I_r)\end{aligned} \quad (3)$$

2 Main Results

We use power series expansion to analyse the asymptotic behavior with respect to parameter $\varepsilon > 0$ of the perturbed System (2) including spatial dynamics. We conclude that the solutions **S** and **I** of the system can be approximated by the solutions of the limit System (3). Indeed, it follows from our previous work [3] (also [1,2]) that the convergence is uniform in finite time with order $O(\varepsilon)$.

Regarding the numerical simulations, we made some considerations about the initial value of the parameters in order to apply an algorithm to fit the model to dengue data of some pairs of cities in Rio de Janeiro state, which were chosen due to previous evidence of human mobility acting as a virus spread factor. Our results (Fig 1) showed that the reduced model is capable of properly reproducing the number of infected individuals for all pairs of cities. The fit also obtained reasonable values for the parameters, including the ones simulating mobility. The model could capture the real movement between the locations. The results give us an indication that human mobility actually has influence on the spread of dengue and bring us perspectives for future studies combining more complex mobility networks and asymptotic techniques.

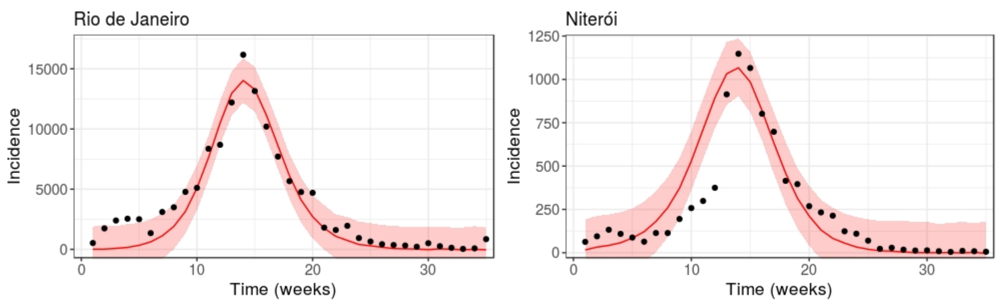

Figure 1: Result of fitting the infected equations to dengue incidence data from Rio de Janeiro and Niterói.

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A MODIFIED SEAIR MODEL WITH VACCINATION USING FUNCTIONAL DIFFERENTIAL EQUATIONS WITH CONSTANT DELAYS FOR COVID-19 SPREAD

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Abstract

We considered a modified version of a time-delay SEAIR (susceptible, exposed, asymptomatic, symptomatic, and recuperated) model for COVID-19 spread adding the vaccination process. The modified version introduced two main modifications in relation to the original model: we considered the vaccination and the possibility of reinfection of the recuperated population.

1 Introduction

Over the past two years, several researchers have introduced mathematical models for the spread of the COVID-19 pandemic that considers the existence of a time delay between the exposition and the first symptoms. Basically, the related works proposed modified versions of classical models for other diseases adding the discoveries of this new infection as reported in [1] using delayed functional differential equation systems in your mathematical formulations. Considering this scenario in this report we presented a modified version of the SEAIR model described in [1] considering the vaccination process and the possibility of reinfection as related in the scientific literature.

2 Description of the Modified Model

Let be S, E, A, I, R, and V the number of susceptible, exposed, asymptomatic, symptomatic, recuperated, and vaccinated individuals of COVID-19 in a specific population supposed constant of size N in the time t.
We introduced in the SEAIR model described in [1] a vaccinated compartment V, and beyond the original assumptions, we supposed that (a) the susceptible population is vaccinated with a constant rate v and all individuals in the recuperated compartment R can be reinfected after τ_1 days and, for simplicity, the effect of the vaccination is considered as instantaneous. In addition, we supposed that the vaccine presents an efficacy rate constant and equal to $\epsilon > 0$, and we did not distinguish between the different vaccines available.
The mathematical formulation of the model with these modifications is given by the System 1 where p, μ, β, α, γ_{as}, γ_s, γ_a, ϵ, v, and ϵ are the fraction of population that is eligible to vaccination, the constant rate of death, the force of the infection, the fraction of exposed that manifests the disease, the rate of conversion of asymptomatic to symptomatic, the rate of recuperation of the asymptomatic, the rate of recuperation of the symptomatic, and the efficacy rate of vaccines are positive parameters, respectively. As the population is supposed constant we have $\Lambda = (\mu - p)N - vS$.

A graphical illustration of the model can be shown in the Figure (1) where the dashed line indicates the modifications. The constant delays τ and τ_1 are the time (in days) of the disease incubation and, the necessary interval for that recuperated individuals can be reinfected.

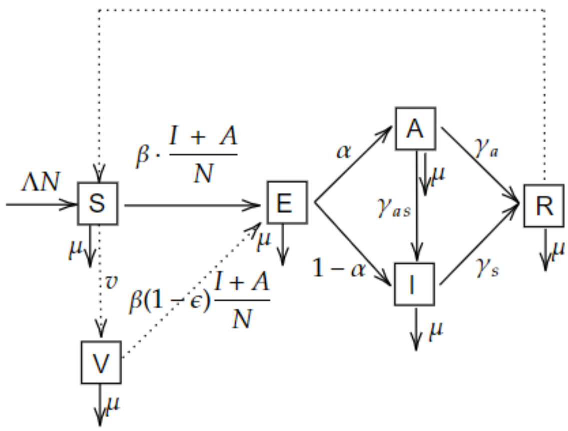

Figure 1: Model SEAIR with modifications indicated to dashed lines.

$$\begin{cases} S' = \Lambda - R(t-\tau_1) - \beta \cdot \left(\dfrac{A+I}{N}\right) \cdot S - \mu \cdot S \\ V' = p \cdot N + v \cdot S - \beta \cdot (1-\epsilon) \cdot \left(\dfrac{A+I}{N}\right) \cdot V - \mu \cdot V \\ E' = \beta \cdot \left(\dfrac{A+I}{N}\right) \cdot S + \beta \cdot (1-\epsilon) \cdot \left(\dfrac{A+I}{N}\right) \cdot V - \beta \cdot e^{-\mu\tau} \cdot \left(\dfrac{A(t-\tau)+I(t-\tau)}{N}\right) \cdot S(t-\tau) - \mu \cdot E \\ A' = \alpha \cdot \beta \cdot e^{-\mu\tau} \cdot \left(\dfrac{A(t-\tau)+I(t-\tau)}{N}\right) \cdot S(t-\tau) - (\gamma_{as} + \gamma_a + \mu) \cdot A \\ I' = \beta \cdot (1-\alpha) \cdot e^{-\mu\tau} \cdot \left(\dfrac{A(t-\tau)+I(t-\tau)}{N}\right) \cdot S(t-\tau) - (\gamma_s + \mu) \cdot I + \gamma_{as} \cdot A \\ R' = \gamma_A \cdot A + \gamma_S \cdot I - \mu \cdot R - R(t-\tau_1) \end{cases}, \quad (1)$$

3 Final Remarks

A qualitative analysis of the original model is presented in [1]. For the adapted model considering the new assumptions described this analysis is in progress. Particularly we are investigating the theoretical implications of the modifications that were introduced and are interested in numerical simulations considering the official data of Brazil and São Paulo State.

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BIFURCATION POINTS FOR FUNCTIONAL VOLTERRA STIELJES INTEGRAL EQUATIONS

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Abstract

In this work we study the existence of a bifurcation point with respect to the trivial solution of the Volterra-Stieltjes functional integral equations. The tool used to obtain our result is the Leray-Schauder degree theory.

1 Introduction

This presentation is based on the work [1]. In this research we are interested in the following type of integral equations

$$\begin{cases} x(t) = \phi(0) + \int_{\tau_0}^{t} a(t,s) f(x_s, s) dg(s), t \geq \tau_0, \\ x_{\tau_0} = \phi, \end{cases} \tag{1}$$

where the integral in the right-hand side is understood in the sense of Henstock-Kurzweil-Stieltjes, $+\infty \geq d > \tau_0 \geq t_0 \geq 0$, $r > 0$, $\phi \in G([-r,0],\mathbb{R}^n)$, $f : G([-r,0],\mathbb{R}^n) \times [t_0,d) \to \mathbb{R}^n$, $x_s : [-r,0] \to \mathbb{R}^n$ is given by $x_s(\theta) = x(s+\theta)$ for $s \in [t_0,d)$, and we assume the following conditions on the functions f, a and g:

(A1) The function $g : [t_0,d) \to \mathbb{R}$ is nondecreasing and left-continuous on (t_0,d).

(A2) The function $a : [t_0,d)^2 \to \mathbb{R}$ is nondecreasing and continuous with respect to the first variable and regulated with respect to the second variable.

(A3) The Henstock-Kurzweil-Stieltjes integral

$$\int_{\tau_1}^{\tau_2} a(t,s) f(x_s,s) dg(s)$$

exists for each compact interval $[\tau_0, \tau_0+\sigma] \subset [t_0,d)$, all $x \in G([\tau_0 - r, \tau_0+\sigma],\mathbb{R}^n)$, $t \in [t_0,d)$ and all $\tau_0 \leq \tau_1 \leq \tau_2 \leq \tau_0+\sigma$.

(A4) There exists a locally Henstock-Kurzweil-Stieltjes integrable function $M : [t_0,d) \to \mathbb{R}^+$ with respect to g such that for each compact interval $[\tau_0,\tau_0+\sigma] \subset [t_0,d)$, we have

$$\left\| \int_{\tau_1}^{\tau_2} (c_1 a(\tau_2,s) + c_2 a(\tau_1,s)) f(x_s,s) dg(s) \right\| \leq \int_{\tau_1}^{\tau_2} |c_1 a(\tau_2,s) + c_2 a(\tau_1,s)| M(s) dg(s),$$

for all $x \in G([\tau_0 - r,\tau_0+\sigma],\mathbb{R}^n)$, all $c_1, c_2 \in \mathbb{R}$ and all $\tau_0 \leq \tau_1 \leq \tau_2 \leq \tau_0+\sigma$. In particular, we have that

$$\left| \int_{\tau_1}^{\tau_2} a(\tau,s) f(x(s),s) dg(s) \right| \leq \int_{\tau_1}^{\tau_2} |a(\tau,s)| M(s) dg(s),$$

and

$$\left| \int_{\tau_1}^{\tau_2} (a(\tau_2,s) - a(\tau_1,s)) f(x(s),s) dg(s) \right| \leq \int_{\tau_1}^{\tau_2} |a(\tau_2,s) - a(\tau_1,s)| M(s) dg(s),$$

for all $x \in G([\tau_0,\tau_0+\sigma],\mathbb{R}\diagdown)$, and all $\tau,\tau_1,\tau_2 \in [\tau_0,\tau_0+\sigma]$.

(A5) There exists a locally regulated function $L : [t_0, d) \to \mathbb{R}^+$ such that for each compact interval $[\tau_0, \tau_0+\sigma] \subset [t_0, d)$ we have

$$\left\| \int_{\tau_1}^{\tau_2} a(\tau_2, s)[f(x_s, s) - f(z_s, s)]dg(s) \right\| \leq \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| L(s) \|x_s - z_s\|_\infty dg(s),$$

for all $x, z \in G([\tau_0, \tau_0 + \sigma], \mathbb{R})$ and all $[\tau_1, \tau_2] \subset [\tau_0, \tau_0 + \sigma]$.

A branch point with respect to the trivial solution of problem (3) is given by the following definition.

Definition 1.1. *A pair $(\lambda_0, 0) \in \Lambda \times \bar{\Omega}$ we say a bifurcation point of the equation $\Psi(\lambda, x)(t) = 0$, if every neighborhood of $(\lambda_0, 0) \in \Lambda_0 \times \bar{\Omega}$ contains a solution (λ, x) of the equation $\Psi(\lambda, x)(t) = 0$ such that $x \neq 0$., where the Ψ operator is defined by $\Psi(\lambda, x)(t) = x(t) - x(\tau_0 + T) - \lambda \int_{\tau_0}^{t} a(t, s) f(x_s, s) dg(s)$.*

2 Main Results

Our main results are the following.

Proposition 2.1. *Suppose for each $\lambda \in \Lambda_0$. Assume $[\lambda_1, \lambda_2] \subset \Lambda_0$ contains no bifurcation point for equation $\Psi(\lambda, x) = 0$. Then, there exists $\delta > 0$ such that for each $\lambda \in [\lambda_1, \lambda_2]$ and each $x \in B(0, \delta) \cap \bar{\Omega}$ if we have*

$$x = \mathcal{N}(\lambda, x)$$

then $x = 0$, with the operator $\mathcal{N}$ is defined by $\mathcal{N}(\lambda, x)(t) = x(\tau_0 + T) + \lambda \int_{\tau_0}^{t} a(t, s) f(x_s, s) dg(s),\ t \in [\tau_0, \tau_0 + T]$.

Theorem 2.1. *Let $\Psi : \Lambda_0 \times \bar{\Omega} \to G$ and $\mathcal{N} : \Lambda_0 \times \bar{\Omega} \to G$. If we have $[\lambda_1, \lambda_2] \subset \Lambda_0$ and*

$$ind_{LS}[I - \mathcal{N}(\lambda_1, \cdot), 0] \neq ind_{LS}[I - \mathcal{N}(\lambda_2, \cdot), 0], \tag{2}$$

then there exists $\lambda_0 \in [\lambda_1, \lambda_2]$ such that $(\lambda_0, 0)$ is a bifurcation point of equation $\Psi(\lambda, x) = 0$.

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EXISTENCE OF POSITIVE SOLUTIONS FOR A CRITICAL NONLOCAL ELLIPTIC SYSTEM

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Abstract

In this work, the existence of positive solution to the critical nonlocal elliptic system

$$(S) \qquad \begin{cases} (-\Delta)_p^s u + a(x)|u|^{p-2}u + c(x)|v|^{p-2}v = \dfrac{1}{p_s^*}K_u(u,v) \ \text{ in } \ \mathbb{R}^N, \\ (-\Delta)_p^s v + c(x)|u|^{p-2}u + b(x)|v|^{p-2}v = \dfrac{1}{p_s^*}K_v(u,v) \ \text{ in } \ \mathbb{R}^N, \\ u, v > 0 \ \text{ in } \ \mathbb{R}^N, \\ u, v \in D^{s,p}(\mathbb{R}^N), \quad N > ps, \ s \in (0,1), \end{cases}$$

is established. Here $(-\Delta)_p^s$ denotes the fractional $p-$Laplacian, a, b and c are suitable functions and K is a p_s^*-homogeneous function, $p_s^* = \frac{pN}{N-ps}$, $N > ps$. One of the main tool is to apply the global compactness result for the associated energy functional similar to that due to Struwe in [3] combined with some information on a limit system of (S) with $a = b = c = 0$, the concentration compactness due to P. L. Lions [3] and the Brouwer degree theory.

1 Introduction

Let $\mathbb{R}_+^2 := [0,\infty) \times [0,\infty)$. We state our main hypotheses on the function $K \in C^2(\mathbb{R}_+^2, \mathbb{R})$ as follows.

(K_0) K is p_s^*-homogeneous, that is,

$$K(\lambda\alpha, \lambda\beta) = \lambda^{p_s^*} K(\alpha,\beta) \text{ for each } \lambda > 0, \ (\alpha,\beta) \in \mathbb{R}_+^2;$$

(K_1) there exists $c_1 > 0$ such that

$$|K_\alpha(\alpha,\beta)| + |K_\beta(\alpha,\beta)| \le c_1 \left(\alpha^{p_s^*-1} + \beta^{p_s^*-1}\right) \text{ for each } (\alpha,\beta) \in \mathbb{R}_+^2;$$

(K_2) $K(\alpha,\beta) > 0$ for each $\alpha, \beta > 0$;

(K_3) $\nabla K(0,1) = \nabla K(1,0) = (0,0)$;

(K_4) $K_\alpha(\alpha,\beta), K_\beta(\alpha,\beta) \ge 0$ for each $(\alpha,\beta) \in \mathbb{R}_+^2$;

(K_5) the $1-$homogeneous function $G : \mathbb{R}_+^2 \to \mathbb{R}$ given by $G(\alpha^{p_s^*}, \beta^{p_s^*}) := K(\alpha,\beta)$ is concave.

The hypotheses on the functions $a, b, c : \mathbb{R}^N \mapsto \mathbb{R}^+$ are given by:

(A_1) The functions a, b, c are positive in a same set of positive measure.

(A_2) $a, b, c \in L^q(\mathbb{R}^N)$ for all $q \in [p_1, p_2]$ with $1 < p_1 < \dfrac{N}{ps} < p_2$ and $p_2 < \dfrac{N(p-1)}{p^2 s - N}$ if $N < p^2 s$.

(A_3) $\alpha_o^p |a|_{L^{N/ps}(\mathbb{R}^N)} + \beta_o^p |b|_{L^{N/ps}(\mathbb{R}^N)} + \alpha_o^{p-1}\beta_o |c|_{L^{N/ps}(\mathbb{R}^N)} + \alpha_o \beta_o^{p-1} |c|_{L^{N/ps}(\mathbb{R}^N)} < \widetilde{S}_{K,s}(p^{ps/N} - 1)$.

2 Main Results

Theorem 2.1. *Assume that* $(A_1)-(A_3)$ *and* $(K_0)-(K_5)$ *hold. Then,* (S) *has a positive solution* $(u_0, v_0) \in D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ *with*

$$\frac{s}{N}\widetilde{S}_{K,s}^{N/ps} < I(u_0, v_0) < \frac{ps}{N}\widetilde{S}_{K,s}^{N/ps}.$$

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SUPERLINEAR FRACTIONAL ELLIPTIC PROBLEMS VIA THE NONLINEAR RAYLEIGH QUOTIENT

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Abstract

It is establish existence of weak solutions for nonlocal elliptic problems driven by the fractional Laplacian where the nonlinearity is indefinite in sign. More specifically, we shall consider the following nonlocal elliptic problem

$$\begin{cases} (-\Delta)^s u + V(x)u &= \mu a(x)|u|^{q-2}u - \lambda |u|^{p-2}u \text{ in } \mathbb{R}^N, \\ \quad u \in H^s(\mathbb{R}^N), \end{cases}$$

where $s \in (0,1), s < N/2, N \geq 1$ and $\mu, \lambda > 0$. The potentials $V, a : \mathbb{R}^N \to \mathbb{R}$ satisfy some extra assumptions. The main feature is to find sharp parameters $\lambda > 0$ and $\mu > 0$ where the Nehari method can be applied. In order to do that we employ the nonlinear Rayleigh quotient together a fine analysis on the fibering maps.

1 Introduction

In the present work we shall consider nonlocal elliptic problems driven by the fractional Laplacian defined in the whole space where the nonlinearity is superlinear at infinity and at the origin. Namely, we shall consider the following nonlocal elliptic problem

$$\begin{cases} (-\Delta)^s u + V(x)u &= \mu a(x)|u|^{q-2}u - \lambda |u|^{p-2}u \text{ in } \mathbb{R}^N, \\ \quad u \in H^s(\mathbb{R}^N), \end{cases} \tag{1}$$

where $s \in (0,1), s < N/2, N \geq 1$. Furthermore, we assume that $2 < q < p < 2^*_s = 2N/(N-2s)$ and $\mu, \lambda > 0$. Assume also that $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function and $a : \mathbb{R}^N \to \mathbb{R}$ is nonnegative measurable function. It is important to recall that the main difficult in order to consider weak solutions for Problem (3) comes from the fact that the nonlinear term $g_{\lambda,\mu}(x,t) = \mu a(x)|t|^{q-2}t - \lambda|t|^{p-2}t, x \in \mathbb{R}^N, t \in \mathbb{R}$ is indefinite in sign.

2 Main Results

In the present work we shall consider existence and nonexistence of nontrivial weak solutions for the Problem (3) looking for the parameters $\lambda > 0$ and $\mu > 0$. Throughout this work we assume the following assumptions:

- (Q) It holds $\mu, \lambda > 0$ and $2 < q < p < 2^*_s = 2N/(N-2s)$;
- (V_0) The potential $V : \mathbb{R}^N \to \mathbb{R}$ is continuous function and there exists a constant $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^N$;
- (V_1) For each $M > 0$ it holds that $|\{x \in \mathbb{R}^N : V(x) \leq M\}| < +\infty$.
- (a_0) It holds that $a \in L^\infty(\mathbb{R}^N)$ where $a(x) > 0$ a. e. in $x \in \mathbb{R}^N$.

It is important to mention that the working space X is a Hilbert space. It is worthwhile to emphasize that that the energy functional $E_{\lambda,\mu} : X \to \mathbb{R}$ associated to Problem (3) is given by

$$E_{\lambda,\mu}(u) = \frac{1}{2}||u||^2 - \frac{\mu}{q}\|u\|^q_{q,a} + \frac{\lambda}{p}\|u\|^p_p,\ u \in X, \tag{1}$$

where $\|u\|^q_{q,a} = \int_{\mathbb{R}^N} a(x)|u|^q dx$ and $\|u\|^p_p = \int_{\mathbb{R}^N} |u|^p dx, u \in X$. Recall that a function $u \in X$ is a critical point for the functional $E_{\lambda,\mu}$ if and only if u is a weak solution to the elliptic Problem (3). Now, by using the same ideas introduced in [1, 2, 3], we shall consider the Nehari method for our main Problem (3) as follows

$$\mathcal{N}_{\lambda,\mu} \quad := \quad \left\{ u \in X \setminus \{0\} : ||u||^2 + \lambda \int_{\mathbb{R}^N} |u|^p dx = \mu \int_{\mathbb{R}^N} a(x)|u|^q dx \right\}. \tag{2}$$

At this stage, using some ideas introduced in [4], we also consider

$$\mu_n := \inf_{u \in X\setminus\{0\}} \inf_{t>0} R_n(tu) \quad \text{and} \quad \mu_e := \inf_{u \in X\setminus\{0\}} \inf_{t>0} R_e(tu). \tag{3}$$

As a product, we shall state our first main result as follows:

Theorem 2.1. *Suppose (Q), $(V_0) - (V_1)$ and (a_0). Then for each $\lambda > 0$ we obtain that $0 < \mu_n < \mu_e < \infty$. Furthermore, there exists $\lambda_* > 0$ such that for each $\mu > \mu_n$ Problem (3) admits at least a weak solution $u_{\lambda,\mu} \in X$ whenever $\lambda \in (0, \lambda_*)$ which it satisfies the following statements:*

i) $E''_\lambda(u_{\lambda,\mu})(u_{\lambda,\mu}, u_{\lambda,\mu}) < 0$, that is, $u_{\lambda,\mu} \in \mathcal{N}^-_{\lambda,\mu}$; ii) There exists $D_\mu > 0$ such that $E_{\lambda,\mu}(u_{\lambda,\mu}) \geq D_\mu$ and $u_{\lambda,\mu} \to 0$ in X as $\mu \to \infty$.

Now, we shall assume the following hypothesis:

(a_1) It holds that $a \in L^\infty(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ with $r = (p/q)' = p/(p-q)$ and $a(x) > 0$ a.e. in $x \in \mathbb{R}^N$.

Theorem 2.2. *Suppose (Q), $(V_0) - (V_1)$ and (a_1). Then for each $\lambda > 0$ we obtain that $0 < \mu_n < \mu_e < \infty$. Furthermore, there exits $\lambda^* > 0$ such that for each $\mu > \mu_n$ Problem (3) admits at least a ground state solution $v_{\lambda,\mu} \in X$ taking into account one of the following conditions:*

a) $\mu \in [\mu_e, \infty), \lambda > 0$; b) $\mu \in (\mu_n, \mu_e)$ and $\lambda \in (0, \lambda^*)$.

Moreover, the weak solution $v_{\lambda,\mu}$ satisfies the following assertions:

i) It holds that $E''_\lambda(v_{\lambda,\mu})(v_{\lambda,\mu}, v_{\lambda,\mu}) > 0$, that is, $v_{\lambda,\mu} \in \mathcal{N}^+_{\lambda,\mu}$. Furthermore, $\|v_{\lambda,\mu}\| \to \infty$ in X as $\mu \to \infty$.

ii) For each $\mu \in (\mu_n, \mu_e)$ we obtain that $E_{\lambda,\mu}(v_{\lambda,\mu}) > 0$; (iii) For $\mu = \mu_e$ it follows that $E_\lambda(v_{\lambda,\mu}) = 0$; (iv) For each $\mu > \mu_e$ we obtain also that $E_\lambda(v_{\lambda,\mu}) < 0$.

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THE EXTREMAL PROBLEM FOR SOBOLEV INEQUALITIES WITH UPPER ORDER REMAINDER TERMS

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Abstract

Given a smooth compact Riemannian n-manifold (M, g), we prove existence of extremal functions for sharp Sobolev inequalities which are closely related to the embedding of $H^{1,q}(M)$ into $L^{qn/(n-q)}(M)$ where the L^q remainder term is replaced by upper order terms.

1 Introduction

$$||u||^q_{L^{q^*}(M)} \leq K(n,q)^q ||\nabla_g u||^q_{L^q(M)} + B_0(p,n,g)||u||^q_{L^p(M)} . \tag{1}$$

The constant $B_0(p,n,g)$ depends only on p and (M,g) and

$$K(n,q) := \sup \left\{ \frac{||u||_{L^{q^*}(\mathbb{R}^n)}}{||\nabla u||_{L^q(\mathbb{R}^n)}} : \ u \in L^{q^*}(\mathbb{R}^n) \setminus \{0\}, \ |\nabla u| \in L^q(\mathbb{R}^n) \right\} .$$

for $n \geq 3$ and $q^* = qn/(n-q)$, $K(n,q)$ is the classical Euclidean Sobolev best constant,

Special attention has also been paid to the existence problem of extremal functions to (1). A non-zero function $u_0 \in C^\infty(M)$ is said to be an extremal to (1), if

$$||u_0||^q_{L^{q^*}(M)} = K(n,q)^q ||\nabla_g u_0||^q_{L^q(M)} + B_0(p,n,g)||u_0||^q_{L^q(M)} .$$

Our goal is to discuss the existence and compactness of extremal functions to two classes of Sobolev type inequalities modeled on smooth compact Riemannian manifolds, precisely sharp Riemannian Sobolev-Poincaré inequalities involving upper order remainder terms.

We here are interested in discussing the extremal problem related to the inequalities (1) regarding upper order remainder terms, i.e. $p \geq 2$. Let now $B_0(p,n,g)$ be the best possible constant in (1), i.e.

2 Main Results

Denote by $E_p(g)$ the set of the extremal functions to (1) with unit L^{q^*}-norm.

Our main results in this paper are summarized in the next theorem.

Theorem 2.1. *Let (M,g) be a smooth compact Riemannian n-manifold without boundary of dimension $n \geq 4$ such that the inequality (1) is true. Then the set $E_p(g)$ is non-empty for any $1 < q < p < q^*$.*

Note that the conclusion of Theorem 2.1 does not depend on the geometry of the manifold.

The tools are based on blow-up techniques, concentration analysis and PDE estimates. What happens is that each proof has its specific technical difficulties inherent to the problem addressed. For instance, here part them are caused by by the range of values of p in our inequalities.

The ideas of the proofs are mainly inspired in the works of Aubin [1]. The key points are the so-called L^p concentration estimates. The outline of the existence part is as follows. We begin by constructing minimizers for certain functionals, related to our geometric Sobolev inequalities, which converges weakly to a nonnegative function $u \in C^\infty(M)$. Our aim now is to show that u is non-zero, since this conclusion easily implies $||u||_{L^{q^*}} = 1$. Assuming then that $u = 0$ on M, we perform a comprehensive study of blow-up, concentration and a priori estimates on the generated family of minimizers in order to obtain a contradiction. With a few adaptations of the proof of existence, we easily achieve compactness.

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NO-FLUX BOUNDARY PROBLEM INVOLVING P(X)-LAPLACIAN-LIKE OPERATORS WITH CRITICAL EXPONENTS

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Abstract

The purpose of this article is to obtain weak solutions for a class nonlinear elliptic problem for the $p(x)$-Laplacian-like operators under no-flux boundary conditions, where the nonlinearity has a critical growth. To overcome the lack of compactness we use a fixed point result due to Carl and Heikkilä and the theory of the variable exponent Sobolev spaces.

1 Introduction

The purpose of this work is to investigate the existence of weak solutions for the following nonlinear elliptic problem for the $p(x)$-Laplacian-like operators originated from a capillary phenomena

$$
\begin{aligned}
-M\Big(L(u)\Big)\Big[\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u+\frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}})-|u|^{p(x)-2}u\Big] &= a|u|^{t(x)-2}+|u|^{r(x)-2}+h \\
&\text{in } \Omega,\\
u = constant \qquad\qquad & \text{on } \partial\Omega, \qquad (1)\\
\int_{\partial\Omega}\Big(|\nabla u|^{p(x)-2}+\frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\frac{\partial u}{\partial \nu}d\Gamma = 0. &
\end{aligned}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, and $N\geq 1$, $p,r,t\in C_+(\overline{\Omega})=\{f : f\in C(\overline{\Omega}), f(x)>1 \text{ for any } x\in\overline{\Omega}\}$; a is a positive parameter; $M:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function, $L(u)=\int_\Omega \frac{|\nabla u|^{p(x)}+\sqrt{1+|\nabla u|^{2p(x)}}+|u|^{p(x)}}{p(x)}dx$ is a $p(x)$-Laplacian type operator and

$$1<p^-:=\min_{\overline{\Omega}} p(x)\leq p^+:=\max_{\overline{\Omega}} p(x)<N \quad \text{for every } p\in C_+(\overline{\Omega}).$$

In recent years, Kirchhoff type equations involving the $p(x)-$Laplace operator with critical growth have attracted an increasing attention.(see for example [3,4,5]). Due to the lack of compactness of the embedding $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{p^*(x)}(\Omega)$, many authors have used the concentration compactness principle for variable exponents proved by Bonder and Silva [1]. In this research, to deal with this difficulty we will use a fixed theorem in [2].

2 Assumptions and Main Result

Throughout this paper, let

$$V=\{u\in W^{1,p(x)}(\Omega) : u|_{\partial\Omega}=\text{constant}\},$$

where $W^{1,p(x)}(\Omega)$ ($p \in C(\overline{\Omega})$, $2 \le p(x) < +\infty$) is the well known variable exponent Sobolev space.

The space V is a closed subspace of the separable and reflexive Banach space $W^{1,p(x)}(\Omega)$, so V is also separable and reflexive Banach space with the usual norm of $W^{1,p(x)}(\Omega)$.

We will assume

(A_1) $M : [0, +\infty[\to [m_0, +\infty[$ is a continuous and increasing function; $m_0 > 0$.

(A_2) $h \in W^{-1,p'(x)}(\Omega)$, $p^*(x) = \frac{Np(x)}{N-p(x)}, \forall x \in \overline{\Omega}$ and

$$A = \left\{ x \in \overline{\Omega} : t(x) = p^*(x) \right\} \quad \text{is nonempty.}$$

The main result in this work can be stated as follows.

Theorem 2.1. *Suppose (A_1) - (A_2) hold. Then problem (1.1) has a weak solution $u \in V$, provided that constant a is sufficiently small.*

Proof We use a fixed point theorem for increasing self-mappings in Banach semilattice (See [2]).

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EXISTENCE OF CRITICAL POINTS AT PRESCRIBED ENERGY LEVELS FOR A CLASS OF FUNCTIONALS

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Abstract

In this talk we present a general approach based on the fibering method and Nonlinear Rayleigh Quotient to prove existence of critical points for a suitable family of functionals depending on a parameter. Under suitable assumptions we show existence of infinitely many critical points at a fixed energy level and also bifurcation results are obtained.

1 Introduction

Many problems in nonlinear pdes can be formulated as a critical point equation

$$\Phi'(u) = 0, \tag{1}$$

where Φ' is the Fréchet derivative of a certain functional Φ, the so-called *energy functional*. The search of critical points is then addressed by variational methods. For physical reasons, the partial differential equation is often coupled to some additional constraint on the variable u (e.g. a *sign constraint* $u > 0$ or a *mass constraint* $\|u\| = m$) and a huge bibliography is available on this subject.

The aim here is to investigate (1) under a different constraint, namely, the *level (or energy) constraint* $\Phi(u) = c$, where $c \in \mathbb{R}$ is given a priori. Motivated by several nonlinear elliptic problems, we shall consider a class of functionals depending on a real parameter, namely

$$\Phi_\mu := I_1 - \mu I_2,$$

where $\mu \in \mathbb{R}$, $I_1, I_2 \in C^1(X)$, and X is an infinite dimensional Banach space. Then, given $c \in \mathbb{R}$ we consider the problem

$$\Phi'_\mu(u) = 0, \quad \Phi_\mu(u) = c, \tag{2}$$

i.e. we look for couples $(\mu, u) \in \mathbb{R} \times X \setminus \{0\}$ which solve the system. We shall follow the *nonlinear generalized Rayleigh quotient* method introduced by Y. Ilyasov [1] and show that this method is suitable to investigate the structure of the *solution set* of (2)

$$\mathcal{S} := \{(\mu, c) \in \mathbb{R}^2 : \Phi_\mu \text{ has a critical point at the level } c\}.$$

2 Main abstract results

Before stating our result let us show how the *nonlinear generalized Rayleigh quotient* method applies to the system (2). Assume that $I_2(u) \neq 0$ for every $u \in X \setminus \{0\}$ (this is often the case in many elliptic problems), so that one can solve the level constraint explicitely in μ:

$$\Phi_\mu(u) = c \quad \Longleftrightarrow \quad \mu = \mu(c, u) := \frac{I_1(u) - c}{I_2(u)}$$

and we see that for any c the functional $u \mapsto \mu(c,u)$ satisfies the following relation,

$$\frac{\partial \mu}{\partial u}(c,u) = \frac{\Phi'_{\mu(c,u)}(u)}{I_2(u)}, \quad \forall u \in X \setminus \{0\}.$$

Here $\frac{\partial \mu}{\partial u}(c,u)$ denotes the Fréchet derivative of the functional $u \mapsto \mu(c,u)$. Then we have the equivalence:

$$\Phi'_\mu(u) = 0, \quad \Phi_\mu(u) = c \quad \Longleftrightarrow \quad \mu = \mu(c,u), \quad \frac{\partial \mu}{\partial u}(c,u) = 0,$$

i.e. (2) can be completely solved by understanding the set of critical points (and critical values) of the functional $u \mapsto \mu(c,u)$. In conclusion (2) is solvable if and only if μ is a critical value (and u an associated critical point) of the latter functional. Thus, denoting by $\mathcal{K}(c)$ the set of these critical values, we immediately find a sufficient and necessary condition for the solvability of (2) obtaining an existence result.

Theorem 2.1. *For a given $c \in \mathbb{R}$ the problem (2) has a solution (μ, u) if, and only if, $\mu \in \mathcal{K}(c)$ and u is the associated critical point. In particular, if $u \mapsto \mu(c,u)$ has a ground state (or least energy) level $\mathfrak{gs}(c)$ then* (2) *has no solution for $\mu < \mathfrak{gs}(c)$.*

Under some technical conditions on the family of functionals Φ_μ we prove also that there exist infinitely many pairs $(\mu_{n,c}, \pm u_{n,c})$ solving (2). This multiplicity result, which involves some technical staff to state it rigorously, is proved via the Ljusternik-Schnirelman theory, and is established not only for a single value of c, but for c lying in an open interval $\mathcal{I} \subset \mathbb{R}$. Consequently it makes sense to study also the behaviour of $\mu_{n,c}$ and $u_{n,c}$ with respect to $c \in \mathcal{I}$. In many cases the values $\mu_{n,c}$ depend continuously on c, so that letting c vary we shall obtain a family of *energy curves* $\{(\mu_{n,c}, c); c \in \mathcal{I}\}_{n\in\mathbb{N}}$. The properties of these *energy curves* give information on the bifurcation analysis of the unconstrained problem $\Phi'_\mu(u) = 0$. In particular, this procedure shall allow us to deduce several bifurcation and multiplicity results for this problem. Applications are given to important partial differential equations.

The results are taken from a join work with Humberto R. Quoirin and Kaye Silva [2].

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REACTION-DIFFUSION PROBLEM IN A THIN DOMAIN WITH OSCILLATING BOUNDARY AND VARYING ORDER OF THICKNESS

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Abstract

We study a reaction-diffusion problem in a thin domain with varying order of thickness. Motivated by the applications, we assume the oscillating behavior of the boundary and prescribe the Robin-type boundary condition simulating the reaction catalyzed by the upper wall. Using the appropriate functional setting and the unfolding operator method, we rigorously derive lower-dimensional approximation of the governing problem. Five different limit problems have been obtained by comparing the magnitude of the reaction mechanism with the variation in domain's thickness.

1 Introduction

We consider a thin two-dimensional domain whose order of thickness is not fixed, but it can vary. Inspired by the microfluidic applications, we also allow the oscillating behaviour of the upper boundary. More precisely, let us consider the following family of open sets

$$R^\varepsilon = \operatorname{int}\left(\overline{R_1^\varepsilon} \cup \overline{R_2^\varepsilon} \cup \overline{R_3^\varepsilon}\right)$$

where

$$R_i^\varepsilon = \left\{(x,y) \in \mathbb{R}^2 : x \in (0,a),\ 0 < y < \delta_i(\varepsilon) g_i(x/\varepsilon)\right\}, \quad \delta_1(\varepsilon) = \delta_3(\varepsilon) = \varepsilon, \quad \delta_2(\varepsilon) = \varepsilon^\beta.$$

Notice that parameter $\beta > 1$ sets the greatest order of the compression, and the L_i-periodic functions $g_i : \mathbb{R} \mapsto \mathbb{R}$ with $i = 1, 2, 3$ the profile of the regions. We assume g_i are strictly positive and Lipschitz. Figure 1 below illustrates our thin channel with different orders of thickness and with the roughness modeled by the periodic functions g_i.

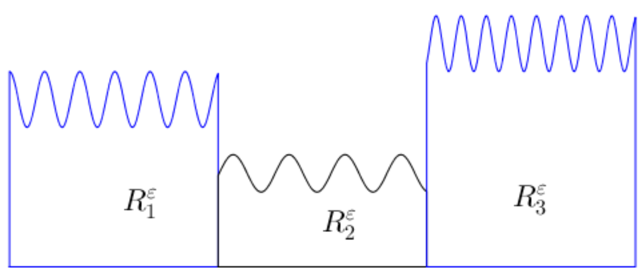

Figure 1: A thin domain with different orders of compression and oscillating boundary.

We are interested in analyzing the asymptotic behaviour of the solutions of the following elliptic boundary-value problem

$$\begin{cases} -\Delta u^\varepsilon + u^\varepsilon = f^\varepsilon \quad \text{in} \quad R^\varepsilon\,, \\ \dfrac{\partial u^\varepsilon}{\partial \eta^\varepsilon} = \varepsilon^\alpha (h - u^\varepsilon) \quad \text{on} \quad \Gamma^\varepsilon\,, \\ \dfrac{\partial u^\varepsilon}{\partial \eta^\varepsilon} = 0 \quad \text{on} \quad \partial R^\varepsilon \backslash \Gamma^\varepsilon\,, \end{cases}$$

where Γ^ε denotes the upper rough boundary of R^ε, the family of forcing terms f^ε are uniformly bounded that converges to f in an appropriate sense and $h : (0,1) \mapsto \mathbb{R}$ is assumed to be a one variable function in $H^1(0,1)$.

2 Main Results

Theorem 2.1. *There exists a unique limit function $u \in H^1(0,1)$ satisfying*

$$\alpha \geq \beta : \begin{cases} -q_1 u'' + r_1 u = \bar{f}_1 \quad in \quad (0,a) \\ -q_2 u'' + r_2 u = \bar{f}_2 + \bar{h}_2 \quad in \quad (a,b) \\ -q_3 u'' + r_3 u = \bar{f}_3 \quad in \quad (b,1) \\ u'(0) = u'(1) = 0 \\ q_1 u'_-(a) = q_2 u'_+(a) \\ q_2 u'_-(b) = q_3 u'_+(b) \end{cases} \qquad 1 \leq \alpha < \beta : \begin{cases} -q_1 u'' + r_1 u = \bar{f}_1 + \bar{h}_1 \quad in \quad (0,a) \\ -q_3 u'' + r_3 u = \bar{f}_3 + \bar{h}_3 \quad in \quad (b,1) \\ u'(0) = u'(1) = 0 \\ u(a) = h(a) \\ u(b) = h(b) \end{cases}$$

$$q_1 = \frac{1}{L_1}\int_{Y_1^*} (1 - \partial_1 X_1) dY, \quad r_1 = \langle g_1 \rangle_{(0,L_1)}, \quad \bar{f}_1 = \frac{1}{L_1}\int_{Y_1^*} \hat{f}_1 dY, \quad q_2 = \frac{1}{\langle 1/g \rangle_{(0,L_2)}}, \quad \bar{f}_2 = \frac{1}{L_2}\int_{Y_2^*} \hat{f}_2 dY,$$
$$q_3 = \frac{1}{L_3}\int_{Y_3^*} (1 - \partial_1 X_3) dY, \quad r_3 = \langle g_3 \rangle_{(0,L_3)}, \quad \bar{f}_3 = \frac{1}{L_3}\int_{Y_3^*} \hat{f}_3 dY.$$

$$\textit{If} \quad \alpha = \beta, r_2 = \langle g_2 \rangle_{(0,L_2)} + \frac{1}{L_2}, \quad \bar{h}_2 = \frac{1}{L_2} h, \quad \textit{and if} \quad \alpha > \beta, r_2 = \langle g_2 \rangle_{(0,L_2)}, \quad \bar{h}_2 = 0,$$

where $\langle g_i \rangle_{(0,L_i)}$ denotes the average of g_i in the interval $(0, L_i)$. Moreover, $X_j \in H^1_\#(Y_i^*)$, $j = 1,3$, with $\int_{Y_j^*} X dY = 0$, is the unique solution of the auxiliary problem

$$\int_{Y_j^*} \nabla X_j \nabla \psi dY = \int_{Y_j^*} \partial_1 \psi dY, \quad \forall \psi \in H^1_\#(Y_j^*),$$

$dY = dy_1 dy_2$ and

$$Y_i^* = \{(y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < L_i \text{ and } 0 < y_2 < g_i(y_1)\}.$$

Finally, if $\alpha < 1 < \beta$

$$\frac{1}{[\delta_i(\varepsilon)]^{1/2}} ||u^\varepsilon - h||_{L^2(R_i^\varepsilon)} \to 0, \qquad as\ \varepsilon \to 0.$$

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THE METHOD OF THE ENERGY FUNCTION AND APPLICATIONS

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Abstract

In this work, we establish a new method to find critical points of differentiable functionals defined in reflexive Banach spaces which belong to an appropriated class $(\mathcal{J})$ of functionals. As a consequence, we solve some variational elliptic problems, whose associated energy functional belongs to $(\mathcal{J})$ and provide a version of the mountain pass theorem for functionals in the class $(\mathcal{J})$.

1 Introduction

This work, which is based in [3], establishes an alternative method that allows us to complement some known results of the literature. In fact, this referred method, which we will call *Method of the energy function*, has proved to be effective to treat some classes of relevant elliptic partial differential problems, for which, some progress is made in the present article. The idea of trying to relate the energy functional to a real function was inspired in [1], where the authors study a degenerate Kirchhoff problem of the form

$$\begin{cases} -m(\|u\|)\Delta u = f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

with $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a smooth bounded domain and the functions $m : [0,\infty) \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous satisfyng suitable ocnditions.

In fact, in [1], the authors were inspired by an argument used in [2, Proposition 9] to prove the differentiability of a certain real function α. Another important detail is that, in [1], the function α was used to show that the weak solution found to the problem (1) is nontrivial and no relation was established between critical points of a real function involving α and critical points of the energy functional associated to the problem (1). In the present paper we use a similar strategy, however in our case we introduce a function $\zeta : [0,\infty) \to \mathbb{R}$ (see (2) that is differentiable and we prove a strong connexion between the critical points of J and ζ.

In order to give a more clear idea about the subject we are going to treat, let us consider, throughout this notes, the following situation:

Let E be a reflexive Banach space. We say that a functional J belongs to class $(\mathcal{J})$ in E, if $J : E \to \mathbb{R}$, $J = \Psi - \Phi$ and $\Psi, \Phi \in C^1(E,\mathbb{R})$ satisfy the following hypotheses:

(Ψ_1) Ψ is weakly lower semicontinuous and coercive;

(Ψ_2) $\tilde{\mathcal{A}}_r = \{u \in E;\ \Psi(u) \leq r\}$ for all $r \geq 0$, $\tilde{\mathcal{A}}_0 = \{0\}$ and $0 \in \operatorname{int}(\tilde{\mathcal{A}}_r)$ for each $r > 0$;

(Ψ_3) $\Psi'(u)u \neq 0$ for all $u \in \partial\tilde{\mathcal{A}}_r$, where $\partial\tilde{\mathcal{A}}_r = \{u \in E;\ \Psi(u) = r\}$;

(Ψ_4) For each $u \neq 0$ and $r > 0$, there exists a unique $t_u(r) > 0$ such that $t_u(r)u \in \partial\tilde{\mathcal{A}}_r$;

(Ψ_5) If $u_n \rightharpoonup u_0$ in E and $\Psi(u_n) \to \Psi(u_0)$, then $u_n \to u_0$ in E;

(Φ_1) Φ and $u \mapsto \Phi'(u)u/\Psi'(u)u$ are weakly upper semicontinuous;

(Φ_2) If $\Phi'(u) = 0$, then $\Phi(u) \leq 0$;

(Φ_3) There exists a sequence $\{u_n\} \subset E$ such that $u_n \to 0$ and $\Phi(u_n) > 0$, for all $n \in \mathbb{N}$.

We call *energy function* to the function $\zeta : [0, \infty) \to \mathbb{R}$ defined by

$$\zeta(r) := r - \varphi(r), \tag{2}$$

where $\varphi(r) := \max\limits_{u \in \partial\tilde{\mathcal{A}}_r} \Phi(u)$ if $r > 0$ and $\varphi(0) = \Phi(0)$.

We point out that it is not obvious or intuitive to think that ζ is differentiable, or even continuous regarding to r, mainly because the maximum point of Φ on $\partial\tilde{\mathcal{A}}_r$ cannot be unique. For this reason, next proposition is a nontrivial and very technical result.

We say that a function $u_r \in \mathcal{G}_r$ is an *energy maximum type point* if

$$\frac{\Phi'(u_r)u_r}{\Psi'(u_r)u_r} = \max_{u \in \mathcal{G}_r} \frac{\Phi'(u)u}{\Psi'(u)u}.$$

Observe that an energy maximum type point $u_r \in \partial\tilde{\mathcal{A}}_r$ maximizes at the same time Φ in $\tilde{\mathcal{A}}_r$ and $u \mapsto \frac{\Phi'(u)u}{\Psi'(u)u}$ in $\mathcal{G}_r$, and, for sure, it is nontrivial when $r > 0$.

2 Main Results

Theorem 2.1. *Let E be a reflexive Banach space and J be a functional of class $(\mathcal{J})$ in E. A maximum energy type point $u_r \in \mathcal{G}_r$ is a critical point of the functional J, for some $r > 0$ if, and only if, r is a critical point of the energy function ζ.*

Theorem 2.2. *Let E be a reflexive Banach space and J be a functional of class $(\mathcal{J})$ in E. Suppose that there exist $\alpha, \rho > 0$ and $w \in E$, with $w \in \partial\tilde{\mathcal{A}}_R$, such that*

(H_1) *$J(u) \geq \alpha > J(0)$ for all $u \in \partial\tilde{\mathcal{A}}_\rho$;*

(H_2) *$J(w) < \alpha$, with $R > \rho$.*

Then there holds the inequality below

$$c_* = \max_{r \in [0,R]} \min_{u \in \partial\tilde{\mathcal{A}}_r} J(u) \leq \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)) = c,$$

where $\Gamma = \{\gamma \in C([0,1], E); \gamma(0) = 0, \gamma(1) = w\}$. Moreover, $c_* > \max\{J(0), J(w)\}$ is a critical value of J.

In order to demonstrate the efficacy of the method, some applications of the previous theorems will be provided, by considering different types of elliptic partial differential equations, resulting in some advances in the literature.

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CLOSED RANGE ESTIMATES FOR $\bar{\partial}_B$ ON CR-MANIFOLDS OF HYPERSURFACE TYPE

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Abstract

The purpose of this presentation is to establish sufficient conditions for closed range estimates on $(0, q)$-forms, for some *fixed* q, $1 \leq q \leq n-1$, for $\bar{\partial}_b$ in both L^2 and L^2-Sobolev spaces in embedded, *not necessarily pseudoconvex* CR manifolds of hypersurface type. The condition, named weak $Y(q)$, is both more general than previously established sufficient conditions and easier to check. Applications of our estimates include estimates for the Szegö projection as well as an argument that the harmonic forms have the same regularity as the complex Green operator. We use a microlocal argument and carefully construct a norm that is well suited for a microlocal decomposition of form. We do not require that the CR manifold is the boundary of a domain. Finally, we provide an example that demonstrates that weak $Y(q)$ is an easier condition to verify than earlier, less general conditions.

1 Introduction

A CR manifold is essentially the generalization of a real hypersurface into a complex manifold; hence odd-dimensional, where its tangent bundle is split into a complex subbundle, which is the sum of holomorphic and anti holomorphic directions, and another bundle, which is the totally real part. A CR manifold is called of *hypersurface* type if the totally real part of the tangent bundle is a line bundle. The CR manifold $M \subset C^N$ is endowed with a subbundle $T^{1,0}(M)$, called the CR *structure*, of its complexified tangent bundle $\mathbb{C}TM = TM \otimes \mathbb{C}$ of M which does not intersect its complex conjugate $T^{0,1}(M)$ (so being possible to define a Hermitian metric such that $T^{1,0}(M) \perp T^{0,1}(M)$), and with an *integrability condition*, that is $T^{1,0}(M)$ is preserved by Lie bracket ($[L_1, L_2] \in T^{1,0}(M)$ for any $L_1, L_2 \in T^{1,0}(M)$). This structure on the CR manifold allow us to define the bundle of $(0, q)$-forms $\Lambda^{0,q}(M)$, and also there define the *tangential Cauchy Riemann* operator $\bar{\partial}_b$ as the natural restriction of the De Rham exterior derivative on the bundle of $(0, q+1)$-forms. The inner product on $(0, q)$-forms on $\mathbb{C}^N$ induces a pointwise inner product on $(0, q)$-forms on M. This product allow us to define a L^2 norm $||\cdot||_0$ on $\Lambda^{0,q}(M)$, and its completion produces a Hilbert space $L^2_{0,q}(M)$. The L^2 closure of $\bar{\partial}_b$, denoted also by $\bar{\partial}_b$, defines a densely defined closed operator. In this work we explore the L^2-theory of the $\bar{\partial}_b$-equation,

$$\bar{\partial}_b u = f. \tag{1}$$

The study of existence and regularity of solutions of (1) is reduced to study of the operator $\Box_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b$ ($\bar{\partial}_b^*$ is the L^2 adjoint of $\bar{\partial}_b$) called the Kohn Laplacian operator, and the operator which invert $\Box_b$, if it exists, is called by the complex Green operator. Although $\Box_b$ is a Laplace type operator, it is neither elliptic nor constant coefficient, so its analysis is quiet intricate. Moreover, in general $\Box_b$ is not invertible - there must be some geometric or potential theoretic structure. A standard condition generally considered is the *Pseudoconvexity*, which is a curvature condition that is measured by an object called the Levi form γ and it is equivalent to existence of complex Green at all (possible) form levels of the $\bar{\partial}_b$-complex [1,2]. However, it is not a natural condition for the analysis of $(0, q)$-forms when $q \geq 1$ is fixed.

Definition 1.1. *For $1 \leq q \leq n-1$ we say M satisfies* $Z(q)$-weakly *if there exists a real $\Upsilon \in T^{1,1}(M)$ satisfying*

1. $|\theta|^2 \geq (i\theta \wedge \bar{\theta})(\Upsilon) \geq 0$ *for all* $\theta \in \Lambda^{1,0}(M)$

2. $\mu_1 + \mu_2 + \cdots + \mu_q - i < d\gamma_x, \Upsilon > \geq 0$ *where* $\mu_1, ..., \mu_{n-1}$ *are the eigenvalues of the Levi form at* x *in increasing order.*

3. $\omega(\Upsilon) \neq q$ *where* ω *is the* $(1,1)$*-form associated to the induced metric on* $\mathbb{C}T(M)$*.*

We say that M satisfies *weak* $Y(q)$ if M satisfies both $Z(q)$-weakly and $Z(n-q-1)$-weakly. If M is pseudoconvex, the definition above is satisfied choosing $\Upsilon = 0$. This definition generalizes its previous versions given on [3,4].

Theorem 1.1. *Let* M^{2n-1} *be an embedded* C^∞*, compact, orientable CR-manifold of hypersurface type that satisfies weak* $Y(q)$ *for some fixed* q*,* $1 \leq q \leq n-2$*. Then the following hold:*

1. *The operators* $\bar{\partial}_b : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ *and* $\bar{\partial}_b : L^2_{0,q-1}(M) \to L^2_{0,q}(M)$ *have closed range;*

2. *The operators* $\bar{\partial}_b^* : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$ *and* $\bar{\partial}_b^* : L^2_{0,q}(M) \to L^2_{0,q-1}(M)$ *have closed range;*

3. *The Kohn Laplacian* $\Box_b$ *has closed range on* $L^2_{0,q}(M)$*;*

4. *The complex Green operator* G_q *exists and is continuous on* $L^2_{0,q}(M)$*;*

5. *The canonical solution operators,* $\bar{\partial}_b^* G_q : L^2_{0,q}(M) \to L^2_{0,q-1}(M)$ *and* $G_q \bar{\partial}_b^* : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$ *are continuous;*

6. *The canonical solution operators,* $\bar{\partial}_b G_q : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ $G_q \bar{\partial}_b : L^2_{0,q-1}(M) \to L^2_{0,q}(M)$ *are continuous;*

7. *The space of the harmonic forms* $\mathcal{H}_{0,q}(M)$*, defined to be the (0,q)-forms annihilated by* $\bar{\partial}_b$ *and* $\bar{\partial}_b^*$*, is finite dimensional;*

8. *If* $\tilde{q} = q$ *or* $q+1$ *and* $\alpha \in L^2_{0,\tilde{q}}$*, then there exists* $u \in L^2_{0,\tilde{q}-1}$ *so that* $\bar{\partial}_b u = \alpha$ *and* $\|u\|_0 \leq C\|\alpha\|_0$ *for some constant* C *independent of* α*;*

9. *The Szegö projections* $S_q = I - \bar{\partial}_b^* \bar{\partial}_b G_q$ *and* $S_{q-1} = I - \bar{\partial}_b^* G_q \bar{\partial}_b$ *are continuous on* $L^2_{0,q}(M)$*.*

The outline of the argument to prove this result is as follows: we start by proving a basic identity that is well suited to the geometry of M. The problem with basic identities for $\bar{\partial}_b$ is that the Levi form appears with in a term that also contains the derivative in the totally real direction T. We apply a microlocal argument to control this term, specifically, we construct a norm based on a microlocal decomposition of our form which allows us to use a version of the sharp Gårding's inequality and eliminate the T from the inner product term. This allows us to prove a basic estimate from the basic identity and the main results are due to careful applications of the basic estimate.

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SPECTRUM OF DIFFERENTIAL OPERATORS WITH ELLIPTIC ADJOINT ON A SCALE OF LOCALIZED SOBOLEV SPACES

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Abstract

In this work we provide a complete study of the spectrum of a constant coefficients differential operator on a scale of localized Sobolev spaces, $H^s_{loc}(I)$, which are Fréchet spaces. This is quite different from what we find in the literature, where all the relevant results are concerned with spectrum on Banach spaces.

Our aim is to understand the behavior of all the three types of spectrum (point, residual and continuous) and the relation between them and those of the dual operator. The main result we present shows that there is no complex number in the resolvent set of such operators, which suggest a new way to define spectrum if we want to reproduce the classical theorems of the Spectral Theory in Fréchet spaces.

1 Introduction

In this work we present a complete study about the spectrum of a constant coefficients differential operator of order $m \in \mathbb{N}$, $a(D)$, whose adjoint $a(D)^*$ is elliptic, seen as a pseudo-differential operator on a interval $I \subset \mathbb{R}$, that is, seen as $a(D) : H_0^{s+m}(I) \subset H^s_{loc}(I) \longrightarrow H^s_{loc}(I)$, $s \in \mathbb{R}$. Here, $H^s_{loc}(I)$ is endowed with the topology generated by a family of seminorms $\left(p_j^{(s)}\right)_{j\in\mathbb{N}}$ given by $p_j^{(s)}(f) \doteq \|\varphi_j f\|_{H^s(\mathbb{R})}$, $f \in H^s_{loc}(I)$, where, for each $j \in \mathbb{N}$, $I_j = (a_j, b_j)$ is such that $[a_j, b_j] \subset (a_{j+1}, b_{j+1})$, with $I = \bigcup_{j\in\mathbb{N}}[a_j, b_j]$, and $\varphi_j \in C_c^\infty(I_{j+1})$ satisfies $\varphi_j = 1$ in $[a_j, b_j]$.

When we indicate $a(D)$ as above, we mean that in $H_0^{s+m}(I)$ we consider the topology induced from $H^s_{loc}(I)$.

This study was developed inspired by what happen with the Laplace operator on $L^2(I)$. Here, we replace $L^2(I)$ by $H^s_{loc}(I)$ and $H_0^1(I) \cap H^2(I)$ by $H_0^{m+s}(I)$, as suggested by the definitions we found in [3].

The best conclusions we obtain are when we consider the Laplace operator on an interval I as $\Delta : H_0^2(I) \subset L^2_{loc}(I) \longrightarrow L^2_{loc}(I)$. For it, we calculate its closure and compare its spectrum in three stages:

(1) When it is defined on $H_0^2(I)$.

(2) When its domain is $H_0^1(I) \cap H^2(I)$, where we call it Δ_{L^2}; and

(3) When it is defined on $H^2_{loc}(I)$. This, as we are going to see, is the domain of the closure $\overline{\Delta}$.

In particular, we prove that $\sigma_c(\Delta) = \sigma_r(\Delta^*) = \sigma_p(\overline{\Delta}) = \mathbb{C}$ and $\sigma_c(\Delta_{L^2}) = \mathbb{C} \setminus \left\{-\frac{\pi^2 n^2}{l(I)^2} : n \in \mathbb{N}\right\}$, where $l(I)$ is the length of I.

2 Main Results

2.1 Spectrum of differential operators with elliptic dual

Consider a symbol $a \in S^m(\mathbb{R})$ given by $a(\xi) = \sum_{k=0}^m a_k \xi^k$, $m \in \mathbb{N}$, $a_k \in \mathbb{C}$, and the differential operator $a(D) = \sum_{k=0}^m (2\pi i)^{-k} a_k \frac{d^k}{dx^k}$, determined by it, defined on the following scales $a(D) : H_0^{s+m}(I) \subset H^s_{loc}(I) \to H^s_{loc}(I)$, $s \in \mathbb{R}$. Our goal is to compare its spectrum with that from its dual $a(D)^* : D(a(D)^*) \subset H_c^{-s}(I) \to H_c^{-s}(I)$ where $D(a(D)^*) \doteq \left\{g \in H_c^{-s}(I); g \circ a(D) : H^{s+m}_{loc}(I) \subset H^s_{loc}(I) \to \mathbb{C} \text{ is continuous}\right\}$ and it satisfies the relation

$\langle u, a(D)^*\psi\rangle = = \langle a(D)u, \psi\rangle = \left\langle \sum_{j=0}^{m} (2\pi i)^{-j} a_j \frac{d^j u}{dx^j}, \psi \right\rangle = \left\langle u, \sum_{j=0}^{m} (-2\pi i)^{-j} a_\alpha \frac{d^j \psi}{dx^j} \right\rangle$ for $u \in H^{s+m}_{loc}(I)$ and $\psi \in C^\infty_c(I)$.

Theorem 2.1. *Let $a(D) : H^{s+m}_0(I) \subset H^s_{loc}(I) \longrightarrow H^s_{loc}(I)$ be a differential operator of order m with hypoelliptic formal transpose $a(D)'$. There exists $0 < \delta \leq 1$ such that $H^{-s+m}_c(I) \subset D\left[a(D)^*\right] \subset H^{-s+\delta m}_c(I)$.*

Now we compare the sets $\sigma(a(D))$ and $\sigma(a(D)^*)$, where $a(D) : H^{s+m}_0(I) \subset H^s_{loc}(I) \to H^s_{loc}(I)$ is a differential operator with constant coefficients and $a(D)^* : D(a(D^*)) \subset H^{-s}_c(I) \to H^{-s}_c(I)$ is its hypoelliptic dual.

Theorem 2.2. *Under the hypotheses of the last theorem with $a(D)'$ elliptic, $a(D)$ and its adjoint $a(D)^*$ both have empty resolvent set and, independently of $s \in \mathbb{R}$, their types of spectrum are classified as follows: $\sigma_p(a(D)) = \sigma_p(a(D)^*) = \emptyset$, $\sigma_r(a(D)) = \sigma_c(a(D)^*) = \emptyset$ and $\sigma_c(a(D)) = \sigma_r(a(D)^*) = \mathbb{C}$.*

2.2 Closure of a Differential Operator on a Fréchet Space

Here we determine the closure of a differential operator with constant coefficients $a(D)$ of order $m \geq 1$ on $H^s_{loc}(I)$. That will allow us to obtain a more precise analysis of the spectrum.

Theorem 2.3. *If $a(D) : H^{s+m}_0(I) \subset H^s_{loc}(I) \to H^s_{loc}(I)$ is an elliptic differential operator, with constant coefficients, given by $\sum_{j=1}^{m} (-2\pi i)^k a_k u^{(k)}$ where $s \in \mathbb{N}$, then its closure is given by $\overline{a(D)} : H^{s+m}_{loc}(I) \subset H^s_{loc}(I) \longrightarrow H^s_{loc}(I)$ with $\overline{a(D)}(u) = \sum_{j=1}^{m} (-2\pi i)^k a_k u^{(k)}$.*

2.3 Spectrum of the Laplace operator on a Fréchet Space

In this section we apply the results obtained in the previous section to the Laplacian operator.

Corollary 2.1. *The Laplace operator, seen as a pseudodiferencial operator $\Delta : H^2_0(0,\pi) \subset L^2_{loc}(0,\pi) \longrightarrow L^2_{loc}(0,\pi)$ and its adjoint $\Delta^* : H^2_c(0,\pi) \subset L^2_c(0,\pi) \longrightarrow L^2_c(0,\pi)$, both have resolvent set empty and their spectra are classified as follows: $\sigma_p(\Delta) = \sigma_p(\Delta^*) = \emptyset$, $\sigma_r(\Delta) = \sigma_c(\Delta^*) = \emptyset$, and $\sigma_c(\Delta) = \sigma_r(\Delta^*) = \mathbb{C}$.*

Proof This corollary follows immediately from the ellipticity of Δ and Δ^* and Theorem 2.2. □

Denote by Δ_{L^2} the Laplacian defined on the domain $\Delta_{L^2} : H^1_0(I) \cap H^2(I) \subset L^2_{loc}(I) \longrightarrow L^2_{loc}(I)$.

We have $\sigma(\Delta) = \sigma(\Delta_{L^2}) = \sigma(\overline{\Delta}) = \mathbb{C}$.

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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR A SUPERCRITICAL NONLINEAR SCHRÖDINGER EQUATION

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Abstract

We show existence of solution for a supercritical nonlinear Schrödinger equation on the whole $\mathbb{R}^N$ by means of an approximation scheme. We prove a Sobolev embedding corresponding to the variable exponent of the equation.

1 Introduction and main results

Assume for a moment that p and q are constants. The nonlinear Schrödinger equation

$$i\varphi_t - \Delta\varphi = a(x)|\varphi|^{q-2}\varphi + |\varphi|^{p-2}\varphi \quad \text{in } \mathbb{R}^N \times \mathbb{R} \tag{1}$$

has standing wave solutions

$$\varphi(x,t) = e^{-i\zeta^2 t}\zeta^{\frac{2}{p-2}}u(\zeta x) \tag{2}$$

where $\zeta > 0$,

$$-\Delta u + u = \lambda a(x)|u|^{q-2}u + |u|^{p-2}u \quad \text{in } \mathbb{R}^N \tag{3}$$

and $\lambda = \zeta^{\frac{2(q-p)}{p-2}}$.

We are interested in finding positive solutions $u(r)$, $r = |x|$, with radial symmetry of the equation

$$-\Delta u + u = \lambda a(r)u^{q(r)-1} + u^{p(r)-1} \quad in \quad \mathbb{R}^N \tag{4}$$

where $N \geq 3$ and the exponents $p(r)$ and $q(r)$ are functions fulfilling

$$1 < q(r) < 2 \tag{5}$$

and

$$p(r) = \frac{2N}{N-2} + h(r). \tag{6}$$

Here $\lambda > 0$ is a parameter and a, h, p, q are positive radially symmetric continuous functions, other assumptions will be timely presented. Clearly (2) is still standing wave for (4) if $p(r)$ is constant and $q(r)$ is a function.

- h is a function with the following properties.

$$h : [0,\infty) \to [0,\infty) \;\; \textit{is continuous}, \;\; \textit{bounded}, \;\; h(0) = 0 \;\; \textit{and} \;\; h(r) > 0 \;\; \textit{for all} \;\; r > 0; \tag{7}$$

$$\textit{there is a constant} \;\; \beta > 2 \;\; \textit{such that} \; h(r) \leq c|\log r|^{-\beta} \;\; \textit{for} \;\; r \;\; \textit{close to} \;\; 0. \tag{8}$$

We denote by $H_r^1(\mathbb{R}^N)$, $N \geq 3$, the closed subspace of $H^1(\mathbb{R}^N)$ composed by radially symmetric functions on $\mathbb{R}^N$, i.e.,

$$H_r^1(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : u = u(r), \; r = |x|\},$$

endowed with the usual norm

$$\|u\|_{H^1(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2\right) dx\right)^{\frac{1}{2}}. \tag{9}$$

2 Main Results

Theorem 2.1. *Let $p(r) = 2^* + h(r)$, $2^* = 2N/(N-2)$ and h satisfying (7)-(8). Then*

$$\sup\left\{\int_{\mathbb{R}^N} |u(x)|^{p(r)}dx : \quad u \in H^1_r(\mathbb{R}^N), \quad \|u\|_{H^1(\mathbb{R}^N)} = 1\right\} < \infty. \tag{1}$$

The continuity of the embedding reads as follows.

Corollary 2.1. *Let $p(r) = 2^* + h(r)$ with h satisfying (7)-(8). Then the following embedding is continuous*

$$H^1_r(\mathbb{R}^N) \hookrightarrow L_{p(r)}(\mathbb{R}^N). \tag{2}$$

We turn our attention to the supercritical elliptic equation which is slightly more general than (4), namely

$$\begin{cases} -\Delta u + u = \lambda a(r)u^{q(r)-1} + f(r,u) & \text{in} \quad \mathbb{R}^N \\ u > 0 & \text{in} \quad \mathbb{R}^N. \end{cases} \tag{3}$$

Theorem 2.2. *If $\lambda > 0$ is a constant, q and a are radially symmetric continuous functions, $r = |x|$, such that $1 < q_- \leq q(r) \leq q_+ < 2$, $q_-, q_+ \in \mathbb{R}$, $a \in L^{\frac{2}{2-q(r)}}(\mathbb{R}^N)$, $a(r) > 0$ in $\mathbb{R}^N$, $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function, radially symmetric in the first variable and satisfying the growth condition*

$$0 \leq f(r,t)t \leq b_1 t^{p(r)} \text{ for every } r \in \mathbb{R} \text{ and } t \geq 0, \tag{4}$$

where $b_1 > 0$ is a constant, $p(r) = 2^* + h(r)$, with h satisfying (7)-(8), $2^* = 2N/(N-2)$. Then there exists $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$ problem (P) possesses at least a positive radial solution $u_\lambda \in H^1(\mathbb{R}^N)$. Furthermore, $\|u_\lambda\|_{\mathbb{R}^N} \to 0$ as $\lambda \to 0$.

We would like to mention that problem (P) was studied in [1] for the unit ball case.

Corollary 2.2. *The solution u_λ decays to zero, in the sense that $u_\lambda(r) \to 0$ as $r \to \infty$. Moreover, there is a constant $C > 0$ such that $0 \leq \min\{r^{\frac{N-2}{2}}, r^{\frac{N-1}{2}}\}u_\lambda(r) \leq C$ for every $r \in \mathbb{R}$.*

For a particular f, the nonexistence of solution for equation (4) reads as follows.

Proposition 2.1. *Assume the hypotheses of Theorem 2.2. If $f(r,t) = t^{p(r)-1}$ and $\lambda > 0$ is sufficiently large, then equation (4) has no positive solution in $H^1(\mathbb{R}^N)$. In other words $\lambda^* < \infty$.*

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LOG-SINGULAR ELLIPTIC EQUATIONS IN THE PLANE WITH NONLINEARITIES OF EXPONENTIAL GROWTH

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Abstract

We show the existence of a solution for an equation where the nonlinearity is logarithmically singular at the origin, namely $-\Delta u = (\log u + f(u))\chi_{\{u>0\}}$ in $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary condition. The function f has exponential growth, which can be subcritical or critical with respect to the Trudinger-Moser inequality. In the critical context, we obtain an admissibility condition for problems of the form $-\Delta u = (\log u + \lambda f(u))\chi_{\{u>0\}}$ in $\Omega \subset \mathbb{R}^2$.

1 Introduction

This work consists in studying the problem

$$\begin{cases} -\Delta u = (\log u + f(u))\chi_{\{u>0\}} \text{ in } \Omega \\ u \geq 0,\ u \not\equiv 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, $\chi_{\{u>0\}}$ denotes the characteristic function corresponding to the set $\{x \in \Omega : u(x) > 0\}$ and we tacitly assume $(\log u)\chi_{\{u>0\}} = 0$ if $u = 0$.

The function f can be allowed to have either subcritical or critical growth in the sense of Trudinger-Moser, that is,

Definition 1.1. *The function f has subcritical growth if*

$$\lim_{s\to\infty} \frac{|f(s)|}{\exp(\alpha s^2)} = 0 \text{ for all } \alpha > 0. \tag{2}$$

Definition 1.2. *We say that f has critical growth if there exists $\alpha > 0$ such that*

$$\lim_{s\to\infty} \frac{|f(s)|}{\exp(\kappa s^2)} = \infty \text{ for all } 0 < \kappa < \alpha, \text{ and } \lim_{s\to\infty} \frac{|f(s)|}{\exp(\kappa s^2)} = 0 \text{ for all } \kappa > \alpha. \tag{3}$$

Under certain assumptions of f, we are able to obtain a nontrivial function $u \in H_0^1(\Omega)$ that solves problem (1). The main difficulties are the singularity of log at the origin and the critical behaviour of f. Indeed, in [1] the authors obtained results only in the subcritical context. In [2], the results of [1] were extended and the critical case was addressed. Our approach is variational, and we use an approximation scheme. We study the energy functional I_ϵ corresponding to the perturbed equation $-\Delta u + g_\epsilon(u) = f(u)$, where g_ϵ is well defined at 0 and approximates $-\log u$. We show that I_ϵ has a critical point (which is a Mountain Pass) u_ϵ in $H_0^1(\Omega)$, which converges to a nontrivial nonnegative solution of (1) as $\epsilon \to 0$.

2 Main Results

The following result was proven in [2, Chapter 5] and concerns the subcritical case.

Theorem 2.1. *Problem (1) is solvable for a large variety of functions f, including*
- $f(s) = e^s$;
- $f(s) = s^k e^s$ *with* $k > 1$;
- $f(s) = e^s - \mu$ *with* $\mu > 0$.

The critical case was treated in [2, Chapter 6]. The following results were obtained. They can be extended for a larger class of functions f with critical growth.

Theorem 2.2. *Let* $f(s) = s^k e^{\alpha s^2}$ *with* $k > 1$. *There exists* $\alpha_0 > 0$ *such that problem (1) has a nontrivial nonnegative solution provided* $0 < \alpha < \alpha_0$.

Theorem 2.3. *Let* $\alpha \geq 3/4$ *and assume that* $f(s) = \lambda s e^{\alpha s^2}$. *There exists* $\lambda_0 > 1$ *such that problem (1) has a nontrivial nonnegative solution for* $\lambda > \lambda_0$ *and* $|\Omega| < c_{\lambda,\alpha}$, *where*

$$c_{\lambda,\alpha} = \frac{\pi}{2}\left(\frac{e}{2\lambda e^{1+4/\alpha} + \alpha}\right).$$

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EXISTENCE RESULT FOR FRACTIONAL P-LAPLACIAN PROBLEM WITH MULTIPLE NONLINEARITIES AND HARDY POTENTIAL

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Abstract

In this work, we study the existence of weak solution to a quasilinear elliptic problem involving the fractional p-Laplacian operator, a Hardy potential and multiple critical Sobolev nonlinearities with singularities,

$$(-\Delta_p)^s u - \mu\frac{|u|^{p-2}u}{|x|^{ps}} = \frac{|u|^{p_s^*(\beta)-2}u}{|x|^{\beta}} + \frac{|u|^{p_s^*(\alpha)-2}u}{|x|^{\alpha}},$$

where $x \in \mathbb{R}^N$, $u \in D^{s,p}(\mathbb{R}^N)$, $0 < s < 1$, $1 < p < +\infty$, $N > sp$, $0 < \alpha < sp$, $0 < \beta < sp$, $\beta \neq \alpha$, $\mu < \mu_H \equiv \inf_{u\in D^{s,p}(\mathbb{R}^N)\setminus\{0\}}[u]_{s,p}^p/||u||_{s,p}^p > 0$. To prove our result we formulate a refined version of the concentration-compactness principle and, as an independent result, we show that the extremals for the Sobolev inequality are attained.

1 Introduction

In this work, we consider the quasilinear elliptic problem involving the fractional p-Laplacian operator with a Hardy potential and multiple critical nonlinearities with singularities at the origin,

$$(-\Delta_p)^s u - \mu\frac{|u|^{p-2}u}{|x|^{ps}} = \frac{|u|^{p_s^*(\beta)-2}u}{|x|^{\beta}} + \frac{|u|^{p_s^*(\alpha)-2}u}{|x|^{\alpha}} \qquad (x \in \mathbb{R}^N) \tag{1}$$

where $0 < s < 1$, $1 < p < +\infty$, $N > sp$, $0 < \alpha < sp$, $0 < \beta < sp$, $\beta \neq \alpha$, $\mu < \mu_H \equiv \inf_{u\in D^{s,p}(\mathbb{R}^N)\setminus\{0\}}[u]_{s,p}^p/||u||_{s,p}^p > 0$ and $p_s^*(\alpha) = (p(N-\alpha)/(N-ps)$; in particular, if $\alpha = 0$, then $p_s^*(0) = p_s^* = Np/(N-p)$. Recall that the fractional p-Laplacian operator is a non-linear and non-local operator defined for differentiable functions $u\colon \mathbb{R}^N \to \mathbb{R}$ by

$$(-\Delta_p)^s u(x) := 2\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\,dy,$$

where $x \in \mathbb{R}^N$, $p \in (1, +\infty)$, $s \in (0, 1)$ and $N > sp$.

Let $\Omega \subset \mathbb{R}^N$ be an open, bounded subset with differentiable boundary. We consider tacitly that all the functions are Lebesgue integrable and we introduce the fractional Sobolev space $W_0^{s,p}(\Omega)$ and the fractional homogeneous Sobolev space $D^{s,p}(\mathbb{R}^N)$, respectively, by

$$W_0^{s,p}(\Omega) \equiv \left\{u \in L^1_{\text{loc}}(\mathbb{R}^N)\colon [u]_{s,p} < +\infty;\ u \equiv 0 \text{ a.e. } \mathbb{R}^N\setminus\Omega\right\},$$
$$D^{s,p}(\mathbb{R}^N) \equiv \left\{u \in L^{p_s^*}(\mathbb{R}^N)\colon [u]_{s,p} < \infty\right\} \supset W_0^{s,p}(\Omega).$$

In these definitions, the symbol $[u]_{s,p}$ stands for the Gagliardo seminorm, defined by

$$u \longmapsto [u]_{s,p} = \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\right)^{1/p} \qquad (u \in C_0^\infty(\mathbb{R}^N)).$$

The variational structure of problem (1) is established through the Hardy-Sobolev inequality: Let $0 < s < 1$, $1 < p < +\infty$ and $0 \leqslant \alpha < sp < N$; then there exists a positive constant $C \in \mathbb{R}_+$ such that

$$\left(\int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}\,dx\right)^{1/p_\alpha^*} \leqslant C\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{1/p}$$

for every $u \in W_0^{s,p}(\Omega)$. The parameter $p_s^*(\alpha)$ is the critical fractional exponent of the Hardy-Sobolev embeddings $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N; |x|^{-sp})$ where the Lebesgue space $L^p(\mathbb{R}^N; |x|^{-sp})$ is equipped with the norm

$$||u||_{L^p(\mathbb{R}^N;|x|^{-sp})} \equiv \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{sp}}\,dx\right)^{1/p}.$$

The embeddings $W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega; |x|^\alpha)$ are continuous for $0 \leqslant \alpha \leqslant ps$ and for $1 \leqslant q \leqslant p_s^*(\alpha)$; and these embeddings are compact for $1 \leqslant q < p_s^*(\alpha)$. Moreover, the best constants of these embeddings are positive numbers, that is,

$$\mu_H \equiv \inf_{\substack{u \in D^{s,p}(\mathbb{R}^N)\\ u \neq 0}} \frac{[u]_{s,p}^p}{\|u\|_{L^p(\mathbb{R}^N;|x|^{-sp})}^p} > 0.$$

We say that the function $u \in D^{s,p}(\mathbb{R}^N)$ is a weak solution to problem (1) if

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} &\frac{J_p(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy - \mu\int_{\mathbb{R}^N} \frac{J_p(u)\varphi(x)}{|x|^{ps}}\,dx \\ &= \int_{\mathbb{R}^N} \frac{J_{p_s^*(\beta)}(u)\varphi(x)}{|x|^\beta}\,dx + \int_{\mathbb{R}^N} \frac{J_{p_s^*(\alpha)}(u)\varphi(x)}{|x|^\alpha}\,dx\end{aligned}$$

for every function $\varphi \in D^{s,p}(\mathbb{R}^N)$; given $1 < m < +\infty$, the function $J_m \colon \mathbb{R} \to \mathbb{R}$ is defined by $J_m(t) = |t|^{m-2}t$. A weak solution to problem (1) corresponds to a critical point to the energy functional $\Phi \colon \mathbb{R}^N \to \mathbb{R}$ defined by

$$\Phi(u) \equiv \frac{1}{p}[u]_{s,p}^p - \frac{\mu}{p}\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{ps}}\,dx - \frac{1}{p_s^*(\beta)}\int_{\mathbb{R}^N} \frac{|u|^{p_x^*(\beta)}}{|x|^\beta}\,dx - \frac{1}{p_s^*(\alpha)}\int_{\mathbb{R}^N} \frac{|u|^{p_x^*(\alpha)}}{|x|^\alpha}\,dx.$$

In other terms, $u \in D^{s,p}(\mathbb{R}^N)$ is a weak solution to problem (1) if, and only if, $\Phi'(u) = 0$.

2 Main result

Theorem 2.1. *Let $0 < s < 1$, $1 < p < +\infty$, $N > sp$, $0 < \alpha < sp$, $0 < \beta < sp$, $\beta \neq \alpha$, $\mu < \mu_H$. Then there exists a weak solution $u \in D^{s,p}(\mathbb{R}^N)$ to problem (1).*

To prove Theorem 2.1 we cannot apply the harmonic extension of the fractional Laplacian because this idea is valid only for $p = 2$; moreover, since the Hardy-Sobolev embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N; |x|^{-sp})$ is non compact, we face several difficulties to prove that bounded Palais-Smale sequences in the reflexive Sobolev space $D^{s,p}(\mathbb{R}^N)$ have at least a strongly convergent subsequence. The presence of multiple critical Sobolev nonlinearities also contributes to the difficulties in the proof of the theorem. We also mention that due to the Hardy potential, the functional $u \longmapsto ([u]_{s,p}^p - \mu\int_{\mathbb{R}^N} |u|^p/|x|^{sp}\,dx)^{1/p}$ does not define a norm in $D^{s,p}(\mathbb{R}^N)$; as a consequence, the energy functional Φ is not lower semicontinuous. With the help of some carefully proved estimates, we managed to overcome these difficulties and prove a refined version of the concentration-compactness principle.

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FOURTH-ORDER SUPERLINEAR ELLIPTIC PROBLEMS INTERACTING WITH HIGH EIGENVALUES

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Abstract

It is established existence of weak solutions for a class of superlinear elliptic involving a fourth-order elliptic problem under Navier conditions on the boundary. Here we do not apply the well known Ambrosetti-Rabinowitz condition at infinity. Instead of we assume that the nonlinear term is a nonlinear function satisfying the well known nonquadraticity condition at infinity. Using a Local Linking Theorem we get our main results without any restrictions on the first eigenvalue for the linear problem. Namely, the first eigenvalue can be negative or positive. Furthermore, we consider nonlinear terms interacting at high eigenvalues

1 Introduction

In this work we consider the fourth-order elliptic problem

$$\begin{cases} \alpha\Delta^2 u + \beta\Delta u = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Delta^2 = \Delta \circ \Delta$ is the biharmonic operator, $N \geq 4, \Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\alpha > 0, \beta \in \mathbb{R}$. The problem (1) is named fourth-order elliptic problem under Navier boundary conditions. Throughout this work λ_1 denotes the first eigenvalue for the linear eigenvalue problem associated to Laplacian operator. The nonlinear term f is a continuous function which is superlinear at infinity and at the origin. Later on, we shall consider the assumptions on the nonlinear term f.

The fourth-order elliptic problems are modeled in the functional space $\mathcal{H} = \mathrm{H}_0^1(\Omega) \cap \mathrm{H}^2(\Omega)$. The weak solutions for problem (1) are precisely the critical points for the functional of C^1 class $I : \mathcal{H} \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\int_\Omega (\alpha\Delta u\Delta v - \beta\nabla u\nabla v)\,dx - \int_\Omega F(x,u)\,dx, \tag{2}$$

where the primitive for f.

In this work we shall consider that $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$. Moreover, we assume the following hypotheses:

(f_0) There exist $a_1 > 0$ and $p \in (2, 2_*)$ such that

$$|f(x,t)| \leq a_1(1 + |t|^{p-1}), \text{ for any } (x,t) \in \Omega \times \mathbb{R}$$

where $2_* = \dfrac{2N}{N-4}$ for each $N \geq 5$ and $2_* = \infty$ for $N = 4$.

(f_1) $\displaystyle\lim_{|t|\to\infty} \frac{f(x,t)}{t} = +\infty$ uniformly in Ω.

(f_2) There exist $k \in \mathbb{N}$ and $f_0 \in (\mu_k, \mu_{k+1})$, where μ_k and μ_{k+1} are two consecutive eigenvalues for the problem associated to (1), such that

$$\lim_{|t|\to 0} \frac{f(x,t)}{t} = f_0 \quad \text{uniformly in } \Omega.$$

(NQ) setting $H(x,t) := f(x,t)t - 2F(x,t)$, we have that

$$\lim_{|t|\to\infty} H(x,t) = +\infty, \text{ uniformly for } x \in \Omega.$$

2 Main Results

Using a Local Linking Theorem we get our main results without any restrictions on the first eigenvalue for the linear problem. Namely, the first eigenvalue can be negative or positive.

Theorem 2.1. *Suppose that f satisfies $(f_0), (f_1), (f_2)$ and (NQ). Assume also that one of the following conditions*

i) the first eigenvalue μ_1 is positive;

ii) the first eigenvalue μ_1 is negative and $0 \in I_k$;

iii) the first eigenvalue μ_1 is negative and $0 \notin \overline{I}_k$.

Then problem (1) admits at least one nontrivial solution.

In order to prove the theorem, we use the Linking Geometry and prove that the functional I satisfies the Cerami Condition at any level $c \in \mathbb{R}$.

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LEBESGUE SOLVABILITY OF ELLIPTIC HOMOGENEOUS LINEAR EQUATIONS ON MEASURES

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Abstract

In this talk we present new results on solvability of the equation $A^*(D)f = \mu$ for $f \in L^p(\mathbb{R}^n)$ and positive measure data μ associated to an elliptic homogeneous differential operator $A(D)$ of order m. Our method is based on controlling the $(m,p)-$energy of μ giving a natural characterization for solutions when $1 \leq p < \infty$. We also obtain sufficient conditions in the limiting case $p = \infty$ using new L^1 estimates on measures for elliptic and canceling operators.

1 Introduction

N. C. Phuc and M. Torres in [6] characterized the existence of solutions in Lebesgue spaces for the divergence equation

$$\text{div } f = \mu, \tag{1}$$

where $\mu \in \mathcal{M}_+(\mathbb{R}^N)$, the set of positive measures on $\mathbb{R}^N$, and $f \in L^p(\mathbb{R}^N, \mathbb{R}^N)$. The method is based on controlling the $(1,p)-$energy of μ defined by $\|I_1\mu\|_{L^p}$, for $1 \leq p < \infty$, where I_1 is the Riesz potential of order 1 (see [6, Theorems 3.1 and 3.2]). The previous results do not cover the case $p = \infty$, however, in Theorem 3.3 they show that an L^∞ solution for (1) exists if and only if μ is a $(N-1)-$Ahlfors regular measure, *i.e.* μ satisfies

$$\mu(B(x,r)) \leq Cr^{N-1},$$

where the constant $C > 0$ is independent of $x \in \mathbb{R}^N$ and $r > 0$.

Let $A(D)$ be a homogeneous differential operator on $\mathbb{R}^N$, $N \geq 2$ with constant coefficients, of order m, from a finite dimensional complex vector space E to a finite dimensional complex vector space F given by

$$A(D) = \sum_{|\alpha|=m} a_\alpha \partial^\alpha : C_c^\infty(\mathbb{R}^N, E) \to C_c^\infty(\mathbb{R}^N, F), \quad a_\alpha \in \mathcal{L}(E, F).$$

In this work, we study the Lebesgue solvability for the equation

$$A^*(D)f = \mu, \tag{2}$$

where $A^*(D)$ is the (formal) adjoint operator associated to the homogeneous differential operator $A(D)$. In what follows, for an open subset $\Omega \subseteq \mathbb{R}^N$ and a finite dimensional complex vector space X, $\mathcal{M}_+(\Omega, X)$ denotes the set of all X-valued complex measures on Ω such that the real and imaginary parts of each component are positive Radon measures. We also define the $(m,p)-$energy of μ by $\|I_m\mu\|_{L^p}$, for $1 \leq p < \infty$, where I_m is the Riesz potential of order m.

2 Main Results

Our first result is a slight improvement on [6, Theorems 3.1 and 3.2].

Theorem 2.1. *Let $A(D)$ be a homogeneous linear differential operator of order $1 \leq m < N$ on $\mathbb{R}^N$, $N \geq 2$, from E to F and $\mu \in \mathcal{M}_+(\mathbb{R}^N, E)$.*

(i) If $1 \leq p \leq N/(N-m)$ and $f \in L^p(\mathbb{R}^N, F)$ is a solution for (2) then $\mu \equiv 0$.

(ii) If $N/(N-m) < p < \infty$ and $f \in L^p(\mathbb{R}^N, F)$ is a solution for (2), then μ has finite (m,p)-energy. Conversely, if μ has finite (m,p)-energy and $A(D)$ is elliptic, then there exists a function $f \in L^p(\mathbb{R}^N, F)$ solving (2).

We recall that ellipticity means that the symbol $A(\xi) : E \to F$ given by

$$A(\xi) := \sum_{|\alpha|=m} a_\alpha \xi^\alpha$$

is injective for $\xi \in \mathbb{R}^N \setminus \{0\}$. In particular, we recover Theorems 3.1 and 3.2 from [6] taking $A(D) = \nabla$ with $E = \mathbb{R}$ and $F = \mathbb{R}^N$. Our second and main result deals with the case $p = \infty$.

Theorem 2.2. *Let $A(D)$ be a homogeneous linear differential operator of order $1 \leq m < N$ on $\mathbb{R}^N$ from E to F and $\mu \in \mathcal{M}_+(\mathbb{R}^N, E)$. If $A(D)$ is elliptic and canceling, and μ satisfies both*

$$|\mu|(B(0,r)) \leq C_1 r^{N-m}, \tag{1}$$

and the following control of the truncated Wolff's potential

$$\int_0^{|y|/4} \frac{|\mu(B(y,r))|}{r^{N-m+1}}\, dr \leq C_2, \text{ uniformly on } y, \tag{2}$$

then there exists $f \in L^\infty(\mathbb{R}^N, F)$ solving (2).

The canceling property means $\bigcap_{\xi \in \mathbb{R}^N \setminus 0} A(\xi)[E] = \{0\}$. The main ingredient in the proof of Theorem 2.2 is to investigate the sufficient conditions on μ in order to obtain

$$\left| \int_{\mathbb{R}^N} u(x)\, d\mu(x) \right| \leq C \|A(D)u\|_{L^1}, \qquad \forall\, u \in C_c^\infty(\mathbb{R}^N, E).$$

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A SINGULAR PERTURBATION PROBLEM GOVERNED BY NORMALIZED $P(X)$-LAPLACIAN

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Abstract

In this work we will focus our attention on finding a family of solutions (in the viscosity sense) $(u_\varepsilon)_{\varepsilon>0}$ for a singularly perturbed problem driven governed by normalized $p(x)$-Laplacian. Assuming suitable assumptions on the data we will show that such solutions enjoy certain properties, such as uniform boundedness, local Lipschitz regularity and non-degeneracy in a smooth domain $\Omega \subset \mathbb{R}^n$. Furthermore, by using the stability of notion of viscosity solutions, we will show, up to a subsequence, that $\lim_{j\to\infty} u_{\varepsilon_j} = u_0$, which becomes a solution of a one-phase free problem of Bernoulli type. Our results are natural extension to the ones in [2] and [3].

1 Introduction

In last years researches about Tug-of-war game with varying probabilities have appeared in the modern interplay of Probability Theory and Elliptic PDEs models. Precisely, in [1] the authors studied a two player zero-sum tug-of-war game with varying probabilities that depend on the game location $x \in \Omega$. Particularly, they show that the value of the game converges to a solution of the normalized $p(x)-$Laplacian.

Thus, motivated by modern issues of singular PDEs in non-variational models (cf. [2]), we would like to study:

$$\begin{cases} \Delta^{\mathrm{N}}_{p_\varepsilon(x)} u_\varepsilon(x) = \zeta_\varepsilon(u_\varepsilon) + f_\varepsilon(x) & \text{in} \quad \Omega, \\ u_\varepsilon(x) = g(x) & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where the Normalized $p(x)-$Laplacian is defined as follows:

$$\Delta^{\mathrm{N}}_{p(x)} u = \frac{1}{p(x)}\Delta u + \frac{p(x)-2}{p(x)}\Delta^{\mathrm{N}}_{\infty} u = \frac{1}{p(x)}\Delta u + \frac{p(x)-2}{p(x)}\frac{\langle D^2u\cdot Du, Du\rangle}{|Du|^2}.$$

Furthermore, we will assume $p_\varepsilon \in C^{0,1}(\Omega)$, $f_\varepsilon \in C^\infty(\Omega)$ and there exist positive constants $\mathcal{A}, \mathcal{B}, p^+, p^-, p_l$ such that $1 < p^- \le p_\varepsilon(x) \le p^+ < \infty$, $|Dp_\varepsilon(x)| \le p_l$ and $\mathcal{A} \le f_\varepsilon(x) \le \mathcal{B}$ for every $x \in \Omega$ and all $\varepsilon > 0$. Moreover, $\Omega \subset \mathbb{R}^n$ is a bounded and open domain, $0 \le g \in C(\partial\Omega)$ and $\zeta_\varepsilon(t) = \frac{1}{\varepsilon}\zeta(\frac{t}{\varepsilon})$, with $0 \le \zeta_\varepsilon \in C^\infty_0(\Omega)$.

Definition 1.1. *A function $u \in C(\Omega)$ is said a viscosity sub-solution (super-solution) of*

$$\Delta^{\mathrm{N}}_{p(x)} u = h(x, u(x)) \quad \textit{com} \quad h \in C^0(\Omega \times \mathbb{R}_+) \tag{2}$$

if and only if for each $x_0 \in \Omega$ e $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local maximum (minimum) at x_0, then

(1) $\Delta^{\mathrm{N}}_{p(x_0)}\phi(x_0) \le h(x_0, \phi(x_0))$ (resp. $\ge h(x_0, \phi(x_0))$) if $D\phi(x_0) \neq 0$.

(2) $\frac{1}{p(x_0)}\Delta\phi(x_0) + \frac{p(x_0)-2}{p(x_0)}\lambda_{\max}\left(D^2\phi(x_0)\right) \le h(x_0, \phi(x_0))$ (resp. $\ge h(x_0, \phi(x_0))$) if $D\phi(x_0) = 0$ and $p(x) \ge 2$.

(3) $\frac{1}{p(x_0)}\Delta\phi(x_0) + \frac{p(x_0)-2}{p(x_0)}\lambda_{\min}\left(D^2\phi(x_0)\right) \le h(x_0, \phi(x_0))$ (... $\ge h(x_0, \phi(x_0))$) if $D\phi(x_0) = 0$ and $p(x) \in (1, 2)$.

Finally, u is a solution of (2) in the viscosity sense if it is a sub and super-solution.

In order to show the existence of solutions for (1), we choose an appropriate approximation of (1) by a "regularized problem": for fixed $\delta > 0$ and $\varepsilon > 0$, there are solutions (in the classical sense) regular enough $u_{\varepsilon,\delta}$ to

$$\left\{ \begin{array}{rclll} \displaystyle\sum_{i,j=1}^{n} a_{ij}^{p_\varepsilon(x),\delta}(Du)u_{ij} & = & \zeta_\varepsilon(u) + f_\varepsilon(x) & \text{in} & \Omega \\ u(x) & = & g(x) & \text{on} & \partial\Omega, \end{array} \right. \tag{3}$$

where the uniformly elliptic operator $\nu \mapsto a_{ij}^{p_\varepsilon(x),\delta}(\cdot)$ is given by:

$$a_{ij}^{p_\varepsilon(x),\delta}(\nu) = \frac{1}{p_\varepsilon(x)}\delta_{ij} + \frac{p_\varepsilon(x)-2}{p_\varepsilon(x)}\frac{\nu_i\nu_j}{|\nu|^2+\delta^2}, \ \forall \nu \in \mathbb{R}^n.$$

By sing auxiliary results, e.g., the Aleksandrov-Bakelman-Pucci's Maximum Principle, Harnack Inequality and the existence of Perron-type solutions, we conclude that, up to a subsequence $(u_{\varepsilon,\delta})_{\delta>0}$ converges local uniformly for a non-negative family of solutions and uniformly bounded.

2 Main Results

Our first result establishes local Lipschitz regularity to solution of (1) (cf. [1], [2] and [3])

Theorem 2.1 (**Lipschitz regularity- cf. [1]**). *Let $(u_\varepsilon)_{\varepsilon>0}$ be viscosity solutions of (1) and $\Omega' \Subset \Omega$, then there exist a constant $\mathrm{C} > 0$ depending on universal parameters and $\Omega' \subseteq \Omega$ such that*

$$\|u_\varepsilon\|_{C^{0,1}(\Omega')} = \|u_\varepsilon\|_{L^\infty(\Omega')} + \|Du_\varepsilon\|_{L^\infty(\Omega')} \leq \mathrm{C}.$$

Furthermore, by invoking a suitable barrier and other tools, we will prove the following result:

Theorem 2.2 (**Strong No-degeneracy - cf. [1]**). *Let $(u_\varepsilon)_{\varepsilon>0}$ be Perron's solutions of (1). Then, there exists a constant $\mathrm{L} > 0$ such that, for every $x_0 \in \{u_\varepsilon > \varepsilon\}$ and $\varepsilon < d_\varepsilon(x_0) := dist(x_0, \{u_\varepsilon \leq \varepsilon\}) \ll 1$ there holds*

$$u_\varepsilon(x_0) \geq \mathrm{L}(universal)d_\varepsilon(x_0) \qquad \Rightarrow \qquad \sup_{B_r(x_0)} u_\varepsilon(x) \geq \mathrm{L}_0(universal)r.$$

Finally, the family $(u_\varepsilon)_{\varepsilon>0}$ is pre-compact in the topology $C^{0,1}_{\text{loc}}(\Omega)$. Hence, we are able to conclude:

Theorem 2.3 (**Limiting profile - cf. [1]**). *Let $(u_\varepsilon)_{\varepsilon>0}$ be solutions of (1). Then, up to a subsequence, $(u_\varepsilon)_{\varepsilon>0}$ converges local uniformly to a solution u_0 of one-phase free boundary problem:*

$$\left\{ \begin{array}{rclll} \Delta^{\mathrm{N}}_{p_0(x)}u_0(x) & = & f_0(x) & in & \{u_0>0\}\cap\Omega, \\ u_0(x) & = & g(x) & on & \partial\Omega. \end{array} \right. \tag{1}$$

where $f_\varepsilon \to f_0$ and $p_\varepsilon \to p_0$ local uniformly in Ω. Furthermore, $\|u_\varepsilon\|_{C^{0,1}(\Omega')} \leq \mathrm{C}(universal)$.

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ON THE ANTIPLANE FRICTIONAL CONTACT PROBLEM OF $P(X)$-KIRCHHOFF TYPE WITH CONVECTION TERM

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Abstract

In this paper, we deal with a class of frictional contact problem of $p(x)$-Kirschhoff type with convection term. Due to the lack of a variational structure the well-known variational methods are not applicable. By means of the topological degree theory for (S_+) type mappings (See Skrypnik, 1994) we establish the existence of weak solutions.

1 Introduction

The purpose of this work is to investigate the existence of weak solutions for the following boundary value problem

$$\begin{array}{rcll} -M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)div\Big(|\nabla u|^{p(x)-2}\nabla u\Big) &=& f_1(x,u,\nabla u) & \text{in} \quad \Omega \\ u &=& 0 & \text{on} \quad \Gamma_1 \\ M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} &=& f_2(x,u) & \text{in} \quad \Gamma_2 \\ \Big|M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu}\Big| &\leq& g, & \\ M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} &=& -g\dfrac{u}{|u|}, & \text{if} \quad u\neq 0 \quad \text{in} \quad \Gamma_3 \end{array} \tag{1}$$

where $\Omega \subseteq \mathbb{R}^2$ is a bounded domain with smooth enough boundary Γ, partitioned in three parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that meas $(\Gamma_i) > 0$, $(i = 1, 2, 3)$; $f_1 : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $f_2 : \Gamma_2 \times \mathbb{R} \to \mathbb{R}$, $g : \Gamma_3 \to \mathbb{R}$ and $M : \mathbb{R}^+ \to \mathbb{R}^+$ are given functions, $p \in C(\overline{\Omega})$.

Many scholars make efforts to investigate $p(x)$-Kirchhoff type equations with nonlinear boundary condition of different class in recent years (See e.g. [3,5]) and indeed, important results were obtained. Motivated by the ideas in [1] we consider problem (1.1) (which has already been treated for constant exponent, with $M(s) = 1, f_1(x, u, \nabla u) = f_1(x), f_2(x, u) = f_2(x)$ in [2] by means of an abstract Lagrange multiplier technique) with M a nonconstant continuous function in the setting of the variable exponent spaces. Unlike in [2], we will apply the topological degree introduced by Skrypnik to solve our problem.

2 Assumptions and Main Result

First, we introduce the space

$$X = \{u \in W^{1,p(x)}(\Omega) : \gamma u = 0 \quad \text{on} \quad \Gamma_1\}$$

herein $W^{1,p(x)}(\Omega)$ $(p \in C(\overline{\Omega}),\ 2 \leq p(x) < +\infty)$ is the well known variable exponent Sobolev space.

(A_1) $M : [0, +\infty[\to [m_0, +\infty[$ is a continuous and increasing function; $m_0 > 0$.

(A_2) $f_1 : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a Caratheodory function such that

$$|f_1(x, \eta, \xi)| \leq c_3 \Big(k(x) + |\eta|^{q(x)-1} + |\xi|^{q(x)-1} \Big)$$

for almost all $x \in \Omega$ and all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$, $c_3 > 0$, q is a continuous function such that $1 < q(x) < p(x)$ and $k \in L^{p'(x)}(\Omega)$.

(A_3) $f_2 : \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function satisfying the following conditions

$$|f_2(x, s)| \leq c_1 + c_2 |s|^{\alpha(x)-1}), \quad \forall x \in \Gamma_2, s \in \mathbb{R},$$

for some $\alpha \in C_+(\Omega)$ such that $1 < \alpha(x) < p^*(x)$ for $x \in \overline{\Omega}$ and c_1, c_2 are positive constants.

(A_4) $g \in L^{p'(x)}(\Gamma_3), g(x) \geq 0$ a.e on Γ_3

So, our main result can be stated as follows.

Theorem 2.1. *Suppose* (A_1) - (A_4) *hold. Then problem (1.1) has a solution* $u \in X$.

Proof We apply the topological degree theory for (S_+) type mappings (See [4]).

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STABILIZATION OF A THERMOELASTIC BRESSE SYSTEM WITH NONLINEAR DISSIPATION AT THE BOUNDARY

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Abstract

In this work, we study the existence and stabilization of the solution of a Bresse Thermoelastic System with nonlinear dissipation at the boundary. We will initially prove the existence through the Theory of Semigroups of Nonlinear Operators. Later, to analyze the stabilization, we used the method of multipliers.

1 Introduction

Our objective, inspired by the works of [3] and [5] was to study the following thermoelastic Bresse system

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) = 0 \quad \text{em } (0,L)\times(0,\infty), \tag{1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\theta_x = 0 \quad \text{em } (0,L)\times(0,\infty), \tag{2}$$

$$\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) = 0 \quad \text{em } (0,L)\times(0,\infty), \tag{3}$$

$$\theta_t - k_1\theta_{xx} + m\psi_{xt} = 0 \quad \text{em } (0,L)\times(0,\infty), \tag{4}$$

with initial conditions

$$\begin{aligned} \varphi(\cdot,0) &= \varphi_0, \quad \varphi_t(\cdot,0) = \varphi_1\\ \psi(\cdot,0) &= \psi_0, \quad \psi_t(\cdot,0) = \psi_1\\ w(\cdot,0) &= w_0, \quad w_t(\cdot,0) = w_1,\\ \theta(\cdot,0) &= \theta_0 \end{aligned} \tag{5}$$

and with boundary conditions

$$\varphi(0,t) = \psi(0,t) = w(0,t) = \theta(0,t) = \theta(L,t) = 0 \tag{6}$$

and

$$\begin{aligned} k(\varphi_x + \psi + lw)(L,t) &= -g_(\varphi_t(L,t))\\ b\psi_x(L,t) &= -g_2(\psi_t(L,t))\\ k_0 l(w_x - l\varphi)(L,t) &= -g_3(w_t(L,t)) \end{aligned} \tag{7}$$

The coefficients $\rho_1, \rho_2, k, k_0, k_1, b, \gamma, l$ and m are positive constants and g_1, g_2, e g_3 represent nonlinear dissipative terms on the boundary satisfying

H_g**:** The functions $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3$ satisfy the following hypotheses:

(i): g_i is continuous, monotonous increasing.

(ii): $g_i(s)s \geq 0$ for $s \neq 0$

(iii): There are constant m and M constants such that $0 < m < M$ and $ms^2 \leq g_i(s)s \leq Ms^2$, $|s| > 1$.

The energy of the system (1)-(7) is given by

$$E(t) = \frac{1}{2}\int_0^L \left(\rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + \rho_1|w_t|^2 + b|\psi_x|^2 + k|\varphi_x + \psi + lw|^2 + k_0|w_x - l\varphi|^2 + \frac{\gamma}{m}|\theta|^2\right) dx$$

The phase space H is given by $H = V_0 \times V_0 \times V_0 \times L^2(0,L) \times L^2(0,L) \times L^2(0,L) \times L^2(0,L)$.

2 Main Results

To show the existence of a solution to the system (1)-(7), we use the theory found in [1] and [4].

Our main result is that for the system(1)-(7), using the hypotheses for$g$ mentioned above and some more functions also defined by Lasiecka e Tataru in [3] and with the multipliers technique. we were able to prove the following theorem.

Theorem 2.1. *Let g_i be the functions satisfying the hypotheses $(i), (ii)$ and (iii) of H_g, if $U = (\varphi, \psi, w, \varphi_t, \psi_t, w_t, \theta)$ is solution of system (1)-(7), then for some $T_0 > 0$*

$$E(t) \leq S\left(\frac{t}{T_0} - 1\right), \quad \forall t > T_0.$$

For the proof we follow the same reasoning used in [5] by Salinas.

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GLOBAL EXISTENCE OF SOLUTIONS TO A NONLOCAL $(P_1(X), P_2(X))$-LAPLACE EQUATION WITH CONVECTION TERM AND NONLINEAR BOUNDARY CONDITION

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Abstract

The object of this work is to study the existence of weak solutions for $(p_1(x), p_2(x))$-Laplacian parabolic Kirchhoff equation . We establish our results by using the degree theory for operators of the form $T + S + C$, where T is a maximal monotone, S is bounded pseudomonotone and C is is compact con $D(T) \subseteq D(C)$ and satisfies a sublinearity condition, in the framework of variable exponent Sobolev spaces.

1 Introduction

In this research, we focus on the following nonlocal parabolic problem

$$u_t - M_1(L_1(u))(\mathrm{div}(|\nabla u|^{p_1(x)-2}\nabla u) - |u|^{p_1(x)-2}u) - M_2(L_2(u))(\mathrm{div}(|\nabla u|^{p_2(x)-2}\nabla u) - |u|^{p_2(x)-2}u) + f(x,t,u,\nabla u) = h(x,t) \quad \text{in } \Omega \times (0,T) \tag{1}$$

$$\Big(M_1(L_1(u))|\nabla u|^{p_1(x)-2} + M_2(L_2(u))|\nabla u|^{p_2(x)-2}\Big)\frac{\partial u}{\partial \nu} + g(x,u) = 0 \qquad \text{on } \partial\Omega \times (0,T),$$
$$u(x,0) = u(x,T), \qquad x \in \Omega. \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $p_i(x) \in C(\overline{\Omega})$ with $p_i(x) > 1$ for any $x \in \overline{\Omega}$ i=1,2 , $L_i(u) = \int_\Omega \frac{1}{p_i(x)}(|\nabla u|^{p_i(x)} + |u|^{p_i(x)})\,dx$, and M_i, f are functions that satisfy conditions which will be stated later.
In literature, considerable amount of work has been done by many researches for elliptic $p(x)$-Kirchhoff-Laplacian equations, see [2-4]. As far as the parabolic type $p(x)$-Laplacian equations are concerned, few articles have appeared, we refer the reader to [5-6]. In this work, unlike the previous ones we use the theory of degree for operators of type $T + S + C$, where T is a maximal monotone, S is bounded pseudomonotone and C is is compact con $D(T) \subseteq D(C)$ and satisfies a sublinearity condition.

2 Assumption and Main Result

Throughout this paper $W^{1,p(x)}(\Omega)$ ($p \in C(\overline{\Omega})$, $2 \le p(x) < +\infty$) is the well known variable exponent Sobolev space.
We give the following hypotheses.

(A_1) $M_i : [0, +\infty[\to [m_0, +\infty[$ is a continuous and nondecreasing function, i=1,2; $m_0 > 0$.

(A_2) $f : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a Caratheodory function such that
$|f(x,t,\eta,\xi)| \le c_1 k(x,t), \quad k \in L^\infty(0,T; L^{p'(x)}(\Omega))$;
$f(x,t,\eta,\xi)\eta \ge |\eta|^{p(x)}$, for all $(x,t) \in \Omega \times (0,T), \eta \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$

(A_3) $h \in \mathcal{H} = \big(L^{p(x)}(0,T; W^{1,p(x)}(\Omega))\big)'$; $|g(x,u)| \le c_2|u|^{r(x)-1}$ for a.e. $x \in \Gamma$ and all $u \in \mathbb{R}_+, r^+ < p^-$.

3 Main Result

We are ready to state and prove the main result of the present paper.

Theorem 3.1. *Suppose* (A_1) - (A_3) *hold. Then (1) admits at least one weak solution.*

Proof We transform (1) into an equivalent problem of type $Tu + Su + Cu = h$. Then, we apply a result in [1]. □.

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ON THE VARIATIONAL INEQUALITY FOR A BEAM NON LINEAR EQUATION

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Abstract

In this paper we investigate the existence and uniqueness of global solutions for the variational inequality for the beam non linear equation

$$u_{tt} - \Delta u + \Delta^2 u + |u|^\rho = 0 \text{ in } \Omega \times (0,\infty)$$
$$u = 0, \ \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma \times (0,\infty) u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), x \in \Omega,$$

where Ω is a bounded domain of $\mathbb{R}^n$, $\rho > 0$ is a real number, $\nu(x)$ is the exterior unit normal vector at $x \in \Gamma$. Our result is obtained using the Galerkin method with a special basis, the Tartar argument and the compactness approach. Uniqueness is also studied.

In Medeiros et al [9] was investigated the existence and uniqueness of global solutions for the problem

$$u_{tt} - \Delta u + |u|^\rho = f \text{ in } \Omega \times (0,\infty),$$
$$u = 0 \text{ on } \Gamma \times (0,\infty), u(x,0) = u_0(x), \ \ u'(x,0) = u_1(x), x \in \Omega.$$

There, Galerkin method an Tartar argument [8] were applied.
In Milla et al. [1] the authors investigate the existence and uniqueness of global solutions of the initial value problem for the nonlinear mixed problem

$$\begin{aligned} &u_{tt} - \Delta u + \Delta^2 u + |u|^\rho = 0 \text{ in } \Omega \times (0,\infty), \\ &u = 0, \ \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma \times (0,\infty), u(x,0) = u_0(x), \ \ u'(x,0) = u_1(x), x \in \Omega. \end{aligned} \tag{1}$$

By applying the Galerkin method with a special basis, a modification of the Tartar approach [8] and compactness method, they get their result. A nonlinear perturbation of problem $((1))_1$ is given by $u_{tt} - \Delta u + \Delta^2 u + |u|^\rho \geq 0$. This inequality is satisfied in a certain sense. In the present work we investigated the existence of global solutions for the unilateral problem associated with this perturbation.

Unilateral problem is very interesting too, because in general, dynamic contact problems are characterized by nonlinear hyperbolic variational inequalities. For contact problem on elasticity and finite element method see Kikuchi-Oden [2] and reference there in. For contact problems on viscoelastic materials see [3]. For contact problems on Klein-Gordon operator see [4]. For contact problems on Oldroyd Model of Viscoelastic fluids see [5]. For contact problems on Navier-stokes Operator with variable viscosity see [6]. For contact problems on viscoelastic plate equation see [7]. We formulate the unilateral problem as follows. Let $K = \{v \in H_0^1(\Omega); v \geq 0 \text{ a.e. in } \Omega\}$ a closed and convex subset of $H_0^1(\Omega)$, the variational problem is to find a solution $u(x,t)$ satisfying

$$\int_Q (u_{tt} - \Delta u + \Delta^2 u + |u|^\rho)(v - u_t) \geq 0, \forall v \in K, \tag{2}$$

with $u_t(x,t) \in K$ a. e. on $[0,T]$ and talking the initial and boundary data

$$u = 0, \ \ u_t = 0 \text{ on } \Sigma, u = \Delta u = 0 \text{ on } \Sigma, u(.,0) = u_0, \ \ u_t(.,0) = u_1 \text{ in } \Omega.$$

Next we shall state the main results of this paper.

Theorem 0.2. *Assume that* $\frac{1}{2}|u_1|^2 + \frac{1}{2}\|u_0\|^2 + \frac{1}{2}\|u_0\|^2_{H_0^2(\Omega)} + \frac{1}{\rho+1}\int_\Omega |u_0|^{\rho+1}u_0 dx < N < \frac{1}{4}(\lambda^*)^2$ *(note that* $N < \frac{1}{4}(\lambda^*)^2$ *implies* $(4N)^{1/2} < \lambda^*$*),* $\|u_0\|_{H_0^2(\Omega)} < \lambda^* = \frac{\rho+1}{4k_0}$*, consider the space* $H_\Gamma^4(\Omega) = \{u \in H^4(\Omega)|u = \Delta u = 0 \text{ on } \Sigma\}$ *and similarly* $H_\Gamma^3(\Omega) = \{u \in H^3(\Omega)|u = \Delta u = 0 \text{ on } \Sigma\}$, $u_0 \in H_\Gamma^4(\Omega)$ *and* $u_1 \in H_0^2(\Omega) \cap L^2(\Omega)$*. Then there exists a function* $u : [0,T] \to L^2(\Omega)$ *in the class*
$u \in L^\infty(0,T; H_0^1(\Omega) \cap H_0^2(\Omega) \cap H_\Gamma^3(\Omega)), u_t \in L^\infty(0,T; L^2(\Omega) \cap H_0^1(\Omega) \cap H^2(\Omega)), u_{tt} \in L^\infty(0,T; L^2(\Omega))$
$u_t(t) \in K$ *a.e. in* $[0,T]$*, satisfying*

$$\int_Q (u_{tt} - \Delta u + \Delta^2 u + |u|^\rho)(v - u_t) \geq 0, \forall v \in L^\infty(0,T; H_0^1(\Omega)),$$

$v(t) \in K$ *a.e. in* $t, u(0) = u_0, \ \ u_t(0) = u_1$

The proof of Theorem 2.1 is made by the penalization method. It consists in considering a perturbation of the problem (1) adding a singular term called penalty, depending on a parameter $\epsilon > 0$. We solve the mixed problem in Q for the penalization operator and the estimates obtained for the local solution of the penalized equation, allow to pass to limits, when ϵ goes to zero, in order to obtain a function u which is the solution of our problem.
The Penalized Problem associated with the variational inequality (2), consists in given $0 < \epsilon < 1$, find u^ϵ satisfying

$$u^\epsilon_{tt} - \Delta u^\epsilon + \Delta^2 u^\epsilon + \frac{1}{\epsilon}(\beta(u^\epsilon_t)) = |u^\epsilon|^\rho \text{ in } Q$$
$$u^\epsilon = 0 \text{ on } \Sigma, \ u^\epsilon_t = 0 \text{ on } \Sigma, \ u^\epsilon(x,0) = u^\epsilon_0(x), \ u^\epsilon_t(x,0) = u^\epsilon_1(x) \text{ in } \Omega.$$

In order to prove Theorem 2.1, we first prove the Penalized Problem. The existence of solutions will be given by using Faedo-Galerkin approximations with a special basis, the Tartar argument and the compactness approach. Uniqueness is studied via the energy method.

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SEMILINEAR EFFECTIVELY DAMPED WAVE MODELS WITH GENERAL RELAXATION FUNCTION

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Abstract

In this talk, we study the global (in time) existence of small data solutions to the Cauchy problem for semilinear effective damped wave models with general relaxation function in the source term. Our goal is to generalize some known results for special nonlinear memory terms, where the convolution is given with respect to the time variable. We first present auxiliary estimates for integrals. Then we prove results on global (in time) existence of small data Sobolev solutions for different classes of data.

1 Introduction

Recently in [1] and [2] the authors studied the following Cauchy problem for the classical damped wave equation with nonlinear memory:

$$\begin{cases} u_{tt} - \Delta u + (1+t)^r u_t = g * |u|^p, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases}$$

where $r \in (-1,1)$ and the relaxation function $g = g(t) = t^{-\gamma}$ with $\gamma \in (0,1)$. The convolution with respect to the time variable is defined in the nonlinear term as follows:

$$(g * |u|^p)(t,x) := \int_0^t g(t-\tau)|u(\tau,x)|^p d\tau.$$

The goal of this work is to generalize results proved previously in [2] in such a way that we can treat the set of effective dissipation terms. Namely, we treat the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + b(t) u_t = g * |u|^p, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where the dissipation term $b(t)u_t$ is effective in the sense of [3].

Example 1.1. *Typical examples are*

$$b(t) = \frac{\mu}{(1+t)^r}, \qquad b(t) = \frac{\mu}{(1+t)^r}(\log(e+t))^\gamma, \qquad b(t) = \frac{\mu}{(1+t)^r(\log(e+t))^\gamma}$$

for some $\mu > 0$, $\gamma > 0$ *and* $r \in (-1,1)$.
Further examples are

$$b(t) = \mu(1+t)(\log(e+t))^{1-\gamma}$$

with $\mu > 0$ *and* $\gamma \geq 0$.

On the other hand the relaxation function $g = g(t)$ is supposed to satisfy the following properties:

(P1) $g = g(t)$ is defined on $(0, \infty)$ and positive there,

(P2) $g = g(t)$ is continuous and strictly decreasing on $(0, \infty)$ with $\lim_{t\to\infty} g(t) = 0$,

(P3) $g \in L^1\big((0, T)\big)$ for all $T > 0$.

Let us introduce the function $G = G(t)$ by

$$G : t \in (0, \infty) \to G(t) = \int_0^t g(s)\, ds.$$

From $(P1)$ to $(P3)$ one can get immediately the following properties:

$$G(0) = 0,\ \ G(t) > 0 \ \text{ on } \ (0, \infty),\ \ G \ \text{ is increasing on } \ (0, \infty).$$

Example 1.2. *Typical examples are*

- $g(t) = e^{-t}$,
- $g(t) = (1 + t)^{-\gamma}$ *for* $\gamma > 0$,
- $g(t) = t^{-\gamma}$ *for* $\gamma \in (0, 1)$,
- $g(t) = (1 + t)^{-1} \log(e + t)$,
- $g(t) = \big((1 + t) \log(e + t)\big)^{-1}$.

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REMARKS ON THIN QUASILINEAR PLATES WITH MIXED BOUNDARY CONDITIONS

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Abstract

This paper deals with initial-boundary value problems for a damped thin quasilinear plate. With restriction on the norms of the initial data it will be established global weak and global strong solutions. It will also be shown that the strong solution is uniformly stable and unique. Furthermore, using a weak internal damping mechanism, an exponential decay estimate for the energy of weak solutions is established.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and connected set situated locally on the one side of its smooth boundary Γ. $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. $x = (x_1, x_2) \in \mathbb{R}^2$, $\nu = (\nu_1, \nu_2)$ is the exterior unit normal vector at each point $x \in \Gamma$, $\tau = (-\nu_2, \nu_1)$ is the unit tangential vector defined at each point $x \in \Gamma$ oriented in the positive direction of Γ. The normal and tangential derivative are denoted by ∂_ν and ∂_τ, respectively.

The mathematical deduction of vertical deflections phenomena, such as strings or plates with non-homogeneous material, establishes models with variable coefficients, and so a thin plate problem could be of the type,

$$\begin{aligned}
&\partial_t^2 u + \Delta^2 u - M\left(\cdot, \cdot, |u|^2\right) \Delta u = 0 \ \text{ in } \ \Omega \times (0,\infty),\\
&u = \partial_\nu u = 0 \ \text{ on } \ \Gamma_0 \times (0,\infty),\\
&\Delta u + (1-\mu) B_1 u = 0 \ \text{ on } \ \Gamma_1 \times (0,\infty),\\
&\partial_\nu \Delta u + (1-\mu)\partial_\tau B_2 u - M\left(\cdot, \cdot, |u|^2\right) \partial_\nu u = 0 \ \text{ on } \ \Gamma_1 \times (0,\infty),\\
&u(x,0) = u_0(x), \ \partial_t u(x,0) = u_1(x) \ \text{ in } \ \Omega,
\end{aligned} \tag{1}$$

where $u = u(x,t)$ is the displacement of the point x of the plate at time t, $|\cdot|$ is the L^2 norm in the variable x, $B_1 u = 2\nu_1\nu_2 \partial^2_{x_1x_2} u - \nu_1^2 \partial^2_{x_2} u - \nu_2^2 \partial^2_{x_1} u$, $\quad B_2 u = \nu_1\nu_2\left[\partial^2_{x_2} u - \partial^2_{x_1} u\right] + \left(\nu_1^2 - \nu_2^2\right)\partial^2_{x_1x_2} u$ and $0 < \mu < 1/2$ is the Poisson constant.

The two Hilbert spaces $V = \left\{v \in H^2(\Omega);\, v = \partial_\nu v = 0 \text{ on } \Gamma_0\right\}$ and $W = \left\{v \in V;\, \Delta^2 v \in L^2(\Omega)\right\}$ have an important role in this work to obtain the main results, which are:

- Problem (1) is ill-posed in the J. Hadamard sense. That is, only the existence of weak solution will be established. Moreover, with addition of the weak internal damping, $\partial_t u$, in equation $(1)_1$ we show that the total energy has an exponential decay rate.
- With addition of the strong internal damping, $\Delta\partial_t u$, in equation $(1)_1$ we show that such problem is well-posed in the J. Hadamard sense. This means: there exists a unique global strong solution and this solution is uniformly stable, that is, small perturbations in the initial data produce small variations in the solutions.

2 Main Results

Theorem 2.1. *With restriction on the norms of the initial data and some assumptions on the function M it will be established global weak and global strong solutions for problem (1). Thus,*

1. *If $u_0 \in V$, $u_1 \in L^2(\Omega)$ there exists at least one function u, that is, a weak solution of problem (1) such that*

$$u \in L^\infty(0,\infty;V) \quad \text{and} \quad \partial_t u \in L^\infty(0,\infty;L^2(\Omega)),$$
$$-\int_0^\infty (u'(t),\varphi)\theta'(t)dt + \int_0^\infty ((u(t),\varphi))\theta(t)dt + \int_0^\infty \left(M(t,|u(t)|^2)\nabla u(t),\nabla\varphi\right)\theta(t)dt +$$
$$\int_0^\infty \left(\nabla M(t,|u(t)|^2)\nabla u(t),\varphi\right)\theta(t)dt = 0 \ \text{for all} \ \varphi \in V \ \text{and} \ \theta \in \mathcal{D}(0,\infty),$$
$$u(x,0) = u_0(x) \ \text{and} \ \partial_t u(x,0) = u_1(x) \ \text{in} \ \Omega.$$

2. *With addition of the weak internal damping, $\partial_t u$, in equation $(1)_1$ we show that there are real constants κ_1 and κ_2 s.t. the energy, $E(t) = \frac{1}{2}\left\{|u'(t)|^2 + ||u(t)||^2 + \int_\Omega M\left(\cdot,t,|u(t)|^2\right)|\nabla u(t)|_{\mathbb{R}}^2\,dx\right\}$, defined by the global weak solution satisfies*

$$E(t) \le 2\kappa_1 E(0)\exp\left\{-\frac{\zeta}{\kappa_1}t\right\}.$$

3. *With addition of the strong internal damping, $\Delta\partial_t u$, in equation $(1)_1$ and if $u_0 \in W$ and $u_1 \in V$ with*

$$\gamma_0 u_0 = 0 \ \text{on} \ \Gamma_1 \ \text{and} \ \gamma_1 u_0 - M\left(\cdot,0,|u_0|^2\right)\partial_\nu u_0 - \zeta\partial_\nu u_1 = 0 \ \text{on} \ \Gamma_1,$$

then there is a global strong solution, u, of this problem with strong damped in the class

$$u \in L^\infty(0,\infty;V) \cap L^2_{loc}(0,\infty;W), \ \ \partial_t u \in L^\infty\left(0,\infty;L^2(\Omega)\right) \cap L^\infty_{loc}(0,\infty;V), \ \ \partial_t^2 u \in L^\infty_{loc}\left(0,\infty;L^2(\Omega)\right)$$

and such that

$$\begin{aligned}
&\partial_t^2 u + \Delta^2 u - M\left(\cdot,\cdot,|u|^2\right)\Delta u - \zeta\Delta\partial_t u = 0 \ \text{in} \ L^2_{loc}(0,\infty;L^2(\Omega),\\
&\gamma_0 u = 0 \ \text{in} \ L^2_{loc}(0,\infty;H^{-1/2}(\Gamma_1)),\\
&\gamma_1 u - M\left(\cdot,\cdot,|u|^2\right)\partial_\nu u - \zeta\partial^2_{\nu t} u = 0 \ \text{in} \ L^2_{loc}(0,\infty;H^{-3/2}(\Gamma_1)),\\
&u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x) \ \text{in} \ \Omega.
\end{aligned}$$

4. *Furthermore, the global strong solution is uniformly stable and, consequently, unique.*

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ABOUT THE CAUCHY PROBLEM FOR THE SYSTEM WITH THREE SCHRODINGER EQUATIONS WITH QUADRATIC NONLINEARITY

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Abstract

We study the Good Local Placement of the Cauchy Problem described by the coupling of three Schrodinger equations in the continuous case in dimension 1 presented in (1).

This work is inspired by the results obtained in [1] for a coupled system of two Schrodinger equations with quadratic nonlinearity. This work aims to study the Cauchy problem for a coupled system of equations type Schrodinger on the real straight.

To establish the main result of this work, we obtain inequalities in Bougain spaces.

1 Introduction

This work is dedicated to studying the Cauchy problem for a system of equations that arises in nonlinear optics problems. More precisely, let us study the following mathematical model:

$$\begin{cases} iw_t + w_{xx} - w + \bar{w}v + \bar{v}u = 0 \\ 2iv_t + v_{xx} - \beta v + \frac{1}{2}w^2 + \bar{w}u = 0 \\ 3iu_t + u_{xx} - \beta_1 u + \chi vw = 0 \\ w(x,0) = w_0(x) \in H^r(\mathbb{R}),\ v(x,0) = v_0(x) \in H^s(\mathbb{R}) \text{ e } u(x,0) = u_0(x) \in H^k(\mathbb{R}), \end{cases} \tag{1}$$

where w, v, and u are functions that take on complex values and represent the fundamental harmonic, second, and third harmonic, respectively, and β, β_1, and χ are real numbers that represent the physical parameters of the system. For more physical information on this model, we recommend [2].

In [3], results of global local good placement are established for the system (1) in the cases $r = s = k = 0$ or 1, that is, in L^2 and H^1. In addition, stability results are obtained for traveling wave solutions. Finally, our result extends the region of local well-posed obtained in [3] concerning the Sobolev indices.

2 Main Results

Theorem 2.1. *Given $(w_0, v_0, u_0) \in H^r(\mathbb{R}) \times H^s(\mathbb{R}) \times H^k(\mathbb{R})$ with $(r, s, k) \in \mathcal{R} \subset \mathbb{R}^3$. The Cauchy problem (1) is locally well-posed in $H^r(\mathbb{R}) \times H^s(\mathbb{R}) \times H^k(\mathbb{R})$ in the following way: for each $\rho > 0$, exist $T = T(\rho) > 0$ and $b > 1/2$ such that for all initial data with $\|w_0\|_{H^r} + \|v_0\|_{H^s} + \|u_0\|_{H^k} < \rho$, there is only one solution (w, v, u) for (1) satisfying the following conditions:*

$$\psi_T(t)w \in X^{r,b}, \quad \psi_T(t)v \in X_2^{s,b} \quad \text{and} \quad \psi_T(t)w \in X_3^{k,b},$$
$$w \in C([0,T]; H^r), \quad v \in C([0,T]; H^s) \quad \text{and} \quad u \in C\left([0,T]; H^k\right).$$

In addition, the data-solution application is locally Lipschitzian.

Proof The proof of the Theorem above simulates the proof presented in section IV of [1].

The key results for establishing this Theorem and delimiting the $\mathcal{R}$ region of the Main Theorem are:

(i) $\|\overline{w}\cdot v\|_{X^{r,-d}} \leq C\|w\|_{X^{r,b}}\cdot\|v\|_{X_2^{s,b}}$, with (r,s,k) satisfying a condition $\mathcal{R}_1$;

(ii) $\|\overline{v}\cdot u\|_{X^{r,-d}} \leq C\|v\|_{X_2^{s,b}}\cdot\|u\|_{X_3^{k,b}}$ with (r,s,k) satisfying a condition $\mathcal{R}_2$;

(iii) $\|w\cdot\tilde{w}\|_{X_2^{s,-d}} \leq C\|w\|_{X^{r,b}}\cdot\|w\|_{X^{r,b}}$ with (r,s,k) satisfying a condition $\mathcal{R}_3$;

(iv) $\|\overline{w}\cdot u\|_{X_2^{s,-d}} \leq C\|w\|_{X^{r,b}}\cdot\|u\|_{X_3^{k,b}}$ with (r,s,k) satisfying a condition $\mathcal{R}_4$ e

(v) $\|w\cdot v\|_{X_3^{k,-d}} \leq C\|w\|_{X^{r,b}}\cdot\|v\|_{X_2^{s,b}}$ with (r,s,k) satisfying a condition $\mathcal{R}_5$.

The region presented in the theorem is $\mathcal{R}=\mathcal{R}_1\cap\mathcal{R}_2\cap\mathcal{R}_3\cap\mathcal{R}_4\cap\mathcal{R}_5$.

The inequalities (i) and (iii) are obtained in [1].

The Bougain space, $X_j^{a,b}$ considered in this work is the completion of $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm:

$$\|f\|_{X_j^{a,b}} = \left\|\langle\xi\rangle^a\langle j\tau+\phi(\xi)\rangle^b\widehat{f}(\tau,\xi)\right\|_{L^2_\tau L^2_\xi},$$

we consider $X_1^{a,b}=X^{a,b}$.

The other inequalities are under review, therefore, this work is under writing for submission.

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UNIQUE CONTINUATION OF THE CAUCHY PROBLEM OF NONLINEAR INTERACTIONS OF THE SCHRODINGER TYPE

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Abstract

In this work, we will show that if (u, v) are sufficiently smooth solutions to the Cauchy problem associated with a system of Schrodinger equations with nonlinear quadratic interactions such that there are $a,\ b \in \mathbb{R}$ with $\text{supp}u(t_j) \subset (a,\ \infty)$ and $\text{supp}v(t_j) \subset (b,\ \infty)$ for $j = 1$ or 2 ($t_1 \neq t_2$) then $u \equiv v \equiv 0$.

1 Introduction

This work is dedicated to studying the Unique Continuation problem of smooth solutions of the Cauchy problem for a system of equations that arises in the context of nonlinear optics problems. More precisely, let's study the following mathematical model:

$$\begin{cases} i\partial_t u(x,t) + p\partial_x^2 u(x,t) - \theta u(x,t) + \bar{u}(x,t)v(x,t) = 0, \quad x \in \mathbb{R},\ t \geq 0, \\ i\sigma\partial_t v(x,t) + q\partial_x^2 v(x,t) - \alpha v(x,t) + \frac{a}{2}u^2(x,t) = 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{1}$$

where u and v are functions that take on complex values and α, θ and $a := 1/\sigma$ are real numbers that represent physical parameters of the system, where $\sigma > 0$ and $p,\ q\ = \pm 1$. The model (1) is given by a nonlinear coupling of two Schrodinger equations with quadratic terms of the type

$$N_1(u, v) = \overline{u} \cdot v \ \text{ and } N_2(u, v) = \frac{1}{2} u \cdot v. \tag{2}$$

The model above was studied, in the period case, by Angulo and Linares in [2]. And in [1], we established results of local and global well-posed for (1).

This work follows the ideas presented in [4] to establish the main result.

The key result for this work is presented in [3] which we present below:

Theorem 1.1. *Given $w \in L^1([0,T] : L^\infty(\mathbb{R})) \cap L^\infty_{loc}([0,T] \times \mathbb{R})$ and suppose $\|w\|_{L^1_t L^\infty_x(\{|x|>R\})} \longrightarrow 0$ when $R \longrightarrow \infty$.*

Assume that $u \in C([0,T] : H^1(\mathbb{R}))$, $\partial_t u,\ \partial_x^2 u \in L^2_{loc}(\mathbb{R})$ is a solution of the equation

$$i\partial_t u + \partial_x^2 u = wu, \quad (t,x) \in [0,T] \times \mathbb{R}, \tag{3}$$

and there are $c, d \in \mathbb{R}$, with $c \neq 0$ such that **supp** *$u(0)$,* **supp** *$u(T) \subset \{x \in \mathbb{R} :\ xc < d\}$, then $u \equiv 0$.*

2 Main Results

The main result of this work is

Theorem 2.1. *Given $(u, v) \in C([0,T] : H^3 \times H^3) \cap C^1([0,T] : L^2(\mathbb{R}) \times L^2(\mathbb{R}))$ strong solution of (1). Suppose there are $a, b \in \mathbb{R}$ with* **supp** *$u(0)$,* **supp** *$u(T) \subset (a, \infty)$ and* **supp** *$v(0)$,* **supp** *$v(T) \subset (b, \infty)$, then $u \equiv v \equiv 0$. in $[0,T] \times \mathbb{R}$.*

Proof :

The proof of the same follows from the fact of revisiting the Theorem 2.1 and obtaining as a result the equation (3) with $\overline{u}$ in place of u.

With a suitable change of we variables can assume $\theta = \alpha = 0$ in the system (1).

Lemmas 2.1 and 2.2 of [4] allow us to conclude that if $(u, v) \in H^3 \times H^3$ is a solution of (1) then $w = v$ satisfies the hypotheses of Theorem 2.1 and, as a hypothesis, u satisfies the hypotheses of the same theorem, it follows that $u \equiv 0$.

Finally, it follows from the Theorem 2.1 that $\sigma i\partial_t v + \partial_x^2 v = 0, \quad (t, x) \in [0, T] \times \mathbb{R}$ with **supp** $v(0)$, **supp** $v(T) \subset (b, \infty)$ implies that $v \equiv 0$.

Finalizing the result.

Note: The theorem above remains valid for $(u, v) \in H^s(\mathbb{R}) \times H^\kappa(\mathbb{R})$ with $s = \ \kappa \geq 3$ and (u, v) solutions to the Cauchy problem (1). In [1], results of good local placement are presented in regions described for the cases $\sigma < 2$, $\sigma = 2$, and $\sigma > 2$.

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EXISTENCE OF A GLOBAL ATTRACTOR FOR THE HEAT EQUATION WITH DEGENERATE MEMORY

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Abstract

We analyze the long-time behavior of the semigroup $S(t)$ generated by a heat equation with degenerate past history and a nonlinear heat supply posed in a three dimensional bounded domain Ω. Assuming that the degeneracy occurs in a positive measure subset of Ω, we prove the existence and regularity of the global attractor associated to this semigroup.

1 Introduction

We consider the heat equation with past history posed on a bounded domain $\Omega \subset \mathbb{R}^3$,

$$\begin{cases} \theta_t - k_0\Delta\theta - \int_{-\infty}^{t} k(t-s)div[a(x)\nabla\theta]ds + f(\theta) = 0, & (x,t) \in \Omega \times (0,\infty); \\ \theta(x,t) = 0, & \partial\Omega \times \mathbb{R}; \\ \theta(x,s) = \theta_0(x,s), & x \in \Omega \times (-\infty, 0]. \end{cases} \tag{1}$$

Here k represents the memory kernel, $a(x) \geq 0$ is a smooth function such that $a = 0$ in A, $A \subset \Omega$, and $\theta_0 : \Omega \times (-\infty, 0] \to \mathbb{R}$ is the prescribed past history of θ.

According to Dafermos [3] and following Giorgi, Marzocchi and Pata [5], let us define a new variable η corresponding to the relative displacement history,

$$\eta^t(x,s) = \int_0^s \theta^t(x,\tau)d\tau = \int_{t-s}^{t} \theta(x,\tau)d\tau.$$

being

$$\theta^t(x,s) = \theta(x, t-s).$$

Suppose that $k(\infty) = 0$. Then, after performing a change of variables, a formal integration by parts lead us to

$$\int_{-\infty}^{t} k(t-s)div[a(x)\nabla\theta]ds = \int_0^\infty k'(s)div[a(x)\nabla\eta^t(x,s)]ds.$$

Therefore, system (1) is rewritten as

$$\begin{cases} \theta_t - \Delta\theta - \int_0^\infty (-k'(s))div[a(x)\nabla\eta^t(x,s)]ds) + f(\theta) = 0, & (x,t,s) \in \Omega \times (0,\infty) \times (0,\infty); \\ \eta_t^t(x,s) = \theta(x,t) - \partial_s\eta^t(x,s), & \Omega \times (0,\infty) \times (0,\infty); \\ \theta(x,t) = 0, & \partial\Omega \times (0,\infty); \\ \eta^t(x,s) = 0, & \partial\Omega \times (0,\infty) \times (0,\infty); \\ \theta(x,0) = \theta_0(x), & x \in \Omega; \\ \eta^0(x,s) = \eta_0(x,s), & \Omega \times (0,\infty). \end{cases} \tag{2}$$

Under appropriate assumptions we prove that this problem is well-posed and its solution operator $S(t)$ is a nonlinear C_0-semigroup. After that, following some ideas from [1,2] we establish the existence and regularity of the global attractor associated to this semigroup, which is the main result of this work.

2 Main Results

In order to state our results, we first need to introduce some spaces which will be useful in our analysis. Consider

$$H_a^1(\Omega) = \{u \in L^2(\Omega), \sqrt{a}\nabla u \in L^2(\Omega), u|_{\partial\Omega} = 0\}.$$

Let $\mathcal{E}$ be the operator $\mathcal{E} = -div[a\nabla]$, whose domain is such that $D(\mathcal{E}) \subset M = \{u \in H_a^1(\Omega), a\nabla u \in H^1(\Omega)\} \subset H_a^1(\Omega)$. For $r \in \mathbb{R}$, we define the scale of compactly nested Hilbert spaces

$$\mathbb{H}^r = D(\mathcal{E}^{\frac{r}{2}}),$$

with inner products given by $\langle u, v\rangle_r = \langle u, v\rangle + \langle \mathcal{E}^{\frac{r}{2}} u, \mathcal{E}^{\frac{r}{2}} v\rangle$. Particularly, we have $\mathbb{H}^0 = L^2(\Omega)$, $\mathbb{H}^1 = H_a^1(\Omega)$. For $r \in \mathbb{R}$, we introduce the so-called history spaces

$$\mathcal{M}^r = L^2_{(-k')}(\mathbb{R}^+, \mathbb{H}^{r+1})$$

endowed with the weighted L^2- inner products $< \eta, \xi >_{\mathcal{M}^r} = \int_0^\infty (-k'(s)) < \eta, \xi >_{r+1} ds$. Finally, consider

$$\mathcal{H}^r = \mathbb{H}^r \times \mathcal{M}^r \qquad \text{and} \qquad E_r(t) = \frac{1}{2}\|\theta\|_r^2 + \frac{1}{2}\int_0^\infty (-k'(s))\|\sqrt{a}\nabla\eta\|_r^2 ds$$

the $r-$phase space and the r-energy associated with the problem (P), respectively.

Now we are able to state our results.

Proposition 2.1. *There exist $C_0 > 0$ and $\epsilon_0 > 0$ such that the energy associated with problem (P) satisfies*

$$E_r(t) \leq 2E(0)e^{-\epsilon_0 t} + C_0,$$

for all $(\theta, \eta) \in \mathcal{H}^0$.

Proposition 2.2. *For every $t \geq 0$, there exists a closed and bounded set $B(t) \subset \mathcal{H}^{\frac{1}{3}}$ such that $dist_{\mathcal{H}}(S(t)B_0, B(t)) \leq \Lambda e^{-\omega t}$, for constants $\Lambda \geq 0$ and $\omega > 0$.*

Proposition 2.3. *For every $t \geq 0$, there exists a compact set $K(t) \subset \mathcal{H}^0$ such that*

$$dist_{\mathcal{H}}(S(t)B_0), K(t)) \leq \Gamma e^{-\omega t}$$

for some $\Gamma \geq 0$, and $\omega > 0$ from the previous proposition.

Theorem 2.1. *The global attractor A of $S(t)$ is bounded in $\mathcal{H}^0$.*

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NONHOMOGENEOUS CELL-FLUID NAVIER-STOKES MODEL WITH INCLUSION OF CHEMOTAXIS

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Abstract

This work is concerned with the mathematical study of the general cell-fluid Navier-Stokes model with inclusion of chemotaxis proposed by [4] in a particular case. More precisely, we investigate the existence of solutions in a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, when the fluids are supposed to have divergence free velocity vectors and constant volume fraction.

1 Introduction

The model proposed by [4] is a general cell-fluid model that consists of two mass balance equations, two momentum balance equations for the cell and fluid phase and a convection-diffusion-reaction equation for the oxygen. The authors include the chemotaxis stress term in the pressure difference. This general model can represent the Chemotaxis-Navier-Stokes model in a special case, in particular, the authors recover the Chemotaxis-Navier-Stokes equations by considering that the fluids are incompressible among others assumptions.

The purpose of this work is to investigate a particular case of the model proposed by [4] from a mathematical point of view. We consider the non-homogeneous case when the fluids have divergence free velocity vectors and constant volume fraction:

$$\frac{1}{2}(\rho_c)_t + \frac{1}{2}\nabla\cdot(\rho_c u_c) = 0, \tag{1}$$

$$\frac{1}{2}(\rho_\omega)_t + \frac{1}{2}\nabla\cdot(\rho_\omega u_\omega) = 0, \tag{2}$$

$$\frac{1}{2}(\rho_c u_c)_t + \nabla\cdot\left(\frac{1}{2}\rho_c u_c\otimes u_c\right) + \frac{1}{2}\nabla\left(P_c + \Lambda(c)\right) = \zeta(u_\omega - u_c) + \frac{1}{2}\rho_c g + \varepsilon_c\nabla\cdot(\rho_c D u_c), \tag{3}$$

$$\nabla\cdot u_c = 0, \tag{4}$$

$$\frac{1}{2}(\rho_\omega u_\omega)_t + \nabla\cdot\left(\frac{1}{2}\rho_\omega u_\omega\otimes u_\omega\right) + \frac{1}{2}\nabla P_\omega = \zeta(u_c - u_\omega) + \frac{1}{2}\rho_\omega g + \varepsilon_\omega\nabla\cdot(\rho_\omega D u_\omega), \tag{5}$$

$$\nabla\cdot u_\omega = 0, \tag{6}$$

$$c_t + \nabla\cdot(c u_\omega) = \nabla\cdot(D_c\nabla c) - \frac{\kappa}{2}c, \tag{7}$$

where $Du = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$, ρ_c and ρ_ω are the cell and the fluid densities; u_c and u_ω are the velocities of the cell and the fluid; P_c and P_ω represent the cell and the fluid pressures; c is the oxygen concentration; g is the gravity constants; the constants ϵ_c and ϵ_ω are the effective viscosities cell and fluid; ζ is a constant that represents the cell-fluid interaction; the constant κ is the consumption rate; D_c is a constant related to the diffusion coefficient; and $\Lambda(c) = \Lambda_0 - \Lambda_1 c$, where Λ_0 and Λ_1 are constants, is the chemotaxis stress.

We supplement the system (1)-(7) with the following initial and boundary conditions

$$u_c = u_\omega = 0,\ \frac{\partial c}{\partial\eta} = 0,\ \text{ in } \partial\Omega\times(0,\infty), \tag{8}$$

$$\rho_c(x,0) = \rho_{c,0},\ \rho_\omega(x,0) = \rho_{\omega,0},\ u_c(x,0) = u_{c,0},\ u_\omega(x,0) = u_{\omega,0},\ c(x,0) = c_0, \tag{9}$$

in Ω, where Ω is a bounded domain of $\mathbb{R}^n$, $n = 2, 3$, with smooth boundary $\partial\Omega$.

To state the existence of weak solutions, we apply the Semi-Galerkin method and we follow the idea of [1,3], for example. And to estate the existence and uniqueness of strong solution we follow the idea of [2]. In this case, we start by linearizing problem (1)-(7) and prove the existence of strong solution for this linearized problem. Next, we construct a sequence of approximate solutions in a inductively way and prove that this sequence is bounded independently of the index. Finally, we show that the whole sequence converges to a strong solution of (1)-(7) in a sufficiently strong sense.

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A COMPARISON PRINCIPLE FOR P−LAPLACIAN EVOLUTION TYPE EQUATION

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Abstract

In this work we provide a comparison principle for the weak solutions $u(\cdot,t), v(\cdot,t)$ of two similar evolution p-Laplacian equations, both with source terms in a divergent and non-divergent form. The initial conditions $u(\cdot,0)$ and $v(\cdot,0)$ are supposed to belong to the space $L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. Once we treat with signal solutions defined in all space $\mathbb{R}^n$, for all t in a maximal existence interval $[0,T_*)$, the arguments presented here differ from the ones used to prove the comparison principle in bounded domains. We suppose $p\geq n$, $p>2$ and also consider some additional natural assumptions.

1 Introduction

The main objective of this work is to present a comparison principle (proved in [2]), for the solutions $u(\cdot,t)$ and $v(\cdot,t)$, defined in a maximal existence interval $[0,T_*)$ and $[0,T_{**})$ respectively, of two similar initial value problems for evolution p-laplacian equations of the type

$$u_t + \mathrm{div} f(x,t,u) = \mathrm{div}\,(|\nabla u|^{p-2}\,\nabla u) + g(x,t,u) + h(x,t), \tag{1}$$

$$u(\cdot,0) = u_0 \in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n), \tag{2}$$

and

$$v_t + \mathrm{div} f(x,t,v) = \mathrm{div}\,(|\nabla v|^{p-2}\,\nabla v) + G(x,t,v) + H(x,t), \tag{3}$$

$$v(\cdot,0) = v_0 \in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n), \tag{4}$$

where $p>2$ is a constant, $p\geq n$, such that $g(x,t,\mathrm{u})\leq G(x,t,\mathrm{u})$ and $h(x,t)\leq H(x,t)$ $\forall\, x\in\mathbb{R}^n$ $0\leq t\leq T$ and $|u|\in\mathbb{R}$.

We also require that $f=(f_1,f_2,\dots,f_n)$, g, G, h and H are given continous functions that satisfy the following basic hypothesis:

$$|f(x,t,u)|\leq K_f(M,T)\,|u| \;\text{ and }\; |g(x,t,u)|\leq K_g(M,T)\,|u|\;\forall\, x\in\mathbb{R}^n, 0\leq t\leq T, |u|\leq M, \tag{5}$$

$$|G(x,t,v)|\leq K_G(M,T)\,|v| \;\;\forall\; x\in\mathbb{R}^n,\;\; 0\leq t\leq T,\;\; |v|\leq M, \tag{6}$$

with constants $K_f(M,T), K_g(M,T), K_G(M,T)$ depending upon the values of M,T and where $|\cdot|$ denotes the absolute value (in case of scalars) or the Euclidean norm (in case of vectors), and $h(\cdot,t)\in L^1_{loc}([0,\infty),L^1(\mathbb{R}^n))$ and $H(\cdot,t)\in L^1_{loc}([0,\infty),L^1(\mathbb{R}^n))$. We also need this additional assumption about the functions f, g and G :

$$|f(x,t,u)-f(x,t,v)|\leq L_f(M,T)|u-v| \tag{7}$$

$$|g(x,t,u)-g(x,t,v)|\leq L_g(M,T)|u-v| \tag{8}$$

$$|G(x,t,u) - G(x,t,v)| \leq L_G(M,T)|u-v|, \tag{9}$$

for all $x \in \mathbb{R}^n$, $t \in [0,T]$ for any $T > 0$ such that both solutions are defined. Besides that, $|u| \leq M$, $|v| \leq M$, where $M = M(T)$ is a constant that depends on T.

In this work we are considering a weak solution in some time interval $[0, T_*)$, any function $w(\cdot,t) \in S$, satisfying the problem (1) with some initial condition $w(\cdot,0) = w_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $S = L^\infty_{loc}([0,T_*), L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)) \cap C^0([0,T_*), L^1_{loc}(\mathbb{R}^n)) \cap L^p_{loc}((0,T_*), W^{1,p}_{loc}(\mathbb{R}^n))$, for $p > 2$ and $p \geq n$. For the local (in time) existence of solutions, see e.g. [3,4,5,6,7]. For the global (in time) existence of solutions, see e.g. [1,3].

2 Main Results

Our purpose is to show the following comparison theorem for the solutions $u(\cdot,t)$ and $v(\cdot,t)$ of the problems $(1a)-(1b)$ and $(2a)-(2b)$ respectively.

Theorem 2.1. *Let $p \geq n$, $p > 2$ and $u(\cdot,t), v(\cdot,t) \in S$ solutions of (1)-(2) and (3)-(2) respectively, such that $g(x,t,u) \leq G(x,t,u)$ and $h(x,t) \leq H(x,t)$ $\forall$ $x \in \mathbb{R}^n$ $0 \leq t \leq T$ and $|u| \in \mathbb{R}$. Under the assumptions (5) - (9) above, if $u_0 \leq v_0$, then we have*

$$u(\cdot,t) \leq v(\cdot,t) \quad \forall \quad 0 < t \leq T$$

for any $T > 0$ such that both solutions are defined in $[0,T]$.

Proof The idea of the proof is first to demonstrate, under the hyphotesis of the theorem, that

$$\int_{\mathbb{R}^n} (u(x,t) - v(x,t))_+ dx \leq e^{L_G(M,T)t} \int_{\mathbb{R}^n} (u_0(x) - v_0(x))_+ dx$$

and

$$\int_{\mathbb{R}^n} (u(x,t) - v(x,t))_- dx \leq e^{L_G(M,T)t} \int_{\mathbb{R}^n} (u_0(x) - v_0(x))_- dx,$$

$\forall$ $0 \leq t \leq T < T_*$. Then, if $u_0 \leq v_0$, the results follows.

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HIERARCHICAL EXACT CONTROLLABILITY OF HYPERBOLIC EQUATIONS WITH BOUNDARY CONTROLS

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Abstract

This paper deals with the Stackelberg-Nash strategies for boundary control problems for linear and semilinear wave equations. Assuming that we can act on the system through a hierarchy of controls, to each leader we associate a Nash equilibrium (two followers) corresponding to a bi-objective optimal control problem. Then we look for a leader that solves an exact boundary controllability problem.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary of class C^2, let us assume that $T > 0$ and let us inttroduce nonempty open sets $\Gamma_0, \Gamma_1, \Gamma_2 \subset \partial\Omega$. We will set $Q := \Omega \times (0,T)$, $\Sigma := \Gamma \times (0,T)$ and $\Sigma_i := \Sigma_i$ for $i = 0, 1, 2$ and we will denote by $\nu = \nu(x)$ the outwards unit normal to Ω at the point $x \in \partial\Omega$.

Consider the following semilinear system:

$$\begin{cases} y_{tt} - \Delta y + a(x,t)\, y = F(y) & \text{in} \quad Q, \\ y = f\, 1_{\Gamma_0} + v^1\, 1_{\Gamma_1} + v^2\, 1_{\Gamma_2} & \text{on} \quad \Sigma, \\ y(\cdot\,,0) = y^0, \quad y_t(\cdot\,,0) = y^1 & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where $a \in L^\infty(Q)$, $f \in L^2(\Sigma_0)$, $v^i \in L^2(\Sigma_i)$ $(i = 1,\, 2)$, $F : \mathbb{R} \mapsto \mathbb{R}$ is a C^1 function, $(y^0,\, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and 1_A stands for the characteristic function of A.

The main goal of this article is to give a positive answer to a question left open in [1]. More precisely, the objective is to get hierarchical exact controllability results for (1) following a Stackelberg-Nash strategy.

Let us mention that these results extend the results of [5] in two directions: we get exact and not approximate control and, moreover, we work with several followers.

For simplicity, we will assume in the sequel that only three controls are applied (one leader and two followers), but very similar considerations hold for systems with a higher number of controls.

Let us define the following main cost functional

$$J(f) := \frac{1}{2} \iint_{\Sigma_0} |f|^2 \, d\Gamma\, dt$$

and the secondary cost functionals

$$J_i(f;\, v^1,\, v^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d} \times (0,T)} |y - y_{i,d}|^2 \, dx\, dt + \frac{\mu_i}{2} \iint_{\Sigma_i} \left|v^i\right|^2 d\Gamma\, dt, \quad i = 1,\, 2, \tag{2}$$

where the $\mathcal{O}_{i,d} \subset \Omega$ are nonempty open sets, the $y_{i,d} \in L^2(\mathcal{O}_{i,d} \times (0,T))$ are given functions, the α_i and μ_i are positive constants and y is the solution to (1).

1.1 Main results

Let us present the main results of this paper.

Let $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ be given and let us introduce the set

$$\Gamma_+ := \{x \in \Gamma;\ (x - x_0) \cdot \nu(x) > 0\},$$

the function $d : \overline{\Omega} \mapsto \mathbb{R}$ with $d(x) = |x - x_0|^2$ for all $x \in \overline{\Omega}$ and the quantities

$$R_0 := \min\{\sqrt{d(x)};\ x \in \overline{\Omega}\} \quad \text{and} \quad R_1 := \max\{\sqrt{d(x)};\ x \in \overline{\Omega}\}.$$

In the sequel, we will impose the assumptions

$$\Gamma_0 \supset \Gamma_+ \ \text{ and } \ T > 2R_1. \tag{3}$$

Definition 1.1. *Let $f \in L^2(\Sigma_0)$ be given. It will be said that the pair (v^1, v^2) is a Nash quasi-equilibrium for the functionals J_i associated to f if*

$$J_i'(f;\, v^1,\, v^2) \cdot \hat{v}^i = 0 \qquad \forall\, \hat{v}^i \in L^2(\Sigma_i), \quad i = 1,\, 2. \tag{4}$$

The following result holds in the semilinear case:

Theorem 1.1. *Assume that F is C^1 and globally Lipschitz-continuous and the $\mu_i > 0$ $(i = 1,\, 2)$ are large enough, depending on Ω, $\mathcal{O}$, Γ_i, $\mathcal{O}_{i,d}$, T, R_1 and $\|F\|_{W^{1,\infty}}$. Let $(\bar{y}_0, \bar{y}_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ be given and let $\bar{y}$ be an associated trajectory. Then, for any $(y^0,\, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exist a control $f \in L^2(\Sigma_0)$ and an associated Nash quasi-equilibrium $(v^1(f),\, v^2(f))$ such that the corresponding solution to (1) satisfies*

$$y(\cdot\,, T) = \bar{y}(\cdot\,, T), \quad y_t(\cdot\,, T) = \bar{y}_t(\cdot\,, T) \qquad in \qquad \Omega.$$

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DYNAMICAL AND VARIATIONAL PROPERTIES OF NLS-δ'_S EQUATION ON THE STAR GRAPH

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Abstract

Firstly, we are looking for the ground states of NLS-δ'_s equation on the star graph. Secondly, we study spectral/orbital instability of the critical points of an associated action functional.

1 Introduction

Let Γ be a star graph identified with the disjoint union of the intervals $I_j = (0, \infty)$, $j = 1, \ldots, N$, augmented by the central vertex $\mathbf{v} = 0$. In [3] we study the following focusing nonlinear Schrödinger equation on Γ with δ'_s coupling:

$$i\partial_t u(t,x) = -\Delta_\beta u(t,x) - |u(t,x)|^{p-1} u(t,x), \quad (t,x) \in \mathbb{R} \times \Gamma, \tag{1}$$

where $\beta \in \mathbb{R} \setminus \{0\}$, $p > 1$, $u : \mathbb{R} \times \Gamma \to \mathbb{C}^N$, and $(-\Delta_\beta v)(x) = (-v''_j(x))$, with

$$\operatorname{dom}(\Delta_\beta) = \Big\{ v \in H^1(\Gamma) : v'_1(0) = \ldots = v'_N(0), \quad \textstyle\sum_{j=1}^N v_j(0) = \beta v'_1(0) \Big\}.$$

The Schrödinger operator $-\Delta_\beta$ with δ'_s coupling has the precise interpretation as the self-adjoint operator on $L^2(\Gamma)$ uniquely associated with the closed semibounded quadratic form $F_\beta(v) = \|v'\|_2^2 + \dfrac{1}{\beta}\left|\sum\limits_{j=1}^N v_j(0)\right|^2$ on $H^1(\Gamma)$. It belongs to N^2-parametric family of self-adjoint extensions of the minimal symmetric operator $(-\Delta_{\min} v)(x) = \left(-v''_j(x)\right)_{j=1}^N$,

$$\operatorname{dom}(\Delta_{\min}) = \left\{ v \in H^2(\Gamma) : v_1(0) = \ldots = v_N(0) = 0, \quad v'_1(0) = \ldots = v'_N(0) = 0 \right\}.$$

It is a difficult problem to understand which of self-adjoint conditions are physically relevant (self-adjointness is just a necessary physical requirement to ensure conservation of the probability current at the vertex).

Among all matching conditions, the most popular are the Kirchhoff ones: $v_1(0) = \ldots = v_N(0)$, $\sum_{j=1}^N v'_j(0) = 0$. Justifications of the Kirchhoff conditions on different types of metric graphs have been obtained in many physical experiments involving wave propagation in thin waveguides and quantum nanowires. These arguments led to the fact that the Kirchhoff conditions have been assumed the most natural, hence they become the most widely studied.

The systematic study of nonlinear evolution equations on metric graphs dates back to the nineties. The nonlinear PDEs on graphs, mostly the nonlinear Schrödinger equation (NLS), have been studied in the past decade in the context of existence, stability, and propagation of solitary waves (see [6] for the references). Two fields where NLS equation appears as a preferred model are optics of nonlinear Kerr media and dynamics of Bose-Einstein condensates involving application to graph-like structures. The extensive study of existence of ground states (as minimizers of energy functional under fixed mass) for the NLS models on metric graphs was carried out in the presence of the Kitchhoff conditions at the vertices of the graph (see [2] and references therein). The existence and the stability of solitary waves for different types of graphs with the Kirchhoff conditions at the vertices were treated in numerous papers (see [5] for the references).

Rigorous study of the NLS models on graphs in presence of impurities is related to a so-called δ coupling. On Γ it is defined by: $v_1(0) = \ldots = v_N(0)$, $\sum_{j=1}^N v'_j(0) = \alpha v_1(0)$, $\alpha \in \mathbb{R} \setminus \{0\}$. The δ coupling is the most studied non-Kirchhoff condition.

If we drop the continuity condition $v_1(0) = \ldots = v_N(0)$, the next more general class of self-adjoint conditions which seems natural for applications consists of those which are permutation invariant at each vertex. The δ'_s

coupling belongs to this class. It is a natural generalization of the δ' coupling on the line. Its systematic investigation appears in [1]. The authors prove the existence of the minimizer of the action functional S_ω on the Nehari manifold for attractive coupling. It appears that the δ' coupling gives rise to a rich structure of a family of ground states, including a pitchfork bifurcation with symmetry breaking. Mathematically, the main advantage of studying δ' and δ'_s coupling (on $\mathbb{R}$ and Γ respectively) is the existence of an explicit nontrivial family of soliton profiles.

2 Main Results

In [3] we extend the results from [1] for the NLS model on Γ. Firstly, we deal with the existence of the ground states as minimizers of the action functional S_ω restricted to the Nehari manifold. We prove that for an attractive and a sufficiently weak δ'_s coupling the minimizer exists. The principal step in the proof is to compare our minimization problem with the one for $\beta = \infty$. This problem involves the technique of symmetric rearrangements on Γ.

Secondly, we are looking for the candidates to be the minimizers, i.e. critical points to S_ω of the form $e^{i\theta}\phi(x)$, where $\phi(x)$ is a real-valued profile. It appears that for $\omega > \frac{p+1}{p-1}\frac{N^2}{\beta^2}$ the whole family of such critical points consists of N profiles (one is symmetric ϕ_β and $N-1$ are asymmetric profiles ϕ_k). We show that for $\omega > \frac{p+1}{p-1}\frac{N^2}{\beta^2}$ the minimizer is given by the asymmetric tail-profile ϕ_1, moreover we conjecture that for ω below $\frac{p+1}{p-1}\frac{N^2}{\beta^2}$ the symmetric profile ϕ_β is the minimizer.

Lastly, we study instability properties of the family of the critical points mentioned above. Namely, using the Grillakis/Jones Instability Theorem (see [4]), we have proved spectral instability of the asymmetric critical points ϕ_k for $\beta < 0, k \geq 2$ and $\beta > 0, N-k \geq 4$. For $p > 2$, using C^2 regularity of the mapping data-solution, we have shown orbital instability of ϕ_k.

We also concertize the instability results by proving strong instability (by blow up) of the symmetric profile ϕ_β in the supercritical case $p > 5$. The proof essentially uses variational characterization of ϕ_β.

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CONTROLLABILITY OF SEMILINEAR SYSTEMS FOR BEAMS

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Abstract

We consider the dynamical one-dimensional semilinear systems for beams and we analyze how its controllability properties depend on the modulus of elasticity in shear k. This joint work with Fágner Araruna (UFPB) and Diego Souza (Universidad de Sevilla).

1 Introduction

The Mindlin-Timoshenko system for thin beams and plates is a widely used model in applications because it takes into account the vertical displacement and the transverse shear effects. The one-dimensional version reads as:

$$\begin{cases} \dfrac{\rho h^3}{12} u_{tt} - u_{xx} + k(u + v_x) = 0 & \text{in} \quad Q, \\ \rho h v_{tt} - k(u + v_x)_x + f(v) = 0 & \text{in} \quad Q. \end{cases} \tag{1}$$

Here, $Q := (0, L) \times (0, T)$, being $L > 0$ the length of the beam and $T > 0$ a given time. The angle of rotation and the vertical displacement at time t of the cross section located at x units from the origin are represented by $u = u(x, t)$ and $v = v(x, t)$, respectively. The function f is a nonlinear term and the constant $h > 0$ represents the thickness of beam which, for this model, is arbitrarily small compared to L. The constant ρ is mass density per volume unit of the beam and the parameter k is the modulus of elasticity in shear. For more details about the Mindlin-Timoshenko parameters and governing equations see [1,2].

Another model used to describe vertical displacement of beams is the so-called Kirchhoff equation. It is modelled when the linear filament of beam remains perpendicular to the deformed middle surface, see [2]. The semilinear Kirchhoff equation reads as:

$$\rho h v_{tt} - \frac{\rho h^3}{12} v_{xxtt} + v_{xxxx} + f(v) = 0 \quad \text{in} \quad Q. \tag{2}$$

2 Main Results

In this paper, we analyze the boundary exact controllability problem for semilinear Mindlin–Timoshenko and Kirchhoff beam models by using only one boundary control.

Firstly, we will analyze the controllability problem for the semilinear Mindlin–Timoshenko:

$$\begin{cases} \dfrac{\rho h^3}{12} u_{tt} - u_{xx} + k(u + v_x) = 0 & \text{in} \quad Q, \\ \rho h v_{tt} - k(u + v_x)_x + f(v) = 0 & \text{in} \quad Q, \\ u(0, \cdot) = 0, \quad u(L, \cdot) = 0 & \text{in} \quad (0, T), \\ v_x(0, \cdot) = \Theta_k, \quad v_x(L, \cdot) = 0 & \text{in} \quad (0, T). \\ u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = u_1 & \text{in} \quad (0, L), \\ v(\cdot, 0) = v_0, \quad v_t(\cdot, 0) = v_1 & \text{in} \quad (0, L). \end{cases} \tag{1}$$

These boundary conditions means that the angle of rotation is kept fixed at both $x = 0$ and $x = L$, a boundary control Θ_k acts on the slope of the vertical displacement at the extreme $x = 0$ and no slope is considered at $x = L$.

Secondly, we will deal with the controllability problem for the semilinear Kirchhoff equation:

$$\begin{cases} \rho h v_{tt} - \dfrac{\rho h^3}{12} v_{xxtt} + v_{xxxx} + f(v) = 0 & \text{in} \quad Q, \\ v_x(0,\cdot) = v_x(L,\cdot) = 0 & \text{in} \quad (0,T), \\ v_{xxx}(0,\cdot) = \Upsilon, \quad v_{xxx}(L,\cdot) = 0 & \text{in} \quad (0,T), \\ v(\cdot,0) = v_0, \quad v_t(\cdot,0) = v_1 & \text{in} \quad (0,L). \end{cases} \tag{2}$$

In this case, the boundary conditions means there are no slopes of the vertical displacement at the extremes $x = 0$ and $x = L$, a boundary control Υ enters this system through the third derivative of the displacement at $x = 0$, which is interpreted as the shearing force on the beam at $x = 0$, and no shearing forces are exerted on the beam at $x = L$. Shearing forces on the beam are unaligned vertical forces pushing one part of the beam in one direction, and another part of the beam in the opposite direction.

On the other hand, throughout this paper, the nonlinearity f satisfies the following conditions:

$$f \in L^\infty_{loc}(\mathbb{R}), \quad f' \in L^\infty(\mathbb{R}), \quad \lim_{|s|\to+\infty} \frac{f(s)}{s} = \alpha, \tag{3}$$

where α is a real number. The last condition above means that f behaves as αs, when $|s| \to +\infty$, and this kind of nonlinearity is known as *asymptotically linear nonlinearity.* It is worth to mentioning that many problems which involve asymptotically linear nonlinearities have physically significance. For example, nonlinearities of the form

$$f(s) = \frac{|s|^2}{1+\gamma|s|^2} s, \quad \gamma > 0,$$

were found to describe the variation of the dielectric constant of gas vapors where a laser beam propagates, those of the form

$$f(s) = \left(1 - \frac{1}{e^{\gamma|s|^2}}\right) s, \quad \gamma > 0,$$

were used in the context of laser beams in plasma.

We have our first main result:

Theorem 2.1. *Let $k \geq 1$ be fixed and $T > 2L \max\{\sqrt{\rho h^3/12}, \sqrt{\rho h/k}\}$. Then, for every data (u_1, u_0, v_1, v_0) and $(\overline{u}_1, \overline{u}_0, \overline{v}_1, \overline{v}_0)$ in $H^{-1}(0,L) \times L^2(0,L) \times [H^1(0,L)]' \times L^2(0,L)$, there exists a boundary control $\Theta_k \in H^{-1}(0,T)$ such that the associated solution (u,v) to (1) satisfies*

$$(u(\cdot,T),\, u_t(\cdot,T),\, v(\cdot,T),\, v_t(\cdot,T)) = (\overline{u}_0, \overline{u}_1, \overline{v}_0, \overline{v}_1) \quad \textit{in} \quad (0,L). \tag{4}$$

Our second main result is the following:

Theorem 2.2. *Let $T > 2L\sqrt{\rho h^3/12}$. Then, for every (v_0, v_1) and $(\overline{v}_0, \overline{v}_1)$ in $H^1(0,L) \times L^2(0,L)$, there exists a boundary control $\Upsilon \in H^{-1}(0,T)$ such that the associated solution v to (2) satisfies*

$$(v(\cdot,T),\, v_t(\cdot,T)) = (\overline{v}_0, \overline{v}_1) \quad \textit{in} \quad (0,L). \tag{5}$$

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PREDATOR-PREY DYNAMICS WITH HUNGER STRUCTURE

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Abstract

We present, analyse and simulate a model for predator-prey interaction with hunger structure. The model consists of a nonlocal transport equation for the predator, coupled to an ODE for the prey. We deduce a system of 3 ODEs for some integral quantities of the transport equation, which generalises some classical Lotka-Volterra systems. By taking an asymptotic regime of fast hunger variation, we find that this system provides new interpretations and derivations of several variations of the classical Lotka–Volterra system, including the Holling-type functional responses. We next establish a well-posedness result for the nonlocal transport equation by means of a fixed-point method. Finally, we show that in the basin of attraction of the nontrivial equilibrium, the asymptotic behaviour of the original coupled PDE-ODE system is completely described by solutions of the ODE system. The present work is published in [1].

1 Introduction

The aim of the present work is to introduce a new model for predator-prey dynamics, which includes effects due to predator hunger. Specifically, in the context of the classical Lotka-Volterra predator-prey dynamics with logistic prey dynamics,

$$\begin{cases} u' = \alpha u w - u \\ w' = \beta w(1 - w - u) \end{cases} \tag{1}$$

(where u is the predator and w is the prey), it is assumed that predator efficiency, represented by the term αuw, depends only on the values of u and w, and a nondimensional constant α. The same remark is true even in the case of models allowing for more realistic functional responses than just αuw.

In nature, however, it is to be expected that the level of hunger of an individual (in whatever way that is defined) will induce it to behave in distinct ways: when hunger is low, the predator feels sated and may allow itself to rest, as is observed for instance in lions; in contrast, when hunger level is high, we may expect that a predator will try to increase its efficiency – for instance, by hunting less desirable prey, or trying harder to capture prey – but only up to a certain point, after which hunger will translate into an actual weakening of the individual and thus a reduction of its fitness. Conversely, a sated individual can be expected to have a greater reproductive potential, and so it is natural to assume that low values of hunger correlate with a greater fitness.

This leads us to propose the following system, which consists of a transport equation for the predator $\rho(t, \lambda)$, whose population is structured by a hunger variable $\lambda \in [0, 1]$, coupled with an ODE for the prey population $w(t)$. See [1] for more details.

$$\begin{cases} \partial_t \rho + \partial_\lambda \big((\alpha_1 - \alpha_2 w \lambda) \rho \big) = \rho \alpha_3 \beta(z) \\ \\ w'(t) = w(1 - w) - w u \gamma(z), \end{cases} \tag{2}$$

$$u(t) = \int_0^{+\infty} \rho(t, \lambda)\, d\lambda, \quad z(t) = \frac{1}{u(t)} \int_0^{+\infty} \lambda \rho(t, \lambda)\, d\lambda,$$

From eq. (2), we can deduce that the integral quantities of the system satisfy the following system,

$$\begin{cases} u' = \alpha_3 u \beta(z) \\ z' = \alpha_1 - \alpha_2 w z \\ w' = w(1-w) - wu\gamma(z) \end{cases} \tag{3}$$

where u is the predator, w the prey, z is the mean hunger, and in this work we explore the relationship of this system to well-known predator-prey models from the literature.

2 Main Results

The definition of solution to the equation (2) is

Definition 2.1. *A solution of problem (2) is a pair ρ, w satisfying for almost every $t > 0$, $\lambda \in \mathbb{R}$,*

$$(1+|\lambda|)\rho(t,\lambda) \in L^1(\mathbb{R}), \qquad w \in C^1(\mathbb{R}_+),$$

$$\begin{cases} \rho(t,\lambda) = \rho_0(\lambda^{\downarrow t}) \exp\Big(\int_0^t \alpha_3 \beta\big(z(s)\big) + \alpha_2 w(s)\, ds \Big), \\ w'(t) = w(1-w) - wu(t)z(t), \\ \rho_0(\lambda) \geq 0, \quad \mathrm{supp}\rho_0 \subset (0,+\infty), \qquad u(0) > 0, \qquad w(0) = w_0 > 0, \end{cases} \tag{1}$$

where $\lambda^{\downarrow t}$ denotes the action of the diffeomorphism generated by the characteristics. With this definition we can prove the following well-posedness result:

Theorem 2.1. *Suppose that $w(0) > 0$, $\rho_0 \in C^1(\mathbb{R})$ such that $\int_{\mathbb{R}} \rho_0 \, d\lambda > 0$, $(1+|\lambda|^2)\rho_0'(\lambda) \in L^1(\mathbb{R})$, and* $\mathrm{supp}\rho_0 \subset (0,+\infty)$. *Let $\beta(z)$ and $\gamma(z)$ verify some growth assumptions. Then, there exists a unique solution of the system (2) in the sense of Definition 2.1.*

The proof, found in [1], follows a fixed point method, and special care must be taken due to the nonlocal character of the equations.

We further prove a result about the asymptotic behavior of the solutions, showing that they concentrate (under some circumstances) on dirac deltas at some explicitly defined values, and provide numerica experiments to illustrate our results.

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ON THE REGULARITY OF A SEMILINEAR HEAT EQUATION WITH DYNAMIC BOUNDARY CONDITIONS

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Abstract

We consider the heat equation with dynamic boundary conditions. Using complex interpolation results by Seeley and Grisvard, we characterize the typical interpolation spaces asssociated to the problem. Then we provide sufficient conditions to define Lipschitz functions on those spaces. We apply our results to improve the regularity of solutions and global attractors that can be found in the literature.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with C^∞ boundary $\Gamma = \partial\Omega$. In this talk, we will be concerned with the following equation:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u - \lambda_1 u + f(u), && (t,x) \in \mathbb{R}_+ \times \Omega,\\
\frac{\partial u}{\partial t} &= \Delta_\Gamma u - \frac{\partial u}{\partial \nu} - \lambda_2 u + g(u), && (t,x) \in \mathbb{R}_+ \times \Gamma,\\
u &= u_0, && t = 0, x \in \overline{\Omega},
\end{aligned} \tag{1}$$

where $\mathbb{R}_+ = [0,\infty)$, $\lambda_1, \lambda_2 \geq 0$ and either λ_1 or λ_2 is strictly larger than 0. The operator Δ is the Laplacian, Δ_Γ is the Laplace-Beltrami operator acting on Γ and ν is the outward unit normal vector field on the boundary. Due to the presence of the Laplace-Beltrami, we say that the boundary condition is of *reactive-diffusive* type.

In our presentation:

- We first recall the typical function spaces that are used in the study of (1). We show how to define an analytic semigroup and how to use it to write the equation as an abstract parabolic equation.
- We provide new interpolation formulas and we show how to define Lipschitz continuous functions on the interpolation spaces.
- We provide new abstract results, based on standard arguments that, under suitable conditions on f and g, can be used to obtain higher order regularity of the solutions and of the global attractor.
- Finally we show how similar problems can be treated similarly. In particular, we compare with the case where we do not have the Laplace-Beltrami term, that is, the boundary condition is of *pure reactive type*. For this case, some small, but important changes are necessary.

2 Main Result

In the presentation, our main aim is to explain the following result. In the presentation, our main aim is to explain the following result.

Theorem 2.1 (Main Theorem)**.** *Let $s \in [-1/p, 1-1/p)$, $p \in (n, \infty)$ and $n \geq 2$. If $f \in C^{3,1}(\mathbb{R})$ satisfies a dissipative condition and $g = f$, then the solutions of (1) belong to $C^{3,\alpha}(\overline{\Omega}) \cap C^{4,\alpha}(\Gamma)$ for each $t > 0$ and $u_0 \in H^1_p(\Omega) \cap B^s_{pp}(\Gamma)$. Moreover, a semigroup on $H^1_p(\Omega) \cap B^s_{pp}(\Gamma)$ can be defined with a global attractor contained in $C^{3,\alpha}(\overline{\Omega}) \cap C^{4,\alpha}(\Gamma)$.*

In the main theorem, $C^{k,\alpha}$ is used to denote a function that is of class C^k, whose the kth-derivative is Hölder continous.

Besides the main theorem, we show how some results found in the literature can be easily obtained, due to our complete characterization of the interpolation spaces.

Finally, we also compare our results with the works of Gal, Meyries, Sprekels and Wu.

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DYNAMIC CONTROL ON THE BOUNDARY FOR A NONLINEAR PROBLEM

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Abstract

In this paper, we are interested in studying theoretical analysis for the stability of a vibrating beam of finite length which is fixed at one end and free at the other end and with a dynamical boundary control. The position u(x, t) of the point x of the beam, at the instant t, is governed by the following wave equation:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + M\left(u(x,t)\right) = 0,\ 0 < x < 1,\ t > 0 \\ u(0,t) = 0,\ u_x(1,t) = -\xi(t),\ t > 0 \\ \xi_t(t) + \xi(t) = u_t(1,t),\ t > 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ 0 < x < 1,\ \xi(0) = \xi_0 \in \mathbb{R} \end{cases}$$

Where $M : \mathbb{R} \to \mathbb{R}$, be such that $s \cdot M(s) \geq 0$, Lipschitziana and derivable except a finite number.
The solution of problem is a pair of functions $\{u, \zeta\}$, where u = u(x, t) depending on time t and spatial variable x and $\xi = \xi(t)$ is dependent only on t. The $\xi(t)$ function denotes the dynamic control.

1 Introduction

In recent years, the study of mathematical models related to flexible structures subject to vibrations have been significantly driven by an increasing number of issues of practical interest.
Among these models, those related to structural engineering stand out, which require active control mechanisms to stabilize inherently unstable structures or have a natural damping.
When a vibrating source (deformation) disturbs the medium, one can control these vibrations by adding several dampers, one option is dynamic damping.
The concept of dynamic control was introduced by automatists in the finite case dimensional (see Francis `\cite{3}`).
In the case of infinite dimension, the concept of dynamics of controls is considered as an indirect damping mechanism proposed by Russell `\cite{5}` and, since then, it attracted the attention of many authors.
In `\cite{6}`, they theoretically studied the stabilization of one-dimensional wave equations with dynamic limit control, and establishing, by a multiplier method, an optimal polynomial which gives an energy decay rate of the type $1/(t+1)$. A numerical analysis for a locally damped equation was made in `\cite{2}`.
Introduce the energy space

$$H^1_E(0,1) = \{\chi \in H(0,1),\ \chi(0) = 0\}$$

We observe

$$H^1_E(0,1) \subset C[0,1]$$

Let us represent by G the function

$$G(s) = \int_0^s M(r)dr$$

The energy of system is defined by

$$E(t) = \frac{1}{2}\left(\int_0^1 (|u_x(x,t)|^2 + |u_t(x,t)|^2 + 2\int_0^1 G(u(t))dx + |\xi(t)|^2\right),$$

2 Main Results

Theorem 2.1. *Assume that* $y_0 \in H_E^1(0,1)\cap H^2(0,1),\ y_1 \in H_E^1(0,1)\cap H^2(0,1),\ G(y_0) \in L^1(0,1)$ *and* $\xi_0 = -y_{0x}(1)$. *Then, for each* $T > 0$, *there exist a unique solution to the problem under the following regularity*

$$y \in L^\infty\left(0,T,H_E^1(0,1)\cap H^2(0,1)\right)$$

$$y_t \in L^\infty\left(0,T,L^2(0,1)\right)\cap L^2\left(0,T,H_E^1(0,1)\right)$$

$$y_{tt} \in L^\infty\left(0,T,L^2(0,1)\right)\cap L^2\left(0,T,H_E^1(0,1)\right)$$

$$\eta,\eta_t \in L^\infty(0,T)\cap H^1(0,1)$$

$$y_{tt} - y_{xx} + M(y) = 0,\ \ a.e.\ in\ (0,1)\times(0,T)$$

$$y(0,t) = 0,\ \ y_x(1,t) = -\eta(t),$$

$$\eta_t(t) + \eta(t) = y_t(1,t),$$

$$y(0) = y_0,\ \ y_t(0) = y_1\ in\ (0,1)$$

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THE WAVE EQUATION EQUATION UNDER EFFECTS OF LOGARITHMIC-LAPLACIAN DISPERSION WITH STRONG DAMPING

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Abstract

We consider the wave equation with a type of logarithmic dispersion under effects of a strong damping. This research is a part of a family of wave equations type that was initiated by Charão-Ikehata [1], Charão-D'Abbicco-Ikehata considered in [2] depending on a parameter $\theta \in (0, 1/2)$ and Piske-Charão-Ikehata [4] for small parameter.

1 Introduction

We consider in this work strongly damped wave equation under effects of a logarithmic dispersion depending on a parameter θ as follows

$$u_{tt} - \Delta u + m^2 L_\theta u - \Delta u_t = 0, \quad t > 0, \quad x \in \mathbf{R}^n \tag{1}$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbf{R}^n, \tag{2}$$

for $\theta \in (0, 1]$ and $m > 0$. In this connection, $\theta = 0$ case is already studied by D'Abbicco-Ikehata [5], so one should restrict the parameter θ to the case $\theta > 0$. In this sense, this study is a kind of generalization of [5] to the general $\theta \in (0, 1)$ through the logarithmic Laplacian type of dispersion term. Asymptotic behavior of the solution and its optimal decay/growth property has been already discussed in [3,4] considering the operator logarithmic-Laplacian included in the equation. Without loss of generalization we consider $u_0 = 0$.
The linear operator $L_\theta : D(L_\theta) \subset L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n), \quad \theta > 0$ is defined as follows

$$D(L_\theta) := \left\{ f \in L^2(\mathbf{R}^n) \;\middle|\; \int_{\mathbf{R}^n} (\log(1 + |\xi|^{2\theta}))^2 |\hat{f}(\xi)|^2 d\xi < +\infty \right\} = Y_\theta^2$$

and $\quad (L_\theta f)(x) := \mathcal{F}^{-1}_{\xi \to x} \left(\log(1 + |\xi|^{2\theta}) \hat{f}(\xi) \right)(x)$, for $f \in D(L_\theta)$.

2 Main Results

The asymptotic profile as $t \to \infty$ of the solution we consider to problem (1.1)-(1.2) is

$$\varphi(t, \xi) = P_1 e^{-\frac{|\xi|^2}{2} t} \frac{\sin\left(t\sqrt{|\xi|^2 + m^2 \log(1 + |\xi|^{2\theta})}\right)}{\sqrt{|\xi|^2 + m^2 \log(1 + |\xi|^{2\theta})}}. \tag{3}$$

The main result of this work is given by the following theorem.

Theorem 2.1. *Let $\|u_1\|_{1,\theta} := \|(1 + |\cdot|^\theta) u_1\|_1$, $n \geq 1$ and $0 < \theta < 1$. Let φ the function defined in (3). Choose $u_0 = 0$, and $u_1 \in L^{1,\theta}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$. Then there exists a positive constant $C = C_{n,\delta_0,\theta}$ such that the mild solution $u \in C([0, \infty); H^1(\mathbf{R}^n)) \cap C^1([0, \infty); L^2(\mathbf{R}^n))$ to problem (1.1)-(1.2) satisfies*

$$\int_{\mathbf{R}^n} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi \leq C_{n,\delta_0,\theta} \Big(m^{-2} P_1^2 t^{-\frac{n+8-6\theta}{2}} + m^{-2} P_1^2 t^{-\frac{n+4-4\theta}{2}} + m^{-2} ||u_1||^2_{1,\theta} t^{-\frac{n}{2}} + P_1^2 e^{-\gamma t}$$
$$+ ||u_1||_1^2 t^2 e^{-\gamma t} + ||u_1||_1^2 t^2 e^{-\alpha t} + ||u_1||_1^2 t^2 e^{-\beta t} \Big), \quad t \geq 0,$$

where $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ are constants and $P_1 = \int_{\mathbb{R}^n} u_1(x) dx$.

As a result of Theorem 2.1, the following optimal L^2-estimates to the solution of problem (1.1)-(1.2) hold.

Theorem 2.2. *Under the same assumptions as in Theorem 2.1 the following statements are true.*

(i) *for* $n = 1$ *with* $0 < \theta < \frac{1}{2}$, $n = 2$ *with* $0 < \theta < 1$ *and* $n \geq 3$ *with* $0 < \theta \leq 1$, *there exist positive constants* C_1, C_2 *depending on* n, θ *and* $m > 0$ *such that*

$$C_1|P_1|t^{-\frac{n-2\theta}{4}} \leq \|u(t,\cdot)\| \leq C_2\Big(\|u_1\|_1 + \|u_1\|_{1,\theta}\Big)t^{-\frac{n-2\theta}{4}}, \quad t \gg 1,$$

(ii) *for* $n = 1$ *and* $\frac{1}{2} < \theta \leq 1$, *there exist positive constants* C_1, C_2 *such that*

$$\frac{C_1}{(1+m^2)^{\frac{1}{4\theta}}}|P_1|t^{\frac{2\theta-1}{2\theta}} \leq \|u(t,\cdot)\| \leq \frac{C_2}{m}\frac{1}{\sqrt{2\theta-1}}\Big(\|u_1\|_1 + \|u_1\|_{1,\theta}\Big)t^{\frac{2\theta-1}{2\theta}}, \quad t \gg 1,$$

(iii) *and for* $n = 1$ *and* $\theta = \frac{1}{2}$ *or* $n = 2$ *and* $\theta = 1$, *there exist positive constants* C_1, C_2 *such that*

$$\frac{C_1}{\sqrt{2+m^2}}|P_1|\sqrt{\log t} \leq \|u(t,\cdot)\| \leq \frac{C_2}{m}\Big(\|u_1\|_1 + \|u_1\|_{1,\theta}\Big)\sqrt{\log t}, \quad t \gg 1$$

Remark 2.1. The result of (i) in Theorem 2.2 implies decay estimates, and this comes from the stronger effect with decay order $t^{-\frac{n}{4}}$ from the Gauss kernel $e^{-t|\xi|^2/2}$ in the low frequency zone, and the osscilation part gives a small effect with growth order $t^{\frac{\theta}{2}}$, while (ii) and (iii) of Theorem 1.2 reflect a stronger effect of the osscilation part

$$\frac{\sin\left(t\sqrt{|\xi|^2 + m^2\log(1+|\xi|^{2\theta})}\right)}{\sqrt{|\xi|^2 + m^2\log(1+|\xi|^{2\theta})}},$$

of the profile, and the diffusion part is no effective. As a result one can get infinite time blow up results as $t \to \infty$ in (ii) and (iii). This reflects a singularity included in the solution itself. This kind of growth property has been newly discovered to the equation (1.1).

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BLOW-UP FOR A β-1D SUPERCRITICAL TRANSPORT EQUATION

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Abstract

In this manuscript we investigate a nonlocal transport β-1D model with supercritical dissipation in which velocity is coupled via a composition of the Hilbert transform and Riesz potentials. We show blow-up in finite time in part of the supercritical range based on suitable weighted estimates.

1 Introduction

In the present text we consider the initial value problem (IVP) for the β-1D transport equation with nonlocal velocity

$$\begin{cases} \partial_t\theta - (\Lambda^{-\beta}\mathcal{H}\theta)\partial_x\theta + \mu\Lambda^{\gamma}\theta = 0 \quad \text{in} \quad \mathbb{T}\times(0,\infty) \\ \theta(0,x) = \theta_0(x) \quad \text{in} \quad \mathbb{T}, \end{cases} \tag{1}$$

where $0 < \beta < 1$, $0 < \gamma < 2$, $\mu > 0$, and $\mathbb{T}$ is the 1D torus. The study of the IVP (1) is divided into three basic cases that reflect the balance between the nonlinearity and dissipation: subcritical $1-\beta < \gamma < 2$, critical $\gamma = 1-\beta$ and supercritical $\gamma < 1-\beta$.

This IVP was initially considered by Bae, Granero-Belinchón e Lazar [1] that proved the global existence of weak solutions in the critical and subcritical cases with nonnegative initial data $\theta_0 \in L^1 \cap L^\infty$. Silvestre and Vicol [4] obtained blow-up of solutions with $\mu = 0$ and $0 < \beta < 1$. In addition, they analyzed some possible Hölder regularization effects and their consequences to and (1) with $\mu > 0$. Li and Rodrigo [3] obtained new pointwise and weighted estimates for the Hilbert transform as well as a number of nonlinear versions and reobtained blow-up of solutions for (1) with $\mu = 0$ and $0 < \beta < 1$.

In this work we focus on supercritical values of γ contained in the range $0 < \gamma < \frac{1-\beta}{2}$ with $0 < \beta < 1$ and we conclude blow-up of solutions in finite time via an approach inspired by [3].

2 Main Results

Here, for simplicity, we consider $\mu = 1$.

Theorem 2.1. *Let $0 < \beta < 1$ and $0 < \gamma < 1-\beta$. Assume that $0 < \gamma < \frac{1-\beta}{2}$ and let the initial data θ_0 be an even Schwartz function. There exists a constant $A(\beta,\gamma) > 0$ depending only on β and γ such that if*

$$\int_0^\infty \frac{\theta_0(0)-\theta_0(x)}{x} e^{-x}dx \geqslant A(\beta,\gamma)\left(\|\theta_0\|_{L^\infty}+1\right), \tag{1}$$

then the corresponding solution θ of (1) blows up in finite time.

Firstly, we recalling two weighted estimates whose proofs can be found in [3]. Let $g:\mathbb{R}\to\mathbb{R}$ be an even Schwartz function. Then

$$-\int_0^\infty \frac{\Lambda^{-\beta}\mathcal{H}g(x)\cdot g'(x)}{x}e^{-x}dx \geqslant C_1(\beta)\int_0^\infty \frac{(g(0)-g(x))^2}{x^{2-\beta}}dx - C_2(\beta)\|g\|_{L^\infty}^2 \quad 0<\beta<1 \tag{2}$$

and

$$\left|\int_0^\infty \frac{\Lambda^\gamma g(0)-\Lambda^\gamma g(x)}{x}e^{-x}dx\right| \leqslant C_3(\gamma)\int_0^\infty \frac{|g(0)-g(x)|}{x^{1+\gamma}}\log\left(10+\frac{1}{x}\right)dx \quad 0<\gamma<1, \tag{3}$$

where $C_1(\beta)$, $C_2(\beta)$ and $C_3(\gamma)$ are positive constants depending only on β or γ.

Proof Applying (3) and using assumption $0<\gamma<\frac{1-\beta}{2}$, we can estimate

$$\begin{aligned}
\left|\int_0^\infty \frac{\Lambda^\gamma \theta(0,t)-\Lambda^\gamma \theta(x,t)}{x}e^{-x}dx\right| &\leqslant C(\gamma)\int_0^\infty \frac{|\theta(0,t)-\theta(x,t)|}{x^{1+\gamma}}\log\left(10+\frac{1}{x}\right)dx \\
&\leqslant C(\gamma)\|\theta_0\|_{L^\infty}+C(\gamma)\left(\int_0^1 \frac{(\theta(t,0)-\theta(t,x))^2}{x^{2-\beta}}e^{-x}dx\right)^{\frac12}\left(\int_0^1 \frac{e^x}{x^{2\gamma+\beta}}\left(\log\left(10+\frac{1}{x}\right)\right)^2dx\right)^{\frac12} \\
&\leqslant C(\gamma)\|\theta_0\|_{L^\infty}+C(\beta,\gamma)\left(\int_0^1 \frac{(\theta(t,0)-\theta(t,x))^2}{x^{2-\beta}}e^{-x}dx\right)^{\frac12},
\end{aligned} \tag{4}$$

where $C(\gamma)>0$ depending only γ and $C(\beta,\gamma)>0$ depending only on β and γ.

Consider $J(t)=\int_0^\infty \frac{\theta(0,t)-\theta(x,t)}{x}e^{-x}dx$. Computing $J'(t)$ and applying (2) and (4), we conclude

$$\begin{aligned}
J'(t) \geqslant &\frac{C_1(\beta)}{2}\int_0^\infty \frac{(\theta(0,t)-\theta(x,t))^2}{x^{2-\beta}}dx - C_2(\beta)\|\theta_0\|_\infty^2 - C(\gamma)\|\theta_0\|_{L^\infty} \\
&+\frac{C_1(\beta)}{2}\int_0^1 \frac{(\theta(0,t)-\theta(x,t))^2}{x^{2-\beta}}e^{-x}dx - C(\beta,\gamma)\left(\int_0^1 \frac{(\theta(t,0)-\theta(t,x))^2}{x^{2-\beta}}e^{-x}dx\right)^{\frac12}.
\end{aligned}$$

By Hölder inequality, we obtain $C(\beta)[J(t)]^2 \leqslant \int_0^\infty \frac{(\theta(0,t)-\theta(x,t))^2}{x^{2-\beta}}dx$ where $C(\beta)>0$ depends only on β. Since that the function $f(z)=\frac{C_1(\beta)}{2}z-C(\beta,\gamma)\sqrt{z}$ reaches the minimum at $z_0=\left(\frac{C(\beta,\gamma)}{C_1(\beta)}\right)^2$ and $f(z_0)=-\frac{C(\beta,\gamma)^2}{2C_1(\beta)}$, we conclude that

$$J'(t)\geqslant C(\beta)[J(t)]^2 - C(\beta,\gamma)(\|\theta_0\|_{L^\infty}+1)^2, \tag{5}$$

where $C(\beta)>0$ depends only on β and $C(\beta,\gamma)>0$ depends only on β and γ.

Now, choosing $A(\beta,\gamma)>\sqrt{\frac{C(\beta,\gamma)}{C(\beta)}}$ and considering $J(0)\geq A(\beta,\gamma)\left(\|\theta_0\|_{L^\infty}+1\right)$ (see (1)), estimate (5) implies $J'(0)>0$ and that J blows up in finite time. Since $J(t)\leqslant \|\theta_x(t,\cdot)\|_{L^\infty}$, we conclude that $\|\theta_x(t,\cdot)\|_{L^\infty}$ must blow up in finite time, as requested. □

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A NOTE ON THE CRITICAL EXPONENT OF A SEMILINEAR EVOLUTION EQUATIONS WITH EFFECTIVE SCALE-INVARIANT TIME-DEPENDENT DISSIPATION

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Abstract

The goal of this note is to find a critical exponent for a class of a semilinear evolution equations with effective scale-invariant time-dependent dissipation

$$u_{tt} + (-\Delta)^\sigma u + \frac{\mu}{1+t} u_t = |u|^p, \;\; u(0,x) = 0, \;\; u_t(x,0) = u_1(x), \quad t \geq 0, \quad x \in \mathbb{R}^n. \tag{1}$$

where $\mu > 1$, $p > 1$ and $\sigma > 1$. A exponent p_c is critical when it is possible to prove global existence (in time) of small data energy solutions to $p > p_c$ and blow up in a finite time for $1 < p \leq p_c$. To achieve this goal we use a fixed point argument in a special operator defined on a suitable function space. So we shall derive optimal $L^p - L^q$ decay estimates, $1 \leq p \leq 2 \leq q \leq \infty$, for the solutions to (1).

The critical exponent p_c for the global (in time) existence of small data solutions to the Cauchy problem (1) is related to the long time behavior of solutions, which changes accordingly with μ. In this presentation, we consider effective dissipation, this means we suppose $\mu > 1$. Under the assumption of small initial data $u_1 \in L^1 \cap L^2$, we find the critical exponent $p_F(\sigma, n) = 1 + \frac{2\sigma}{n}$, which is known as a Fujita type exponent.

1 Introduction

Let us consider the following Cauchy problem for the semilinear σ-evolution equations with scale-invariant time-dependent effective dissipation

$$u_{tt} + (-\Delta)^\sigma u + \frac{\mu}{1+t} u_t = |u|^p, \quad u(0,x) = 0, \quad u_t(0,x) = u_1(x), \quad t \geq 0, \quad x \in \mathbb{R}^n. \tag{2}$$

where $\mu > 1$, $\sigma > 1$ and $p > 1$. We discuss the global (in time) existence of small data energy solutions and blow up results to (2).

It is well known that the size of the parameter μ is relevant to describe the asymptotic behavior of solutions. When $\mu > 1$, this model is related to the semilinear corresponding heat type equation. This can be explain by the diffusion phenomenon.

For $\sigma = 1$ and constant coefficient case, in [6] the authors proved global existence of small data solutions for the semilinear damped wave equation $u_{tt} - \Delta u + u_t = |u|^p$, $u(0,x) = u_0(x)$, $u_t(0,x) = u_1(x)$, in the supercritical range $p > 1+2/n$, by assuming small initial data with compact support from the energy space. The compact support assumption on the initial data can be weakened. By only assuming initial data in Sobolev spaces, the existence result was proved in space dimensions $n = 1$ and $n = 2$ in [4], by using energy methods, and in space dimensions $n \leq 5$ in [5], by using $L^r - L^q$ estimates, $1 \leq r \leq q \leq \infty$. Nonexistence of the global small data solution is proved in [6] for $1 < p < 1+2/n$ and in [8] for $p = 1+2/n$. The exponent $p_F(n) \doteq 1+2/n$ is well known as Fujita exponent and it is the critical index for the semilinear parabolic problem [3]: $v_t - \triangle v = v^p$ $v(0,x) = v_0(x) \geq 0$. The diffusion phenomenon between linear heat and linear classical damped wave models, see [5], explains the parabolic nature of classical damped wave models with power nonlinearities from the point of view of decay estimates of solutions. In [7] the author considered a more general model $u_{tt} - \Delta u + b(t)u_t = 0$, $u(0,x) = u_0(x)$, $u_t(0,x) = u_1(x)$, with a class of time dependent damping $b(t)u_t$ for which the critical exponent is still Fujita exponent $1 + 2/n$ for the associate semilinear Cauchy problem with power nonlinearity $|u|^p$ (see [2] and [1]).

The main goals in this presentation are to derive $L^p - L^q$ estimates and energy estimates for solutions to the linear Cauchy problem associated to ((2)) and to obtain the critical exponent for the global (in time) existence of small initial data energy solutions.

2 Main Results

The following results show us that for $\mu > 1$ the critical exponent for the Cauchy problem (2) is given by a Fujita type exponent $p_F(\sigma, n) = 1 + \frac{2\sigma}{n}$.

Theorem 2.1. *Let $\sigma > 1$, $n < 2\sigma$ and $\mu > \max\left\{\frac{n}{\sigma} + \frac{2n}{n+2\sigma}; 1\right\}$, $\mu \neq \frac{2n}{\sigma}$ and $\mu \neq \frac{n}{\sigma} + 2$. If $1 + \frac{2\sigma}{n} < p \leq \frac{n}{[2n-\sigma\mu]_+} \doteq \frac{q_0}{2}$, then there exists $\epsilon > 0$ such that for any initial data $u_1 \in \mathcal{A} = L^2(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$, $||u_1||_{\mathcal{A}} \leq \epsilon$, there exists a unique energy solution $u \in C([0,\infty), H^\sigma(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)) \cap C^1([0,\infty), L^2(\mathbf{R}^n))$ to (2). Moreover, for $2 \leq q \leq q_0$ the solution satisfies the following estimates*

$$||u(t,\cdot)||_{L^q} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(1-\frac{1}{q}\right)}||u_1||_{\mathcal{A}}, \;\; ||u(t,\cdot)||_{L^\infty} \lesssim (1+t)^{-\min\left\{\frac{n}{2\sigma}, \frac{\mu}{2}\right\}}||u_1||_{\mathcal{A}},$$

$$\|u(t,\cdot)\|_{\dot{H}^\sigma} + \|\partial_t u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\min\left\{\frac{n}{2\sigma}+1, \frac{\mu}{2}\right\}}||u_1||_{\mathcal{A}}, \quad \forall t \geq 0.$$

For the sake of simplicity, in the next result we restrict our analysis for integer σ.

Theorem 2.2. *Let $\sigma \in \mathbb{N}$, $\mu > 1$ and $1 < p \leq 1 + \frac{2\sigma}{n}$. If $u_1 \in L^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} u_1(x)\,dx > 0$, then there exists no global (in time) weak solution $u \in L^p_{loc}([0,\infty) \times \mathbb{R}^n)$ to (2).*

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STABILITY ANALYSIS OF THE FRACTIONAL DIFFUSION EQUATION WITH DIMENSIONAL CORRECTION

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Abstract

This work aims to present a study of the dimensional correction in fractional order diffusion equations by the insertion of a new parameter. Fractional versions of classical integer-order equations are obtained by including a fractional-order derivative or by replacing an integer-order derivative with a fractional one. Such procedures generate an imbalance in the modeling that needs to be corrected.

1 Introduction

In the literature there are several well-established formulations for fractional derivatives, for example those of Riemann-Liouville, Grünwald-Letnikov, Caputo, Riesz, among others [1]. The appropriate choice of the fractional derivative in a given model involves a detailed analysis of the operators and their adequacy to the characteristics of the phenomena studied. However, in order to choose the formulation, it is not enough to introduce a fractional derivative, or substitute an integer derivative for a non-integer one, in the classical equation under examination. The fractional approach also requires considering the necessary adjustments in order to maintain the correct dimension of the equations. [2].

The fractional differential equation (2) is somehow proposed from the classical equation of integer-order diffusion

$$\frac{\partial u(x,t)}{\partial t} = \mathcal{K}\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad 0 < x < L, \ \ 0 < t \leq T, \tag{1}$$

where $\mathcal{K}$ is the thermal diffusivity, and this parameter in the international system (SI) has a dimension equal to $[\mathcal{K}] = m^2/s$.

The formulation of the fractional diffusion equation (2) is done by applying the fractional derivative to the diffusion term, as follows:

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{\tau^{-\alpha}}\frac{\partial^\alpha}{\partial t^\alpha}\left(\mathcal{K}\frac{\partial^2 u(x,t)}{\partial x^2}\right) + f(x,t), \quad 0 < x < L, \ \ 0 < t \leq T, \tag{2}$$

where the unit of the parameter τ is time.

The introduction of the fractional order α in (2) generates a dimensional imbalance, since the integer-order differential operators $\frac{d}{dt}$, $\frac{d^2}{dx^2}$, and the arbitrary $\frac{d^\alpha}{dt^\alpha}$ in the SI have respectively the dimensions $\left[\frac{d}{dt}\right] = s^{-1}$, $\left[\frac{d^2}{dx^2}\right] = m^{-2}$ and $\left[\frac{d^\alpha}{dt^\alpha}\right] = s^{-\alpha}$. Therefore, the dimensional adjustment performed in (2) was performed by introducing $\frac{1}{\tau^{-\alpha}}$ together with the fractional derivative operator, which generates dimension cancellation fraction in the equation.

2 Main Results

A numerical approach by the finite difference method to the problem of initial values and boundary with (2) leads us to the study of the influence of the parameter τ on stability, convergence and uniqueness. Here we will stick to the Fractional Diffusion Equation (FDS) with fractional derivative according to Riemann-Liouville

$$ {}_{RL}D^{\alpha}_{a,t}f(t) := \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-\alpha-1} f(s) ds \, , \tag{3} $$

where m is the positive integer that satisfies $m-1 < \alpha \leq m$.

We apply the backward and centered differences to (2) to approximate the integer derivatives, while the Grünwald-Letnikov operator approximates the Riemann-Liouville operator. We thus obtain a regressive Euler method [4]. The computational mesh in the domain of (4a) is defined by $x_i := i\Delta x, \quad i = 0, 1, \ldots, M, \quad L = M\Delta x$, $t_n := n\Delta t, \quad n = 0, 1, \ldots, N, \quad T = N\Delta t$, being the discrete system of evolution determined with the

$$ \frac{U_i^n - U_i^{n-1}}{\Delta t} = \frac{\mathcal{K}\tau^{\alpha}}{\Delta t^{\alpha}} \sum_{k=0}^{n} \omega(k) \left(\frac{U_{i-1}^{n-k} - 2U_i^{n-k} + U_{i+1}^{n-k}}{\Delta x^2} \right) + f_i^n \quad (n = 0, 1, \ldots, N; \; i = 0, 1, \ldots, M), \tag{4a} $$

$$ U_i^0 = \phi(x_i) \quad (i = 0, 1, \ldots, M), \tag{4b} $$

$$ U_0^n = l(t_n) \quad (n = 0, 1, \ldots, N), \tag{4c} $$

$$ U_M^n = r(t_n) \quad (n = 0, 1, \ldots, N), \tag{4d} $$

where U_i^n is the approximation of $u(x_i, t_n)$, $\omega(k) = (-1)^k \binom{\alpha}{k}$ e $f_i^n = f(x_i, t_n)$.

Theorem 2.1. *The implicit regressive Euler method (4) is unconditionally stable.*

The proof of this theorem was accepted for presentation at the XLI National Congress of Applied and Computational Mathematics (CNMAC 2022). By replacing backward differences with advanced differences in (4a) we have an explicit scheme, the progressive Euler method.

Theorem 2.2. *The explicit progressive Euler-type method is stable if*

$$ \mathcal{K} \frac{\Delta t}{\Delta x^2} \left(\frac{\tau}{\Delta t} \right)^{\alpha} \leq \frac{1}{2^{1+\alpha}}. $$

The proof of this theorem was submitted to the XLIII Ibero-Latin American Congress on Computational Methods in Engineering (CILAMCE 2022).

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HAUSDORFF DIMENSION OF SELF-AFFINE FRACTALS

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Abstract

In this work, we will present the self-affine fractals, introduced in [1], as a generalization of the concept of self-similar fractals, discussed in [2], and then we will state Falconer's theorem, which presents a method to calculate the Hausdorff dimension of self-affine fractals (demonstrated in [1]). Finally, as a main result of this work, we will also show a python code that facilitates the computational implementation of the method described by Falconer and we will apply it to the example of Barnsley's fern.

1 Introduction: self-affine fractals

A **iterated function system** (IFS) consists of a complete metric space (X, d) equipped with a finite collection $\{S_1, \dots, S_k\}$ of contractions $S_j \colon X \to X$, with $j = 1, \dots, k$, and will be denoted by $\{(X, d); S_1, \dots, S_k\}$. In general, **fractals** are defined as fixed points of a IFS. Defining fractals using IFS makes it easier to calculate the dimension, since many fractals are made up of a collection of copies of themselves. Associated with an IFS, we will consider the **Hutchinson operator** $\mathcal{H} \colon \mathcal{K}(X) \to \mathcal{K}(X)$, where $\mathcal{K}(X) = \{A \subset X : A \text{ is compact}\}$, and, for each $K \in \mathcal{K}(X)$, $\mathcal{H}(K) := \bigcup_{j=1}^{k} S_j(K)$. An elegant application of Banach's fixed point theorem shows that there exists a unique non-empty compact set $\mathcal{F} \subset \mathbb{R}^n$ such that $\mathcal{F} = \bigcup_{i=1}^{k} S_i(\mathcal{F})$. The set $\mathcal{F}$ is called an invariant set (**self-similar fractal**) by IFS. The Moran-Hutchinson theorem assures us that if $S_1, \dots, S_k$ are similarities that satisfy the open set condition (see [1]), then the similarity dimension — which is the unique real number s such that $\sum_{i=1}^{k} \lambda_i = 1$ — coincides with the Hausdorff dimension of $\mathcal{F}$, which we will denote by $\dim_H \mathcal{F}$.

In order to generalize this result, instead of considering only similarities, we will consider that the applications S_i are affines, that is, that, for each $x \in \mathbb{R}^n$, $S_i(x) = T_i(x) + a_i$, where $S_i \colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, and a_i is a vector of $\mathbb{R}^n$. In this context, for each $a := (a_1, a_2, \dots, a_k) \in \mathbb{R}^{nk}$, the invariant set $F(a)$ (which is the **self-affine fractal**) is such that $F(a) = \bigcup_{i \in J_\infty} x_i(a)$, where $J_\infty := \{(i_1, i_2, i_3, \dots) : \forall\, l \in \mathbb{N},\ 1 \leq i_\ell \leq k\}$, and, for each $i \in J_\infty$, $x_i(a) := \lim_{r \to \infty} (T_{i_1} + a_{i_1})(T_{i_2} + a_{i_2}) \dots (T_{i_r} + a_{i_r})(0)$.

2 Falconer's theorem: the method for calculating the Haudorff dimension of self-affine fractals

In this section, we will enunciate important definitions and results for understanding Falconer's theorem. In what follows, we will consider linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$, which we will assume to be non-singular. We will denote the singular values of a linear transformation $T \in \mathcal{L}(\mathbb{R}^n)$ by $\alpha_1, \dots, \alpha_k$. Furthermore, we will adopt the convention that $1 > \alpha_1 \geq \dots \geq \alpha_k > 0$ to sort the k singular values of the transformation. For $r \geq 1$, let $J_r := \{(w_1, \dots, w_r) : 1 \leq w_\ell \leq k\}$ be the set of sequences of size r formed by the integers from 1 to k. We denote the set of all finite sequences by J (note that $J := \bigcup_{r=0}^{\infty} J_r$), the member $(w_1, \dots, w_r)$ of J by w and the number of terms in $w \in J$ by $|w|$. Next, we will define the singular value function, which will be important to construct an exterior measure on the set J_∞.

Definition 2.1. *The* ***singular value function*** $\phi^{(\cdot)}(T)\colon [0,+\infty[\to \mathbb{R}_+$ *is defined so that, for every* $s \in [0,+\infty[$,

$$\phi^s(T) = \begin{cases} 1, \ \textit{if } s = 0 \\ \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{m-1} \cdot \alpha_m^{s-m+1}, \ \textit{where } m \in \mathbb{N} \textit{ is such that } m-1 < s \leq m, \ \textit{if } 0 < s \leq n \\ (\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n)^{\frac{s}{n}}, \ \textit{if } s > n \end{cases} . \tag{1}$$

From the singular value function, it is possible to construct an outer measure in J_∞ in a similar way to the construction of the Hausdorff measure. Let $u \in J_r$, $u = (u_1, u_2, \ldots, u_r)$, $r \in \mathbb{N}$. We define the **cylinder** N_u of size u as follows: $N_u := \{j \in J_\infty; u < j\}$ ($u \in J$ is said to be a shortening of $j \in J_\infty$ if there exists a $j' \in J_\infty$ such that $j = u \cdot j'$, in this case, we also say that $u < j$); furthermore, we say that a set of finite sequences A is a cover for J_∞ if $J_\infty \subset \bigcup_{u\in A} N_u$. Fixed s, and given $U \in \mathcal{P}(J_\infty)$ (where $\mathcal{P}(J_\infty)$ is the set of parts of J_∞), for each $r \in \mathbb{Z}$, define $\mathcal{M}_r^s(U) := \inf\left\{\sum_u \phi^s(T_u) : U \subset \bigcup_{u\in A} N_u, |u| \geq r\right\}$, where $T_u := T_{u_1} \circ T_{u_2} \circ \ldots \circ T_{u_r}$. We will use the outer measure $\mathcal{M}_r^s$ to define a dimension $d(T_1, \ldots, T_k)$, called the **Falconer dimensio**n of $\mathcal{F}$.

Proposition 2.1. *The following numbers exist and are all the same.*

1. $\inf\{s : \mathcal{M}_r^s(J_\infty) = 0\} = \sup\{s : \mathcal{M}_r^s(J_\infty) = \infty\}$,
2. *The unique* $s > 0$ *such that* $\lim_{r\to\infty}[\sum_{u\in J_r} \phi^s(T_u)]^{\frac{1}{r}} = 1$,
3. $\inf\{s : \sum_{u\in J} \phi^s(T_u) < \infty\} = \sup\{s : \sum_{u\in J} \phi^s(T_u) = \infty\}$.

We denote this common value by $d(T_1, \ldots, T_k)$.

Theorem 2.1. ***(Falconer).*** *Suppose that* $||T_\ell|| < \frac{1}{2}$ *for any* $1 \leq \ell \leq k$. *Then, for almost all* $a \in \mathbb{R}^{nk}$ *(relative to the* $nk-$*dimensional Lebesgue measure),* $\dim_H F(a) = \min\{n, d(T_1, \ldots, T_k)\}$.

3 Application to the Barnsley's fern example

Barnsley's fern is generated by the applications $S_1, \ldots, S_4 \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that, for each $(x, y) \in \mathbb{R}^2$,

$$S_1(x,y) = \begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ S_2(x,y) = \begin{bmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1,6 \end{bmatrix},$$

$$S_3(x,y) = \begin{bmatrix} 0.2 & -0.26 \\ 0.23 & 0.22 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1,6 \end{bmatrix} \text{ e } S_4(x,y) = \begin{bmatrix} -0,16 & 0,28 \\ 0,26 & 0,24 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0.44 \end{bmatrix}.$$

Inserting the values into the implemented algorithm, we observe that, for values of s between 1 and 2, ranging from 0.1, and for k=5, the Hausdorff dimension of the Barnsley's fern converges very fastly to values between 1.7 and 1.8. Taking the values for s between 1.7 and 1.8, varying from 0.01, we conclude that the Hausdorff dimension is between 1.74 and 1.75. Finally, using values of s between 1.74 and 1.75, ranging from 0.001, we conclude that the Hausdorff dimension is approximately 1.742.

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SPACES OF REGULAR MULTILINEAR OPERATORS BETWEEN BANACH LATTICES

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Abstract

The main purpose of this work is to give a detailed proof of the fact that the space of regular multilinear operators between Banach lattices with Dedekind complete target space is a Banach lattice with the usual operator order and the regular norm. In [1] D. H. Fremlin showed that this result holds for spaces of bilinear operators using the (Fremlin) positive projective tensor product. We prove the general case of multilinear operators using a different approach by induction, in which the use of the positive projective tensor product is not necessary.

1 Introduction

For Riesz spaces $E_1, \ldots, E_n$ and F, an n-linear operator $A \colon E_1 \times \cdots \times E_n \longrightarrow F$ is positive if $A(x_1, \ldots, x_n) \geq 0$ for all $x_1, \ldots, x_n \geq 0$. And A is said to be regular if A is the difference of two positive n-linear operators. The vector space of all such regular n-linear operators is denoted by $L_r(E_1, \ldots, E_n; F)$. When $E_1, \ldots, E_n$ and F are normed Riesz spaces, we denote the space of all continuous regular n-linear operators by $\mathcal{L}_r(E_1, \ldots, E_n; F)$, where the norm of $E_1 \times \cdots \times E_n$ is any of the usual equivalent norms $\| \cdot \|_2, \| \cdot \|_1, \| \cdot \|_\infty$.

Fremlin developed in [2] the theory of tensor products of Riesz spaces to show that $L_r(E_1, E_2; F)$ is a Dedekind complete Riesz space when F is Dedekind complete. Since then this space is referred to as the Fremlin tensor product. In [1], under the assumption that E_1, E_2 and F are Banach lattice, he showed that the space $\mathcal{L}_r(E_1, E_2; F)$ of regular bilinear operators is a Banach lattice with the regular norm, which is defined by

$$\|A\|_r := \| \, |A| \, \| = \sup\{\| \, |A|(x_1, x_2)\| : \|x_i\| \leq 1, i = 1, 2\}$$

for any $A \in \mathcal{L}_r(E_1, E_2; F)$. He did so by considering the completion of the Fremlin tensor product endowed with the positive projective tensor norm.

The general case of multilinear operators is also true and it is usually proved through the Fremlin tensor product in the case of Riesz spaces and the positive projective tensor product in the case of Banach lattices. Another way to show these results is apply induction starting with the linear case, as Loane did in [3] to provee that the space of regular multilinear operators between Riesz spaces is a Riesz space. In this work we follow this induction approach to prove that the space of regular multilinear operators between Banach lattices is a Banach lattice with the regular norm. It is worth mentioning that the corresponding result for the space of homogeneous polynomials can also be proved by induction, for details see [4].

2 Main Results

The starting point of our induction process is the following linear result. For a proof, see [1] or [4].

Theorem 2.1. *If E and F are Banach lattices with F Dedeking complete, then the space $\mathcal{L}_r(E; F)$ of regular linear operators is a Banach lattice with the regular norm.*

To pass from the linear case to the multilinear case by induction, the following canonical isomorphism between linear spaces is essential. Let $E_1, \ldots, E_n$ and F be linear spaces. For each fixed $1 \leq i \leq n$ define

$$\Phi_i \colon L(E_1, \ldots, E_n; F) \longrightarrow L(E_i; L(E_1, \ldots, E_{i-1}, E_{i+1}, \ldots, E_n; F)) \ , \ \Phi_i(A) = L_A^i,$$

where L_A^i is the linear operator given by

$$L_A^i(x_i)(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = A(x_1, \ldots, x_n).$$

Recall that a norm $\|\cdot\|$ on a Riesz space E is a lattice norm if the following holds:

$$x, y \in E \text{ and } |x| \leq |y| \Longrightarrow \|x\| \leq \|y\|.$$

In this case, E is called normed Riesz space. If the norm $\|\cdot\|$ is complete, then we say that E is a Banach lattice. The following classical spaces are Banach lattices with their usual orders: $c_0, \ell_p, L_p(\mu), 1 \leq p \leq \infty$, $C(K)$.

Proposition 2.1. *Let $E_1, \ldots, E_n$ be Banach lattices and let F be a normed Riesz space. Then*

$$L_r(E_1, \ldots, E_n; F) = \mathcal{L}_r(E_1, \ldots, E_n; F),$$

that is, every regular multilinear operator $A \colon E_1 \times \cdots \times E_n \longrightarrow F$ is continuous.

Note that if $E_1, \ldots, E_n$ and F are ordered linear spaces, then the linear space $L(E_1, \ldots, E_n; F)$, equipped with the partial order

$$A \leq B \Longleftrightarrow (B - A) \quad \text{is positive},$$

is an ordered linear space.

Corollary 2.1. *Let $E_1, \ldots, E_n$ be Banach lattices and let F be a Dedekind complete normed Riesz space. Then $\mathcal{L}_r(E_1, \ldots, E_n; F)$ is a Dedekind complete Riesz space.*

Theorem 2.2. *Let $E_1, \ldots, E_n$ be Banach lattices and let F be a normed Riesz space. Then*

$$\mathcal{L}_r(E_1, \ldots, E_n; F) \ e \ \mathcal{L}_r(E_1; \mathcal{L}_r(E_2, \ldots, E_n; F))$$

are canonically isomorphic as linear spaces.

Theorem 2.3. *Let $E_1, \ldots, E_n$ and F be Banach lattices with F Dedekind complete. Then $\mathcal{L}_r(E_1, \ldots, E_n; F)$ is a Banach lattice with the regular norm (r-norm)*

$$\|A\|_r := \||A|\| = \sup\{\||A|(x_1, \ldots, x_n)\| \colon \|x_i\| \leq 1, \ 1 \leq i \leq n\}.$$

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ISOMETRIC ACTIONS ON L_P SPACES AND THE PROPERTY OF UNBOUNDED ORBITS

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Abstract

This is a summary of the main results that are being developed in a master's thesis in the field of Functional Analysis. The central direction of study is to detail a theorem and its corollaries from the recent paper [1] by Marrakchi and de la Salle (2020). These authors show that for every locally compact group G, there is a *critical constant* $p_G \in [0, \infty]$ such that G admits a continuous affine isometric action in an L_p-space ($0 < p < \infty$) with unbounded orbits if, and only if, $p \geq p_G$ (or $p > p_G$).

1 Introduction

Studies such as [2],[3],[4] and [5] led us to expect that for a given topological group G, it should be "easier" to act isometrically on an L_p-space when the value of p gets larger. This is what the main theorem in [1] states.

As shown in the paper, for every locally compact group G, there is a *critical constant* $p_G \in [0, \infty]$ such that G admits a continuous affine isometric action in an L_p-space ($0 < p < \infty$) with unbounded orbits if, and only if, $p \geq p_G$ (or $p > p_G$).

Such result is the leading object of study of the master's thesis.

Here, L_p space means $L_p(X, \mu)$, for (X, μ) a fixed standard measure space. Also, isometries are considered affine, not necessarily linear.

2 Main Results

The main following theorem was proved by Marrakchi and de la Salle in [1].

Theorem 2.1. *Let G be a topological group and $0 < p \leq q < \infty$. Then, for every continuous affine isometic action $\alpha : G \curvearrowright L_p$, there is a continuous affine isometric action $\beta : G \curvearrowright L_q$ such that $||\alpha_g(0)||^p_{L_p} = ||\beta_g(0)||^q_{L_q}$, for all $g \in G$.*

Such theorem implies that if a group G has a continuous action by isometries on an L_p space with unbounded (resp. metrically proper) orbits, then it has the same action on the corresponding L_q space for $q > p$.

Corollary 2.1. *Let G be a topological group. Then,*

1. *The set of values of $p \in (0, \infty)$ such that G admits a continuous action by isometries on an L_p space with unbounded orbits is an interval of the form (p_G, ∞) or $[p_G, \infty)$ for some $p_G \in \{0\} \cup [2, \infty]$.*

2. *The set of values of $p \in (0, \infty)$ such that G admits a proper continuous action by isometries on an L_p space is an interval of the form (p'_G, ∞) or $[p'_G, \infty)$ for some $p'_G \in \{0\} \cup [2, \infty]$.*

Theorem 2.1 also can be applied to the group of affine isometries of L_p and we get the following

Corollary 2.2. *Let $0 < p \leq q < \infty$. Then Isom(L_p) is isomorphic as a topological group to a closed subgroup of Isom(L_q).*

For a topological group G and $p > 0$, $K^p(G)$ denotes the set of all continuous functions $\psi : G \to \mathbb{R}_+$ of the form $\psi(g) = ||\alpha_g(0)||^p_{L_p}$ for some continuous affine isometric action α of G on some L_p space.

If $\pi : G \curvearrowright V$ is a continuous linear representation of a topological group G on a topological vector space V, we denote by $Z^1(G, \pi, V)$ the set of all continous 1-cocycles, that means, all continuous maps $c : G \to V$ such that $c(gh) = c(g) + \pi_g(c(h))$, for all $g, h \in G$.

To prove Theorem 2.1, the following proposition is necessary

Proposition 2.1. *Let G be a topological group, $p > 0$ and $\psi \in K^p(G)$. Then, there exists a continuous nonsingular action $\sigma : G \curvearrowright (X, \mu)$ and a cocycle $c \in Z^1(G, \sigma^{p,\mu}, L_p(X, \mu))$ such that $\psi(g) = \|c(g)\|^p_{L_p}$, for all $g \in G$.*

First we prove this proposition for $p = 2$ and then separately for $p \neq 2$. Also, $\sigma^{p,\mu}$ is a continuous linear isometric representation of the group G on L_p which is given by

$$\sigma_g^{p,\mu}(f) = \left(\frac{\mathrm{d}\left[(\sigma_g)_*\mu\right]}{\mathrm{d}\mu}\right)^{\frac{1}{p}} \sigma_g(f)$$

for every $g \in G$ and $f \in L_p(X, \mu)$, where $\frac{\mathrm{d}[(\sigma_g)_*\mu]}{\mathrm{d}\mu}$ is the Rado-Nikodym derivative of the pushforward measure $(\sigma_g)_*\mu$ with respect to μ.

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ABSTRACT DIFFERENTIAL EQUATIONS WITH $L^{Q,\alpha}_{\beta}$-HÖLDER AND L^{Q}_{LIP}-LIPSCHITZ NON-LINEAR TERMS

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Abstract

We introduce a class of $L^{q,\alpha}$-Hölder and $L^{q,\alpha}_{\beta}$-Hölder functions and study the regularity of mild solutions for abstract differential equations assuming that the non-linear term is a function cited above.

1 Introduction

We introduce the class of $L^{q,\alpha}_{\beta}$-Hölder functions and study the regularity of the mild solution to abstract ordinary differential equation described by:

$$u'(t) = Au(t) + f(t),\ t \in [0,a], \quad u(0) = x_0 \in X \tag{1}$$

where X is a Banach space, $A : D(A) \subset X \to X$ is the generator of an analytic C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on X and $f : [0,a] \to X$ is a $L^{p,\alpha}$-Hölder function.

Using the notations above, we admit, for sake of simplicity, that $0 \in \rho(A)$, and for $\eta > 0$ we denote $(-A)^{\eta}$ and X_{η} the η-order fractional power of A and his domain with the norm defined by $\|x\|_{\eta} = \|(-A)^{\eta}\|$, respectively. We also assume that C_i, C_{η} $(i \in \mathcal{N})$ are constants such that $\|A^iT(t)\| \leq \frac{C_i}{t^i}$ and $\|(-A)^{\eta}T(t)\| \leq \frac{C_{\eta}}{t^{\eta}}$ for all $t \in (0,a]$.

2 Main Results

To begin we introduce the mentioned function class, next we state the regularity of mild solution to problem (1).

2.1 $L^{q,\alpha}$-Hölder, $L^{q,\alpha}_{\beta}$-Hölder and L^{q}_{Lip}-Lipschitz functions.

In the next definitions $(Y_i, \|\cdot\|_{Y_i})$, $i = 1,2$ are Banach spaces and $q \geq 1$.

Definition 2.1. *Let $P : [c,d] \times Y_1 \mapsto Y_2$ be a function. Assume that there is $\alpha \in (0,1]$, a integrable function $[P]_{(\cdot,\cdot)} : [c,d] \times [c,d] \to \mathbb{R}^+$ and a non-decreasing function $\mathcal{W}_P : \mathbb{R}^+ \to \mathbb{R}^+$ such that $[P]_{(\cdot,\cdot)}$, $[P]_{(t,\cdot)}$ and $[P]_{(\cdot,0)}$ belongs to $L^p([c,t];\mathbb{R}^+)$ for all $t \in (c,d]$, and*

$$\|P(t,x) - P(s,y)\|_{Y_2} \leq \mathcal{W}_P(\max\{\|x\|_{Y_1}, \|y\|_{Y_1}\})[P]_{(t,s)}(|t-s|^{\alpha} + \|x-y\|_{Y_1}),$$

for all $x,y \in Y_1$ and $c \leq s \leq t \leq d$. If $\alpha \in (0,1)$, we say that $P(\cdot)$ is a $L^{q,\alpha}$-Hölder function and a L^{q}_{Lip}-function if $\alpha = 1$.

Next, we use the notations $L^{p,\alpha}([c,d]\times Y_1; Y_2)$ and $L^{p}_{Lip}([c,d]\times Y_1; Y_2)$ for the sets formed by all the $L^{q,\alpha}$-Hölder functions and by all the L^{q}_{Lip}- Hölder functions defined from $[0,a] \times Y_1$ into Y_2.

Definition 2.2. *Let $P : [c,d] \times Y_1 \mapsto Y_2$ be a function and $\alpha, \theta \in (0,1)$. We said that $P(\cdot)$ is a $L^{q,\alpha}_{\theta}$-Hölder function if there is an integrable function $[P]_{(\cdot,\cdot)} : [c,d] \times [c,d] \to \mathbb{R}^+$ and a non-decreasing function $\mathcal{W}_P : \mathbb{R}^+ \to \mathbb{R}^+$ such that the function $s \mapsto \frac{[P]_{(t,s)}}{s^{\theta}}$ belongs to $L^p([c,t];\mathbb{R}^+)$ and $[P]_{(\cdot,0)} \in L^p([c,t];\mathbb{R}^+)$ for all $t \in (c,d]$, and*

$$\|P(t,x) - P(s,y)\|_{Y_2} \leq \mathcal{W}_P(\max\{\|x\|_{Y_1}, \|y\|_{Y_1}\})\frac{[P]_{(t,s)}}{s^{\theta}}(|t-s|^{\alpha} + \|x-y\|_{Y_1}),$$

for all $x,y \in Y_1$ and $c \leq s \leq t \leq d$.

Next, $L_\theta^{p,\alpha}([c,d]\times Y_1;Y_2)$ denotes the space formed by all the $L_\theta^{q,\alpha}$-Hölder function in $C([c,d]\times Y_1;Y_2)$.

Example 2.1. *We now cite several examples concerning the definitions above.*

- *For $p>1$, $f:[0,a]\mapsto\mathbb{R}$ given by $f(t)=\sqrt[p]{t}$, belongs to $L^q_{Lip}([0,a];\mathbb{R})$ if $q\in(1,p')$.*
- *Let $f:[0,1]\mapsto\mathbb{R}$ be the function defined by $f(t)=t\sin(\frac{1}{\sqrt[p]{t}})$ for $t>0$ and $f(0)=0$ therefore $f\in L^q_{Lip}([0,1];\mathbb{R})$ for all $1<q<p$.*
- *Let $f:[0,1]\mapsto\mathbb{R}$ be the function given by $f(t)=t^{-\frac{1}{p}}$ and $f(0)=0$ with $p>1$. Then $f\in L^{q,\frac{1}{p}}([0,1])$ if $p>2q$.*
- *Assume that $f\in L^{q,\alpha}([a,b])$ and that $G\in C(X;X)$ is locally Lipschitz. Then $H(t,x)=f(t)G(x)\in L^{q,\alpha}([a,b]\times X;X)$.*
- *The sets $L^{q,\alpha}([c,d]\times X;Y)$, $L^{q,\alpha}_\beta([c,d]\times X;Y)$ and $L^q_{Lip}([c,d]\times X;Y)$ are vectorial spaces.*

2.2 Regularity of mild solutions to abstract differential equations with $L^{q,\alpha}$ non-linear terms

To finish we only cite on regularity result that we proved and is going to be submitted to publication soon.

Consider the abstract ordinary differential equation

$$u'(t)=Au(t)+f(t), t\in[0,a],\ u(0)=x_0\in X. \tag{1}$$

If $f\in L^{q,\alpha}([0,a];X)\cap L^p([0,a];X)$, it obvious that there exists a unique mild solution $u\in C([0,a]:X)$ of the problem (1) defined on $[0,a]$.

Proposition 2.1. *Let $f\in L^{q,\alpha}([0,a];X)$ and $u(\cdot)$ be the mild solution of (1) on $[0,a]$.*

(a) *If $x_0\in X$ and $\sup_{t\in[0,a]}\|\frac{[f]_{(t,\cdot)}}{(t-\cdot)^{1-\alpha}}\|_{L^1([0,t])}<\infty$, then $u(\cdot)$ is a strong solution and $u'\in L^q([0,a];X)$.*

If in addition to the conditions in (**a**), *$f\in C([0,a];X)$, $[f]_{(t,s)}(t-s)^\alpha\to 0$ as $s\uparrow t$ for all $t\in[0,a]$ and $\Lambda:=\sup_{s\in[0,a]}\|[f]_{(s,\cdot)}\|_{L^q([0,s])}$ is finite, then we get:*

(b) *If $\mu=(1-2(1-\alpha)q')>0$ and $AT(\cdot)x_0\in C^\beta([0,a];X)$, then $u(\cdot)$ is a classical solution and $u'\in L^{q,\min\{\beta,\alpha,1-\alpha,\frac{\nu}{q'}\}}_{\min\{\alpha,\beta\}}([0,a];X)$.*

(c) *If $\mu=(1-2(1-\alpha)q')>0$, $x_0\in D(A)$ and $Ax_0+f(0)\in X_\beta$ for some $\beta\in(0,1)$, then $u(\cdot)$ is a strict solution on $[0,a]$ and $u'\in L^{q,\min\{\beta,\alpha,1-\alpha,\frac{\nu}{q'}\}}([0,a];X)$.*

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ESTIMATING THE NUMBER OF SOLUTIONS FOR A SYSTEM OF THE TYPE SCHRÖDINGER-BOPP-PODOLSKY USING THE LJUSTERNIK-SCHNIRELMANN CATEGORY

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Abstract

In this work we found an estimate for the number of solutions for a Schrödinger-Bopp-Podolsky system

$$\begin{cases} -\varepsilon^2\Delta w + V(x)w + \psi w = f(w) \\ -\varepsilon^2\Delta\psi + \varepsilon^4\Delta^2\psi = 4\pi\varepsilon w^2 \end{cases} \qquad (P_\varepsilon)$$

where $\varepsilon > 0$, and $V : \mathbb{R}^3 \to \mathbb{R}, f : \mathbb{R} \to \mathbb{R}$ satisfying certain conditions. Using variational techniques,we prove that the number of solutions is rated lower by the Ljusternik-Schnirelmann category of M, i.e., the set of minima of the potential V.

1 Introduction

In this work, using the original ideas developed in [1], [2] and [3], we show the existence and multiplicity of solutions for the following problem in $\mathbb{R}^3$

$$\begin{cases} -\Delta u + V(\varepsilon x)u + \phi u = f(u), \\ -\Delta\phi + \Delta^2\phi = 4\pi\varepsilon u^2. \end{cases} \qquad (P_\varepsilon^*)$$

which is obtained after a change of variables.

The hypotheses of the problem about the potential V and the non linearity f is

V1 $V : \mathbb{R}^3 \to \mathbb{R}$ a continuous function such that

$$0 < \min_{\mathbb{R}^3} V := V_0 < V_\infty := \liminf_{|x|\to+\infty} V \in (V_0, +\infty],$$

with $M = \{x \in \mathbb{R}^3 : V(x) = V_0\}$ smooth and bounded,

f1 $f : \mathbb{R} \to \mathbb{R}$ a C^1-function and $f(t) = 0$ for $t \leq 0$,

f2 $\lim_{t\to 0} \frac{f(t)}{t} = 0$,

f3 exists $q_0 \in (3, 2^* - 1)$ such that $\lim_{t\to+\infty} \frac{f(t)}{t^{q_0}} = 0$, where $2^* = 6$,

f4 exist $K > 4$ such that $0 < KF(t) := K\int_0^t f(\tau)d\tau \leq tf(t)$ for all $t > 0$,

f5 the function $t \mapsto \frac{f(t)}{t^3}$ is strictly increasing in $(0, +\infty)$.

Such hypotheses are very common for working with variational methods, the Nehari manifold and Palais-Smale sequences.

2 Main Results

Theorem 2.1. *Under the assumptions (V1), (f1)-(f5), there exists an $\varepsilon^* > 0$ such that for every $\varepsilon \in (0, \varepsilon^*]$, problem ($P_{varepsilon}$) possesses at least cat$M$ solutions. Moreover, if cat$M > 1$, then (for a suitably small ε) there exist at least cat$M + 1$ solutions.*

Proof The first step to demonstrate the Theorem 2.1 is the definition of a functional associated to the problem $(P_{varepsilon})$ for the search of weak solutions, namely

$$\mathcal{I}_\varepsilon(u, \phi) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int V(\varepsilon x)u^2 + \frac{1}{2}\int \phi u^2 - \frac{1}{16\pi}\|\nabla \phi\|_2^2 - \frac{1}{16\pi}\|\Delta \phi\|_2^2 - \int F(u),$$

and the next step is through a usual reduction argument, treat the functional as a functional of just one "variable"

$$I_\varepsilon(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int V(\varepsilon x)u^2 + \frac{1}{4}\int \phi_u u^2 - \int F(u).$$

After that, several properties of compactness and the existence of solutions for the functionals involved are demonstrated, where one of the most important results is the existence of a *ground state* solution for the problem (P_ε).

An important application in the proof of the 2.1 theorem is the barycenter map

$$\beta_\varepsilon(x) := \frac{\int \chi(\varepsilon x)u^2(x)}{\int u^2} \in \mathbb{R}^3,$$

which will serve in the analysis of certain sections of the Nehari manifold and in the homotopic equivalence relation between M and a given set containing M, denoted M^+.

Using results from the Ljusternik-Schnirelmann theory, together with the validity of the Palais-Smale sequences, the existence of $cat(M)$ critical points of I_ε restricted to the Nehari manifold of the problem, guarantees the existence of at least $cat(M)$ solutions of (P_ε).

Finally, considering $cat(M) > 1$, the existence of another critical point of I_ε comes from showing the existence of a set that is not contractible in one sublevel of the Nehari manifold, but contractible in another level. Which guarantees the existence of at least $cat(M) + 1$ solutions of (P_ε).

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NUMERICAL BEHAVIOR OF THE SET OF EQUILIBRIA FOR A NONLINEAR PARABOLIC PROBLEM WITH TERMS CONCENTRATED AT THE BOUNDARY

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Abstract

In this work, we analyze the behavior of the solutions of a nonlinear parabolic problem, when some reaction terms are concentrated in a neighborhood of the domain boundary and this neighborhood shrinks to the boundary as a parameter goes to zero. More precisely, we prove the continuity of the equilibrium set of the nonlinear parabolic problem. The equilibrium points are the solutions of a nonlinear elliptic problem associated to the parabolic problem. Moreover, some numerical simulations will be presented to illustrate the behavior of these solutions.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open bounded smooth set with a C^2 boundary $\partial\Omega$. We define the strip with width ϵ and base $\partial\Omega$, as

$$\omega_\epsilon = \{x - \sigma\vec{n}(x) : \quad x \in \partial\Omega \quad \text{e} \quad \sigma \in [0, \epsilon)\}$$

for ϵ small enough, say $0 < \epsilon \leq \epsilon_0$, where $\vec{n}(x)$ is the outward unit normal vector at $x \in \partial\Omega$.

The set ω_ϵ has Lebesgue measure $|\omega_\epsilon| = O(\epsilon)$ with $|\omega_\epsilon| \leq k\epsilon|\partial\Omega|$, for some $k > 0$ independent of ϵ. Also, for small ϵ, ω_ϵ is a neighborhood of the boundary $\partial\Omega$ in $\overline{\Omega}$, that shrinks to $\partial\Omega$ when the parameter $\epsilon \to 0$, see Figure 1.

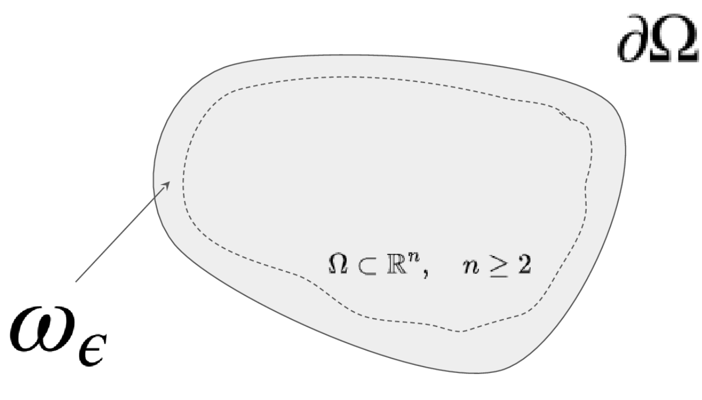

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Figure 1: The bounded open set Ω and the ϵ-strip ω_ϵ.

In [1,2] was studied the behavior, for small ϵ, of the solutions of the nonlinear parabolic problem with homogeneous Neumann boundary conditions, given by

$$\begin{cases} \dfrac{\partial u^\epsilon}{\partial t} - \Delta u^\epsilon + \lambda u^\epsilon = f(x,u^\epsilon) + \dfrac{1}{\epsilon}\chi_{\omega_\epsilon}(x) g(x,u^\epsilon), & \text{in } (0,\infty)\times\Omega \\ \dfrac{\partial u^\epsilon}{\partial \vec{n}} = 0, & \text{on } (0,\infty)\times\partial\Omega \\ u^\epsilon(0) = \varphi_0 \in H^1(\Omega), \end{cases} \tag{1}$$

where $\lambda > 0$ and χ_{ω_ϵ} is the characteristic function of the set ω_ϵ.

We refer to the term $\frac{1}{\epsilon}\chi_{\omega_\epsilon}(x)g(x,u^\epsilon)$ as the reaction (which is not linear) concentrated in the strip ω_ϵ.

Assuming that the nonlinearities $f, g : \overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ satisfy critical growth, sign and dissipative conditions given in [1,2], the authors proved that the limiting problem of the concentrated problem (1) is given by the following parabolic problem with nonlinear Neumann boundary conditions

$$\begin{cases} \dfrac{\partial u^0}{\partial t} - \Delta u^0 + \lambda u^0 = f(x,u^0), & \text{in } (0,\infty)\times\Omega \\ \dfrac{\partial u^0}{\partial \vec{n}} = g(x,u^0), & \text{on } (0,\infty)\times\partial\Omega \\ u^0(0) = \varphi_0 \in H^1(\Omega). \end{cases} \tag{2}$$

The authors showed the global existence and uniqueness of solutions u^ϵ of (1) and (2), $0 \le \epsilon \le \epsilon_0$, in Sobolev space $H^1(\Omega)$ with initial condition $\varphi_0 \in H^1(\Omega)$.

Thus, we pretend to continue the work [1,2], proving the continuity of the family of equilibria in $H^1(\Omega)$, which we will denote by $\{\mathcal{E}_\epsilon\}_{\epsilon\in[0,\epsilon_0]}$, of the nonlinear parabolic problems (1) and (2). Moreover, some numerical simulations will be presented to illustrate the behavior of the set $\{\mathcal{E}_\epsilon\}_{\epsilon\in[0,\epsilon_0]}$ as ϵ goes to zero. As far as we know, these simulations are the first of their kind and represent one of the great contributions in this research. This work is in partnership with Gleiciane da Silva Aragão.

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KIRCHHOFF-CHOQUARD EQUATIONS WITH INDEFINITE INTERNAL POTENTIAL

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Abstract

In the present work we are concerned with the following Kirchhoff-Choquard-type equation

$$-M(||\nabla u||_2^2)\Delta u + Q(x)u + \mu(V(|\cdot|) * u^2)u = f(u) \text{ in } \mathbb{R}^2,$$

for $M:\mathbb{R}\to\mathbb{R}$ given by $M(t)=a+bt$, $\mu>0$, V a sign-changing and possible unbounded potential, Q a continuous external potential and a nonlinearity f with exponential critical growth. We prove existence and multiplicity of solutions in the *nondegenerate* case and guarantee the existence of solutions in the *degenerate* case.

1 Introduction

Equations of Kirchhoff type have been exaustively studied by mathematicians since its importance and applications. In this work we combine Kirchhoff and Choquard equations and consider a general indefinite internal potential than the logaritmic one. We will denote $\mathbb{R}^+=\{t\in\mathbb{R}\ ;\ t>0\}$, $V^-=\max\{-V,0\}$, $V^+=\max\{V,0\}$ and, in order to prove the existence and multiplicity results, we ask for:

(M) $M:\mathbb{R}\to\mathbb{R}$ given by $M(t)=a+bt$, for all $t\in\mathbb{R}$, with $a>0$ and $b\geq 0$ or $a=0$ and $b>0$.

(V_1) There are real functions $a_1,a_2:\mathbb{R}^+\to\mathbb{R}$ such that $a_2\in L^\infty(\mathbb{R}^+), a_{1,0}=\inf\limits_{t\geq 2} a_1(t)>0, a_{2,0}=\inf\limits_{t\in\mathbb{R}^+} a_2(t)>0$ and

$$a_1(t)\ln(1+t)\leq V^+(t)\leq a_2(t)\ln(1+t), \forall\, t>0.$$

(V_2) There exists a real function $a_3:\mathbb{R}^+\to\mathbb{R}$ such that $a_3(t)>0$ in a subset of $\mathbb{R}^+$ with positive measure,

$$V^-(t)\leq \frac{a_3(t)}{t} \quad \forall\, t>0 \quad \text{and} \quad \begin{cases} a_3\in L^\infty(\mathbb{R}), \\ \text{or} \\ a_3(t)=t^{-\lambda}, \text{ for some } \lambda\in[1,3) \text{ and for all } t>0, \end{cases}$$

(V_3) There exists an open subset $\mathcal{I}\subset\mathbb{R}^+$ such that $V(t)<0$ for all $t\in\mathcal{I}$.

(Q) $Q\in C(\mathbb{R}^2,\mathbb{R})$, $\inf\limits_{x\in\mathbb{R}^2} Q(x)=Q_0>0$ and there exists $p\in(1,\infty]$ such that $Q\in L^p(\mathbb{R}^2)$.

(f_1) $f\in C(\mathbb{R},\mathbb{R}), f(0)=0$ and has critical exponential growth with $\alpha_0=4\pi$.

(f_2) $\lim\limits_{|t|\to 0}\dfrac{|f(t)|}{|t|^\tau}=0$, for some $\tau>1$.

(f_3) There exists $\theta\geq 4$ such that $f(t)t\geq\theta F(t)>0$, for all $t\in\mathbb{R}\setminus\{0\}$.

(f_4) There exist $q>4$ and $C_q>0$ such that $F(t)\geq C_q|t|^q$, for all $t\in\mathbb{R}$.

Then, in order to get multiple solutions for (P), we are going to apply a symmetric version of mountain pass theorem. To do so, we need to change condition (f_1) by the following.

(f_1') $f\in C(\mathbb{R},\mathbb{R}), f(0)=0, f$ is odd and has critical exponential growth with $\alpha_0=4\pi$.

In the *degenerate* case we also need some changes in the hypotheses for f.

(f_2') $\lim\limits_{|t|\to 0}\dfrac{|f(t)|}{|t|^\tau}=0$, for some $\tau>3$.

(f_3') There exists $\theta\geq 8$ such that $f(t)t\geq\theta F(t)>0$, for all $t\in\mathbb{R}\setminus\{0\}$.

2 Main Results

In this section we present our main results, concerning the existence and multiplicity results for equation

$$-M(||\nabla u||_2^2)\Delta u + Q(x)u + \mu(V(|\cdot|) * u^2)u = f(u) \text{ in } \mathbb{R}^2 \tag{1}$$

in the cases nondegenerate and degenerate, respectively. The full proof for these results can be found in [1].

Theorem 2.1. *Suppose* $(V_1) - (V_3)$, (Q), $(f_1) - (f_4)$, $a > 0$, $b \geq 0$, $\mu > 0$, $q > 4$ *and* $C_q > 0$ *sufficiently large. Then,*
(a) *problem (P) has a nontrivial solution at the mountain pass level, that is, there exists* $u \in X \setminus \{0\}$ *such that* u *is a critical point for* I *and* $I(u) = c_{mp}$, *where*

$$c_{mp} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)), \tag{2}$$

with $\Gamma = \{\gamma \in C([0, 1], X)\ ;\ \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\}$.
(b) *Problem (P) has a nontrivial ground state solution, in the sense that, there is* $u \in X \setminus \{0\}$ *that is a critical point to* I *and satisfies*

$$I(u) = c_g = \inf\{I(v)\ ;\ v \in \mathcal{K}\}, \quad \text{where } \mathcal{K} = \{v \in X \setminus \{0\}\ ;\ I'(v) = 0\}.$$

Theorem 2.2. *Suppose* $(V_1) - (V_3)$, (Q), (f_1'), $(f_2) - (f_4)$, $a > 0$, $b \geq 0$, $\mu > 0$, $q > 4$ *and* $C_q > 0$ *sufficiently large. Then, problem (P) has infinitely many solutions.*

Theorem 2.3. *Suppose* $(V_1) - (V_3)$, (Q), $(f_1), (f_2'), (f_3'), (f_4)$, $a = 0$, $b > 0$, $q > 4$ *and* $C_q > 0$ *sufficiently large. Then,*
(a) *there exists a value* $\mu_* > 0$ *such that, for all* $\mu \in (0, \mu_*)$, *problem (P) has a nontrivial solution at the mountain pass level, i.e., there exits* $u \in X \setminus \{0\}$ *a critical point for* I *satisfying* $I(u) = c_{mp}$, *where*

$$c_{mp} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

with $\Gamma = \{\gamma \in C([0, 1], X)\ ;\ \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\}$.
(b) *There exists a value* $\mu_{**} \in (0, \mu_*]$ *such that, for all* $\mu \in (0, \mu_{**})$, *problem (P) has a nontrivial ground state solution, in the sense that, there is* $u \in X \setminus \{0\}$ *that is a critical point to* I *and satisfies*

$$I(u) = c_g = \inf\{I(v)\ ;\ v \in \mathcal{K}\}, \quad \text{where } \mathcal{K} = \{v \in X \setminus \{0\}\ ;\ I'(v) = 0\}.$$

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POSITIVE SOLUTIONS FOR KIRCHHOFF ELLIPTIC PROBLEMS VIA RAYLEIGH QUOTIENT IN THE WHOLE SPACE $\mathbb{R}^N$

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Abstract

In this work we consider existence of positive solutions for the following nonlocal elliptic problem:

$$\begin{cases} -m\left(\|\nabla u\|_2^2\right)\Delta u + V(x)u = \lambda a(x)|u|^{q-2}u + b(x)|u|^{p-2}u \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1}$$

where $N \geq 3$, $\lambda > 0$, $1 \leq q < 2; 2(\sigma+1) < p < 2^* = 2N/(N-2)$, $a \in L^{r_1}(\mathbb{R}^N)$, $b \in L^{r_2}(\mathbb{R}^N)$ where $a(x), b(x) > 0$ in $\mathbb{R}^N$ and $r_1, r_2 > 1$ are suitable exponents. The potential $V : \mathbb{R}^N \to \mathbb{R}$ is bounded from below by a positive constant and $m(t) = \alpha_1 + \alpha_2 t^\sigma$ with $\alpha_1, \alpha_2 > 0$, $t \in \mathbb{R}^+$ and $\sigma \in (0, 2/(N-2))$. Hence, by using the nonlinear Rayleigh quotient our main objective is to apply the minimization method on the Nehari manifold finding at least two positive solutions for our main problem whenever $\lambda \in (0, \lambda^*)$ for some suitable $\lambda^* > 0$. In fact, λ^* is the greatest positive number where the Nehari method can be applied for $\lambda \in (0, \lambda^*)$.

1 Introduction

The main objective in the present work is to investigate existence of positive solutions for elliptic problems with concave-convex nonlinearities involving Kirchhoff equations. More specifically, we consider the elliptical problem given in (1). As a consequence, we will show that there exists at least one ground state solution, that is, a solution with the minimal energy among any other nontrivial solutions for the Problem (1). Furthermore, we shall prove existence of another solutions which is a bound state solution, that is, a solution with finite energy for the Problem (1). The main idea is to introduce the concepts of the nonlinear Rayleigh Quotient and the Nehari manifold. To this end, we shall consider the following hypotheses:

(m_1) The function $m(t) = \alpha_1 + \alpha_2 t^\sigma$ with $t \in \mathbb{R}^+$ and $\alpha_1, \alpha_2 > 0$;

(a_1) $1 \leq q < 2; 2(\sigma + 1) < p < 2^*$, where $2^* = \dfrac{2N}{N-2}$, $N \geq 3$ and $0 < \sigma < \dfrac{2}{N-2}$;

(a_2) The functions $a, b : \mathbb{R}^N \to \mathbb{R}$ are defined such that $a \in L^{r_1}(\mathbb{R}^N)$ and $b \in L^{r_2}(\mathbb{R}^N)$ where $\left(\dfrac{2^*}{q}\right)' < r_1 \leq \left(\dfrac{2}{q}\right)'$, $\left(\dfrac{2^*}{p}\right)' < r_2 \leq \left(\dfrac{2}{p}\right)'$ and $a(x), b(x) > 0$ a.e. in $\mathbb{R}^N$;

(V_1) There exists a constant $V_0 > 0$ such that $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^N$.

It is important to mention that the working space is defined by

$$X := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < +\infty \right\}.$$

Throughout the work, we shall use the following inner product and the following norm in the space X defined as

$$\langle u, \varphi \rangle = \int_{\mathbb{R}^N} [\alpha_1 \nabla u \nabla \varphi + V(x) u \varphi] \, dx \quad \text{and} \quad \|u\| := \left(\int_{\mathbb{R}^N} \left[\alpha_1 |\nabla u|^2 + V(x) u^2\right] dx \right)^{\frac{1}{2}}.$$

Through variational methods, we can define the energy functional $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ associated to Problem (1) is given by

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{\alpha_2}{2(\sigma+1)}\|\nabla u\|_2^{2(\sigma+1)} - \frac{\lambda}{q}\int_{\mathbb{R}^N} a(x)|u|^q dx - \frac{1}{p}\int_{\mathbb{R}^N} b(x)|u|^p dx. \tag{2}$$

It is important to emphasize that, $u \in X$ is a critical point of J if, and only if, u is a critical point for J. In view of hypothesis (V_1) it follows that the embedding $X \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for each $r \in [2, 2^*)$. Here we borrow some ideas discussed in [2] and [3]. Hence we consider the nonlinear generalized Rayleigh quotients $R_n, R_e : X\backslash\{0\} \to \mathbb{R}$ associated with the parameter $\lambda > 0$ in the following form:

$$R_n(u) := \frac{\|u\|^2 + \alpha_2\|\nabla u\|_2^{2(\sigma+1)} - \|u\|_{p,b}^p}{\|u\|_{q,a}^q}, \ u \in X\backslash\{0\} \tag{3}$$

and

$$R_e(u) := \frac{\frac{1}{2}\|u\|^2 + \frac{\alpha_2}{2(\sigma+1)}\|\nabla u\|_2^{2(\sigma+1)} - \frac{1}{p}\|u\|_{p,b}^p}{\frac{1}{q}\|u\|_{q,a}^q}, \ u \in X\backslash\{0\}. \tag{4}$$

Define $S_n(u) := \sup_{t>0} R_n(tu)$ and $S_e(u) := \sup_{t>0} R_e(tu)$. As a consequence, we shall consider the extreme as follows:

$$\lambda^* := \inf_{u \in X\backslash\{0\}} S_n(u) \quad \text{and} \quad \lambda_* := \inf_{u \in X\backslash\{0\}} S_e(u). \tag{5}$$

2 Main Results

Theorem 2.1. *Suppose that (m_1), (a_1)-(a_2) and (V_1) hold. Then $0 < \lambda_* < \lambda^* < +\infty$ and for each $\lambda \in (0, \lambda^*)$ the Problem (1) admits at least two distinct positive solutions $u, v \in X\backslash\{0\}$ satisfying the following properties: $J''(u)(u,u) > 0$, $J''(v)(v,v) < 0$, $J(u) < 0$, $u \in \mathcal{N}^+$ and $v \in \mathcal{N}^-$. Moreover, u is a ground state solution and v is a bound state solution which it satisfies:*

(a) For each $\lambda \in (0, \lambda_*)$, we deduce that $J(v) > 0$;

(b) For $\lambda = \lambda_*$ it holds that $J(v) = 0$;

(c) For each $\lambda \in (\lambda_*, \lambda^*)$, we obtain that $J(v) < 0$.

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SOLUTIONS TO CONSTRAINED SCHRÖDINGER–BOPP–PODOLSKY SYSTEMS IN $\mathbb{R}^3$

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Abstract

We are interested in the following constrained Schrödinger–Bopp–Podolsky system in $\mathbb{R}^3$,

$$\begin{cases} -\Delta u + \omega u + \phi u = u|u|^{p-2}, \\ -\Delta \phi + a^2 \Delta^2 \phi = 4\pi u^2, \\ \|u\|_{L^2} = \rho, \end{cases} \tag{SBP$_{a,\rho}$}$$

where $a, \rho > 0$ are fixed parameters and our unknowns are $\omega \in \mathbb{R}$ and $u, \phi \colon \mathbb{R}^3 \to \mathbb{R}$. We prove that if $2 < p < 3$ (resp., $3 < p < 10/3$) and $\rho > 0$ is sufficiently small (resp., sufficiently large), then $(SBP_{a,\rho})$ admits a least energy solution. Moreover, we show that if $2 < p < 14/5$ and $\rho > 0$ is sufficiently small, then least energy solutions are radially symmetric up to translation and as $a \to 0$, they converge to a least energy solution of the constrained Schrödinger–Poisson–Slater system.

1 Introduction

The wave function $\psi \colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$ of a free quantum particle of mass m and electrical charge q under electrostatic self-interaction solves the following nonlinear Schrödinger equation:

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \Delta_x \psi + q\phi\psi - \frac{2}{p} \psi |\psi|^{p-2},$$

where $\hbar, p > 0$ are fixed and ϕ denotes the corresponding electrostatic potential. Under convenient normalizations, *standing waves*, i.e., wave functions of the form $\psi(x,t) = u(x)e^{i\omega t}$, which solve this equation subject to an L^2-norm constraint and such that ϕ satisfies Maxwell's electromagnetic theory (resp., Bopp–Podolsky theory) can be obtained by searching for solutions to the Schrödinger–Poisson–Slater system (resp., to $(SBP_{a,\rho})$),

$$\begin{cases} -\Delta u + \omega u + \phi u = u|u|^{p-2}, \\ -\Delta \phi = 4\pi u^2, \\ \|u\|_{L^2} = \rho. \end{cases} \tag{SPS$_\rho$}$$

In this context, we aim to use variational methods to prove that some well known facts about (SPS$_\rho$) also hold for $(SBP_{a,\rho})$. Indeed, $(SBP_{a,\rho})$ admits the following variational characterization: u is a weak solution to $(SBP_{a,\rho})$ in $H^1(\mathbb{R}^3)$ if, and only if, u is a critical point of $\mathcal{J}_a \colon H^1(\mathbb{R}^3) \to \mathbb{R}$ restricted to $S_\rho := \{u \in H^1(\mathbb{R}^3) : \|u\|_{L^2} = \rho\}$, where $\mathcal{J}_a$ is given by

$$\mathcal{J}_a(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 + \frac{1}{4}\int \phi_a^v v^2 - \frac{1}{p}\|v\|_{L^p}^p;$$

ϕ_a^v is the unique weak solution to $-\Delta v + a^2\Delta^2 v = 4\pi v^2$ in $\mathcal{X}(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3), \mathcal{X}(\mathbb{R}^3)$ are obtained as the respective Hilbert space completions of $C_c^\infty(\mathbb{R}^3)$ with respect to the inner products $\langle u, v\rangle_{H^1} := \int \langle \nabla u, \nabla v\rangle + uv$ and $\langle u, v\rangle_{\mathcal{X}} := \int \langle \nabla u, \nabla v\rangle + \Delta u \Delta v$. We remark that it is proved in [2, Appendix A.1] that weak solutions in $H^1(\mathbb{R}^3) \times \mathcal{X}(\mathbb{R}^3) \times \mathbb{R}$ are, in fact, classical solutions to $(SBP_{a,\rho})$.

2 Main Results

We say that $(u, \phi, \omega) \in H^1(\mathbb{R}^3) \times \mathcal{X}(\mathbb{R}^3) \times \mathbb{R}$ is a *least energy solution* to $(SBP_{a,\rho})$ when (u, ϕ, ω) is a solution to $(SBP_{a,\rho})$ and $\mathcal{J}_a(u) = \inf_{S_\rho} \mathcal{J}_a$. Therefore, it suffices to obtain relatively compact minimizing sequences of $\mathcal{J}_a|_{S_\rho}$ in order to obtain such solutions. The challenge behind this procedure resides on the fact that $\mathcal{J}_a$ is translation invariant, so it follows from Lions' concentration-compactness principle that if $(u_n)_{n\in\mathbb{N}}$ is a bounded minimizing sequence for $\mathcal{J}_a|_{S_\rho}$, then *dichotomy* could occur, i.e., $u_n \rightharpoonup \overline{u} \neq 0$ and $\|\overline{u}\|_{L^2} < \rho$, or the sequence could *vanish*, i.e., $u_n \to 0$.

To avoid these phenomena, we use Bellazzini and Siciliano's abstract framework for minimization introduced in [3] to obtain the existence of least energy solutions to $(SBP_{a,\rho})$:

Theorem 2.1 ([1, Theorems A, B)**.** *] If $2 < p < 3$ (resp., $3 < p < 10/3$), then there exists $R_p > 0$ such that given $a > 0$ and $\rho \in]0, R_p[$ (resp., $\rho > R_p$), the system $(SBP_{a,\rho})$ admits a least energy solution.*

Slight changes to the arguments in [4] allow us to conclude that least energy solutions are radial whenever $2 < p < 14/5$:

Theorem 2.2 ([1, Theorem C)**.** *] Given $p \in]2, 14/5[$, there exists $R_p > 0$ such that if $a > 0$, $\rho \in]0, R_p[$ and $(u, \phi, \omega) \in S_\rho \times \mathcal{X}(\mathbb{R}^3) \times \mathbb{R}$ is a least energy solution to $(SBP_{a,\rho})$, then u is radially symmetric up to translation.*

To finish, we prove that (SPS_ρ) admits a least energy solution to which least energy solutions to $(SBP_{a,\rho})$ converge as $a \to 0$:

Theorem 2.3 ([1, Theorem D)**.** *] If $2 < p < 14/5$, then there exists $R_p > 0$ such that given $\rho \in]0, R_p[$ and a set*

$$\{(u_a, \phi_a, \omega_a) \in H^1(\mathbb{R}^3) \times \mathcal{X}(\mathbb{R}^3) \times \mathbb{R} : a > 0 \text{ and } (u_a, \phi_a, \omega_a) \text{ is a least energy solution to } (SBP_{a,\rho})\},$$

we conclude that (SPS_ρ) admits a least energy solution, $(u_0, \phi_0, \omega_0) \in H^1_{\mathrm{rad}}(\mathbb{R}^3) \times \mathcal{D}^{1,2}_{\mathrm{rad}}(\mathbb{R}^3) \times]0, \infty[$, such that

$$(u_a, \phi_a, \omega_a) \to (u_0, \phi_0, \omega_0) \text{ in } H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathbb{R} \text{ as } a \to 0$$

up to translations and subsequences, where $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is the Hilbert space completion of $C_c^\infty(\mathbb{R}^3)$ with respect to $\langle u, v \rangle_{\mathcal{D}^{1,2}} := \int \langle \nabla u, \nabla v \rangle$.

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ELLIPTIC EQUATIONS INVOLVING SUPERCRITICAL SOBOLEV GROWTH WITH MIXED DIRICHLET-NEUMANN BOUNDARY CONDITIONS

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Abstract

Our work concerns the existence of solutions to a class of elliptic problems involving supercritical Sobolev growth without the (AR) condition and with a mixed boundary Dirichlet-Neumann type condition. We also study non-existence of solutions for a class of elliptic problems and present a comparison result inspired by the case of Dirichlet boundary conditions.

1 Introduction

What we present here was taken from our submitted paper [2]. Consider the following elliptic problem

$$\begin{cases} -\Delta u = \lambda u^{q-1} + f(u), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ B(u) = 0, & x \in \partial\Omega, \end{cases} \tag{P}$$

where $1 < q < 2$, $\lambda > 0$, $\Omega \subset \mathbb{R}^N$ is a bounded domain, with $N \geq 3$. The boundary condition is given by the mixed operator

$$B(u) = u\chi_{\Sigma_1} + \frac{\partial u}{\partial \nu}\chi_{\Sigma_2}, \tag{BC}$$

where both Σ_1, Σ_2 are smooth (N-1)-dimensional submanifolds of $\partial\Omega$ with positive measure and such that $\overline{\Sigma_1} \cup \overline{\Sigma_2} = \partial\Omega$, $\Sigma_1 \cap \Sigma_2 = \emptyset$ and $\overline{\Sigma_1} \cap \overline{\Sigma_2} = \Gamma$ is a smooth (N-2)-dimensional submanifold. Furthermore, ν is the outward unitary normal vector to the boundary $\partial\Omega$ and χ_A is the characteristic function of the set A.

We will consider here f to be a continuous function satisfying the following conditions:

(H_1) It has the sign property, namely:

$$0 \leq f(t)t \quad , \quad t \in \mathbb{R}\, ;$$

(H_2) It has a critical or supercritical growth at infinite, in the sense that

$$\liminf_{t\to\infty} \frac{f(t)}{t^r} = \infty \quad , \qquad \forall \quad r \in \left(1, \frac{N+2}{N-2}\right]\, ;$$

(H_3) We assume that there exists a number $\theta > 0$ such that

$$\limsup_{t\to\infty} \frac{f(t)}{t^{2^*-1+\theta}} < \infty\, ;$$

(H_4) At last, we assume that there exists a sequence (M_n) with $M_n \to \infty$ and such that, for every $r \in (0, \frac{N+2}{N-1})$,

$$t \in [0, M_n] \;\Rightarrow\; \frac{f(t)}{t^r} \leq \frac{f(M_n)}{(M_n)^r}.$$

The suitable choice for the space in which we look for solutions in this mixed boundary formulation is

$$E_{\Sigma_1}(\Omega) := \left\{ v \in H^1(\Omega); v = 0 \text{ on } \Sigma_1 \right\},$$

which can also be identified as the closure of $C_c^1(\Omega \cup \Sigma_2)$ with the norm of $H^1(\Omega)$.

2 Main Results

Our main result is this

Theorem 2.1. *If $f : [0, \infty) \longrightarrow \mathbb{R}$ is a continuous function satisfying the growth conditions (H_1) - (H_4), then there exists $\gamma > 0$ and $\Lambda > 0$ such that problem (P) has a weak solution $u_\lambda \in E_{\Sigma_1}(\Omega) \cap W^{2,\frac{2^*}{2^*-1}}(\Omega)$ whenever $0 < \theta < \gamma$ and $0 < \lambda < \Lambda$.*

Additionally, we have obtained the following

Theorem 2.2. *If f is a continuous function satisfying the supercrititcal growth of (H_3) and it is above the quadratic power function, meaning that*

$$s^2 \ \leq \ f(s),$$

then the set of parameters λ for which problem (P) has a solution is bounded from above.

Lastly, we would like to mention the quite interesting comparison lemma we have obtained, adapting for our purposes the previous result found in [3].

Theorem 2.3. *If $u, v \in E_{\Sigma_1}(\Omega)$ are, respectively, a weak supersolution and a weak subsolution to the problem*

$$\begin{cases} -\Delta u = g(u), & \text{in} \quad \Omega, \\ u > 0, & \text{in} \quad \Omega, \\ B(u) = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

with g satisfying $g(s) \geq 0$ for $s \geq 0$ and $g(s)/s$ is a decreasing function, then $u \geq v$ in Ω.

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MULTIPLICITY OF SOLUTIONS FOR FRACTIONAL P-LAPLACIAN PROBLEM WITH SIGN CHANGING NONLINEARITY VIA RAYLEIGH QUOTIENT

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Abstract

It is establish existence and multiplicity of solutions to the nonlocal elliptic problem with sign changing nonlinearities given by

$$\begin{cases} (-\Delta)_p^s u \;+\; V(x)\,|u|^{p-2}\,u \;=\; \lambda f(x)\,|u|^{q-2}\,u \;+\; g(x)\,|u|^{r-2}\,u, \text{ in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N), \end{cases}$$

where $\lambda \in (0,\lambda^*), \lambda^* > 0$, $N > ps$ with $s \in (0,1)$ fixed, $1 < q < p < r < p_s^*$ and $p_s^* = \frac{Np}{N-ps}$. The potential V is a continuous function. Here we consider the functions f and g that can be sign changing functions. Hence, we prove existence and multiplicity of solutions via nonlinear Rayleigh quotient. More precisely, there exists $\lambda^* > 0$ such that our main problem has at least two solutions for each $\lambda \in (0,\lambda^*)$. The λ^* parameter is optimal in the sense that we can apply the Nehari method.

1 Introduction

In this work, we consider the elliptical problem given by

$$\begin{cases} (-\Delta)_p^s u \;+\; V(x)\,|u|^{p-2}\,u \;=\; \lambda f(x)\,|u|^{q-2}\,u \;+\; g(x)\,|u|^{r-2}\,u, \text{ in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N), \end{cases} \tag{1}$$

where $\lambda \in (0,\lambda^*), \lambda^* > 0$, $N > ps$ with $s \in (0,1)$ fixed, $1 < q < p < r < p_s^*$ and $p_s^* = \frac{Np}{N-ps}$. Since f and g can be sign changing functions some considerable difficulties in order to apply the Rayleigh method is verified. This fact occurs due to the fact that the Raleigh quotient may not be well defined for all functions. The main idea here is to consider a open set where the quotients are well defined. Hence we prove that the fibering maps have exactly two critical points in the appropriate open where $\lambda \in (0,\lambda^*)$. Throughout this work we assume the following assumptions:

(F) There holds $f \in L^{\tilde{q}}(\mathbb{R}^N)$, with $\tilde{q} = \frac{r}{r-q}$;

(G) Suppose that $g \in L^\infty(\mathbb{R}^N)$;

(A_1) There exists an open $\Omega \subset \mathbb{R}^N$ such that f and g are positive, for all $x \in \Omega$;

(V_1) $V \in C(\mathbb{R}^N,\mathbb{R})$ and there exists a constant $V_0 > 0$ such that $V_0 = \inf\limits_{x\in\mathbb{R}^N} V(x)$;

(V_2) There holds $\mu\{x \in \mathbb{R}^N \setminus V(x) \le M\} < \infty$ for each $M > 0$.

Let $X = \{u \in W^{s,p}(\mathbb{R}^N); \int_{\mathbb{R}^N} V(x)|u|^p dx < \infty\}$ be the working space. Consider the energy functional $J : X \to \mathbb{R}$ given by

$$J(u) = \frac{1}{p}\|u\|^p - \frac{\lambda}{q}\int_{\mathbb{R}^N} f(x)|u|^q dx - \frac{1}{r}\int_{\mathbb{R}^N} g(x)|u|^r dx$$

Under our hypotheses we observe that J is well defined and it belongs $C^1(X,\mathbb{R})$. Note that X is our working space and given $u \in X$ it follows that u is a critical point of J if and only if u is a weak solution for (1). In order to use the Rayleigh Quotient method [1,2,3], we introduce the important set

$$A = \left\{u \in X \setminus \{0\} : \int_{\mathbb{R}^N} f(x)|u|^q dx > 0, \int_{\mathbb{R}^N} g(x)|u|^r dx > 0\right\}.$$

In order state our main results we need to consider auxiliaries functionals, $R_n, R_e : A \to \mathbb{R}$ associated with the parameter λ as follows:

$$R_n(u) = \frac{\|u\|^p - \displaystyle\int_{\mathbb{R}^N} g(x)|u|^r dx}{\displaystyle\int_{\mathbb{R}^N} f(x)|u|^q dx}, \; R_e(u) = \frac{\dfrac{1}{p}\|u\|^p - \dfrac{1}{r}\displaystyle\int_{\mathbb{R}^N} g(x)|u|^r dx}{\dfrac{1}{q}\displaystyle\int_{\mathbb{R}^N} f(x)|u|^q dx}. \tag{2}$$

Define $\Lambda_n(u) := \sup_{t>0} R_n(tu)$ and $\Lambda_e(u) = \sup_{t>0} R_e(tu)$. As a consequence, we have

$$\lambda^* = \inf_{u\in A} \Lambda_n(u), \; \lambda_* = \inf_{u\in A} \Lambda_e(u).$$

2 Main Results

Theorem 2.1. *Suppose (F), $(G), (V_1), (V_2)$ and (A_1). Then $0 < \lambda_* < \lambda^* < \infty$. Furthermore, assume that $\inf_{w\in\mathcal{N}^-\cap A} J(w) < \inf_{w\in\mathcal{N}^-\cap\partial A} J(w)$. Therefore, for each $\lambda \in (0,\lambda^*)$ the Problem (1) has at least one solution $v \in \mathcal{N}^- \cap A$ satisfying the following properties:*

- *i) For each $\lambda \in (0,\lambda_*)$, we obtain that $J(v) > 0$;*
- *ii) For each $\lambda = \lambda_*$, we see that $J(v) = 0$;*
- *iii) For each $\lambda \in (\lambda_*,\lambda^*)$, we deduce that $J(v) < 0$.*

Theorem 2.2. *Assume $(F), (G), (V_1), (V_2)$ and (A_1). Then $0 < \lambda_* < \lambda^* < \infty$. Furthermore, suppose that $\inf_{w\in\mathcal{N}^+\cap A} J(w) < \inf_{w\in\mathcal{N}^+\cap\partial A} J(w)$. Therefore, for each $\lambda \in (0,\lambda^*)$ the Problem (1) has at least one solution $u \in \mathcal{N}^+ \cap A$ such that $J(u) < 0$.*

Corollary 2.1. *Assume (V_1) and (V_2). Moreover, suppose $f \geq 0, f \not\equiv 0, f \in L^{\tilde{q}}(\mathbb{R}^N)$, $g > 0, g \in L^\infty(\mathbb{R}^N)$. Then, for each $\lambda \in (0,\lambda^*)$, the Problem (1) has at least two solutions. Furthermore, assuming that $\lambda \in (0,\lambda_*)$ hods, Problem (1) admits at least two positive solutions.*

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UNIFORM STABILITY OF A THERMOVISCOELASTIC PLATE MODEL

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Abstract

We study a thermoviscoelastic plate system with modificated Timoshenko equation. We prove unicity and exponential stability of total energy as time approaches to infinity.

1 Introduction

In this paper we consider three scalar functions $u(x,t), q(x,t)$ and $\theta(x,t)$ satifying the coupled system

$$\begin{cases} u_{tt} - \mu u_{xxtt} + u_{xxxx} - M\left(\displaystyle\int_0^L u_x^2 dx\right) u_{xx} + \delta\theta_{xx} = 0, 0 < x < L; t > 0 \\ \theta_t + kq_x - \delta u_{xxt} = 0, 0 < x < L; t > 0 \\ \tau q_t + q + k\theta_x = 0, 0 < x < L; t > 0 \end{cases} \tag{1}$$

in $\Omega = (0, L)$ with initial conditions

$$u(x,0) = u_0(x); u_t(x,0) = u_1(x); \theta(x,0) = \theta_0(x); q(x,0) = q_0(x) \tag{2}$$

and boundary conditions

$$u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = \theta(0,t) = \theta(L,t) = 0, \tag{3}$$

such that μ, δ, k and τ are positive constants experimentally provided and $u := u(x,t)$ com $x \in \mathbb{R}$ e $t \in [0, \infty]$. Furthermore, the constants we are considering in (1) are usually associated with the following: τ is the "relaxation" time, δ is a coupling constant for (1) and θ and q denotes the difference to a fixed temperature.

The total energy is associated to (1) is

$$E(t) = \left(\int_0^L u_t^2 + \mu u_{xt}^2 + u_{xx}^2 + \theta^2 + \tau q^2\right) + \mathcal{M}\left(\int_0^L u_x^2 dx\right) \tag{4}$$

where $\mathcal{M} = \displaystyle\int_0^\lambda M(s)ds$ for all $s \geq 0$ with $M(s) \geq 0$ is a $C^1(\Omega)$ real function.
The model (1) describes thermoviscoelastic deformations of a linear plate equation under the presence of thermal effects modeled by Cattaneo's Law (see [1] and [3]). Our main result says that the total energy given in [4] decays exponentially as time approaches to ∞.

2 Main Results

We consider the following spaces: $\mathbb{D} = \mathbb{H}^2(0,L) \times \mathbb{H}^1(0,L) \times L^2(0,L) \times L^2(0,L)$ if $\mu \neq 0$. Now, if $\mu = 0$ we have $\mathbb{D} = \mathbb{H}^2 \times L^2 \times L^2 \times L^2$.
Using the semigroup theory (see for instance [2]) techniques we prove the following theorem

Theorem 2.1. *Let us consider the Cauchy problem described in (1) given initial date* $U_0 = \{u_0, v_0, \theta_0, q_0\} \in \mathbb{D}$ *then there exists a unique function* $U(t) = \{u, v, \theta, q\} \in C([0, +\infty] \cap \mathbb{D})$.

Using a convenient Lyapunov function we prove the following result.

Theorem 2.2. *Consider the global solution of problem (1) - (3) given by theorem* **(2.1)**. *Then the total energy given in* (2) *satisfies:*

- $E(t) \leq C \cdot E(0)e^{-\gamma t}$ *, if* $\mu \geq 0$ *for all*$t > 0$, *where* C *and* γ *are positive constants independently of initial date.*
- *For the case* $\mu = 0$ *we have:*

$$E(t) \leq \frac{C}{t} \cdot E(0)$$

for all $t \geq 0$ *and* C *a positive constant independent of initial date.*

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TWO SOLUTIONS FOR A PROBLEM IN UPPER-HALF SPACE WITH CONCAVE-CONVEX NONLINEARITIES ON THE BOUNDARY

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Abstract

In this work, we study a critical elliptic problem in $\mathbb{R}^N_+$ with concave-convex nonlinearities on the boundary. For solving this problem, we use weighted Sobolev spaces which has compact embeddings in weighted Lebesgue spaces. We obtain two nonnegative and nontrivial solutions for our equation.

1 Introduction

We consider the problem

$$\begin{cases} -\Delta u - \dfrac{1}{2}\left(x \cdot \nabla u\right) = 0, \quad \text{in} \quad \mathbb{R}^N_+ \\ \dfrac{\partial u}{\partial \eta} = \mu a(x')|u|^{q-2}u + b(x')|u|^{2_*-2}u, \quad \text{on} \quad \mathbb{R}^{N-1} \end{cases} \tag{1}$$

where $1 < q < 2$, $2_* := 2(N-1)/(N-2)$, $\mu > 0$ is a real parameter and $a, b : \mathbb{R}^{N-1} \to \mathbb{R}$ are potentials satisfying mild conditions.

This type of problem arises naturally when looking for self-similar solutions to the heat equation [2]. The boundary condition in the problem (1) portrays a concave-convex behavior, since it contains sublinear and superlinear terms. Ambrosetti et al [1] studied the effects of these two nonlinearities in the interior of a bounded domain and obtained a result of multiplicity of solutions. Also in the bounded domain case, Furtado et al [6] obtained two nonnegative solutions when the nonlinearities were at the boundary. Furtado et al [5] considered the problem in $\mathbb{R}^N$. For the upper-half space case we can cite the work of Furtado and Sousa [4] where they obtained infinitely many solutions under some symmetry assumptions.

2 Main Results

The conditions for potentials a and b are similar to those used in [4,5,6], namely

(a_1) $a \in L_K^{\sigma_q}(\mathbb{R}^{N-1})$, where

$$\left(\frac{p}{q}\right)' < \sigma_q \leq \left(\frac{2}{q}\right)', \tag{1}$$

(a_2) $\Omega_a^+ = \{x \in \mathbb{R}^{N-1} : a(x) > 0\}$ has interior point,

(b_1) $b \in L^\infty(\mathbb{R}^{N-1})$,

(b_2) $\Omega_b^+ = \{x \in \mathbb{R}^{N-1} : b(x) > 0\}$ has interior point.

(b_3) there exists $\delta > 0$ such that $B_\delta^{N-1}(0) := B_\delta(0) \cap \mathbb{R}^{N-1} \subset \Omega_a^+ \cap \Omega_b^+$ and

$$|b|_{L^\infty(\mathbb{R}^{N-1})} - b(x') \leq M|x'|^\gamma, \tag{2}$$

for a.e. x' in $B_\delta^{N-1}(0)$, with $M > 0$ and $\gamma > N-1$.

We shall prove the following:

Theorem 2.1. *If a, b satisfy the conditions $(a_1) - (a_2)$ and $(b_1) - (b_3)$, respectively, then problem (1) has at least two nontrivial and nonnegative solutions if $\mu > 0$ is small.*

Proof (Sketch) After a modification on the problem, we show that the nonnegative solutions of (1) are given by critical points of class C^1 functional:

$$I(u) = \int_{\mathbb{R}_+^N} K(x)|\nabla u|^2 dx - \frac{\mu}{q}\int_{\mathbb{R}^{N-1}} K(x',0)a(x')(u^+)^q dx' - \frac{1}{2_*}\int_{\mathbb{R}^{N-1}} K(x',0)b(x')(u^+)^{2_*} dx', \tag{3}$$

where $K(x) = \exp(|x|^2/4)$. The first solution u_1 is obtained by classical arguments of minimization and has negative energy. For the second solution, we argue by contradiction as follows: if the only critical points of I are 0 and u_1, then I satisfies $(PS)_c$ for every level $c < \bar{c}$ given by

$$\bar{c} := I(u_1) + \frac{1}{2(N-1)}\frac{1}{|b|_{L^\infty(\mathbb{R}^{N-1})}^{N-2}} S_{2_*}^{N-1}, \tag{4}$$

where S_{2_*} is the best constant of weighted trace embedding studied in [3]. After that, we show that the mountain pass level of I belongs in range $(-\infty, \bar{c})$, which is contradiction since $I(u_1) < 0$. □

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SYSTEMS OF 1-LAPLACIAN EQUATIONS INVOLVING CRITICAL SOBOLEV EXPONENT DEGREES

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Abstract

In this work, we study the systems of 1−Laplacian equations. In the first part we study the subcritical system and in the second part we deal with a system with homogeneous nonlinearities with critical growth. In both cases the solutions are obtained as limit of solutions to p-Laplacian type problems.

1 Introduction

Problems involving the 1-Laplacian operator have been extensively studied in the last years. The interest in this setting comes, on the one hand, from an optimal design problem in the theory of torsion and related geometrical problems and, on the other, from the variational approach to image restoration . 1- Laplacian problems also appear in game theory.

In the first part of this work, we shall extend the results of [1] for the following system 1-Laplacian equations with subcritical growth.

$$\begin{cases} \text{-div}\left(\frac{Du}{|Du|}\right) = F_u(x,u,v) \quad \text{in} \quad \Omega, \\ \text{-div}\left(\frac{Dv}{|Dv|}\right) = F_v(x,u,v) \quad \text{in} \quad \Omega, \\ u = v = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $N \geq 2$, $\Omega \subset \mathbb{R}^N$ is an open bounded set, and F a function satisfying some hypotheses.

Problems with critical growth has received lots of attention in the last years, beginning by the pioneering work of Brezis and Niremberg [2]. In the second part of this work, we deal with the following system of elliptic equations with critical growth

$$\begin{cases} \text{-div}\left(\frac{Du}{|Du|}\right) = Q_u(u,v) + \frac{2\alpha}{\alpha+\beta} u|u|^{\alpha-2}|v|^{\beta} \quad \text{in} \quad \Omega, \\ \text{-div}\left(\frac{Dv}{|Dv|}\right) = Q_v(u,v) + \frac{2\beta}{\alpha+\beta} |u|^{\alpha} v|v|^{\beta-2} \quad \text{in} \quad \Omega, \\ u = v = 0 \quad \text{on} \quad \partial\Omega, \\ u, v \geq 0; \;\; u, v \neq 0 \;\; \text{in} \;\; \Omega, \end{cases} \tag{2}$$

where $\alpha + \beta = 1^*$, $N \geq 2$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and Q_u, Q_v are the partial derivates of C^1-function Q.

2 Main Results

Our first main result is the following.

Theorem 2.1. *Suppose that F satisfies appropriate growth conditions then, system (1) has a nontrivial solution.*

The proof of Theorem 2.1 follows the method employed by Alexis Molino and Sergio Segura [5].

Our second main result is the following.

Theorem 2.2. *Let Q be a $C^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ function such that*

$$Q(\lambda u, \lambda v) = \lambda Q(u,v) \ \text{for all} \ \lambda > 0; \ u, v \geq 0 \ \text{and} \tag{3}$$

$$Q_u(0,1) = 0, \ \ Q_v(1,0) = 0. \tag{4}$$

Suppose also that

$$\mu_2(1) < \lambda_1(1), \tag{5}$$

where $\mu_2(1) = \max\{Q(u,v) : u, v \geq 0, u + v = 1\}$ *and* $\lambda_1(1) = \inf_{u \in W_0^{1,1}(\Omega)\setminus 0} \dfrac{\int_\Omega |\nabla u| dx}{\int_\Omega |u| dx}$.

Moreover,

$$\tilde{S}_H > 2, \tag{6}$$

where

$$\tilde{S}_H = \inf_{(u,v)\in W_0^{1,1}(\Omega)\times W_0^{1,1}(\Omega)} \frac{\int_\Omega (|\nabla u| + |\nabla v|)dx}{(\int_\Omega \frac{2}{1^*}|u|^\alpha |v|^\beta dx)^{\frac{1}{1^*}}}.$$

Then system (2) has a nontrivial solution.

Our method to prove the Theorem 2.2 is a combination of ideas found in [4] and [3].

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REMARKS ABOUT INSENSITIZING CONTROLS FOR A QUASILINEAR PARABOLIC EQUATION

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Abstract

This work is addressed to showing the existence of insensitizing controls for a quasilinear parabolic equation with homogeneous Dirichlet boundary conditions. We will define a functional associated with the solution of the equation in question and we will look for a control function which is locally insensitive to small perturbations in the initial condition.

1 Introduction

Let $N \in \mathbb{N} \setminus \{0\}$, $T > 0$ and Ω a limited domain of $\mathbb{R}^N$ with boundary $\partial\Omega$ of class C^3. Let us denote by Q the cylinder $\Omega \times (0,T)$ with side boundary $\Sigma = \partial\Omega \times (0,T)$. Assume ω and $\mathcal{O}$ to be two given nonempty open subsets of Ω and by 1_ω the characteristic function of the set ω. We consider the following controlled quasilinear parabolic equation:

$$\begin{cases} y_t - \nabla.(a(x,t;y)\nabla y) = \xi + v1_\omega & \text{in Q}, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y_0 + \tau\hat{y}_0 & \text{in } \Omega. \end{cases} \tag{1}$$

where $a : \overline{\Omega} \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is regular enough and satisfying $0 < m \leq a(x,t;r) \leq M$, for all $(x,t;r) \in \overline{\mathrm{Q}} \times \mathbb{R}$.

The goal of this is to show the existence of insensitizing controls for the quasilinear parabolic equation. When the function a depends only on y, that is, $a = a(y)$ the problem of insensitizing controls was developed in [3]. In this work we intend to generalize this result to $a(x,t;y)$. As usual, this insensitizing problem is reduced to a nonstandard null controllability problem of a nonlinear coupled cascade system governed by a quasilinear parabolic equation and a linear parabolic equation. To establish the null controllability of the nonlinear cascade parabolic system we will use fixed point techniques, nevertheless, to do this, it is first necessary to solve the null controllability of the linearized cascade parabolic system in the framework of classical solutions.

The crucial part is to find a desired control function in a Hölder space for data with certain regularities. More specifically, let's consider $\xi \in C^{\theta,\frac{\theta}{2}}(\overline{\mathrm{Q}})$ and $y_0 \in C^{2+\theta}(\overline{\Omega})$ satisfyng certain conditions and we will assume that the insensitizing control v dependent on ξ and y_0 but independent of τ and $\hat{y}_0$ satisfies the following condition:

(H) There exists a $\tau_0 > 0$ such that for any $|\tau| < \tau_0$ and any $\hat{y}_0 \in C_0^\infty(\Omega)$ with $|\hat{y}_0|_{C^{2+\theta}(\overline{\Omega})} = 1$, the equation (1) admits a unique solution $y(.,.,\tau,v) \in C^{2+\theta,1+\frac{\theta}{2}}(\overline{\mathrm{Q}})$. Moreover,

$$|y|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\mathrm{Q}})} \leq C(N,\Omega,\partial\Omega,T,a)(|\xi|_{C^{\theta,\frac{\theta}{2}}(\overline{\mathrm{Q}})} + |v|_{C^{\theta,\frac{\theta}{2}}(\overline{\mathrm{Q}})} + |y_0 + \tau\hat{y}_0|_{C^{2+\theta}(\overline{\Omega})}).$$

Define the following functional

$$\phi(y) = \frac{1}{2}\int_0^T \int_{\mathcal{O}} |y(x,t;\tau,v)|^2 dxdt, \tag{2}$$

where $y = y(x,t;\tau,v)$ is the corresponding solution of (1) associated with τ and v. For $\xi \in C^{\theta,\frac{\theta}{2}}(\overline{\mathrm{Q}})$ and $y_0 \in C^{2+\theta}(\overline{\Omega})$, we say that a control function $v \in C^{\theta,\frac{\theta}{2}}(\overline{\mathrm{Q}})$ with supp $u \subseteq \omega \times [0,T]$ insensitizing the functional (2) if v satisfies the condition (H), and

$$\left.\frac{\partial\phi(y(.,.;\tau,v))}{\partial\tau}\right|_{\tau=0} = 0,\ \forall \hat{y}_0 \in C_0^{\infty}(\Omega) \text{ with } |\hat{y}_0|_{C^{2+\theta}(\overline{\Omega})} = 1.$$

2 Main Results

Proposition 2.1. *Assume that* $\xi \in C^{\theta,\frac{\theta}{2}}(\overline{\mathrm{Q}})$ *satisfies*

$$|\xi|_{C^{\theta,\frac{\theta}{2}}(\overline{\mathrm{Q}})} + \left|\exp\left(\frac{M}{t(T-t)}\right)\xi\right|_{L^2(\mathrm{Q})} \leq \delta, \tag{1}$$

and $y_0 = 0$. *If a control function* $v \in C^{\theta,\frac{\theta}{2}}(\overline{\mathrm{Q}})$ *satisfies condition (H) and the corresponding solution* $(\overline{y}, q) \in \left(C^{2+\theta,1+\frac{\theta}{2}}(\overline{\mathrm{Q}})\right)^2$ *of the following nonlinear cascade system:*

$$\begin{cases} \overline{y}_t - \nabla.(a(x,t;\overline{y})\nabla\overline{y}) = \xi + v1_{\omega} & \textit{in } \mathrm{Q}, \\ \overline{y} = 0 & \textit{on } \Sigma, \\ \overline{y}(x,0) = 0 & \textit{in } \Omega \end{cases}$$

and

$$\begin{cases} -q_t - \nabla.(a(x,t;\overline{y})\nabla q) + \partial_r a(x,t;\overline{y})\nabla\overline{y}\nabla q = \overline{y}1_{\mathcal{O}} & \textit{in } \mathrm{Q}, \\ q = 0 & \textit{on } \Sigma, \\ q(x,T) = 0 & \textit{in } \Omega \end{cases}$$

satisfies $q(.,0) = 0$ *in* Ω, *then* v *insensitizes the functional (2).*

Theorem 2.1. *Assume that* $\omega \cap \mathcal{O} \neq 0$ *and* $y_0 = 0$. *Then, there exist two positive constants* M *and* δ *depending only on* $N, \Omega, \partial\Omega, T$ *and* a, *such that for any* $\xi \in C^{\theta,\frac{\theta}{2}}(\overline{\mathrm{Q}})$ *satisfying (1), one can find a control functional* $v \in C^{\theta,\frac{\theta}{2}}(\overline{\mathrm{Q}})$, *which insensitizes the functional (2).*

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NASH EQUILIBRIA FOR QUASI-LINEAR PARABOLIC PROBLEM 2D

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Abstract

In this work, we present a study of Nash Equilibria for quasi-linear parabolic problem 2D via Fixed-Point technique.

1 Introduction

Let us consider $T>0$, $\Omega\subset\mathbb{R}^2$ be a nonempty bounded connected open set whose boundary Γ is regular enough, $\omega_i\subset\Omega$, for $i=1,2$. We consider the following system

$$\begin{cases} u_t-\nabla\cdot(a(u)\nabla u)=v_1\chi_{\omega_1}+v_2\chi_{\omega_2} & \text{in} \quad \Omega\times]0,T[,\\ u(x,t)=0 & \text{on} \quad \Gamma\times]0,T[,\\ u(x,0)=u_0(x) & \text{in} \quad \Omega. \end{cases} \tag{1}$$

The existence and uniqueness results are proved using the same ideas of [2] and [3]. We consider the functionals J_i, with $i=1,2$, given by

$$J_i(v_1,v_2)=\frac{\alpha_i}{2}\int_{O_i}|u(x,T)-u_{id}|^2dx+\frac{\eta_i}{2}\int_0^T\int_{\omega_i}|v_i|^2dxdt. \tag{2}$$

Definition 1.1. *(v_1,v_2) is a Nash equilibrium for functionals J_1,J_2 if*

$$J_1(v_1,v_2)\le J_1(\hat{v_1},v_2),\forall\hat{v_1}\in L^2(\omega_1\times(0,T)), \tag{3}$$
$$J_2(v_1,v_2)\le J_2(v_1,\hat{v_2}),\forall\hat{v_2}\in L^2(\omega_2\times(0,T)). \tag{4}$$

Definition 1.2. *(v_1,v_2) is a quasi-equilibrium for functionals J_1,J_2 if*

$$\frac{\partial J_1}{\partial v_1}(v_1,v_2)(v_1',0)=0,\forall v_1'\in L^2(\omega_1\times(0,T)), \tag{5}$$
$$\frac{\partial J_2}{\partial v_2}(v_1,v_2)(0,v_2')=0,\forall v_2'\in L^2(\omega_2\times(0,T)). \tag{6}$$

Since that

$$\frac{\partial J_1}{\partial v_1}(v_1,v_2)(v_1',0)=\alpha_1\int_{O_1}(u(x,T)-u_{1d})Y^1(T)+\eta_1\int_0^T\int_{\omega_1}v_1v_1', \tag{7}$$
$$\frac{\partial J_2}{\partial v_2}(v_1,v_2)(0,v_2')=\alpha_2\int_{O_2}(u(x,T)-u_{2d})Y^2(T)+\eta_2\int_0^T\int_{\omega_2}v_2v_2'. \tag{8}$$

where Y^i, with $i=1,2$, satisfies

$$\begin{cases} -Y^i_t-\nabla\cdot(a'(u)Y^i\nabla u)-\nabla\cdot(a(u)\nabla Y^i)=v_i'\chi_{\omega_i} & \text{in} \quad \Omega\times]0,T[,\\ Y^i(x,t)=0 & \text{on} \quad \Gamma\times]0,T[,\\ Y^i(x,0)=0 & \text{in} \quad \Omega, \end{cases} \tag{9}$$

consider the system adjoint of (9)

$$\begin{cases} -\varphi^i_t - \nabla\cdot(a(u)\nabla\varphi^i) + a'(u)\nabla u\nabla\varphi^i = 0 & \text{in} \quad \Omega\times]0,T[, \\ \varphi^i(x,t) = 0 & \text{on} \quad \Gamma\times]0,T[, \\ \varphi^i(x,T) = u(T) - u_{id} & \text{in} \quad \Omega, \end{cases} \tag{10}$$

We obtain the optimal system

$$\begin{cases} u_t - \nabla\cdot(a(u)\nabla u) = v_1\chi_{\omega_1} + v_2\chi_{\omega_2} & \text{in} \quad \Omega\times]0,T[, \\ u(x,t) = 0 & \text{on} \quad \Gamma\times]0,T[, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \\ -\varphi^i_t - \nabla\cdot(a(u)\nabla\varphi^i) + a'(u)\nabla u\nabla\varphi^i = 0 & \text{in} \quad \Omega\times]0,T[, \\ \varphi^i(x,t) = 0 & \text{on} \quad \Gamma\times]0,T[, \\ \varphi^i(x,T) = u(T) - u_{id} & \text{in} \quad \Omega, \\ v_1 = -\frac{\alpha_1}{\eta_1}\varphi^1, \\ v_2 = -\frac{\alpha_2}{\eta_2}\varphi^2. \end{cases} \tag{11}$$

2 Main Results

Let $U := L^2(\omega_1 \times (0,T)) \times L^2(\omega_2 \times (0,T))$, define $\Lambda : U \longrightarrow U, \quad \Lambda(\tilde{v}_1, \tilde{v}_2) := (-\dfrac{\alpha_1}{\eta_1}\varphi^1, -\dfrac{\alpha_2}{\eta_2}\varphi^2)$

Lemma 2.1. *Let $\tilde{v}_1$, $\tilde{v}_2$ and $\tilde{u}$ solution of (1) associated $\tilde{v}_1$, $\tilde{v}_2$ and let $\bar{v}_1$, $\bar{v}_2$ and $\bar{u}$ solution of (1) associated $\bar{v}_1$, $\bar{v}_2$ then*

$$|\tilde{u} - \bar{u}|_{L^\infty(0,T,L^2)} + |\tilde{u} - \bar{u}|_{L^2(0,T,H^1_0)} \le C|(\tilde{v}_1, \tilde{v}_2) - (\bar{v}_1, \bar{v}_2)|_U \tag{1}$$

Lemma 2.2. *Under the hypotheses of Lemma 2.1, for $\tilde{\varphi}^i$ solution of (10) associated $\tilde{v}_1$, $\tilde{v}_2$ and $\bar{\varphi}^i$ solution of (10) associated $\bar{v}_1$, $\bar{v}_2$ then*

$$|\tilde{\varphi}^i - \bar{\varphi}^i|_{L^\infty(0,T,L^2)} + |\tilde{\varphi}^i - \bar{\varphi}^i|_{L^2(0,T,H^1_0)} \le C(|\tilde{u}(T) - \bar{u}(T)|_{L^2} + |(\tilde{v}_1, \tilde{v}_2) - (\bar{v}_1, \bar{v}_2)|_U) \tag{2}$$

Theorem 2.1. *The system (1) admit quasi-equilibrium .*

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO WAVE TYPE EQUATIONS

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Abstract

The goal of this work is to discuss an alternative proof to the estimates in [7] and [8] for the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = 0, \qquad x \in \mathbb{R}^n, \quad t > 0 \\ u(0,x) = 0 \\ u_t(0,x) = u_1(x), \end{cases} \tag{1}$$

$u_1 \in L^p$, $p \geq 1$

1 Introduction

The Cauchy problem

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u = 0, \quad x \in \mathbb{R}^n, \quad t > 0 \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases} \tag{2}$$

involves a σ-evolution equation in the sense of Petrovsky, since its principal symbol is $-\tau^2 + |\xi|^{2\sigma}$, whose roots are the distinct real $\tau = \pm|\xi|^\sigma$. If $s \geq \sigma$, $f \in H^s(\mathbb{R}^n)$ and $g \in H^{s-\sigma}(\mathbb{R}^n)$, the Cauchy problem (2) is well-posed, that is, there exists a uniquely determined solution $u \in C([0,T], H^s(\mathbb{R}^n)) \cap C^1([0,T], H^{s-\sigma}(\mathbb{R}^n))$.

The study of the long-time asymptotics of the solution and, more in general, the study of long-time behavior of suitable energies, has been a topic of interest in the recent years. For example, the case $\sigma = 2$ in (2) is an important model in the literature, it is known as Germain-Lagrange operator, as well as beam operator and plate operator in the case of space dimension n = 1 and n = 2, respectively.

If $f \neq 0$ in (2), one may not expect $L^q - L^q$ estimates for $q \neq 2$ for solutions to the wave and Germain-Lagrange equation, neither for solutions to the Cauchy problem for the Schrödinger equation.

The authors in [1] show that, if $\sigma > 1$, $f \equiv 0$ and $g \equiv u_1 \in L^p$, $p \geq 1$, the solution to the Cauchy problem (2) satisfies the following $L^p - L^q$ estimates

$$\|u(t,\cdot)\|_{L^q} \lesssim t^{1-\frac{n}{\sigma}\left(\frac{1}{p}-\frac{1}{q}\right)} \|u_1\|_{L^p(\mathbb{R}^n)}, \qquad \forall t > 0, \tag{3}$$

for every $1 < p \leq q < \infty$ satisfying

$$\frac{n}{\sigma}\left(\frac{1}{p} - \frac{1}{q}\right) + n \max\left\{\left(\frac{1}{2} - \frac{1}{p}\right), \left(\frac{1}{q} - \frac{1}{2}\right)\right\} < 1.$$

Our goal is to derive estimates for the wave equation ($\sigma = 1$) and, using the ideas of [1] and [2], give an alternative proof to the estimates in [7] and [8] (Theorem (2.1)).

2 Main Results

Theorem 2.1. *If $n \geq 3$, the solution u for the Cauchy problem (1) satisfies the $L^p - L^q$ decay estimates*

$$\|u(t,\cdot)\|_{L^q} \lesssim t^{1-n\left(\frac{1}{p}-\frac{1}{q}\right)} \|u_1\|_{L^p(\mathbb{R}^n)}, \tag{4}$$

uniformly for $t > 0$ if, and only if, $(\frac{1}{p}, \frac{1}{q})$ *belongs to the closed triangle with vertices* $P_1 = (\frac{1}{2} + \frac{1}{n+1}, \frac{1}{2} - \frac{1}{n+1})$, $P_2 = (\frac{1}{2} - \frac{1}{n-1}, \frac{1}{2} - \frac{1}{n-1})$ *and* $P_3 = (\frac{1}{2} + \frac{1}{n-1}, \frac{1}{2} + \frac{1}{n-1})$.
If $n = 1$ *or* $n = 2$, *we define* $P_2 = (0, 0)$ *and* $P_3 = (1, 1)$.

We should observe that, in the case $n \geq 3$, the closed triangle implication above can be replaced by the following inequality with $1 \leq p \leq q < \infty$

$$n\left(\frac{1}{p} - \frac{1}{q}\right) + (n-1)\max\left\{\left(\frac{1}{2} - \frac{1}{p}\right), \left(\frac{1}{q} - \frac{1}{2}\right)\right\} \leq 1. \tag{5}$$

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