



# Anais do XIV ENAMA

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## Apoio:



O ENAMA é um encontro científico anual com propósito de criar um fórum de debates entre alunos, professores e pesquisadores de instituições de ensino e pesquisa, tendo como áreas de interesse: Análise Funcional, Análise Numérica, Equações Diferenciais Parciais, Ordinárias e Funcionais.

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O XIV ENAMA é uma realização conjunta do Departamento de Matemática da Universidade Estadual da Paraíba e da Unidade Acadêmica de Matemática da Universidade Federal de Campina Grande. O evento estava previsto para ocorrer em novembro de 2020, porém, considerando a situação sanitária causada pela pandemia da COVID-19, foi adiado para novembro de 2021. Tendo em vista as incertezas quanto ao fim da pandemia no Brasil, o XIV ENAMA será realizado de forma totalmente remota no período de 03 a 05 de novembro de 2021.

Os organizadores do XIV ENAMA expressam sua gratidão aos órgãos e instituições, DM - UEPB e UAMat - UFCG, que apoiam e tornaram possível a realização do XIV ENAMA.

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# ENAMA 2021

## ANAIS DO XIV ENAMA

**03 a 05 de Novembro 2021**

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# ON POSITIVE SOLUTIONS OF ELLIPTIC EQUATIONS WITH OSCILLATING NONLINEARITY IN $\mathbb{R}^N$

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## Abstract

In this paper, we study results of existence and multiplicity of positive solutions for the following semilinear problem

$$\begin{cases} -\Delta u = \lambda P(x)f(u), \text{ in } \mathbb{R}^N \\ \lim_{|x| \rightarrow \infty} u(x) = 0, \end{cases}$$

where  $P \in C(\mathbb{R}^N, \mathbb{R})$  and  $f \in C([0, \infty), \mathbb{R})$  is an oscillating nonlinearity satisfying a sort of area condition. The main tools used are variational methods and sub-supersolution method.

## 1 Introduction

In this work, we study the existence and multiplicity positive solutions for the problem

$$\begin{cases} -\Delta u = \lambda P(x)f(u) \text{ in } \mathbb{R}^N \\ \lim_{|x| \rightarrow \infty} u(x) = 0, \end{cases} \quad (P)$$

where  $f : [0, +\infty) \rightarrow \mathbb{R}^N$  is a continuous function satisfy:

(f<sub>1</sub>)  $f(0) \geq 0$

(f<sub>2</sub>) There exist  $2m - 1$  zeros of  $f$ ,  $0 < a_1 < b_1 < a_2 < b_2 < \dots < b_{m-1} < a_m$  such that for  $k = 1, \dots, m - 1$

$$\begin{cases} f(t) \geq 0, t \in (b_k, a_{k+1}) \\ f(t) \leq 0, t \in (a_k, b_k); \end{cases}$$

(f<sub>3</sub>)  $\int_{a_k}^{a_{k+1}} f(s)ds > 0$ , for all  $k \in \{1, 2, \dots, m - 1\}$ .

Related to  $P$  we assume that it is a  $C_{rad}^+(\mathbb{R}^N, \mathbb{R}^N)$  function and

(P<sub>1</sub>)  $\int_{\mathbb{R}^N} |x|^{2-N} P(x)dx < +\infty$ ;

(P<sub>2</sub>)  $P \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$

and

(P<sub>3</sub>)  $\int_{\mathbb{R}^N} \frac{P(y)}{|x-y|^{N-2}} dy \leq \frac{C}{|x|^{N-2}}$ , for all  $x \in \mathbb{R}^N \setminus \{0\}$ , for some  $C > 0$ .

The existence and multiplicity of solutions to elliptic problems like (P) in bounded domains with oscillating nonlinearities, as in (f<sub>2</sub>), and area condition, like (f<sub>3</sub>), have been vastly studied since the appearance of the pioneering papers by Brown and Budin [1, 2]. In [5], Hess improves the aforementioned Brown and Budin's result, thanks to minimization arguments and Leray-Schauder degree theory. After, in [4], De Figueiredo , using variational techniques showed existence of multiple ordered solutions.

Based in the references aforementioned and in the papers due to Loc and Schmitt [6], Corrêa, Carvalho, Gonçalves and Silva [3], we use Variational Methods and Comparison Principles to study the existence and multiplicity of solutions to (P) in whole  $\mathbb{R}^N$ . We would like to point out that there are some particularities in the fact that we are working in unbounded domains, some these problems can be overcome using the Riesz Potential Theory to solutions of (P).

## 2 Main Results

Our main result are the following:

Firstly, we study the existence and multiplicity to problem  $(P)$

**Theorem 2.1.** *Assume that the function  $f$  satisfies  $(f_1) - (f_3)$  and  $P$  verifies  $(P_1) - (P_3)$ . For all  $\lambda$  sufficiently large,  $(P)$  has at least  $m - 1$  non-negative weak solutions  $\{u_1, \dots, u_{m-1}\} \subset L^\infty(\mathbb{R}^N)$  such that  $a_{k-1} < |u_k|_\infty \leq a_k$ , for  $k = 2, \dots, m$ .*

In the end, we show that condition  $(f_3)$  is a necessary condition to existence of solution to problem  $(P)$ .

**Theorem 2.2.** *Assume that  $f(0) > 0$  and*

$$\begin{cases} -\Delta u = P(x)f(u) \text{ in } \mathbb{R}^N \\ \lim_{|x| \rightarrow \infty} u(x) = 0, \end{cases} \quad (P_0)$$

*has a nonnegative weak solution  $u$  such that  $|u|_\infty \in (a_k, a_{k+1}]$ , for some  $k \in \{1, \dots, m-1\}$ , then for such  $k$*

$$\int_{a_k}^{a_{k+1}} f(s)ds > 0. \quad (1)$$

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**EXISTENCE AND APPROXIMATION OF SOLUTIONS FOR A CLASS OF DEGENERATE  
ELLIPTIC EQUATIONS WITH NEUMANN BOUNDARY CONDITION**

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**Abstract**

In this work we study the equation  $Lu = f$ , where  $L$  is a degenerate elliptic operator, with Neumann boundary condition in a bounded open set  $\Omega$ . We prove the existence and uniqueness of weak solutions in the weighted Sobolev space  $W^{1,2}(\Omega, \omega)$  for the Neumann problem. The main result establishes that a weak solution of degenerate elliptic equations can be approximated by a sequence of solutions for non-degenerate elliptic equations.

## 1 Introduction

In this paper, we prove the existence and uniqueness of (weak) solutions in the weighted Sobolev space  $W^{1,2}(\Omega, \omega)$  for the Neumann problem

$$(P) \begin{cases} Lu(x) = f(x) & \text{in } \Omega, \\ \langle A(x)\nabla u, \vec{\eta}(x) \rangle = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\vec{\eta}(x) = (\eta_1(x), \dots, \eta_n(x))$  is the outward unit normal to  $\partial\Omega$  at  $x$ ,  $\langle \cdot, \cdot \rangle$  denotes the usual inner product in  $\mathbb{R}^n$ , the symbol  $\nabla$  indicates the gradient and  $L$  is a degenerate elliptic operator

$$Lu = - \sum_{i,j=1}^n D_j(a_{ij}D_i u) + \sum_{i=1}^n b_i D_i u + g u + \theta u \omega, \quad (1)$$

with  $D_j = \frac{\partial}{\partial x_j}$ , ( $j = 1, \dots, n$ ),  $\theta$  is positive a constant, the coefficients  $a_{ij}$ ,  $b_i$  and  $g$  are measurable, real-valued functions, the coefficient matrix  $A(x) = (a_{ij}(x))$  is symmetric and satisfies the *degenerate ellipticity condition*

$$\lambda |\xi|^2 \omega(x) \leq \langle A(x)\xi, \xi \rangle \leq \Lambda |\xi|^2 \omega(x) \quad (2)$$

for all  $\xi \in \mathbb{R}^n$  and almost every  $x \in \Omega \subset \mathbb{R}^n$  a bounded open set with piecewise smooth boundary (i.e.,  $\partial\Omega \in C^{0,1}$ ),  $\omega$  is a weight function (that is, locally integrable and nonnegative function on  $\mathbb{R}^n$ ),  $\lambda$  and  $\Lambda$  are positive constants

## 2 Main Results

The main purpose of this paper (see Theorem 2.2) is to establish that a weak solution  $u \in W^{1,2}(\Omega, \omega)$  for the Neumann problem  $(P)$  can be approximated by a sequence of solutions of non-degenerate elliptic equations.

**Theorem 2.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set with boundary  $\partial\Omega \in C^{0,1}$ . Suppose that*

- (H1)  $\omega \in A_2$ ;
- (H2)  $f/\omega \in L^2(\Omega, \omega)$ ;
- (H3)  $b_i/\omega \in L^\infty(\Omega)$  ( $i=1, \dots, n$ ) and  $g/\omega \in L^\infty(\Omega)$ .

*Then, there exists a constant  $\mathbf{C} > 0$  such that for all  $\theta \geq \mathbf{C}$  the Neumann problem  $(P)$  has a unique solution  $u \in W^{1,2}(\Omega, \omega)$ . Moreover, we have that  $\|u\|_{W^{1,2}(\Omega, \omega)} \leq \frac{2}{\lambda} \left\| \frac{f}{\omega} \right\|_{L^2(\Omega, \omega)}$ .*

**Proof** See [2], Theorem 1.  $\square$

**Lemma 2.1.** Let  $\alpha, \beta > 1$  be given and let  $\omega \in A_p$  ( $1 < p < \infty$ ), with  $A_p$ -constant  $C(\omega, p)$  and let  $a_{ij} = a_{ji}$  be measurable, real-valued functions satisfying  $\lambda|\xi|^2\omega(x) \leq \langle A(x)\xi, \xi \rangle \leq \Lambda|\xi|^2\omega(x)$  (see (2)). Then there exist weights  $\omega_{\alpha\beta} \geq 0$  a.e. and measurable real-valued functions  $a_{ij}^{\alpha\beta}$  such that the following conditions are met.

- (i)  $c_1(1/\beta) \leq \omega_{\alpha\beta}(x) \leq c_2 \alpha$  in  $\Omega$ , where  $c_1$  and  $c_2$  depend only on  $\omega$  and  $\Omega$ .
- (ii) There exist weights  $\tilde{\omega}_1$  and  $\tilde{\omega}_2$  such that  $\tilde{\omega}_1 \leq \omega_{\alpha\beta} \leq \tilde{\omega}_2$ , where  $\tilde{\omega}_i \in A_p$  and  $C(\tilde{\omega}_i, p)$  depends only on  $C(\omega, p)$  ( $i = 1, 2$ ).
- (iii)  $\omega_{\alpha\beta} \in A_p$ , with constant  $C(\omega_{\alpha\beta}, p)$  depending only on  $C(\omega, p)$  uniformly on  $\alpha$  and  $\beta$ .
- (iv) There exists a closed set  $F_{\alpha\beta}$  such that  $\omega_{\alpha\beta} \equiv \omega$  in  $F_{\alpha\beta}$  and  $\omega_{\alpha\beta} \sim \tilde{\omega}_1 \sim \tilde{\omega}_2$  in  $F_{\alpha\beta}$  with equivalence constants depending on  $\alpha$  and  $\beta$  (i.e., there are positive constants  $c_{\alpha\beta}$  and  $C_{\alpha\beta}$  such that  $c_{\alpha\beta}\tilde{\omega}_i \leq \omega_{\alpha\beta} \leq C_{\alpha\beta}\tilde{\omega}_i$ ,  $i = 1, 2$ ). Moreover,  $F_{\alpha\beta} \subset F_{\alpha'\beta'}$  if  $\alpha \leq \alpha'$ ,  $\beta \leq \beta'$ , and the complement of  $\bigcup_{\alpha, \beta \geq 1} F_{\alpha\beta}$  has zero measure.
- (v)  $\omega_{\alpha\beta} \rightarrow \omega$  a.e. in  $\mathbb{R}^n$  as  $\alpha, \beta \rightarrow \infty$ .
- (vi)  $\lambda \omega_{\alpha\beta}(x) |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}^{\alpha\beta}(x) \xi_i \xi_j \leq \Lambda \omega_{\alpha\beta}(x) |\xi|^2$ ,  $\forall \xi \in \mathbb{R}^n$  and a.e.  $x \in \Omega$ , and  $a_{ij}^{\alpha\beta}(x) = a_{ji}^{\alpha\beta}(x)$ .
- (vii)  $a_{ij}^{\alpha\beta}(x) = a_{ij}(x)$  in  $F_{\alpha\beta}$ .

**Proof** See [1], Theorem 2.1.  $\square$

**Theorem 2.2.** Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set with boundary  $\partial\Omega \in C^{0,1}$ . Suppose (H1), (H3) and (H2\*)  $f/\omega \in L^2(\Omega, \omega) \cap L^2(\Omega, \omega^3)$ . Then the unique solution  $u \in W^{1,2}(\Omega, \omega)$  of problem (P) is the weak limit in  $W^{1,2}(\Omega, \tilde{\omega}_1)$  of a sequence of solutions  $u_m \in W^{1,2}(\Omega, \omega_m)$  of the problems

$$(P_m) \left\{ \begin{array}{l} L_m u_m(x) = f_m(x), \quad \text{in } \Omega, \\ \langle A^m(x) \nabla u_m, \vec{\eta}(x) \rangle = 0, \quad \text{on } \partial\Omega, \end{array} \right.$$

with  $L_m u_m = - \sum_{i,j=1}^n D_j(a_{ij}^{mm} D_i u_m) + \sum_{i=1}^n b_{mi} D_i u_m + g_m u_m + \theta u_m \omega_m$ ,  $f_m = f(\omega_m/\omega)^{1/2}$ ,  $g_m = g \omega_m/\omega$ ,  $b_{mi} = b_i \omega_m/\omega$  and  $\omega_m = \omega_{mm}$  (where  $\omega_{mm}$ ,  $a_{ij}^{mm}$  and  $\tilde{\omega}_1$  are as Lemma 2.1 and  $A^m(x) = (a_{ij}^{mm}(x))$ ).

**Proof** See [2], Theorem 2.  $\square$

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# ON A PRECISE SCALING TO CAFFARELLI-KOHN-NIRENBERG INEQUALITY

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## Abstract

We analyze the Caffarelli-Kohn-Nirenberg inequality in the Euclidean setting, in the non-sharp case. Due to a new parameter introduced, this inequality presents two distinguishable ranges: in one of them, it is shown to be the interpolation between weighted Hardy and weighted Sobolev inequalities; in the other range, the constant is not necessarily bounded for all value of the parameters. In the former case, it is obtained a constant that depends of the new parameter.

## 1 Introduction

In this work, we consider the general form of Caffarelli-Kohn-Nirenberg inequality in the non-sharp case, as appeared in [2]:

$$\left( \int_{\mathbb{R}^n} \|x\|^{\gamma r} |u|^r dx \right)^{1/r} \leq C \left( \int_{\mathbb{R}^n} \|x\|^{\alpha p} \|\nabla u\|^p dx \right)^{a/p} \left( \int_{\mathbb{R}^n} \|x\|^{\beta q} |u|^q dx \right)^{(1-a)/q}, \quad (1)$$

where the real parameters  $p, q, r, \alpha, \beta, \gamma$ , satisfy

$$p, q \geq 1, \quad r > 0 \quad \text{and} \quad \gamma r, \alpha p, \beta q > -n. \quad (2)$$

From a dimensional balance of (1), it follows that

$$\frac{1}{r} + \frac{\gamma}{n} = a \left( \frac{1}{p} + \frac{\alpha-1}{n} \right) + (1-a) \left( \frac{1}{q} + \frac{\beta}{n} \right), \quad (3)$$

where  $a \in [0, 1]$  and

$$\gamma = a\sigma + (1-a)\beta \quad (4)$$

for some parameter  $\sigma$ . In particular, if  $a > 0$ , then  $\sigma \leq \alpha$ . Moreover, if  $a > 0$  and also

$$\frac{1}{p} + \frac{\alpha-1}{n} = \frac{1}{r} + \frac{\gamma}{n},$$

then  $\sigma \geq \alpha - 1$ . These are necessary and sufficient conditions for (1), as it was proved in [2]. Further, for any compact set in the parameter space, such that, (P), (3) and  $(\alpha-1) \leq \sigma \leq \alpha$ , the positive constant  $C$  in (1) is bounded.

Here the analyze of Caffarelli-Kohn-Nirenberg inequality relies in a suitable introduced parameter  $s$  defined by

$$s := \frac{np}{n-p(\sigma-(\alpha-1))}, \quad (5)$$

and we will be focused on the sufficiency.

## 2 Main Results

The main problem to make an analysis of the inequality (1) is the interpolation between the parameters on the right side of the inequality. The following result simplifies the analysis of that interpolation.

**Proposition 2.1.** *Assume conditions (P) and (4). If there exists a constant  $C > 0$ , such that*

$$\left( \int_{\mathbb{R}^n} \|x\|^{\sigma s} |u(x)|^s dx \right)^{1/s} \leq C \left( \int_{\mathbb{R}^n} \|x\|^{\alpha p} \|\nabla u(x)\|^p dx \right)^{1/p}, \quad (6)$$

*then the Caffarelli-Kohn-Nirenberg inequality (1) holds with the same constant.*

Now, the following result shows the existence of the constant  $C$  for the inequality (6).

**Theorem 2.1.** *Let  $p \geq 1$ ,  $\alpha$ , and  $\sigma$  be such that  $\alpha p > -n$ ,  $\sigma \leq \alpha$ . Consider  $s$  as defined in (5) satisfying  $\sigma s > -n$ . Then, there exists  $C > 0$ , such that (6) holds.*

The proof of this theorem, as appeared in [1], shows that the value of constant  $C$  depends of the values of the parameter  $s$ .

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## FINSLER DOUBLE PHASE PROBLEMS INVOLVING CRITICAL SOBOLEV NONLINEARITIES

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### Abstract

In this talk, we discuss about recent results for double phase problems via variational methods. More precisely, our problems are driven by the so-called *Finsler double phase operator* given by

$$\operatorname{div}(F^{p-1}(\nabla u)\nabla F(\nabla u) + a(x)F^{q-1}(\nabla u)\nabla F(\nabla u)) \quad \text{for } u \in W^{1,\mathcal{H},F}(\Omega),$$

set on an appropriate Musielak-Orlicz-Sobolev space  $W^{1,\mathcal{H},F}(\Omega)$ , with  $F$  a positively homogeneous Minkowski norm,  $1 < p < q < \infty$  and  $a \in L^\infty(\bar{\Omega})$  such that  $a(x) \geq 0$  a.e. in  $\bar{\Omega}$ . For the first time in literature, we deal with critical Sobolev nonlinearities on a double phase setting. These nonlinear terms make the study of the energy functional more intriguing, considering the lack of compactness of the critical Sobolev embedding for  $W^{1,\mathcal{H},F}(\Omega)$ . Under suitable assumptions for weight  $a$ , exponents  $p$  and  $q$ , we are able to provide the existence of at least one solution for our problems.

### 1 Introduction

In the paper [2], C. Farkas and P. Winkert studied for the first time in literature a double phase problem involving a critical Sobolev nonlinearity. More precisely, they considered problem

$$\begin{cases} -\operatorname{div}(F^{p-1}(\nabla u)\nabla F(\nabla u) + a(x)F^{q-1}(\nabla u)\nabla F(\nabla u)) = u^{p^*-1} + \lambda(u^{\gamma-1} + g(x, u)) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where the main differential operator is the so-called *Finsler double phase operator*, with  $F : \mathbb{R}^N \rightarrow [0, \infty)$  a positively homogeneous Minkowski norm. Here, they also assumed that  $\Omega \subset \mathbb{R}^N$  is an open bounded set with Lipschitz boundary,  $\lambda > 0$  is a real parameter,  $p^* = Np/(N-p)$ , exponent  $\gamma \in (0, 1)$ ,  $g$  is a suitable subcritical term, while  $2 \leq p < q < N$  and the following assumption holds true

$$(A) \quad \frac{q}{p} < 1 + \frac{1}{N}, \quad \text{while } a : \bar{\Omega} \rightarrow [0, \infty) \text{ is Lipschitz continuous.}$$

Because of the presence of an operator with non-standard growth, the natural functional space where finding solutions of (P) is the homogeneous Musielak-Orlicz-Sobolev space  $W_0^{1,\mathcal{H},F}(\Omega)$ , set with respect to  $F$  and to function  $\mathcal{H}(x, t) := t^p + a(x)t^q$ , with  $(x, t) \in \Omega \times [0, \infty)$ . In order to handle the critical Sobolev nonlinearity and the non-differentiable singular term in (P), C. Farkas and P. Winkert worked with a local analysis on a suitable closed convex subset of  $W_0^{1,\mathcal{H},F}(\Omega)$ , by strongly assuming that  $2 \leq p < q < N$ .

Following this direction, joint with C. Farkas and P. Winkert, we were able to generalize problem (P) considering a nonlinear boundary condition and above all covering the complete situation  $1 < p < q < N$ . That is, in [1] we dealt with problem

$$\begin{cases} -\operatorname{div}(F^{p-1}(\nabla u)\nabla F(\nabla u) + a(x)F^{q-1}(\nabla u)\nabla F(\nabla u)) + u^{p-1} + a(x)u^{q-1} = u^{p^*-1} + \lambda(u^{\gamma-1} + g_1(x, u)) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ (F^{p-1}(\nabla u)\nabla F(\nabla u) + a(x)F^{q-1}(\nabla u)\nabla F(\nabla u)) \cdot \nu = u^{p^*-1} + g_2(x, u) & \text{on } \partial\Omega, \end{cases} \quad (2)$$

where  $\Omega \subset \mathbb{R}^N$  is still an open bounded set with Lipschitz boundary,  $\lambda > 0$  is a real parameter, exponent  $\gamma \in (0, 1)$ ,  $p^* = Np/(N-p)$  and  $p_* = (N-1)p/(N-p)$ ,  $1 < p < q < N$  and we have

$$(\tilde{A}) \quad q < p^*, \quad \text{while } a : \overline{\Omega} \rightarrow [0, \infty) \text{ with } a \in L^\infty(\overline{\Omega}).$$

In order to cover the complete situation, here  $F : \mathbb{R}^N \rightarrow [0, \infty)$  is a positively homogeneous Minkowski norm satisfying

$$(F) \quad \text{the reversibility } r_F = \max_{\xi \neq 0} \frac{F(-\xi)}{F(\xi)} \text{ is finite.}$$

Since we look for positive weak solutions of (2), here  $g_1 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  and  $g_2 : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$  are Carathéodory functions verifying

$$(G) \quad g_1(x, t) = g_2(x, t) = 0 \text{ for all } t \leq 0 \text{ and for a.e. } x \in \Omega \text{ and } x \in \partial\Omega, \text{ respectively. Furthermore, there exist } \theta_1 \in (1, p), r_1 \in [p, p^*), r_2 \in (p, p_*) \text{ as well as nonnegative constants } a_1, a_2 \text{ and } b_1 \text{ such that}$$

$$\begin{aligned} g_1(x, t) &\leq a_1 t^{r_1-1} + b_1 t^{\theta_1-1} && \text{for a.e. } x \in \Omega \text{ and for all } t \geq 0, \\ g_2(x, t) &\leq a_2 t^{r_2-1} && \text{for a.e. } x \in \partial\Omega \text{ and for all } t \geq 0. \end{aligned}$$

## 2 Main Results

In this talk, I will introduce the existence result for (2), proved in [1] and stated below.

**Theorem 2.1.** *Let  $\Omega \subset \mathbb{R}^N$  be an open bounded set with Lipschitz boundary, and let  $\gamma \in (0, 1)$ . Let  $1 < p < q < N$  and  $a(\cdot)$  satisfy  $(\tilde{A})$ . Let (F) and (G) hold true. Then, there exists  $\lambda_* > 0$  such that for any  $\lambda \in (0, \lambda_*)$  problem (2) admits a positive weak solution.*

Inspired by [2], for the proof of myth 2.1 we used a minimization argument on a suitable closed convex subset of Musielak-Orlicz-Sobolev space  $W^{1,\mathcal{H},F}(\Omega)$ . However, in order to cover the situation  $1 < p < q < N$ , we exploited also a truncation argument which forces the new assumption (F).

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# SCHRÖDINGER EQUATIONS WITH VANISHING POTENTIALS INVOLVING BREZIS-KAMIN TYPE PROBLEMS

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## Abstract

We prove the existence of a bounded positive solution for the following stationary Schrödinger equation

$$-\Delta u + V(x)u = f(x, u), \quad x \in \mathbb{R}^n, \quad n \geq 3,$$

where  $V$  is a vanishing potential and  $f$  has a sublinear growth at the origin (for example if  $f(x, u)$  is a concave function near the origin). For this purpose we use a Brezis-Kamin argument included in [3]. In addition, if  $f$  has a superlinear growth at infinity, besides the first solution, we obtain a second solution. For this we introduce an auxiliar equation which is variational, however new difficulties appear when handling the compactness. For instance, our approach can be applied for nonlinearities of the type  $\rho(x)f(u)$  where  $f$  is a concave-convex function and  $\rho$  satisfies the (H) property introduced in [3]. We also note that we do not impose any integrability assumptions on the function  $\rho$ , which is imposed in most works.

## 1 Introduction

We study existence of positive solutions for the semilinear Schrödinger equations

$$-\Delta u + V(x)u = f(x, u) \text{ in } \mathbb{R}^n, \quad n \geq 3, \tag{P}$$

where  $V$  is a continuous and nonnegative vanishing potential, that is,  $\lim_{|x| \rightarrow \infty} V(x) = 0$ , and  $f(x, u)$  is a Carathéodory function. The main models of  $f(x, u)$  studied here are

$$\text{I. } \rho(x)u^q \quad \text{II. } \lambda\rho(x)(u+1)^p \quad \text{and} \quad \text{III. } \lambda\rho(x)(u^q + u^p),$$

where  $0 < q < 1 < p < \frac{n+2}{n-2}$  and  $\rho$  satisfies the property (H) introduced by Brezis and Kamin [3]: *a function  $\rho \in L^\infty_{loc}(\mathbb{R}^n)$ ,  $\rho \geq 0$ , has the property (H) if the linear problem*

$$-\Delta u = \rho \text{ in } \mathbb{R}^n \tag{1}$$

has a bounded solution.

### 1.1 Two solutions involving nonlinearities of type II

Assuming  $\rho \in L^\infty(\mathbb{R}^n)$ ,  $\rho \geq 0$ ,  $\rho \neq 0$  such that

$$0 < \rho(x) \leq \frac{k}{1 + |x|^\beta} \quad \text{in } \mathbb{R}^n, \tag{H_\rho}$$

for constants  $k > 0$  and  $\beta > 2$  we will establish the existence of at least two solutions for two families of superlinear Schrödinger equations. *We observe that  $\rho$  is integrable only for  $\beta > n$ , but here we also consider  $2 < \beta \leq n$ .*

The first nonlinear Schrödinger equation such that we obtain two positive solutions is the following:

$$\begin{cases} -\Delta u + V(x)u = \lambda\rho(x)(u+1)^p & \text{in } \mathbb{R}^n, \\ u > 0 & \text{in } \mathbb{R}^n, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases} \quad (\mathbf{P}_{\lambda,p})$$

where  $1 < p < 2^* - 1$  and  $2^* := 2n/(n-2)$ ,  $n \geq 3$ , is the critical Sobolev exponent. Our main result concerning Problem  $(\mathbf{P}_{\lambda,p})$  is the following.

**Theorem 1.1.** *Assume that  $\rho$  satisfies  $(H_\rho)$  and  $V$  is a nonnegative and continuous potential such that*

$$\frac{a}{1+|x|^\alpha} \leq V(x) \leq \frac{A}{1+|x|^\alpha}, \quad \text{for all } x \in \mathbb{R}^n, \quad (H_V^\alpha)$$

*for some constants  $a, A > 0$ ,  $\alpha \in (0, 2]$ , with  $\alpha + \beta > 4$ . Then, there exists  $\Lambda > 0$  such that problem  $(\mathbf{P}_{\lambda,p})$  has at least two positive solutions  $u_{1,\lambda} < u_{2,\lambda}$  in  $\mathbb{R}^n$ , for any  $\lambda \in (0, \Lambda)$ . Furthermore*

$$u_{1,\lambda}(x) \leq c_\lambda U(x) \quad \text{for all } x \in \mathbb{R}^n,$$

where  $c_\lambda \rightarrow 0$  as  $\lambda \rightarrow 0$ .

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## PROBLEMAS DO TIPO HÉNON COM O OPERADOR 1-LAPLACIANO

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### Abstract

Estudamos, neste trabalho, uma classe de problemas de Dirichlet que envolve a equação do tipo Hénon com o operador 1– Laplaciano na bola unitária  $B \subset \mathbb{R}^N$ . Para isso, provamos a imersão entre os espaços  $BV_{rad}(B)$  e os espaços  $L^r(B)$  com peso. Através de um método de aproximações do problema original por problemas envolvendo o operador  $p$ – Laplaciano, provamos a existência de soluções radialmente simétricas.

## 1 Introdução

Na referência [1], Hénon propôs o seguinte problema

$$-\Delta u = |x|^\alpha |u|^{q-2}u \text{ em } \mathbb{R}^N, \quad (1)$$

para estudar a estabilidade dos aglomerados globulares, que são agrupamentos de estrelas aproximadamente esféricos, em astrofísica. Desde então, vários pesquisadores estudaram inúmeros tipos de generalizações desta equação. Nossa objetivo aqui é estudar a existência de solução radial para o seguinte problema do tipo Hénon envolvendo o operador 1-Laplaciano:

$$\begin{cases} -\Delta_1 u = |x|^\alpha f(u) & \text{em } B \\ u = 0 & \text{sobre } \partial B, \end{cases} \quad (2)$$

em que  $B = B(0, 1) \subset \mathbb{R}^N$ ,  $N \geq 2$ ,  $\alpha > 0$  e  $f$  é uma função localmente Hölder contínua onde  $f(s) \geq 0$  se  $s > 0$  e que satisfaz:

( $f_1$ ) existe  $a > 0$  tal que

$$\limsup_{s \rightarrow 0} \frac{f(s)}{|s|^a} = 0, \text{ uniformemente em } x \in B,$$

( $f_2$ ) existem  $C > 0$  e  $q \in (0, 1_\alpha^* - 1)$  tais que

$$|f(s)| \leq C(1 + |s|^q), \quad \forall s \in \mathbb{R},$$

onde  $1_\alpha^* = \frac{N+\alpha}{N-1}$ ,

( $f_3$ ) existe  $\kappa > 1$  tal que

$$0 < \kappa F(s) \leq f(s)s,$$

para todo  $s > 0$ , onde  $F(t) = \int_0^t f(s)ds$ .

Por meio de um esquema de aproximação por soluções de problemas envolvendo o operador  $p$ –laplaciano, mostramos a existência de uma solução para o problema (1). Para isso, nos baseamos em [2] e provamos a existência das soluções radiais  $u_p \in W_{0,rad}^{1,p}(B)$  no nível do Passo da Montanha do seguinte problema:

$$\begin{cases} -\Delta_p u = |x|^\alpha f(u) & \text{em } B \\ u = 0 & \text{sobre } \partial B, \end{cases} \quad (3)$$

em seguida, usamos alguns argumentos de [3] e demostramos que  $(u_p)$  converge para  $u$ , quando  $p \rightarrow 1^+$  onde  $u \in BV_{rad}(B)$  satisfaz (1), sendo necessário nesta última argumentação a utilização das imersões de  $BV_{rad}(B)$  em  $L_\alpha^r(B)$ .

## 2 Resultados Principais

**Theorem 2.1.** *Seja  $\alpha > 0$ , então a imersão  $BV_{rad}(B) \hookrightarrow L_\alpha^r(B)$  é contínua para  $1 \leq r \leq 1_\alpha^* = \frac{N+\alpha}{N-1}$  e compacta para  $1 \leq r < 1_\alpha^* = \frac{N+\alpha}{N-1}$ .*

**Theorem 2.2.** *Supondo  $N \geq 2$  e que  $f$  satisfaz as condições  $(f_1) - (f_3)$ , então existe uma solução não-negativa  $u \in BV_{rad}(B)$  de (1).*

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## ON A CLASS OF ELLIPTIC SYSTEMS OF THE HARDY-KIRCHHOFF TYPE IN $\mathbb{R}^N$

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### Abstract

In this work we consider a class of critical variational systems in  $\mathbb{R}^N$  of the Hardy-Kirchhoff type involving the fractional Laplacian operator. By imposing some conditions on the nonlinearity as well as in the potential, we recover the compactness combining arguments used in Alves and Souto [1] and in Brezis and Nirenberg [2]. Only monotonicity conditions are employed, without imposing any coercivity condition on the potential, which can tend to zero at infinity. Our result is closely related to that obtained by Fiscella, Pucci and Zhang [3].

## 1 Introduction

In this work we study a class of critical systems in  $\mathbb{R}^N$  of the Hardy-Kirchhoff type involving the fractional Laplacian operator of the form

$$\begin{cases} M(\|u\|^2)(\mathbf{L}_V u) = \frac{\lambda p}{p+q} K(x)|u|^{p-2}u|v|^q + \frac{\alpha}{2_s^*}|u|^{\alpha-2}u|v|^\beta & \text{in } \mathbb{R}^N, \\ M(\|v\|^2)(\mathbf{L}_V v) = \frac{\lambda q}{p+q} K(x)|u|^p|v|^{q-2}v + \frac{\beta}{2_s^*}|u|^\alpha|v|^{\beta-2}v & \text{in } \mathbb{R}^N, \end{cases} \quad (1)$$

where  $\mathbf{L}_V w \equiv (-\Delta)^s w - \sigma \frac{w}{|x|^{2s}} + V(x)w$ , with  $\sigma > 0$  (to be chosen),  $\lambda > 0$  and  $0 < s < 1$ ,  $N > 2s$ . We assume that  $p, q, \alpha, \beta > 1$  are such that  $4 < p + q < \alpha + \beta = 2_s^* = \frac{2N}{N-2s}$  and suppose that the Kirchhoff function  $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is given by  $M(t) = a + bt$ ,  $a, b > 0$ ,  $K$  and  $V$  are positive continuous functions and  $(-\Delta)^s$  fractional Laplacian, which is defined, up to a normalization constant, as

$$(-\Delta)^s \phi(x) = 2 \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus \{B_\epsilon(x)\}} \frac{\phi(x) - \phi(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N, \phi \in C_0^\infty(\mathbb{R}^N),$$

and

$$\|w\|^2 = C_{N,s} \int \int_{\mathbb{R}^{2N}} \frac{|w(x) - w(y)|^2}{|x-y|^{N+2s}} dx dy - \sigma \int_{\mathbb{R}^N} \frac{|w|^2}{|x|^{2s}} dx + \int_{\mathbb{R}^N} V(x)w^2 dx.$$

Assumptions on  $V$  and  $K$ :

- (i) (sign of  $V$  and  $K$ )  $V, K$  are continuous,  $V, K > 0$  on  $\mathbb{R}^N$  and  $K \in L^\infty(\mathbb{R}^N)$ ;
- (ii) (decay of  $K$ ) If  $\{A_n\}$  is a sequence of Borel sets of  $\mathbb{R}^N$  with  $|A_n| \leq R$  for some  $R > 0$ ,

$$\lim_{r \rightarrow \infty} \int_{A_n \cap B_r^c(0)} K(x) dx = 0, \quad \text{uniformly with respect to } n \in \mathbb{N}. \quad (2)$$

The above type of  $(V, K)$  condition, it was introduced by Alves-Souto [1].

## 2 Main Result

**Theorem 2.1.** *In addition to  $(V, K)$ , suppose  $4 < p + q < 2_s^*$ ,  $\sigma \in (0, \lambda_{N,s})$  with  $N = 3s$ ,  $s \in (0, 1)$ . Then, for every  $\lambda > 0$  the problem (1) possesses a positive solution.*

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A HESSIAN-DEPENDENT FUNCTIONAL WITH FREE BOUNDARIES AND APPLICATIONS TO  
MEAN-FIELD GAMES

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**Abstract**

We study a Hessian-dependent functional driven by a fully nonlinear operator, which associated Euler-Lagrange equation is a fully nonlinear mean-field game with free boundaries. Our findings include the existence of solutions to the mean-field game, together with Hölder continuity of the value function and improved integrability of the density. In addition, we derive a free boundary condition and prove that the reduced free boundary is a set of finite perimeter.

## 1 Introduction

We examine Hessian-dependent functionals of the form

$$\mathcal{F}_{\Lambda,p}[u] := \int_{B_1} F(D^2u)^p dx + \Lambda |\{u > 0\} \cap B_1|, \quad (1)$$

where  $F : S(d) \rightarrow \mathbb{R}$  is a uniformly elliptic operator,  $\Lambda > 0$  is a fixed constant, and  $p > d/2$ . The functional in (1) is inspired by the usual one-phase Bernoulli problem, driven by the Dirichlet energy. To a limited extent, we understand  $\mathcal{F}_{\Lambda,p}$  as a Hessian-dependent counterpart of that problem. See [1]; see also [2].

The analysis of (1) relates closely with the system

$$\begin{cases} F(D^2u) = m^{\frac{1}{p-1}} & \text{in } B_1 \cap \{u > 0\} \\ (F_{i,j}(D^2u)m)_{x_i x_j} = 0 & \text{in } B_1 \cap \{u > 0\}, \end{cases} \quad (2)$$

where  $F_{i,j}(M)$  denotes the derivative of  $F$  with respect to the entry  $m_{i,j}$  of  $M$ . Here, the unknown is a pair  $(u, m)$  solving the problem in a sense we make precise further. In fact, the system in (2) amounts to the Euler-Lagrange equation associated with (1). Furthermore we notice that (2) satisfies an *adjoint structure*. Due to such a distinctive pattern, we refer to (2) as a fully nonlinear mean-field game with free boundary.

The interesting aspect in (2) concerns the appearance of a free boundary. At least heuristically, the game is played only in the regions where the value function is strictly positive. Combined with the free boundary condition, (2) models a game in which players optimize in the region where the value function is positive and might face extinction according to a flux condition endogenously determined.

## 2 Main Results

Since one can state the Euler-Lagrange equation associated with (1) in terms of a fully nonlinear mean-field game system with free boundaries, our analysis of the existence of solutions to (2) relies on the existence of minimizers of (1) and their interplay with the notion of a solution of a fully nonlinear mean-field game. In the sequel we define a solution of the mean-field game (2).

**Definition 2.1** (Solution for the MFG system). *The pair  $(u, m)$  is a weak solution to (2) if the following hold:*

1. We have  $u \in C(B_1) \cap W_g^{1,p}$  and  $m \in L^1(B_1)$ , with  $m \geq 0$ ;

2. The function  $u$  is an  $L^p$ -viscosity solution to

$$F(D^2u) = m^{\frac{1}{p-1}} \quad \text{in } B_1 \cap \{u > 0\};$$

3. The function  $m$  is a weak solution to

$$(F_{ij}(D^2u)m)_{x_i x_j} = 0 \quad \text{in } B_1 \cap \{u > 0\}.$$

The definition of  $L^p$ -viscosity solution is necessary since  $L^p$ -functions might not be defined at the points where the usual conditions must be tested. For a comprehensive account of this notion, we refer the reader to [6].

The first contribution in our recent preprint [7] is to prove the existence of solutions for the mean-field game system (2). We report our findings in the following

**Theorem 2.1** (Existence and regularity of solutions). *Suppose  $F$  is a convex, uniformly elliptic operator, satisfying a suitable growth condition, and  $g \in W^{2,p}$  non-negative. Then there exists a solution  $(u, m)$  to (2). In addition, fix  $\alpha \in (0, 1)$ . We have  $u \in C_{loc}^\alpha(B_1)$  and there exists  $C > 0$  such that*

$$\|u\|_{C^\alpha(B_{1/2})} \leq C \|g\|_{W^{2,p}(B_1)}.$$

The constant  $C > 0$  depends on the exponent  $\alpha$ .

Once we have established the existence of solutions for (2) and produced a regularity result, we examine the free boundary. We resort to a variation of the functional and derive a free boundary condition. We summarize our findings in this direction in the following result.

**Theorem 2.2** (Free boundary condition and finite perimeter). *Let  $u \in W_{loc}^{2,p}(B_1) \cap W_g^{1,p}(B_1)$  be a minimizer for (1), for  $p > d/2$ . Suppose  $F$  is a convex, uniformly elliptic operator, satisfying a suitable growth condition, and  $g \in W^{2,p}$  non-negative. Then  $\partial^* \{u > 0\}$  is a set of finite perimeter. Suppose in addition  $u \in C^2(B_1)$ ; then*

$$\int_{\partial \{u > 0\}} \left( F(D^2u)^{p-1} F_{ij}(D^2u)_{x_i} u_{x_j} - \frac{\Lambda}{2p} \right) \langle \xi, \nu \rangle d\mathcal{H}^{d-1} = 0 \quad (1)$$

for every  $\xi \in C_c^\infty(B_1, \mathbb{R}^d)$ .

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**POSITIVE SOLUTIONS FOR A CLASS OF FRACTIONAL CHOQUARD EQUATION IN  
EXTERIOR DOMAIN**

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**Abstract**

This work concerns with the existence of positive solutions for the following class of fractional elliptic problems,

$$\begin{cases} (-\Delta)^s u + u = \left( \int_{\Omega} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} dy \right) |u|^{p-2} u, & \text{in } \Omega \\ u = 0, & \mathbb{R}^N \setminus \Omega \end{cases} \quad (1)$$

where  $s \in (0, 1)$ ,  $N > 2s$ ,  $\alpha \in (0, N)$ ,  $\Omega \subset \mathbb{R}^N$  is an exterior domain with smooth boundary  $\partial\Omega \neq \emptyset$  and  $p \in (2, 2_s^*)$ . The main feature from problem (1) is the lack of compactness due to the unboundedness of the domain and the lack of the uniqueness of solution of the limit problem. To overcome the loss these difficulties we use splitting lemma combined with careful investigation of limit profiles of ground states of limit problem.

In recent years great attention has been devoted to the study of elliptic equations involving the fractional Laplacian operator. It appears in many models arising from concrete applications in Biology, Physics, Game Theory and Financial Mathematics, see [5, 8].

Recently, fractional elliptic equations like

$$(-\Delta)^s u + \omega u = (\mathcal{K}_\alpha * |u|^p) |u|^{p-2} u, \quad u \in H^s(\mathbb{R}^N), \quad (2)$$

where  $\omega > 0$ ,  $\alpha \in (0, N)$ ,  $p > 1$ ,  $s \in (0, 1)$  and  $\mathcal{K}_\alpha(x) = |x|^{\alpha-N}$  was considered. When  $s = 1/2$ , Frank and Lenzmann [7] have used problem (1) to model the dynamics of pseudo-relativistic boson stars. Indeed they considered the existence of ground state solution of the following equation:  $\sqrt{-\Delta} u + u = (\mathcal{K}_2 * |u|^2) u$ ,  $u \in H^{1/2}(\mathbb{R}^3)$ ,  $u > 0$ . Moreover, in [6] the author showed that the dynamical evolution of boson stars is described by the nonlinear evolution equation  $i\partial_t \psi = \sqrt{-\Delta + m^2} \bar{\psi} - (\mathcal{K}_2 * |\psi|^2) \psi$  ( $m \geq 0$ ) for a field  $\psi : [0, T) \times \mathbb{R}^3 \rightarrow \mathbb{C}$ . In [2], d’Avenia et. al. considered problem (1) and obtained regularity, existence, nonexistence, symmetry and decay properties of the corresponding solutions.

When  $s = 1$ , Moroz and Schaftingen [9] considered the following equation in exterior domains

$$-\Delta u + W(x)u = (\mathcal{K}_\alpha * |u|^p) |u|^{p-2} u, \quad u \in H_0^1(\Omega). \quad (3)$$

They showed that problem (2) does not have nontrivial nonnegative super solutions. Moreover Clapp and Salazar [4], under symmetry conditions on unbounded exterior domain  $\Omega$  and  $W$  established the existence of a positive solution and multiple sign changing solutions for (2). When  $\Omega$  has no symmetry, the study becomes more complicated, see [3]. After a bibliographic review, we have observed, up to our knowledge, that there is no results in the literature, for a version of problem (2), also for the fractional case, without any symmetry conditions.

In the fractional case, recently Alves et. al [1] studied the problem

$$(-\Delta)^s u + u = |u|^{p-2}u, \text{ in } \Omega, \quad u \geq 0, \text{ in } \Omega \text{ and } u \not\equiv 0, \quad u = 0, \text{ in } \mathbb{R}^N \setminus \Omega \quad (4)$$

where  $p \in (2, 2_s^*)$  and  $\Omega$  is an exterior domain with (non-empty) smooth boundary  $\partial\Omega$ . They proved that (3) does not have a ground state solution, which becomes a difficulty in dealing with the problem. As in [3], the authors analyzed the behavior of Palais-Smale sequences, obtaining a precise estimate of the energy levels where the Palais-Smale condition fails, which made possible to show that without any symmetry assumption the problem (3) has at least one positive solution, for  $\mathbb{R}^N \setminus \Omega$  small enough. We note that, a key point to prove the results of existence is the uniqueness up to a translation of positive solution of the equation at infinity associated with (3) given by  $(-\Delta)^s u + u = |u|^{p-2}u$ , in  $\mathbb{R}^N$ . We recall that we did not find in the literature any paper dealing with the existence of non negative solutions for Problem ( $P$ ) in exterior domains. The main feature from problem ( $P$ ) is the lack of compactness due to the unboundedness of the domain and the lack of the uniqueness of solution of the limit problem

$$(-\Delta)^s u + u = \left( \int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} dy \right) |u|^{p-2}u, \quad x \in \mathbb{R}^N. \quad (5)$$

To overcome the loss of uniqueness we investigate limit profiles of ground states of (5) as  $\alpha \rightarrow 0$ . This leads to the uniqueness of ground states when  $\alpha$  is closed to 0.

Our main result is the following.

**Theorem 0.3.** *There is  $\alpha_0 > 0$  small enough and  $\rho > 0$  such that if  $\mathbb{R}^N \setminus \Omega \subset B(0, \rho)$ , problem (1) has at least one positive solution for all  $\alpha \in (0, \alpha_0)$ .*

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CHOQUARD EQUATIONS VIA NONLINEAR RAYLEIGH QUOTIENT FOR CONCAVE-CONVEX  
NONLINEARITIES

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### Abstract

It is established existence of ground and bound state solutions for Choquard equation considering concave-convex nonlinearities in the following form

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u + \lambda|u|^{q-2}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases}$$

where  $\lambda > 0, N \geq 3, \alpha \in (0, N)$ . The potential  $V$  is a continuous function and  $I_\alpha$  denotes the standard Riesz potential. Assume also that  $1 < q < 2$ ,  $2_\alpha < p < 2_\alpha^*$  where  $2_\alpha = (N + \alpha)/N$ ,  $2_\alpha^* = (N + \alpha)/(N - 2)$ . Our main contribution is to consider a specific condition on the parameter  $\lambda > 0$  taking into account the nonlinear Rayleigh quotient. More precisely, there exists  $\lambda^* > 0$  such that our main problem admits at least two positive solutions for each  $\lambda \in (0, \lambda^*)$ . In order to do that we combine Nehari method with a fine analysis on the nonlinear Rayleigh quotient. The parameter  $\lambda^* > 0$  is optimal in some sense which allow us to apply the Nehari method.

## 1 Introduction

It is well known that existence, nonexistence and multiplicity of solutions for nonlocal elliptic problems are related with the behavior of the nonlinearity at the origin and at infinity. In this work we shall consider semilinear elliptic problems driven by the Choquard equation described in the following form:

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u + \lambda|u|^{q-2}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases} \quad (1)$$

where  $\lambda > 0, N \geq 3, \alpha \in (0, N)$ . The potential  $V$  is a continuous function and  $I_\alpha$  denotes the standard Riesz potential. Assume also that  $1 < q < 2$ ,  $2_\alpha < p < 2_\alpha^*$  where  $2_\alpha = (N + \alpha)/N$ ,  $2_\alpha^* = (N + \alpha)/(N - 2)$ . Later on, we shall consider hypotheses on  $V$  and  $\lambda$ . Recall that the Riesz potential can be described in the following form

$$I_\alpha(x) = \frac{A_\alpha(N)}{|x|^{N-\alpha}}, \quad x \in \mathbb{R}^N \text{ and } A_\alpha(N) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^s},$$

where  $\Gamma$  denotes the Gamma function, see [6]. The Choquard equation has many physical applications. For example assuming that  $N = 3, \alpha = 2, p = 2, \lambda = 0$  and  $V \equiv 0$ , Problem (1) was investigated in [8] considering the quantum theory of a polaron at rest. It was pointed in [5] that Choquard problem is also applied in the Hartree-Fock theory of one component plasma. It also arises in multiple particles systems [2] and quantum mechanics [7].

It is important to emphasize that nonlocal elliptic problems involving Choquard equations have been studied in the last years taking into account several different assumptions on the potential  $V$ .

Nonlinear Rayleigh quotient have been studied in the last years, see [3, 2]. The main feature in these works is to guarantee that there exists an extreme value  $\lambda^* > 0$  in such way that the Nehari method can be applied for each  $\lambda \in (0, \lambda^*)$ .

## 2 Main Results

We are concerned with existence of ground and bound states for Problem (1) involving concave-convex nonlinearities. In this case, we need to control the parameter  $\lambda > 0$  getting our main results. In order to overcome this difficulty, we shall consider the nonlinear Rayleigh quotient showing that there exists  $\lambda^* > 0$  such that the Nehari method can be applied for each  $\lambda \in (0, \lambda^*]$ . Throughout this work we assume the following assumptions:

- (Q) It holds  $1 < q < 2$  and  $p \in (2_\alpha, 2_\alpha^*)$  with  $2_\alpha = (N + \alpha)/N$ ,  $2_\alpha^* = (N + \alpha)/(N - 2)$ ;
- (V<sub>1</sub>) The function  $V : \mathbb{R}^N \rightarrow \mathbb{R}$  is continuous and there exists a constant  $V_0 > 0$  such that  $V(x) \geq V_0$  for all  $x \in \mathbb{R}^N$ ;
- (V<sub>2</sub>) It holds  $V^{-1} \in L^1(\mathbb{R}^N)$ , i.e., the function  $V$  satisfies the following integrability condition  $\int_{\mathbb{R}^N} V^{-1}(x)dx < +\infty$ .

Now we consider the working Banach space for our problem defined by  $X = \{v \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)v^2dx < +\infty\}$ .

It is worthwhile to mention that the energy functional associated to Problem (1) is given by

$$E_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx - \frac{\lambda}{q} \int_{\mathbb{R}^N} |u|^q dx, \quad u \in X.$$

Using the embedding  $X \hookrightarrow L^r(\mathbb{R}^N)$  for each  $r \in [1, 2^*]$  it is well known that  $E_\lambda \in C^1(X, \mathbb{R})$ . Namely, we can use the all machinery of variational methods in order to ensure existence and multiplicity of solutions.

In this way, we can state our main result in the following way:

**Theorem 2.1.** *Suppose (Q) and (V<sub>1</sub>) – (V<sub>2</sub>). Then, there are  $0 < \lambda_* < \lambda^* < \infty$  such that for each  $\lambda \in (0, \lambda^*)$  the Problem (1) admits at least two distinct positive solutions  $u_\lambda, v_\lambda \in X$  satisfying the following statements:  $E''_\lambda(u_\lambda)(u_\lambda, u_\lambda) > 0$ ,  $E''_\lambda(v_\lambda)(v_\lambda, v_\lambda) < 0$ ,  $E_\lambda(u_\lambda) < 0$ . Furthermore,  $u_\lambda$  is a ground state solution and  $v_\lambda$  satisfies the following statements:*

- (i) For each  $\lambda \in (0, \lambda_*)$  we obtain that  $E_\lambda(v_\lambda) > 0$ ;
- (ii) For each  $\lambda = \lambda_*$  we deduce that  $E_\lambda(v_\lambda) = 0$ ;
- (iii) For each  $\lambda \in (\lambda_*, \lambda^*)$  we obtain also that  $E_\lambda(v_\lambda) < 0$ .

For more details about our main results, see [1].

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SOBRE A CAMADA DE TRANSIÇÃO INTERNA DE PROBLEMAS SEMILINEARES  
NÃO-HOMOGÊNEOS: A LOCALIZAÇÃO DA INTERFACE

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**Abstract**

O objetivo deste trabalho é estudar a localização de camadas de transição interna para determinadas soluções de uma classe de problemas elípticos não-homogêneos, postos num intervalo da reta, e condições de fronteira de Neumann. Nós generalizamos alguns resultados conhecidos usando técnicas variacionais inspiradas na teoria de  $\Gamma$ -convergência. Como aplicação, apresentamos a localização das camadas de transição interna para problemas postos em algumas variedades Riemannianas simétricas.

## 1 Introdução

Quando uma equação diferencial contém um parâmetro pequeno multiplicando o termo com derivadas espaciais e este este parâmetro vai a zero, grosseiramente falando, dizemos que a família de soluções a este parâmetro desenvolve uma camada de transição interna se ela induz uma partição no domínio em duas regiões onde, exceto por uma região “tubular” – a chamada *interface* da camada de transição – as soluções se aproximam de duas funções pré-determinadas (uma em cada região). Soluções desenvolvendo camadas de transição interna possuem um importante papel em muitas áreas da ciência aplicada, por exemplo: teoria da combustão, transição de fases, formação de padrões, dinâmica populacional, reações químicas, etc.

Neste trabalho contribuímos na tarefa de fornecer a localização exata da interface de determinada classe de soluções do seguinte problema singularmente perturbado

$$\begin{cases} \epsilon^2(k(x)u'(x))' + f(u, x) = 0, & x \in (0, 1), \\ u'(0) = u'(1) = 0, \end{cases} \quad (1)$$

onde  $k(\cdot) \in C^1(0, 1)$  é positivo;  $\epsilon > 0$  é um parâmetro positivo e  $f : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$  é de classe  $C^1$ . Assumimos que

- $f(\cdot, x)$  tem dois zeros  $b_1(x), b_2(x)$  tais que  $b_1, b_2 \in C^1(0, 1)$  e  $b_1(x) < b_2(x)$  para todo  $x \in [0, 1]$ ;
- $\partial_1 f(b_1(x), x) < 0$  e  $\partial_1 f(b_2(x), x) < 0$  para todo  $x \in [0, 1]$ ;
- se

$$F(u, x) = - \int_{b_1(x)}^u f(s, x) ds \quad (2)$$

então  $F(\cdot, x) \geq 0$  para todo  $x \in [0, 1]$  e  $\sqrt{k(\cdot)F(\cdot, \cdot)}$  é Lipschitz contínua.

Um típico exemplo de uma função  $f$  satisfazendo as condições acima é

$$f(u, x) = -(u - b_1(x))(u - a(x))(u - b_2(x)), \quad (3)$$

com  $b_1(\cdot), a(\cdot), b_2(\cdot) \in C^1(0, 1)$  e  $b_1(x) < a(x) < b_2(x)$  (com  $a \geq (b_1 + b_2)/2$ ) para todo  $x \in [0, 1]$ . Esta função está relacionada ao problema de Allen-Cahn não-homogêneo que tem sua origem na *teoria de transição de fases* e é usado como modelo para diversos processos de reação e difusão não-lineares.

## 2 Resultado Principal

As soluções de  $(P)$  são pontos críticos do funcional de energia  $\tilde{J}_\epsilon : H^1(0, 1) \rightarrow \mathbb{R}$  definido por

$$\tilde{J}_\epsilon(u) = \int_0^1 \frac{\epsilon}{2} k(x) |u'|^2 + \frac{1}{\epsilon} F(u, x) dx,$$

onde  $F$  foi definido em  $(2)$ . No entanto, nosso principal resultado requer extender este funcional para  $L^1(0, 1)$ ; i.e. consideramos  $J_\epsilon : L^1(0, 1) \rightarrow \mathbb{R} \cup \{\infty\}$  definido por penalização em  $L^1(0, 1)$  por

$$J_\epsilon(u) = \begin{cases} \tilde{J}_\epsilon(u), & u \in H^1(0, 1), \\ \infty, & u \in L^1(0, 1) \setminus H^1(0, 1). \end{cases} \quad (1)$$

**Definição 2.1.** Uma família  $\{u_\epsilon\}$  de soluções de  $(P)$  em  $C^2(0, 1) \cap C^1[0, 1]$  é dita desenvolver uma camada de transição interna, quando  $\epsilon \rightarrow 0$ , com interface em  $\bar{x} \in (0, 1)$  se

$$u_\epsilon \xrightarrow{\epsilon \rightarrow 0} u_0 := b_2 \chi_{[0, \bar{x}]} + b_1 \chi_{[\bar{x}, 1]} \text{ em } L^1(0, 1). \quad (2)$$

A fim de afirmar nosso resultado principal, definimos a seguinte função  $\Lambda : (0, 1) \rightarrow \mathbb{R}$ ,

$$\Lambda(x) := \int_{b_1(x)}^{b_2(x)} \sqrt{k(s)F(s, x)} ds \quad (3)$$

e o conjunto

$$\mathcal{Q} = \left\{ x \in (0, 1); \int_{b_1(x)}^{b_2(x)} f(s, x) ds = 0 \right\}. \quad (4)$$

O resultado principal é afirmado abaixo.

**Teorema 2.1.** Suponha que uma família  $\{u_\epsilon\}$  de soluções de  $(P)$  desenvolve uma camada de transição interna com interface em  $\bar{x} \in \mathcal{C}$ , onde  $\mathcal{C} \subset \mathcal{Q}$  é a componente conexa de  $\mathcal{Q}$  na qual  $\bar{x}$  está. Então,

- se  $\{u_\epsilon\}$  é uma família de mínimos locais em  $L^1$  de  $J_\epsilon$ ,  $\bar{x}$  é um ponto de mínimo local de  $\Lambda(x)$  em  $\mathcal{C}$ ;
- se  $\{u_\epsilon\}$  é uma família de mínimos globais de  $\tilde{J}_\epsilon$ ,  $\Lambda(\bar{x}) = \min\{\Lambda(x); x \in \mathcal{C}\}$ .

Este conteúdo está presente no trabalho [1].

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# ELLIPTIC SYSTEMS INVOLVING SCHRÖDINGER OPERATORS WITH VANISHING POTENTIALS.

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## Abstract

We prove the existence of a bounded positive solution of the following elliptic system involving Schrödinger operators

$$\begin{cases} -\Delta u + V_1(x)u = \lambda\rho_1(x)(u+1)^r(v+1)^p & \text{in } \mathbb{R}^N \\ -\Delta v + V_2(x)v = \mu\rho_2(x)(u+1)^q(v+1)^s & \text{in } \mathbb{R}^N, \\ u(x), v(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty \end{cases}$$

where  $p, q, r, s \geq 0$ ,  $V_i$  is a nonnegative vanishing potential, and  $\rho_i$  has the property (H) introduced by Brezis and Kamin [3]. As in that celebrated work we will prove that for every  $R > 0$  there is a solution  $(u_R, v_R)$  defined on the ball of radius  $R$  centered at the origin. Then, we will show that this sequence of solutions tends to a bounded solution of the previous system when  $R$  tends to infinity. Furthermore, by imposing some restrictions on the powers  $p, q, r, s$  without additional hypotheses on the weights  $\rho_i$ , we obtain a second solution using variational methods for a gradient system.

## 1 Introdução

We first study the existence of a bounded positive solution of the system

$$\begin{cases} -\Delta u + V_1(x)u = \lambda\rho_1(x)(u+1)^r(v+1)^p & \text{in } \mathbb{R}^N \\ -\Delta v + V_2(x)v = \mu\rho_2(x)(u+1)^q(v+1)^s & \text{in } \mathbb{R}^N, \\ u(x), v(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty \end{cases} \quad (\mathbf{S}_{\lambda,\mu})$$

where  $\lambda, \mu > 0$  and  $p, q, r, s \geq 0$ , and where  $V_i$  is a vanishing potential satisfying

$$\frac{a_i}{1+|x|^\alpha} \leq V_i(x) \leq \frac{A_i}{1+|x|^\alpha} \quad \text{for all } x \in \mathbb{R}^N, \quad (H_V^\alpha)$$

for some constants  $\alpha > 0$  and  $A_i, a_i \geq 0$ ,  $i = 1, 2$ . The weight  $\rho_i$  belongs to  $L^\infty(\mathbb{R}^N)$  and satisfies

$$0 < \rho_i(x) \leq \frac{k_i}{1+|x|^\beta} \quad \text{in } \mathbb{R}^N, \quad (H_\rho)$$

for some constants  $\beta > 2$  and  $k_i > 0$ ,  $i = 1, 2$ . In this work, assuming the conditions  $(H_V^\alpha)$ ,  $(H_\rho)$  and using the upper and lower solutions technique, we first prove the existence of a bounded positive solution of System  $(\mathbf{S}_{\lambda,\mu})$ . As far as we know, the first work for elliptic systems using the ideas of [3], was done by Montenegro [3], where uniqueness of solution in balls also plays an important role. Since System  $(\mathbf{S}_{\lambda,\mu})$  in bounded domains does not have this property, we will have to use an alternative argument that involves minimal solutions.

## 2 Resultados Principais

Let us state our first result.

**Theorem 2.1.** *Assume that  $p, q, r, s \geq 0$  and in addition suppose hypotheses  $(H_p)$  and  $(H_V^\alpha)$  hold with  $\alpha \in (0, 2]$  and  $\alpha + \beta > 4$ . Then, there exists  $\Lambda > 0$  such that System  $(S_{\lambda,\mu})$  has at least one bounded positive solution for every  $0 < \lambda, \mu < \Lambda$ .*

When  $r, s > 1$  we can construct a function that is the border between the region of existence and nonexistence.

**Theorem 2.2.** *Suppose hypotheses  $(H_p)$  and  $(H_V^\alpha)$  hold with  $\alpha \in (0, 2]$  and  $\alpha + \beta > 4$ . Assume also that  $r, s > 1$  and  $p, q \geq 0$ . Then, there is a positive constant  $\lambda^*$  and a nonincreasing continuous function  $\Gamma : (0, \lambda^*) \rightarrow [0, \infty)$  such that if  $\lambda \in (0, \lambda^*)$  then System  $(S_{\lambda,\mu})$ :*

- i) *has at least one bounded positive solution if  $0 < \mu < \Gamma(\lambda)$  ;*
- ii) *has no bounded positive solution if  
 $\mu > \Gamma(\lambda)$ .*

On the other hand, the second positive solution will be obtained employing variational methods. Here we will consider the following gradient system

$$\begin{cases} -\Delta u + V(x)u = \lambda\rho_1(x)(u+1)^r(v+1)^{s+1} & \text{in } \mathbb{R}^N \\ -\Delta v + V(x)v = \lambda\rho_2(x)(u+1)^{r+1}(v+1)^s & \text{in } \mathbb{R}^N, \\ u(x), v(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases} \quad (1)$$

with  $r, s > 1$ ,  $r + s < 2^* - 2$ ,  $\rho_1(x) = (r+1)\rho(x)$  and  $\rho_2(x) = (s+1)\rho(x)$ .

**Theorem 2.3.** *Suppose hypotheses  $(H_p)$  and  $(H_V^\alpha)$  hold with  $\alpha \in (0, 2]$  and  $\alpha + \beta > 4$ ,*

- i) *If  $r, s \geq 0$ , then there exists  $\lambda^* > 0$  such that the gradient System (1) possesses at least one bounded positive solution  $(u_{1,\lambda}, v_{1,\lambda})$  for all  $0 < \lambda < \lambda^*$  while for  $r, s > 1$  and  $\lambda > \lambda^*$  there are no bounded positive solutions.*
- ii) *If  $r, s > 1$  and  $r + s < 2^* - 2$ , then there exists  $0 < \lambda^{**} \leq \lambda^*$  such that the gradient System (1) possesses a second positive solution of the form  $(u_{1,\lambda} + u, v_{1,\lambda} + v)$  for all  $0 < \lambda < \lambda^{**}$ , where  $u, v \in H^1(\mathbb{R}^N)$ .*

We would like to point out that in Theorem 2.3, to show existence of a second solution we will use an auxiliary problem which allow us to avoid imposing additional hypotheses of integrabilities on the weights  $\rho_i$ . We also prove a similar result for a class of Hamiltonian system.

This is a joint work with Juan Arratia (Universidad de Santiago de Chile) and Pedro Ubilla (Universidad de Santiago de Chile) to appear at Discrete and Continuous Dynamical Systems.

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## GROUND STATES FOR FRACTIONAL LINEAR COUPLED SYSTEMS VIA PROFILE DECOMPOSITION

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### Abstract

In this work we study existence of weak and ground state (least energy) solutions for a class of nonlocal linearly coupled elliptic systems. We deal with nonautonomous nonlinearities that may not satisfy any kind of monotonicity, also the related potentials may not have any kind of smoothness. In order to obtain ground states, instead of applying the well known methods of Nehari-Pohozaev manifold, we introduce new arguments and techniques whose are based on a Pohozaev type identity, a concentration–compactness principle and a profile decomposition type result.

### 1 Introduction

In this work we study the following class of linearly coupled fractional systems

$$\begin{cases} (-\Delta)^s u + V_1(x)u = f_1(x, u) + \lambda(x)v, & x \in \mathbb{R}^N, \\ (-\Delta)^s v + V_2(x)v = f_2(x, v) + \lambda(x)u, & x \in \mathbb{R}^N, \end{cases} \quad (\mathcal{S})$$

where  $(-\Delta)^s$  denotes the fractional Laplacian operator for  $s \in (0, 1)$ . Coupled elliptic systems arise in various branches of mathematical physics and nonlinear optics.

Our main motivation to study  $(\mathcal{S})$  is based on the following question: is it possible to develop a general argument to obtain ground states for the class of coupled systems  $(\mathcal{S})$  (in particular the scalar equation  $\lambda(x) \equiv 0$ ), when the involved nonlinear terms does not satisfy any conditions such as

$$(N) \quad \frac{f(t)}{|t|} \text{ is nondecreasing on } t \in \mathbb{R} \setminus \{0\},$$

or the related potentials are not necessarily smooth? See [1, 3, 2] for further discussion.

Our purpose here in this work is to study System  $(\mathcal{S})$  inspired by the above question considering that the nonlinearities are superlinear. Roughly speaking, we replace the use of Nehari-Pohozaev manifold method by the use a technique based on concentration-compactness via profile decomposition for weak convergence in fractional Sobolev spaces and the use of a Pohozaev type identity. In order to approach in this way, we assume the existence of a limit system associated with  $(\mathcal{S})$  as  $|x| \rightarrow \infty$ . More precisely, we first study the following nonlocal elliptic problem

$$\begin{cases} (-\Delta)^s u + V_1(\infty)u = f_1(\infty, u) + \lambda(\infty)v, & x \in \mathbb{R}^N, \\ (-\Delta)^s v + V_2(\infty)v = f_2(\infty, v) + \lambda(\infty)u, & x \in \mathbb{R}^N, \end{cases} \quad (\mathcal{S}_\infty)$$

which is obtained by taking  $|x| \rightarrow \infty$  in  $(\mathcal{S})$  and comparing its minimax level with the one of  $(\mathcal{S}_\infty)$ . Here  $V_1(\infty)$ ,  $V_2(\infty)$  and  $\lambda(\infty)$  are constants with  $f_1(\infty, u)$  and  $f_2(\infty, v)$  being autonomous functions.

Next, for each  $i = 1, 2$  we assume  $0 < s_i < \min\{1, N/2\}$  and the following general hypotheses:

(A<sub>1</sub>)  $V_i(x) \geq 0$  almost everywhere (a.e.) in  $\mathbb{R}^N$ ,  $V_i \in L_{\text{loc}}^{\sigma_i}(\mathbb{R}^N)$ ,  $\sigma_i > N/2s_i$  and

$$\inf_{\{u \in C_0^\infty(\mathbb{R}^N) : \|u\|_2=1\}} \left[ \int_{\mathbb{R}^N} |(-\Delta)^{s_i/2} u|^2 dx + \int_{\mathbb{R}^N} V_i(x) u^2 dx \right] > 0.$$

(A<sub>2</sub>)  $\lambda \in L^\infty(\mathbb{R}^N)$  and there exists  $\delta \in (0, 1)$  such that (s.t.)  $|\lambda(x)| \leq \delta \sqrt{V_1(x)V_2(x)}$  a.e.  $x \in \mathbb{R}^N$ .

(A<sub>3</sub>)  $V_i(\infty) := \lim_{|x| \rightarrow \infty} V(x) > 0$  and  $\lambda(\infty) := \lim_{|x| \rightarrow \infty} \lambda(x)$ .

For a.e.  $x \in \mathbb{R}^N$  we suppose that  $t \mapsto f_i(x, t)$  is  $C^1$  and satisfies the following assumptions:

(H<sub>1</sub>)  $\lim_{t \rightarrow +\infty} f_i(x, t)t^{-1} = +\infty$ , uniformly a.e.  $x \in \mathbb{R}^N$ .

(H<sub>2</sub>) For every compact  $L \subset \mathbb{R}$ , there exists  $C_L > 0$  s.t.  $|f_i(x, t)| \leq C_L$  a.e.  $x \in \mathbb{R}^N$  and  $t \in L$ .

(H<sub>3</sub>) Let  $\mathcal{F}_i(x, t) = (f_i(x, t)t)/2 - F_i(x, t)$ , then

$$\inf_{x \in \mathbb{R}^N} \left[ \inf_{a \leq |t| \leq b} \mathcal{F}_i(x, t) \right] > 0, \quad \forall b > a > 0.$$

(H<sub>4</sub>) There exists  $q_i > N/(2s_i)$ ,  $a_i > 0$  and  $R_i > 0$  such that

$$|f(x, t)|^{q_i} \leq a_i \mathcal{F}_i(x, t) |t|^{q_i}, \quad \forall |t| > R_i.$$

(H<sub>5</sub>) For any given  $\varepsilon > 0$ , there exist  $C_{\varepsilon_i} > 0$  and  $p_{\varepsilon_i} \in (2, 2_{s_i}^*)$  such that

$$\left| \frac{\partial f_i}{\partial t}(x, t) \right| \leq \varepsilon + C_{\varepsilon_i} |t|^{p_{\varepsilon_i}-2}, \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and } \forall t \in \mathbb{R}.$$

(H<sub>6</sub>)  $f_i(\infty, t) := \lim_{|x| \rightarrow \infty} f(x, t)$ , uniformly in compact sets of  $\mathbb{R}$ . We also assume that  $f_i(\infty, t) \in C^1(\mathbb{R})$  holds.

We denote  $I$  and  $I_\infty$  the energy functionals related to (S) and (S <sub>$\infty$</sub> ) respectively, with  $c(I)$  and  $c(I_\infty)$  being the associated mountain pass level. We say that a weak solution  $(u, v) \in H \setminus \{(0, 0)\}$  is a ground state of System (S) when  $I(u, v) \leq I(\hat{u}, \hat{v})$  for any other weak solution  $(\hat{u}, \hat{v}) \in H \setminus \{(0, 0)\}$ , where  $H$  is a suitable Sobolev space.

## 2 Main Results

**Theorem 2.1.** *Assume (A<sub>1</sub>)–(A<sub>3</sub>) and (H<sub>1</sub>)–(H<sub>6</sub>). If either  $c(I) < c(I_\infty)$  or  $I(u, v) \leq I_\infty(u, v)$  hold, then System (S) admits at least one ground state solution  $(u, v)$ . Moreover, if  $I(u, v) \leq I_\infty(u, v)$  then the ground state is at the mountain pass level, that is,  $I(u, v) = c(I)$ .*

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# REGULARIDADE DE INTERIOR PARA SOLUÇÕES DE EQUAÇÕES FRACIONÁRIAS QUE DEGENERAM COM O GRADIENTE

DISSON DOS PRAZERES<sup>1</sup><sup>1</sup>Universidade Federal de Sergipe, Brasil, disson@mat.ufs.br**Abstract**

Nesta palestra iremos falar sobre regularidade interior para “viscosity solutions” de problemas de Dirichlet não-locais que degeneram quando o gradiente da solução se anula. Apresentaremos estimativas Hölder quando a ordem da difusão é menor ou igual a 1 e estimativas Lipschitz quando a ordem da difusão é maior que 1. Além disso, no último caso, discutiremos a possibilidade de obter estimativas Hölder para o gradiente.

## 1 Introdução

Nesta apresentação discutiremos sobre regularidade interior para “viscosity solutions”  $u$  de problemas elípticos não-lineares da forma

$$-|Du(x)|^\gamma I(u, x) = f(x) \quad \text{for } x \in B_1, \quad (1)$$

onde  $\gamma > 0$ ,  $f \in L^\infty(B_1)$ ,  $Du(x)$  é o gradiente de  $u$  em  $x$ , e  $I(u, x)$  é um operador não-local uniformemente elíptico da forma

$$I(u, x) = \inf_i \sup_j I_{K_{ij}}(u, x) \quad (2)$$

onde

$$I_{K_{ij}}(u, x) = \text{P.V.} \int_{\mathbb{R}^N} [u(y) - u(x)] K_{ij}(x - y) dy. \quad (3)$$

Consideramos  $K_{ij} : \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}$  uma família de kernels simétricos tais que

$$\lambda \frac{C_{\sigma,N}}{|x|^{N+\sigma}} \leq K_{ij} \leq \Lambda \frac{C_{\sigma,N}}{|x|^{N+\sigma}}, \quad x \neq 0, \quad (4)$$

onde  $\sigma \in (0, 2)$  and  $0 < \lambda \leq \Lambda < \infty$ .

A principal dificuldade vêm da presença de  $|Du(x)|^\gamma$  em (1), pois quando o gradiente de  $u$  vai para zero a equação degenera. Ou seja, a informação que vêm da equação se perde quando o gradiente se anula.

## 2 Resultados Principais

Adaptamos para o nosso caso degenerado o método de Ishii-Lions não-local a fim de obter o seguinte resultado,

**Teorema 2.1.** *Sejam  $f \in L^\infty(B_1)$  e  $I$  um operador da forma (2). Seja  $u \in L_{loc}^\infty \cap L_\sigma^1$  uma “viscosity solution” do problema (1). Então,*

- *Se  $0 < \sigma < 1$  então  $u \in C^\sigma$  e*

$$[u]_{C^\sigma(B_{1/2})} \leq C(\|u\|_{L_\sigma^1(\mathbb{R}^N)} + \|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}).$$

- *Se  $\sigma = 1$ , então  $u \in C^\alpha$  para todo  $\alpha \in (0, 1)$ , e*

$$[u]_{C^\alpha(B_{1/2})} \leq C_\alpha(\|u\|_{L_\sigma^1(\mathbb{R}^N)} + \|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}).$$

- Se  $1 < \sigma < 2$  então  $u \in C^{0,1}$  e

$$[u]_{C^{0,1}(B_{1/2})} \leq C_0(\|u\|_{L_\sigma^1(\mathbb{R}^N)} + \|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}),$$

e a constante  $C_0$  é uniformemente limitada quando  $\sigma \rightarrow 2^-$ .

Além disto, quando  $\sigma$  está próximo de 2 o problema (1) está suficientemente próximo de um problema de segunda ordem para o qual estimativas  $C^{1,\alpha}$  disponível e podemos obter o seguinte resultado,

**Teorema 2.2.** *Sejam  $f \in L^\infty(B_1)$  e  $I$  como em (2) definido a partir da família de kernels  $\{K_{ij}\}_{ij}$ , adicionalmente satisfazendo a seguinte propriedade: existem um módulo de continuidade  $\omega$  e um conjunto  $\{k_{ij}\}_{i,j} \subset (\lambda, \Lambda)$  tal que*

$$|K_{ij}(x)|x|^{N+\sigma} - k_{ij}| \leq \omega(|x|), \quad |x| \leq 1. \quad (5)$$

Então, existe um  $\sigma_0 \in (1, 2)$  próximo de 2 tal que para  $\sigma_0 < \sigma < 2$  toda “viscosity solution”  $u$  para (1) é  $C^{1,\alpha}$  para algum  $\alpha \in (0, 1)$ , e

$$[u]_{C^{1,\alpha}(B_{1/2})} \leq C_0(\|u\|_\infty + \|f\|_\infty^{\frac{\sigma-1}{1+\gamma}}).$$

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## SUPERLINEAR FRACTIONAL ELLIPTIC PROBLEMS VIA THE NONLINEAR RAYLEIGH QUOTIENT WITH TWO PARAMETERS

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### Abstract

It is establish existence of weak solutions for nonlocal elliptic problems driven by the fractional Laplacian where the nonlinearity is indefinite in sign. More specifically, we shall consider the following nonlocal elliptic problem

$$\begin{cases} (-\Delta)^s u + V(x)u = \mu a(x)|u|^{q-2}u - \lambda|u|^{p-2}u \text{ in } \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \end{cases}$$

where  $s \in (0, 1)$ ,  $s < N/2$ ,  $N \geq 1$  and  $\mu, \lambda > 0$ . The potentials  $V, a : \mathbb{R}^N \rightarrow \mathbb{R}$  satisfy some extra assumptions. The main feature is to find sharp parameters  $\lambda > 0$  and  $\mu > 0$  where the Nehari method can be applied. In order to do that we employ the nonlinear Rayleigh quotient together a fine analysis on the fibering maps associated to the energy functional.

## 1 Introduction

In the present talk we shall consider nonlocal elliptic problems driven by the fractional Laplacian defined in the whole space where the nonlinearity is superlinear at infinity and at the origin. Namely, we shall consider the following nonlocal elliptic problem

$$\begin{cases} (-\Delta)^s u + V(x)u = \mu a(x)|u|^{q-2}u - \lambda|u|^{p-2}u \text{ in } \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \end{cases} \quad (1)$$

where  $s \in (0, 1)$ ,  $s < N/2$ ,  $N \geq 1$ . Furthermore, we assume that  $2 < q < p < 2_s^* = 2N/(N - 2s)$  and  $\mu, \lambda > 0$ . Assume also that  $V : \mathbb{R}^N \rightarrow \mathbb{R}$  is a continuous function and  $a : \mathbb{R}^N \rightarrow \mathbb{R}$  is nonnegative measurable function. It is important to recall that the main difficult in order to consider weak solutions for Problem (1) comes from the fact that the nonlinear term  $g_{\lambda, \mu}(x, t) = \mu a(x)|t|^{q-2}t - \lambda|t|^{p-2}t$ ,  $x \in \mathbb{R}^N$ ,  $t \in \mathbb{R}$  is indefinite in sign. In fact, we observe that

$$\lim_{t \rightarrow 0} \frac{g_{\lambda, \mu}(x, t)}{t} = 0, \quad \lim_{t \rightarrow \infty} \frac{g_{\lambda, \mu}(x, t)}{t} = -\infty \quad (2)$$

and  $g_{\lambda, \mu}(x, t) > 0$  for each  $t \in (0, \delta)$ ,  $x \in \mathbb{R}^N$  for some  $\delta > 0$ . Hence, we obtain that  $g_{\lambda, \mu}(x, t)$  is a sign changing nonlinearity. Semilinear elliptic problems have widely considered in the last years since the seminal work [1].

## 2 Main Results

As was told in the introduction we shall consider existence and nonexistence of nontrivial weak solutions for the Problem (1) looking for the parameters  $\lambda > 0$  and  $\mu > 0$ . The main idea here is to ensure sharp conditions on the parameters  $\lambda$  and  $\mu$  such that the Nehari method together with the nonlinear Rayleigh quotient can be applied, see [2, 3]. Throughout this work we assume the following assumptions:

- (Q) It holds  $\mu, \lambda > 0$  and  $2 < q < p < 2_s^* = 2N/(N - 2s)$ ;
- $(V_0)$  The potential  $V : \mathbb{R}^N \rightarrow \mathbb{R}$  is continuous function such that  $V(x) \geq V_0 > 0$  for all  $x \in \mathbb{R}^N$ ;
- $(V_1)$  For each  $M > 0$  it holds that  $|\{x \in \mathbb{R}^N : V(x) \leq M\}| < +\infty$ .
- $(a_0)$  It holds that  $a \in L^\infty(\mathbb{R}^N)$  where  $a(x) > 0$  a. e. in  $x \in \mathbb{R}^N$ .

It is important to mention that the working space for our work is defined by

$$X = \left\{ v \in H^s(\mathbb{R}^N) : \int V(x)v^2 dx < +\infty \right\}.$$

Notice that  $X$  is a Hilbert space. It is worthwhile to emphasize that the energy functional  $E_{\lambda,\mu} : X \rightarrow \mathbb{R}$  associated to Problem (1) is given by

$$E_{\lambda,\mu}(u) = \frac{1}{2}\|u\|^2 - \frac{\mu}{q}\|u\|_{q,a}^q + \frac{\lambda}{p}\|u\|_p^p, \quad u \in X, \quad (1)$$

where

$$\|u\|_{q,a}^q = \int a(x)|u|^q dx \text{ and } \|u\|_p^p = \int |u|^p dx, \quad u \in X.$$

Under our hypotheses we observe that  $E_{\lambda,\mu}$  belongs to  $C^2(X, \mathbb{R})$  for each  $\lambda > 0$  and  $\mu > 0$ . Moreover, a function  $u \in X$  is a critical point for the functional  $E_{\lambda,\mu}$  if and only if  $u$  is a weak solution to the elliptic Problem (1). Now, by using the same ideas introduced we shall consider the Nehari method for our main Problem (1). As a product, we shall state our first main result as follows:

**Theorem 2.1.** *Suppose (Q),  $(V_0) - (V_1)$  and  $(a_0)$ . Then for each  $\lambda > 0$  we obtain that  $0 < \mu_n < \mu_e < \infty$ . Furthermore, there exists  $\lambda_* > 0$  such that for each  $\mu > \mu_n$  Problem (1) admits at least a weak solution  $u_{\lambda,\mu} \in X$  whenever  $\lambda \in (0, \lambda_*)$  which it satisfies the following assertions:  $E''_{\lambda}(u_{\lambda,\mu})(u_{\lambda,\mu}, u_{\lambda,\mu}) < 0$  and there exists  $D_\mu > 0$  such that  $E_{\lambda,\mu}(u_{\lambda,\mu}) \geq D_\mu$  and  $u_{\lambda,\mu} \rightarrow 0$  in  $X$  as  $\mu \rightarrow \infty$ .*

Now we assume the following hypothesis:

- $(a_1)$  It holds that  $a \in L^\infty(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$  with  $r = (p/q)' = p/(p - q)$  and  $a(x) > 0$  a.e. in  $x \in \mathbb{R}^N$ .

Hence, we can written our next main result in the following form:

**Theorem 2.2.** *Suppose (Q),  $(V_0) - (V_1)$  and  $(a_1)$ . Then for each  $\lambda > 0$  we obtain that  $0 < \mu_n < \mu_e < \infty$ . Furthermore, there exits  $\lambda^* > 0$  such that for each  $\mu > \mu_n$  Problem (1) admits at least a ground state solution  $v_{\lambda,\mu} \in X$  taking into account one of the following conditions:  $\mu \in [\mu_e, \infty)$ ,  $\lambda > 0$  and  $\mu \in (\mu_n, \mu_e)$ ,  $\lambda \in (0, \lambda^*)$ . Moreover, the weak solution  $v_{\lambda,\mu}$  satisfies the following assertions: It holds that  $E''_{\lambda}(v_{\lambda,\mu})(v_{\lambda,\mu}, v_{\lambda,\mu}) > 0$ . Moreover,  $\|v_{\lambda,\mu}\| \rightarrow \infty$  in  $X$  as  $\mu \rightarrow \infty$ . For each  $\mu \in (\mu_n, \mu_e)$  we obtain that  $E_{\lambda,\mu}(v_{\lambda,\mu}) > 0$ . For  $\mu = \mu_e$  it follows that  $E_{\lambda}(v_{\lambda,\mu}) = 0$ . For each  $\mu > \mu_e$  we obtain also that  $E_{\lambda}(v_{\lambda,\mu}) < 0$ .*

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EXISTENCE OF SOLUTIONS FOR A FRACTIONAL CHOQUARD–TYPE EQUATION IN  $\mathbb{R}$  WITH  
CRITICAL EXPONENTIAL GROWTH

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**Abstract**

In this work we study the following class of fractional Choquard–type equations

$$(-\Delta)^{1/2}u + u = \left(I_\mu * F(u)\right)f(u), \quad x \in \mathbb{R},$$

where  $(-\Delta)^{1/2}$  denotes the  $1/2$ –Laplacian operator,  $I_\mu$  is the Riesz potential with  $0 < \mu < 1$  and  $F$  is the primitive function of  $f$ . We use Variational Methods and minimax estimates to study the existence of solutions when  $f$  has critical exponential growth in the sense of Trudinger–Moser inequality.

## 1 Introduction

This talk is based on [4], here we are concerned with existence of solutions for a class of fractional Choquard–type equations

$$(-\Delta)^s u + u = \left(I_\mu * F(u)\right)f(u), \quad x \in \mathbb{R}^N, \tag{1}$$

where  $(-\Delta)^s$  denotes the fractional Laplacian,  $0 < s < 1$ ,  $0 < \mu < N$ ,  $F$  is the primitive function of  $f$ ,  $I_\mu : \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}$  is the Riesz potential defined by

$$I_\mu(x) := \mathcal{A}_\mu \frac{1}{|x|^{N-\mu}}, \quad \text{where } \mathcal{A}_\mu := \frac{\Gamma\left(\frac{N-\mu}{2}\right)}{\Gamma\left(\frac{\mu}{2}\right) \pi^{\frac{N}{2}} 2^\mu},$$

and  $\Gamma$  denotes the Gamma function. We consider the “limit case” when  $N = 1$ ,  $s = 1/2$  and a Choquard–type nonlinearity with critical exponential growth motivated by a class of Trudinger–Moser inequality. The main difficulty is to overcome the “lack of compactness” inherent to problems defined on unbounded domains or involving nonlinearities with critical growth. In order to apply properly the Variational Methods, we control the minimax level with fine estimates involving Moser functions (see [7]), but here in the context of fractional Choquard–type equation.

Nonlinear elliptic equations involving nonlocal operators have been widely studied both from a pure mathematical point of view and their concrete applications, since they naturally arise in many different contexts, such as, among the others, obstacle problems, flame propagation, minimal surfaces, conservation laws, financial market, optimization, crystal dislocation, phase transition and water waves, see for instance [2, 3] and references therein.

Inspired by [1], our goal is to establish a link between Choquard–type equations,  $1/2$ –fractional Laplacian and nonlinearity with critical exponential growth. We are interested in the following class of problems

$$(-\Delta)^{1/2}u + u = \left(I_\mu * F(u)\right)f(u), \quad x \in \mathbb{R}, \tag{P}$$

where  $F$  is the primitive of  $f$ . In order to use a variational approach, the maximal growth is motivated by the Trudinger–Moser inequality first given by T. Ozawa [6] and later extended by S. Iula, A. Maalaoui, L. Martinazzi [5]. Precisely, it holds

$$\sup_{\substack{u \in H^{1/2}(\mathbb{R}) \\ \|(-\Delta)^{1/4}u\|_2 \leq 1}} \int_{\mathbb{R}} (e^{\alpha u^2} - 1) dx \begin{cases} < \infty, & \alpha \leq \pi, \\ = \infty, & \alpha > \pi. \end{cases}$$

In this work we suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function satisfying the following hypotheses:

- ( $f_1$ )  $f(t) = 0$ , for all  $t \leq 0$  and  $0 \leq f(t) \leq Ce^{\pi t^2}$ , for all  $t \geq 0$ ;
- ( $f_2$ ) There exist  $t_0, C_0 > 0$  and  $a \in (0, 1]$  such that  $0 < t^a F(t) \leq C_0 f(t)$ , for all  $t \geq t_0$ ;
- ( $f_3$ ) There exist  $p > 1 - \mu$  and  $C_p = C(p) > 0$  such that  $f(t) \sim C_p t^p$ , as  $t \rightarrow 0$ ;
- ( $f_4$ ) There exists  $K > 1$  such that  $KF(t) < f(t)t$  for all  $t > 0$ , where  $F(t) = \int_0^t f(\tau) d\tau$ ;
- ( $f_5$ )  $\liminf_{t \rightarrow +\infty} \frac{F(t)}{e^{\pi t^2}} = \sqrt{\beta_0}$  with  $\beta_0 > 0$ .

## 2 Main Results

We are in condition to state our main result:

**Theorem 2.1.** *Suppose that  $0 < \mu < 1$  and assumptions  $(f_1) - (f_5)$  hold. Then, Problem (P) has a nontrivial weak solution.*

**Remark 2.1.** *Though there have been many works on the existence of solutions for problem (1), as far as we know, this is the first work considering a fractional Choquard-type equation involving 1/2-Laplacian operator and nonlinearity with critical exponential growth. Particularly, our Theorem 2.1 is a version of Theorem 1.3 of [1] for 1/2-Laplacian operator.*

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## CRITICAL METRICS OF THE $\mathcal{S}^k$ OPERATOR

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### Abstract

Given a smooth compact Riemannian  $n$ -manifold  $(M, g)$  with positive scalar curvature, we prove that any complete critical metric of the  $L^k$ -norm of the scalar curvature, has constant scalar curvature.

## 1 Introduction

Let  $(M^n, g)$ ,  $n \geq 3$ , be a  $n$ -dimensional smooth Riemannian manifold and consider the functional

$$\mathcal{S}^k(g) = \int_M R^k dV_g \quad (1)$$

on the space of Riemannian metrics on  $M^n$ , where  $k \in \mathbb{N}$ ,  $R_g$  and  $dV_g$  denote the scalar curvature and the volume for of  $g$  respectively. In the case  $k = 2$ , Giovanni Catino [4] proved the following theorem

**Theorem 1.1.** *Let  $(M^n, g)$ ,  $n \geq 3$ , be a complete critical metric of  $S^2$  with positive scalar curvature. Then  $(M^n, g)$  has constant scalar curvature.*

Urging for a more general result, we calculated the first variation of  $S^k(g)$ , using derivatives formulas (see [3]) in the direction of  $h$  ( $g(t) = g + th$ )

$$\begin{aligned} \delta S^k(g)[h] &= \int_M (kR^{k-1}\delta R + \frac{1}{2}R^k tr(h))dV_g \\ &= \int_M (-kR^{k-1}\Delta_g tr(h) + kR^{k-1}div^2(h) - kR^{k-1} < Ric, h >_g + \frac{1}{2}R^k tr(h))dV_g \\ &= \int_M (-k\Delta_g R^{k-1}g + k\nabla_g^2 R^{k-1} - kR^{k-1}Ric + \frac{1}{2}R^k g)hdV_g. \end{aligned}$$

**Remark 1.1.** *We take  $h$  with compact support, such that we can apply the Divergence Theorem for  $(2,0)$ -tensor.*

Hence, the Euler Lagrange equation for a critical metric of  $S^k$  in the direction of  $h$  is given by

$$R^{k-1}Ric - \nabla_g^2(R^{k-1}) + \Delta_g(R^{k-1})g = \frac{1}{2}\frac{R^k}{k}g. \quad (2)$$

By induction, we can proof that

$$\nabla_g^2(R^{k-1})(X, Y) = (k-1)R^{k-2}\nabla_g^2R(X, Y) + (k-1)(k-2)R^{k-3}X(R)Y(R). \quad (3)$$

By (2) and (3)

$$\Delta_g R = \left( \frac{(n-2k)}{2k(n-1)(k-1)} \right) R^2 - (k-2)\frac{|\nabla_g R|^2}{R} \quad (4)$$

By above equalities; any critical metric of  $\mathcal{S}^k$  is scalar flat if  $n$  is odd, whereas it is either scalar flat or Einstein if  $n = 2k$ .

In this paper we will focus on complete critical metrics of  $\mathcal{S}^k$ . As far as we know, complete critical metrics of  $\mathcal{S}^k$  were not studied yet. Our main result characterizes critical metrics with positive scalar curvature.

**Theorem 1.2.** Let  $(M^n, g)$ ,  $n \geq 3$ , be a complete critical metric of  $\mathcal{S}^k$  with positive scalar curvature and  $k \geq 2 \in \mathbb{N}$ . Then  $(M^n, g)$  has constant scalar curvature.

**Theorem 1.3.** Let  $(M^n, g)$ ,  $n \geq 3$ , be a complete critical metric of  $\mathcal{S}^k$  with positive scalar curvature and  $k \geq 2 \in \mathbb{N}$ . If  $n < 2k$ , then  $(M^n, g)$  is scalar flat.

In particular, from equations (2) and (3), if  $n \neq 2k$ , there are no complete critical metrics of  $\mathcal{S}^k$  with positive scalar curvature, whereas, every complete  $2k$ -dimensional critical metric  $\mathcal{S}^k$  with positive scalar curvature is either flat or Einstein with positive scalar curvature.

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**EXISTENCE OF SOLUTION FOR IMPLICIT ELLIPTIC EQUATIONS INVOLVING THE  
P-LAPLACE OPERATOR**

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**Abstract**

In this research we will study the existence of weak solutions for a class of implicit elliptic equations involving the  $p$ -Laplace Operator. Using a Krasnoselskii-Schaefer type theorem we establish our result.

## 1 Introduction

In this article we focus on the following boundary problem

$$\begin{aligned} -\Delta_p u &= f(x, u, \nabla u, \Delta_p u) + g(x, u, \nabla u)|u|_s^t && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma \end{aligned} \tag{1}$$

where  $\Omega$  is a bounded domain with smooth boundary  $\Gamma$  in  $\mathbb{R}^n (n \geq 3)$ ,  $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $2 \leq p < +\infty$  and  $s > 1$ ,  $t \geq 0$ .

Implicit elliptic equations have been intensively studied in the literature, see for example [1, 3]. More recently Precup [4] studied the case  $p = 2$ ,  $t = 0$ . Motivated by the above works, we are devoted to study problem (1.1).

## 2 Assumptions and Main Results

We give the following hypotheses.

(A<sub>1</sub>) There exist  $a, b, c \geq 0$  such that

$$|f(x, y, z, w) - f(x, \bar{y}, \bar{z}, \bar{w})| \leq a|y - \bar{y}| + b|z - \bar{z}|^{p-1} + c|w - \bar{w}|, \quad f(\cdot, 0, 0, 0) \in L^p(\Omega)$$

(A<sub>2</sub>) There exist constants  $a_0, b_0 \geq 0$ ,  $\alpha \in [1, p^*/(p^*)']$ ,  $\beta \in [1, p/(p^*)']$ ; and  $h \in L^p(\Omega)$  such that

$$|g(x, y, z)| \leq a_0|y|^\alpha + b_0|z|^\beta + h(x), \quad \forall y \in \mathbb{R}, z \in \mathbb{R}^n \quad \text{and} \quad a.e \ x \in \Omega$$

(A<sub>3</sub>)  $yg(x, y, z) \leq \sigma|y|^p$ ,  $\forall y \in \mathbb{R}, z \in \mathbb{R}^n$  a.e  $x \in \Omega$ , for some  $\sigma < \sigma_0 \lambda_1$ ,  $0 < \sigma_0 < 1$ ,  $\lambda_1$  is the first eigenvalue of  $(-\Delta_p, W_0^{1,p}(\Omega))$

(A<sub>4</sub>)  $\ell := \frac{a}{\lambda_1} + \frac{b}{\lambda_1^{1,p}} + c < 1$ ,  $G_0 = 1 - \ell$

Our main result is the following theorem

**Theorem 2.1.** *Let  $(p^*)' \leq \tau \leq p$ . Suppose (A<sub>1</sub>) - (A<sub>4</sub>) hold. Then (1.1) has at least one weak solution  $u \in W_0^{1,p}(\Omega)$  with  $\Delta_p u \in L^\tau(\Omega)$ .*

**Proof** We transform (1.1) into an equivalent problem of fixed point, where the associated operator is a sum of a contraction with a completely continuous mapping. Then, we apply a result in [2].

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## UM SISTEMA DE TIPO SCHRÖDINGER-BORN-INFELD

GAETANO SICILIANO<sup>1,†</sup><sup>1</sup>Universidade de São Paulo, Instituto de Matemática e Estatística, São Paulo, sicilian@ime.usp.br**Abstract**

Apresentamos um sistema envolvendo a equação de Schrödinger não linear e a equação da electrostática de Born-Infeld e procuramos soluções no caso radial em  $\mathbb{R}^3$ . Dependendo do parâmetro  $p$  da não linearidade técnicas diferentes são usadas para mostrar a existência de soluções.

**1 Introdução e resultado principal**

Consideramos o seguinte sistema não linear de tipo Schrödinger-Born-Infeld

$$\begin{cases} -\Delta u + u + \phi u = |u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\operatorname{div}\left(\frac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right) = u^2 & \text{in } \mathbb{R}^3, \end{cases} \quad (1)$$

com  $p$  dado e nas incógnitas  $u, \phi : \mathbb{R}^3 \rightarrow \mathbb{R}$ .

Esse sistema aparece na busca de soluções estacionárias da equação de Schrödinger acoplada com a teoria eletromagnética de Born-Infeld, em lugar da clássica teoria de Maxwell. A vantagem dessa nova teoria é que elimina o problema da energia infinita que a Teoria de Maxwell associa à uma carga puntual. De fato, na Teoria de Maxwell a busca de ondas estacionárias leva ao sistema

$$\begin{cases} -\Delta u + u + \phi u = |u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \quad (2)$$

e é facil de ver que a solução fundamental  $\Phi$  do Laplaciano satisfaz  $\int_{\mathbb{R}^3} |\nabla\Phi|^2 = +\infty$ , ou seja a energia associada à uma carica puntual é infinita.

Contudo a desvantagem da electrodinâmica de Born-Infeld é que a equação do campo eléctrico, ou seja a segunda em (1), é não linear, quando na teoria de Maxwell por ser a equação de Poisson é muito mais simples.

Em dois trabalhos distintos nós provamos existência de soluções para o sistema (1) que é muito menos estudado do sistema (2). Nós usamos Métodos Variaciones, Teoria do Ponto Crítico e oportunas perturbações no sistema.

Em particular, em [1] com A. Azzollini (Università della Basilicata, IT) and A. Pomponio (Politecnico di Bari, IT) mostramos o seguinte resultado.

**Teorema 1.1.** *Pora cada  $p \in (5/2, 5)$ , o problem (1) possui uma solução radial de ground state, ou seja uma solução  $(u, \phi)$  que minimiza o funcional da ação entre todas as outras soluções.*

No trabalho [3] com Z. Liu (China University of Geosciences) mostramos a existência do ground state também por valores menores de  $p$  cobrindo o caso  $p \in (2, 5/2]$ . Além disso, provamos o resultado em presencia de uma não linearidade com crescimento crítico e abordamos o problema da multiplicidade de soluções encontrando infinitas soluções com níveis de energia que tende para  $+\infty$ .

Destacamos que no trabalho [1] para garantir a geometria do Paso da Montanha foi usado o “monotonicity trick” de Jeanjean, que consiste em introduzir um parâmetro de controle multiplicativo  $\lambda$  em um termo já presente na equação e mostrar que quando  $\lambda$  tende para 1 se obtém uma solução do sistema inicial. Por outro lado essa técnica não funciona por valores menores de  $p$ . Para contornar essa dificuldade, em [3] usamos um diferente método de perturbação que consiste em adicionar dessa vez na equação um termo contendo o parâmetro de controle  $\lambda$  e mandar o  $\lambda$  para 0. Nesse caso as contas são bem mais envolvidas mas mesmo assim conseguimos mostrar a geometria do Passo da Montanha e a condição de compacidade necessária para obter existência de soluções.

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**EQUAÇÕES DE SCHRÖDINGER QUASELINEARES COM POTENCIAIS SINGULARES E SE  
ANULANDO ENVOLVENDO NÃO LINEARIDADES COM CRESCIMENTO CRÍTICO  
EXPONENCIAL**

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**Abstract**

Neste trabalho nós estudamos e estabelecemos resultados de existência de solução fraca e de não existência de solução clássica para a seguinte classe de equações de Schrödinger

$$-\Delta_N u + V(|x|)|u|^{N-2}u = Q(|x|)h(u) \quad \text{em } \mathbb{R}^N,$$

em que  $N \geq 2$ ,  $V$  e  $Q$  são potenciais contínuos que podem ser ilimitados na origem ou se anularem no infinito e  $h$  é uma não linearidade que possui um crescimento crítico exponencial com respeito a desigualdade de Trudinger-Moser. Para atingirmos os nossos objetivos atacamos o problema usando uma abordagem variacional, bem como fizemos uso de uma desigualdade do tipo Trudinger-Moser e de resultados do tipo princípio da criticalidade simétrica.

## 1 Introdução

Aqui estamos interessados em estabelecer resultados de existência e de não existência de solução para a seguinte classe de problemas

$$\begin{cases} -\Delta_N u + V(|x|)|u|^{N-2}u = Q(|x|)h(u), & \text{se } x \in \mathbb{R}^N \\ u(x) \rightarrow 0, & \text{quando } |x| \rightarrow +\infty, \end{cases} \quad (\text{P})$$

em que  $N \geq 2$  e  $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$  denota o operador  $N$ -Laplaciano da função  $u$ . Primeiramente, para o estudo de existência de solução fraca, vamos considerar  $V$  e  $Q$  potenciais contínuos satisfazendo:

( $V_1$ )  $V : (0, +\infty) \rightarrow \mathbb{R}$ ,  $V(r) > 0$  para todo  $r > 0$  e existem constantes  $a > -N$  e  $a_0 > -N$  tais que

$$0 < \liminf_{r \rightarrow 0^+} \frac{V(r)}{r^{a_0}} \quad \text{e} \quad 0 < \liminf_{r \rightarrow +\infty} \frac{V(r)}{r^a}.$$

( $Q_1$ )  $Q : (0, +\infty) \rightarrow \mathbb{R}$ ,  $Q(r) > 0$  para todo  $r > 0$  e existem constantes  $b_0 > -N$  e  $b < a$  tais que

$$\limsup_{r \rightarrow 0^+} \frac{Q(r)}{r^{a_0}} < +\infty \quad \text{e} \quad \limsup_{r \rightarrow +\infty} \frac{Q(r)}{r^b} < +\infty.$$

Também pedimos que a não linearidade  $h : \mathbb{R} \rightarrow \mathbb{R}$  seja contínua e satisfaz:

( $H_1$ ) Existe  $\alpha_0 > 0$  tal que

$$\lim_{s \rightarrow +\infty} \frac{h(s)}{e^{\alpha s^{N/(N-1)}}} = \begin{cases} 0, & \forall \alpha > \alpha_0 \\ +\infty, & \forall \alpha < \alpha_0. \end{cases}$$

( $H_2$ )  $\lim_{s \rightarrow 0} \frac{h(s)}{s^{N-1}} = 0$ ;

(H<sub>3</sub>) Existe  $\mu > N$  tal que

$$0 \leq \mu H(s) := \mu \int_0^s h(t) dt \leq sh(s) \quad \text{para todo } s \in \mathbb{R} \setminus \{0\};$$

(H<sub>4</sub>) Existem  $\xi > 0$  e  $\kappa > N$  tais que

$$H(s) \geq \xi s^\kappa, \quad \forall s \geq 0;$$

(H<sub>5</sub>)  $h(s)/s^N$  é não decrescente para  $s > 0$ .

Assumindo tais hipóteses, definindo um espaço adequado, usando um resultado de imersão, uma desigualdade do tipo Trudinger-Moser, métodos variacionais e um resultado do tipo princípio da criticalidade simétrica temos o seguinte resultado.

**Teorema 1.1.** *Suponha que  $V$  e  $Q$  são potenciais satisfazendo (V<sub>1</sub>) e (Q<sub>1</sub>), respectivamente, e que  $h$  é uma não linearidade obedecendo as condições (H<sub>1</sub>) – (H<sub>4</sub>), então (P) possui uma solução fraca não nula e não negativa. Além disso, se  $h$  também satisfaz (H<sub>5</sub>), temos que (P) admite uma solução ground state.*

Por outro lado, se assumirmos  $V$ ,  $Q$  e  $h$  satisfazendo

( $\widehat{V}$ )  $V : (0, +\infty) \rightarrow \mathbb{R}$  é contínuo,  $V(r) \geq 0$  e existe  $a \in \mathbb{R}$  tal que

$$\limsup_{r \rightarrow +\infty} \frac{V(r)}{r^a} < +\infty;$$

( $\widehat{Q}$ )  $Q : (0, +\infty) \rightarrow \mathbb{R}$  é contínuo,  $Q(r) \geq 0$  e existe  $b \geq a$  tal que

$$\liminf_{r \rightarrow +\infty} \frac{Q(r)}{r^b} > 0;$$

( $\widehat{h}$ )  $h : \mathbb{R} \rightarrow \mathbb{R}$  é contínuo e existem  $\xi > 0$  e  $p \leq N - 1$  tais que

$$h(s) \geq \xi s^p, \quad \text{para todo } s > 0;$$

respectivamente, fazendo uso da formulação em coordenadas radiais do  $N$ -Laplaciano e usando argumentos de contradição obtemos nosso resultado de não existência de solução.

**Teorema 1.2.** *Assuma que as condições ( $\widehat{V}$ ), ( $\widehat{Q}$ ) e ( $\widehat{h}$ ) são satisfeitas. Então, o problema (P) não possui uma solução clássica radial e positiva.*

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THE LIMITING BEHAVIOR OF GLOBAL MINIMIZERS IN NON-REFLEXIVE ORLICZ-SOBOLEV SPACES

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**Abstract**

Let  $\Omega$  be a smooth, bounded  $N$ -dimensional domain. For each  $p > N$ , let  $\Phi_p$  be an N-function satisfying  $p\Phi_p(t) \leq t\Phi'_p(t)$  for all  $t > 0$ , and let  $I_p$  be the energy functional associated with the equation  $-\Delta_{\Phi_p} u = f(u)$  in the Orlicz-Sobolev space  $W_0^{1,\Phi_p}(\Omega)$ . We prove that  $I_p$  admits at least one global, nonnegative minimizer  $u_p$  which, as  $p \rightarrow \infty$ , converges uniformly on  $\overline{\Omega}$  to the distance function to the boundary  $\partial\Omega$ .

## 1 Introduction

Let  $\Omega$  be a smooth, bounded  $N$ -dimensional domain and denote by  $d_\Omega$  the distance function to the boundary  $\partial\Omega$ , defined by

$$d_\Omega(x) := \inf_{y \in \partial\Omega} |x - y|, \quad x \in \overline{\Omega}.$$

For each  $p > N$ , let  $\phi_p : [0, \infty) \rightarrow [0, \infty)$  be an increasing function of class  $C^1$  such that

$$p\Phi_p(t) \leq t\Phi'_p(t) \text{ for all } t > 0,$$

where  $\Phi_p : \mathbb{R} \rightarrow [0, \infty)$  is the N-function defined by

$$\Phi_p(t) := \int_0^t s\phi_p(|s|)ds, \quad t \in \mathbb{R}.$$

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function enjoying the following properties:

(f1)  $f(-t) + f(t) \geq 0$  for all  $t \geq 0$ ,

(f2)  $F$ , the primitive of  $f$  given by  $F(t) = \int_0^t f(s)ds$ , is strictly increasing on  $[0, \|d_\Omega\|_\infty]$ , and

(f3) there exist constants  $a, b, r$  and  $t_0$ , with  $a \geq 0$ ,  $b > 0$  and  $r, t_0 \geq 1$ , such that

$$0 \leq f(t) \leq a + bt^{r-1} \quad \text{for all } t \geq t_0.$$

Let  $W_0^{1,\Phi_p}(\Omega)$  be the Orlicz-Sobolev space generated by  $\Phi_p$  and consider the energy functional

$$I_p(u) := \int_{\Omega} \Phi_p(|\nabla u|)dx - \int_{\Omega} F(u)dx, \quad u \in W_0^{1,\Phi_p}(\Omega),$$

associated with the Dirichlet problem

$$\begin{cases} -\operatorname{div}(\phi_p(|\nabla u|)\nabla u) = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Under the above hypotheses on  $\phi_p$ , the N-function  $\Phi_p$  may grow at infinity faster than any polynomial (see [2]). If this is the case,  $\Phi_p$  does not satisfy the  $\Delta_2$ -condition and, consequently, neither  $W_0^{1,\Phi_p}(\Omega)$  is reflexive nor its modular functional  $u \mapsto \int_{\Omega} \Phi_p(|\nabla u|)dx$  is of class  $C^1$ .

Considering these facts and taking into account that the modular functional is always convex and sequentially lower semicontinuous with respect to the weak-star topology (see [6]) we adopt in this paper the following definition, according to [5]: a function  $u \in W_0^{1,\Phi_p}(\Omega)$  is a critical point of  $I_p$  if  $\int_{\Omega} \Phi_p(|\nabla u|)dx < \infty$  and the variational inequality

$$\int_{\Omega} \Phi_p(|\nabla v|)dx - \int_{\Omega} \Phi_p(|\nabla u|)dx \geq \int_{\Omega} f(u)(v - u)dx \quad (1)$$

holds for all  $v \in W_0^{1,\Phi_p}(\Omega)$ .

We show that  $I_p$  admits at least one global, nonnegative minimizer  $u_p$  and, under the additional assumptions

$$\lim_{p \rightarrow \infty} \Phi_p(1) = 0 \quad \text{and} \quad \lim_{p \rightarrow \infty} (\Phi_p(1))^{\frac{1}{p}} = 1,$$

we prove that  $u_p$  converges uniformly on  $\bar{\Omega}$  to  $d_{\Omega}$ , as  $p \rightarrow \infty$ . This convergence result generalizes the corresponding ones of [1, 2, 5].

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## VARIATIONAL FREE TRANSMISSION PROBLEMS OF BERNOULLI TYPE

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We study functionals of the following type

$$J_{A,f,Q}(v) := \int_{\Omega} A(x,u) |\nabla u|^2 - f(x,u)u + Q(x)\lambda(u) dx$$

here  $A(x,u) = A_+(x)\chi_{\{u>0\}} + A_-(x)\chi_{\{u<0\}}$ ,  $f(x,u) = f_+(x)\chi_{\{u>0\}} + f_-(x)\chi_{\{u<0\}}$  and  $\lambda(x,u) = \lambda_+(x)\chi_{\{u>0\}} + \lambda_-(x)\chi_{\{u\leq 0\}}$ . We assume  $0 < \lambda_- < \lambda_+ < \infty$  and  $0 \leq Q \leq q_2$ . We prove the optimal regularity ( $C^{0,1^-}$ ) of minimizers of the functional indicated above when coefficients  $A_{\pm}$  are continuous functions with  $\mu \leq A_{\pm} \leq \frac{1}{\mu}$ ,  $f \in L^N(\Omega)$  and  $Q$  is bounded and continuous.

**1 Introduction**

In various applied sciences, many phenomena are modelled by transmission problem also known as phase transmission problems. These kind of models naturally appear when we study the diffusion of a quantity through different media.

Let us look at example of the stationary state of the ice-water combination and studying the diffusion of heat (related to the temperature  $T$ )  $T : \Omega \rightarrow \mathbb{R}^N$ ,  $\Omega$  being the domain under study. We can say that in ice the diffusion is determined by an operator corresponding to solid state and in water, the diffusion is determined by an operator corresponding to liquid state. As a combination, above mentioned phenomena can be posed in the following variational setup,

$$\int_{\Omega} \langle A(x)\nabla v, \nabla v \rangle - f(x,v)v + \gamma(x,v) dx \quad (1)$$

with

$$A(x,v) := A_+(x)\chi_{\{v>0\}} + A_-(x)\chi_{\{v\leq 0\}}, f(x,v) := f_+(x)\chi_{\{v>0\}} + f_-(x)\chi_{\{v\leq 0\}}, \gamma(x,v) := \gamma_+(x)\chi_{\{v>0\}} + \gamma_-(x)\chi_{\{v\leq 0\}}$$

The matrices  $A_{\pm}$  satisfy the ellipticity condition for any  $\xi \in \mathbb{R}^N$   $\lambda|\xi|^2 \leq \langle A_{\pm}\xi, \xi \rangle \leq \Lambda|\xi|^2$ ,  $f_{\pm} \in L^N(\Omega)$ ,  $\gamma_{\pm} \in C(\bar{\Omega})$ .

This class of problems has attracted a lot of attention in recent years. For example, the [2] consider the PDE with jump in leading coefficients and show Lipschitz regularity of solution when  $A_{\pm} \in C^{\alpha}(\Omega)$ . Furthermore, [1] and [3] consider the variational setup as in (1). These works prove that the regularity of minimizers tend to  $C^{0,1^-}$  as the jump of the coefficients  $A_+$  and  $A_-$  tend to zero. In our work, we are successful in removing the small jump condition and prove that minimizers are  $C^{0,1^-}$  regular independent of quantity of jump  $|A_+ - A_-|$  in any norm.

**2 Main Results****2.1 Regularity of Elliptic PDE with continuous coefficients**

We are concerned about the regularity of (weak) solutions to the following PDE in  $B_1$

$$\operatorname{div}(A(x)\nabla u) = f. \quad (2)$$

where  $A \in C(\Omega)^{N \times N}(B_1)$  and  $f \in L^N(B_1)$ .

**Proposition 2.1.** Suppose  $u \in H^1(B_1) \cap L^\infty(B_1)$  be a weak solution to (2) in  $B_1$ . Then for any  $0 < \alpha < 1$ , we have  $u \in C^\alpha(B_{1/2})$  with the following estimates

$$[u]_{C^{0,\alpha}(B_{1/2})} \leq C(N, \alpha, \mu, \omega_{a, B_{3/4}}) (\|u\|_{L^\infty(B_1)} + \|f\|_{L^N(B_1)}) . \quad (3)$$

Here  $\omega_{A, B_{3/4}}$  is uniform the modulus of continuity of  $A$  in  $B_{3/4}$ . In particular, we have

$$\|u\|_{C^\alpha(B_{1/2})} \leq C(N, \alpha, \mu, \omega_{a, B_{3/4}}) (\|u\|_{L^\infty(B_1)} + \|f\|_{L^N(B_1)}) . \quad (4)$$

## 2.2 Optimal regularity of minimizers

**Theorem 2.1.** Suppose  $u \in H^1(\Omega)$  be a minimizer of  $J_{A,f,Q}(\cdot, \Omega)$ . Then  $u$  is locally bounded in  $\Omega$  and for every  $\alpha \in (0, 1)$  and  $x_0 \in \Omega$  we have

$$\|u\|_{C^\alpha(B_r(x_0))}^* \leq C(N, p, \alpha, \mu, q_2, \lambda_+, \omega_{A_\pm, B_{2r}(x_0)}) (r + \|u\|_{L^\infty(B_{2r}(x_0))} + r\|f\|_{L^N(B_{2r}(x_0))}) .$$

Here  $r < \frac{d}{4}$  ( $d := \text{dist}(x_0, \partial\Omega)$ ),  $\omega_{A_\pm, B_{2r}(x_0)}$  is the modulus of continuity of  $A_\pm$  in the ball  $B_{2r}(x_0)$ . In particular,

$$[u]_{C^\alpha(B_r(x_0))} \leq \frac{C(N, p, \alpha, \mu, q_2, \lambda_+, \omega_{A_\pm, B_{2r}(x_0)})}{r^\alpha} (r + \|u\|_{L^\infty(B_{2r}(x_0))} + r\|f\|_{L^N(B_{2r}(x_0))}) .$$

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COMPACTNESS WITHIN THE SPACE OF COMPLETE, CONSTANT  $Q$ -CURVATURE METRICS  
 ON THE SPHERE WITH ISOLATED SINGULARITIES

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**Abstract**

In this paper we consider the moduli space of complete, conformally flat metrics on a sphere with  $k$  punctures having constant positive  $Q$ -curvature and positive scalar curvature. Previous work has shown that such metrics admit an asymptotic expansion near each puncture, allowing one to define an asymptotic necksize of each singular point. We prove that any set in the moduli space such that the distances between distinct punctures and the asymptotic necksizes all remain bounded away from zero is sequentially compact, mirroring a theorem of D. Pollack about singular Yamabe metrics. Along the way we define a radial Pohozaev invariant at each puncture and refine some *a priori* bounds of the conformal factor, which may be of independent interest.

## 1 Introduction

In recent years many people have pursued parts of Yamabe's program for other notions of curvature. In the present note, we explore a part of the singular Yamabe program as applied to the fourth order  $Q$ -curvature, which is a higher order analog of scalar curvature. On a Riemannian manifold  $(M, g)$  of dimension  $n \geq 5$ , the  $Q$ -curvature is

$$Q_g = -\frac{1}{2(n-1)}\Delta_g R_g - \frac{2}{(n-2)^2}|\text{Ric}_g|^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2}R_g^2, \quad (1)$$

where  $R_g$  is the scalar curvature of  $g$ ,  $\text{Ric}_g$  is the Ricci curvature of  $g$ , and  $\Delta_g$  is the Laplace–Beltrami operator of  $g$ . After a conformal change, the  $Q$ -curvature transforms as

$$\tilde{g} = u^{\frac{4}{n-4}}g \rightarrow Q_{\tilde{g}} = \frac{2}{n-4}u^{-\frac{n+4}{n-4}}P_g u, \quad (2)$$

where  $P_g$  is the Paneitz operator

$$P_g u = \Delta_g^2 u + \text{div} \left( \frac{4}{n-2} \text{Ric}_g(\nabla u, \cdot) - \frac{(n-2)^2 + 4}{2(n-1)(n-2)} R_g \langle \nabla u, \cdot \rangle \right) + \frac{n-4}{2} Q_g u. \quad (3)$$

The  $Q$ -curvature of the round metric  $\overset{\circ}{g}$  is  $\frac{n(n^2-4)}{8}$ , and setting  $Q_g$  to be this value gives the equation

$$P_g u = \frac{n(n-4)(n^2-4)}{16} u^{\frac{n+4}{n-4}}. \quad (4)$$

Just as in the scalar curvature setting, one can search for constant  $Q$ -curvature metrics in a conformal class by minimizing the total  $Q$ -curvature. However, because of the conformal invariance one encounters the same lack of compactness and presence of singular solutions.

In any event, a complete understanding of the fourth order analog of the Yamabe problem would require an understanding of the following singular problem: let  $(M, g)$  be a compact Riemannian manifold and let  $\Lambda \subset M$  be

a closed subset. A conformal metric  $\tilde{g} = u^{\frac{4}{n-4}} g$  is a singular constant  $Q$ -curvature metric if  $Q_{\tilde{g}}$  is constant and  $\tilde{g}$  is complete on  $M \setminus \Lambda$ . According to (2) we can write this geometric problem as

$$\begin{aligned} P_g u &= \frac{n(n-4)(n^2-4)}{16} u^{\frac{n+4}{n-4}} \text{ on } M \setminus \Lambda, \\ \liminf_{x \rightarrow x_0} u(x) &= \infty \text{ for each } x_0 \in \Lambda. \end{aligned} \quad (5)$$

For the remainder of our work we concentrate on the case that  $(M, g) = (\mathbf{S}^n, \overset{\circ}{g})$  is the round metric on the sphere and  $\Lambda = \{p_1, \dots, p_k\}$  is a finite set of distinct points. Thus we examine, given a singular set  $\Lambda$  with  $\#(\Lambda) = k$ , the set of functions

$$u : \mathbf{S}^n \setminus \Lambda = \mathbf{S}^n \setminus \{p_1, \dots, p_k\} \rightarrow (0, \infty)$$

that satisfy

$$\begin{aligned} \overset{\circ}{P} u &= P_{\overset{\circ}{g}} u = \frac{n(n-4)(n^2-4)}{16} u^{\frac{n+4}{n-4}} \\ \liminf_{x \rightarrow p_j} u(x) &= \infty \text{ for each } j = 1, 2, \dots, k. \end{aligned} \quad (6)$$

For technical reasons we will also require  $R_g \geq 0$ .

Following [1] we define the (unmarked) moduli space

$$\mathcal{M}_k = \left\{ g \in [\overset{\circ}{g}] : Q_g = \frac{n(n^2-4)}{8}, R_g \geq 0, g \text{ is complete on } \mathbf{S}^n \setminus \Lambda, \#(\Lambda) = k \right\}. \quad (7)$$

We equip each moduli space with the Gromov–Hausdorff topology. In the present work we explore some of the structure of  $\mathcal{M}_k$  when  $k \geq 3$ . Let  $\Lambda = \{p_1, \dots, p_k\}$  with  $k \geq 3$  and let  $g = u^{\frac{4}{n-4}} \overset{\circ}{g} \in \mathcal{M}_\Lambda$ . As it happens, the metric  $g$  is asymptotic to a Delaunay metric near each puncture  $p_j$ , and so one can associate a Delaunay parameter  $\epsilon_j(g) \in (0, \epsilon_n]$  to each  $p_j$  and  $g \in \mathcal{M}_\Lambda$ .

## 2 Main Results

Our main compactness theorem is the following.

**Theorem 2.1.** *Let  $k \geq 3$  and let  $\delta_1 > 0, \delta_2 > 0$  be positive numbers. Then the set*

$$\Omega_{\delta_1, \delta_2} = \{g \in \mathcal{M}_k : \text{dist}_{\overset{\circ}{g}}(p_j, p_l) \geq \delta_1 \text{ for each } j \neq l, \epsilon_j(g) \geq \delta_2\}$$

*is sequentially compact in the Gromov–Hausdorff topology.*

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# GEOMETRIC GRADIENT ESTIMATES FOR NONLINEAR PDES WITH UNBALANCED DEGENERACY

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## Abstract

We establish sharp  $C_{\text{loc}}^{1,\beta}$  geometric regularity estimates for bounded solutions of a class of nonlinear elliptic equations with non-homogeneous degeneracy, whose model equation is given by

$$[|Du|^p + \alpha(x)|Du|^q] \Delta u(x) = f(x) \quad \text{in } \Omega,$$

for a bounded and open set  $\Omega \subset \mathbb{R}^N$ , and appropriate data  $p, q \in (0, \infty)$ ,  $\alpha$  and  $f$ . Such regularity estimates simplify and generalize, to some extent, earlier ones via a different modus operandi. In the end, we present some connections of our findings with a variety of relevant nonlinear models in the theory of elliptic PDEs.

## 1 Introduction

In this work we shall derive sharp  $C_{\text{loc}}^{1,\beta}$  geometric regularity estimates for solutions of a class of nonlinear elliptic equations having a non-homogeneous double degeneracy, whose mathematical model is given by

$$[|Du|^p + \alpha(x)|Du|^q] \Delta u(x) = f(x) \quad \text{in } \Omega, \tag{1}$$

for a  $\beta \in (0, 1)$ , a bounded and open set  $\Omega \subset \mathbb{R}^N$  and  $f \in C^0(\Omega) \cap L^\infty(\Omega)$ .

In our studies, we enforce that the diffusion properties of the model (1) degenerate along an *a priori* unknown set of singular points of solutions:

$$\mathcal{S}_0(u, \Omega') = \{x \in \Omega' \Subset \Omega : |Du(x)| = 0\}.$$

In turn, regarding the non-homogeneous degeneracy

$$\mathcal{K}_{p,q,\alpha}(x, |\xi|) = |\xi|^p + \alpha(x)|\xi|^q, \quad \text{for } (x, \xi) \in \Omega \times \mathbb{R}^N.$$

we shall assume that the exponents  $p, q$  and the modulating function  $\alpha(\cdot)$  fulfil

$$0 < p \leq q < \infty \quad \text{and} \quad \alpha \in C^0(\Omega, [0, \infty)). \tag{2}$$

Mathematically, (1) consists of a new model case of a nonlinear elliptic equation enjoying a non-homogeneous degenerate term, which constitutes a non-divergent counterpart of certain variational integrals of the calculus of variations with non-standard growth conditions as follows

$$\left( W_0^{1,p}(\Omega) + u_0, L^m(\Omega) \right) \ni (w, f) \mapsto \min \int_{\Omega} (\mathcal{K}_{p,q,\alpha}(x, |Dw|) - fw) dx, \tag{DPF}$$

where  $\alpha \in C^{0,\alpha}(\Omega, [0, \infty))$ , for some  $0 < \alpha \leq 1$ ,  $1 < p \leq q < \infty$  and  $m \in (N, \infty]$ , see [2] and [5] for enlightening works. Moreover, the Euler-Lagrange equation to (DPF) exhibits a type of non-uniform and doubly degenerate ellipticity, which mixes up two different kinds of  $p$ -Laplacian type operators:

$$-\operatorname{div}(\mathcal{A}(x, \nabla u)) = f(x) \quad \text{with} \quad \mathcal{A}(x, \xi) = p|\xi|^{p-2}\xi + q\alpha(x)|\xi|^{q-2}\xi.$$

## 2 Main Results

We are in a position to state our main results.

**Theorem 2.1 ([3, Theorem 1.1]).** *Assume that assumption (2) there hold. Let  $u$  be a bounded viscosity solution to (1). Then,  $u$  is  $C^{1,\frac{1}{p+1}}$ , at interior points. More precisely, for any point  $x_0 \in \Omega' \Subset \Omega$  there holds*

$$[u]_{C^{1,\frac{1}{p+1}}(B_r(x_0))} \leq C(\text{universal}) \cdot \left( \|u\|_{L^\infty(\Omega)} + 1 + \|f\|_{L^\infty(\Omega)}^{\frac{1}{p+1}} \right) \quad \text{for } r \in \left(0, \frac{1}{2}\right).$$

An interpretation to Theorem 2.2 says that if  $u$  solves (1) and  $x_0 \in \mathcal{S}_{r,\frac{1}{p+1}}(u, \Omega')$ , then near  $x_0$  we obtain

$$\sup_{B_r(x_0)} |u(x)| \leq |u(x_0)| + C \cdot r^{1+\frac{1}{p+1}}, \quad \text{where } \mathcal{S}_{r,\frac{1}{p+1}}(u, \Omega') = \left\{ x_0 \in \Omega' \Subset \Omega : |Du(x_0)| \leq r^{\frac{1}{p+1}} \right\}.$$

On the other hand, from a geometric viewpoint, it is a pivotal qualitative information to obtain the (counterpart) sharp lower bound estimate for such operators with non-homogeneous degeneracy.

**Theorem 2.2 ([3, Theorem 1.2]).** *Suppose that the assumptions of Theorem 2.2 are in force. Let  $u$  be a bounded viscosity solution to (1) with  $f(x) \geq m > 0$  in  $\Omega$ . Given  $x_0 \in \mathcal{S}_{r,\frac{1}{p+1}}(u, \Omega')$ , there exists a constant  $c = c(m, \|a\|_{L^\infty(\Omega)}, N, p, q, \Omega) > 0$ , such that*

$$\sup_{\partial B_r(x_0)} u(x) \geq u(x_0) + c \cdot r^{1+\frac{1}{p+1}} \quad \text{for all } r \in \left(0, \frac{1}{2}\right).$$

Our findings extend/generalize regarding non-variational scenario, former results (Hölder gradient estimates) from [1, Theorem 3.1 and Corollary 3.2] and [1, Theorem 1], and to some extent, those from [3, Theorem 1] by making use of different approaches and techniques adapted to the general framework of the nonlinear and non-homogeneous degeneracy models.

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# UM PROBLEMA ANISOTRÓPICO ENVOLVENDO O OPERADOR 1-LAPLACIANO COM PESOS ILIMITADOS

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## Abstract

Neste trabalho provamos a existência de soluções de variação limitada para problemas elípticos quasilineares envolvendo o operador 1-laplaciano com peso, os quais têm a peculiaridade de que tanto o peso desse operador quanto o da não linearidade são ilimitados. Assim é necessário a definição de um espaço de funções de variação limitada com peso para tratar esse tipo de problemas. Além disso, utilizando uma versão da bem conhecida desigualdade de Caffarelli-Kohn-Nirenberg estabelece-se a compacidade das imersões contínuas e compactas desse espaço em alguns espaços de Lebesgue com peso. Também estende-se a teoria de paridade de Anzellotti e usa-se uma variante do Teorema do Passo da Montanha.

## 1 Introdução

A clássica desigualdade de Caffarelli-Kohn-Nirenberg (ver [5]) dá uma interpolação entre as normas de Lebesgue com peso de funções e suas derivadas, a qual, por sua vez, estabelece a compacidade das imersões contínuas e compactas para os espaços de Sobolev com peso. Sendo esses espaços aplicados para o análises de vários problemas elípticos envolvendo os operadores laplaciano e p-laplaciano com peso, esse último para  $p > 1$  (ver [1, 9, 4, 3, 6]).

No caso de problemas envolvendo o operador 1-laplaciano com peso, cujo peso seja uma função limitada que esteja longe de zero, o espaço natural para analizar esse tipo de problemas é o espaço das funções de variação limitada  $BV$  (ver [8, 7, 10]).

O objetivo deste trabalho é lidar com problemas envolvendo o operador 1-laplaciano com pesos ilimitados, onde tais pesos estão relacionados com os da desigualdade de Caffarelli-Kohn-Nirenberg. Mais precisamente, estudamos a existência de soluções não negativas para o seguinte problema

$$\begin{cases} -\operatorname{div}\left(\frac{1}{|x|^a} \frac{|Du|}{|Du|}\right) = \frac{1}{|x|^b} f(u) & \text{em } \Omega, \\ u = 0 & \text{sobre } \partial\Omega, \end{cases} \quad (1)$$

onde  $\Omega$  é um conjunto aberto e limitado em  $\mathbb{R}^N$  ( $N \geq 2$ ) contendo a origem e com fronteira Lipschitz  $\partial\Omega$ , e os dois parâmetros satisfazem:  $0 < a < N - 1$  e  $a < b < a + 1$ . A função  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfaz:

$$(f_1) \quad f \in C([0, +\infty));$$

$$(f_2) \quad f(0) = 0;$$

$$(f_3) \quad \text{Existem } c_1, c_2 > 0 \text{ e } 1 < q < \frac{N}{N-(1+a-b)}, \text{ tais que}$$

$$|f(s)| \leq c_1 + c_2 s^{q-1}, \quad s \in [0, +\infty);$$

$$(f_4) \quad \text{Existe } \mu > 1 \text{ e } s_0 > 0 \text{ tais que}$$

$$0 < \mu F(s) \leq f(s)s, \quad \forall s \geq s_0,$$

$$\text{onde } F(t) = \int_0^t f(s)ds;$$

( $f_5$ )  $f$  é crescente em  $[0, +\infty)$ .

## 2 Resultados Principais

**Teorema 2.1.** *Suponha que  $f$  satisfaz as condições  $(f_1) - (f_4)$ . Então existe uma solução não trivial e não negativa para o Problema (1). Essa solução é uma solução de menor energia se assume-se também a condição  $(f_5)$ .*

**Prova:** Duas abordagens diferentes serão usadas para provar esse resultado. En cada caso uma adequada variante do Teorema do Passo da Montanha (ver [2]) é aplicado. No primeiro deles, consideramos soluções aproximadas de problemas envolvendo o operador  $p$ -laplaciano e depois, fazemos  $p \rightarrow 1^+$ . No segundo, aplicamos os métodos varacionais versatiles para, além de mostrar a existência de solução, mostrar que a solução possui a menor energia entre todas as demais.

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COMPACT EMBEDDING THEOREMS AND A LIONS' TYPE LEMMA FOR FRACTIONAL  
ORLICZ-SOBOLEV SPACES

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### Abstract

In this paper we are concerned with some abstract results regarding to fractional Orlicz-Sobolev spaces. Precisely, we ensure the compactness embedding for the weighted fractional Orlicz-Sobolev space into the Orlicz spaces, provided the weight is unbounded. We also obtain a version of Lions' "vanishing" Lemma for fractional Orlicz-Sobolev spaces, by introducing new techniques to overcome the lack of a suitable interpolation law. Finally, as a product of the abstract results, we use a minimization method over the Nehari manifold to prove the existence of ground state solutions for a class of nonlinear Schrödinger equations, taking into account unbounded or bounded potentials.

## 1 Introduction

This work is motivated by a very recent trend in the fractional framework, which is to consider a new nonlocal and nonlinear operator, the so-called *fractional  $\Phi$ -Laplacian*. Throughout this work, we shall consider  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  an even function defined by

$$\Phi(t) = \int_0^t s\varphi(s) \, ds,$$

where  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^1$ -function satisfying the following assumptions:

( $\varphi_1$ )  $t\varphi(t)$  is strictly increasing in  $(0, \infty)$  such that  $t\varphi(t) \mapsto 0$ , as  $t \mapsto 0$  and  $t\varphi(t) \mapsto \infty$ , as  $t \mapsto \infty$ ;

( $\varphi_3$ ) there exist  $\ell, m \in (1, N)$  such that  $\ell \leq \frac{t^2\varphi(t)}{\Phi(t)} \leq m < \ell^*$ , for all  $t > 0$ .

For  $s \in (0, 1)$  and  $u$  smooth enough, the *fractional  $\Phi$ -Laplacian operator* is defined as

$$(-\Delta_\Phi)^s u(x) := P.V. \int \varphi(|D_s u|) \frac{D_s u}{|x-y|^{N+s}} \, dy, \quad \text{where } D_s u := \frac{u(x) - u(y)}{|x-y|^s} \tag{1}$$

and *P.V.* denotes the principal value of the integral. Note that if  $\varphi(t) = t^{p-2}$ ,  $p \in (1, N)$  then (1) reduces to the *fractional  $p$ -Laplace operator*. In a similar way, if  $\varphi(t) = t^{p-2} + t^{q-2}$ ,  $1 < p < q < N$ , then we have the fractional  $(p, q)$ -Laplacian operator.

Due to the generality of the fractional  $\Phi$ -Laplacian operator (1) and motivated by the very recent papers, mainly taking into account the work of Bonder and Salort [1], our goal is to study the following class of fractional Schrödinger equations

$$(-\Delta_\Phi)^s u + V(x)\varphi(u)u = f(x, u), \quad x \in \mathbb{R}^N, \tag{P}$$

where  $N > 2s$ ,  $0 < s < 1$ . The potential satisfies the following assumptions:

( $V_0$ ) It holds that  $V(x) \geq V_0$  for any  $x \in \mathbb{R}^N$  where  $V_0 > 0$ ;

( $V_1$ ) The set  $\{x \in \mathbb{R}^N; V(x) < M\}$  has finite Lebesgue measure for each  $M > 0$ .

The nonlinear term  $f$  is of  $C^1$  class and satisfies suitable assumptions.

Due to the presence of the potential  $V(x)$ , we introduce the following suitable weighted fractional Orlicz-Sobolev space

$$X := \left\{ u \in W^{s,\Phi}(\mathbb{R}^N) : \int V(x)\Phi(|u|) dx < +\infty \right\},$$

endowed with the norm

$$\|u\| = [u]_{s,\Phi} + \|u\|_{V,\Phi},$$

where

$$\|u\|_{V,\Phi} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} V(x)\Phi\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\}$$

and the  $(s,\Phi)$ -Gagliardo semi-norm is defined as

$$[u]_{s,\Phi} := \inf \left\{ \lambda > 0 : \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \Phi\left(\frac{|u(x) - u(y)|}{\lambda|x-y|^s}\right) \frac{dx dy}{|x-y|^N} \leq 1 \right\}.$$

## 2 Main Results

Our main contribution

**Theorem 2.1** (Compact embedding). *Assume that  $(\varphi_1) - (\varphi_2)$  and  $(V_0) - (V_1)$  hold. Then, the embedding  $X \hookrightarrow L_\Phi(\mathbb{R}^N)$  is compact.*

**Theorem 2.2** (Compact embedding). *Assume that  $(\varphi_1) - (\varphi_2)$  and  $(V_0) - (V_1)$  hold. Suppose that  $\Phi \prec \Psi \prec\prec \Phi_*$  and the following limit holds*

$$\limsup_{|t| \rightarrow 0} \frac{\Psi(|t|)}{\Phi(|t|)} < +\infty. \quad (1)$$

*Then, the space  $X$  is compactly embedded into  $L_\Psi(\mathbb{R}^N)$ .*

**Theorem 2.3** (Lions' Lemma type result). *Suppose that  $(\varphi_1) - (\varphi_2)$  hold and*

$$\lim_{|t| \rightarrow 0} \frac{\Psi(t)}{\Phi(t)} = 0. \quad (2)$$

*Let  $(u_n)$  be a bounded sequence in  $W^{s,\Phi}(\mathbb{R}^N)$  in such way that  $u_n \rightharpoonup 0$  in  $X$  and*

$$\lim_{n \rightarrow +\infty} \left[ \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} \Phi(u_n) dx \right] = 0, \quad (3)$$

*for some  $r > 0$ . Then,  $u_n \rightarrow 0$  in  $L_\Psi(\mathbb{R}^N)$ , where  $\Psi$  is an  $N$ -function such that  $\Psi \prec\prec \Phi^*$ .*

To prove the above results, we shall introduce new techniques to overcome the lack of a suitable interpolation law. Finally, we shall apply these results to obtain solutions to the Problem  $(P)$ .

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**COUPLED AND UNCOUPLED SIGN-CHANGING SPIKES OF SINGULARLY PERTURBED  
ELLIPTIC SYSTEMS**

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**Abstract**

We study the singularly perturbed system of elliptic equations

$$\begin{cases} -\epsilon^2 \Delta u_i + u_i = \mu_i |u_i|^{p-2} u_i + \sum_{\substack{j=1 \\ j \neq i}}^{\ell} \lambda_{ij} \beta_{ij} |u_j|^{\alpha_{ij}} |u_i|^{\beta_{ij}-2} u_i, \\ u_i \in H_0^1(\Omega), \quad u_i \neq 0, \quad i = 1, \dots, \ell, \end{cases} \quad (1)$$

in a bounded domain  $\Omega$  in  $\mathbb{R}^N$ , with  $N \geq 3$ ,  $\epsilon > 0$ ,  $\mu_i > 0$ ,  $\lambda_{ij} = \lambda_{ji} < 0$ ,  $\alpha_{ij}, \beta_{ij} > 1$ ,  $\alpha_{ij} = \beta_{ji}$ ,  $\alpha_{ij} + \beta_{ij} = p \in (2, 2^*)$ , and  $2^* := \frac{2N}{N-2}$ . If  $\Omega$  is the unit ball, we obtain solutions with a prescribed combination of positive and nonradial sign-changing components exhibiting two different types of asymptotic behavior as  $\epsilon \rightarrow 0$ : solutions whose limit profile is a rescaling of a solution with positive and nonradial sign-changing components of the limit system

$$\begin{cases} -\Delta u_i + u_i = \mu_i |u_i|^{p-2} u_i + \sum_{\substack{j=1 \\ j \neq i}}^{\ell} \lambda_{ij} \beta_{ij} |u_j|^{\alpha_{ij}} |u_i|^{\beta_{ij}-2} u_i, \\ u_i \in H^1(\mathbb{R}^N), \quad u_i \neq 0, \quad i = 1, \dots, \ell, \end{cases} \quad (2)$$

and solutions whose limit profile is a solution of the uncoupled system, i.e., after rescaling and translation, the limit profile of the  $i$ -th component is a positive or a nonradial sign-changing solution to the problem

$$-\Delta u + u = \mu_i |u|^{p-2} u, \quad u \in H^1(\mathbb{R}^N), \quad u \neq 0. \quad (3)$$

## 1 Introduction

System (1) arises as a model for various physical phenomena, in particular in the study of standing waves for a mixture of Bose-Einstein condensates of  $\ell$  different hyperfine states which overlap in space, see for example [1]. Here we consider the case in which the interaction between particles in the same state is attractive ( $\mu_i > 0$ ) and the interaction between particles in any two different states is repulsive ( $\lambda_{ij} < 0$ ). Our main objective is to study the existence and profile of solutions to (1) some of whose components can be positive while others change sign. It is reasonable to expect that there will be solutions with sign-changing spikes, i.e., solutions whose sign-changing components look like rescaling of a sign-changing solution of (3). On the other hand, rescaling the components by  $\tilde{u}_i(x) := u_i(\epsilon x)$  system (1) becomes system (2) in  $\Omega_\epsilon := \{x \in \mathbb{R}^N : \epsilon x \in \Omega\}$  instead of  $\mathbb{R}^N$ . As  $\epsilon \rightarrow 0$  these domains cover the whole space  $\mathbb{R}^N$ . So it is natural to ask if the system (1) has a solution that, after rescaling, approaches a solution to the system (2). One might also expect to obtain solutions with positive and sign-changing components for the system (1) whose limit profile is a solution of the same type for the system (2).

## 2 Main Results

Our main result are read as follows:

**Theorem 2.1.** Let  $N = 4$  or  $N \geq 6$ . Then, for any given  $0 \leq m \leq \ell$ , the system (2) has a solution  $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_\ell)$  whose first  $m$  components  $w_1, \dots, w_m$  are positive and whose last  $\ell - m$  components  $w_{m+1}, \dots, w_\ell$  are nonradial and change sign. Furthermore,  $\mathbf{w}$  satisfies

$$\begin{cases} w_i(z_1, z_2, x) = w_i(e^{i\vartheta} z_1, e^{i\vartheta} z_2, gx) & \text{for all } \vartheta \in [0, 2\pi), g \in O(N-4), i = 1, \dots, \ell, \\ w_i(z_1, z_2, x) = w_i(z_2, z_1, x) \text{ if } i = 1, \dots, m, & w_i(z_1, z_2, x) = -w_i(z_2, z_1, x) \text{ if } i = m+1, \dots, \ell, \end{cases} \quad (1)$$

for all  $(z_1, z_2, x) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{N-4} \equiv \mathbb{R}^N$ , and it has least energy among all nontrivial solutions with these properties.

To illustrate our results, let us focus on the case where  $\Omega$  is the open unit ball  $B_1(0)$  in  $\mathbb{R}^N$  centered at the origin. For  $\epsilon > 0$  and  $u \in H^1(\mathbb{R}^N)$  let  $\|u\|_\epsilon^2 := \frac{1}{\epsilon^N} \int_{\mathbb{R}^N} [\epsilon^2 |\nabla u|^2 + u^2]$  and  $\|u\| := \|u\|_1$ .

**Theorem 2.2.** Let  $N = 4$  or  $N \geq 6$ , and  $\Omega = B_1(0)$ . Then, for any given  $0 \leq m \leq \ell$  and any sequence  $(\epsilon_k)$  of positive numbers converging to zero, there exists solution  $\widehat{\mathbf{u}}_k = (\widehat{u}_{1k}, \dots, \widehat{u}_{\ell k})$  to the system (3) whose first  $m$  components are positive and whose last  $\ell - m$  components are nonradial and change sign, with the following limit profile: There exists a solution  $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_\ell)$  to the system (2) such that, after passing to a subsequence,

$$\lim_{k \rightarrow \infty} \|\widehat{u}_{ik} - w_i(\epsilon_k^{-1} \cdot)\|_{\epsilon_k} = 0 \quad \text{for all } i = 1, \dots, \ell.$$

The first  $m$  components of  $\mathbf{w}$  are positive, its last  $\ell - m$  components are nonradial and change sign, and  $\mathbf{w}$  satisfies (1). Therefore,  $\lim_{k \rightarrow \infty} \sum_{i=1}^{\ell} \|\widehat{u}_{ik}\|_{\epsilon_k}^2 = \sum_{i=1}^{\ell} \|w_i\|^2 =: \widehat{\mathfrak{c}}_m$ .

**Theorem 2.3.** Let  $N \geq 5$  and  $\Omega = B_1(0)$ . Then, for any given  $0 \leq m \leq \ell$  and any sequence  $(\epsilon_k)$  of positive numbers converging to zero, there exists solution  $\mathbf{u}_k = (\mathbf{u}_{1k}, \dots, \mathbf{u}_{\ell k})$  to the system (1) with  $\epsilon = \epsilon_k$  and  $\Omega = B_1(0)$ , whose first  $m$  components are positive and whose last  $\ell - m$  components are nonradial and change sign, with the following limit profile: For each  $i = 1, \dots, \ell$ , there exist a sequence  $(\xi_{ik})$  in  $B_1(0)$  and a solution  $v_i$  to the problem (3) such that, after passing to a subsequence,  $\lim_{k \rightarrow \infty} \epsilon_k^{-1} \text{dist}(\xi_{ik}, \partial B_1(0)) = \infty$ ,  $\lim_{k \rightarrow \infty} \epsilon_k^{-1} |\xi_{ik} - \xi_{jk}| = \infty$  if  $i \neq j$ ,  $\lim_{k \rightarrow \infty} \|u_{ik} - v_i(\epsilon_k^{-1}(\cdot - \xi_{ik}))\|_{\epsilon_k} = 0$ . The functions  $v_1, \dots, v_m$  are positive and radial, while the functions  $v_{m+1}, \dots, v_\ell$  are sign-changing, nonradial and satisfy

$$\begin{cases} v_i(z_1, z_2, x) = v_i(e^{i\vartheta} z_1, e^{i\vartheta} z_2, gx) & \text{for all } \vartheta \in [0, 2\pi), g \in O(N-4), \\ v_i(z_1, z_2, x) = -v_i(z_2, z_1, x), \end{cases} \quad (2)$$

for all  $(z_1, z_2, x) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{N-4} \equiv \mathbb{R}^N$ ,  $i = m+1, \dots, \ell$ . Furthermore,  $\lim_{k \rightarrow \infty} \sum_{i=1}^{\ell} \|u_{ik}\|_{\epsilon_k}^2 = \sum_{i=1}^{\ell} \|v_i\|^2 =: \mathfrak{c}_m$ , satisfies  $\mathfrak{c}_m < \widehat{\mathfrak{c}}_m$ , with  $\widehat{\mathfrak{c}}_m$  as in Theorem 2.2, if  $N \geq 6$ .

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## ON AN AMBROSETTI-PRODI TYPE PROBLEM IN $\mathbb{R}^N$

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### Abstract

In this work we study results of existence and non-existence of solutions for the following Ambrosetti-Prodi type problem

$$\begin{cases} -\Delta u = P(x)(g(u) + f(x)) \text{ in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \lim_{|x| \rightarrow +\infty} u(x) = 0, \end{cases} \quad (P)$$

where  $N \geq 3$ ,  $P \in C(\mathbb{R}^N, \mathbb{R}^+)$ ,  $f \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  and  $g \in C^1(\mathbb{R})$ . The main tools used are the sub-supersolution method and Leray-Schauder topological degree theory.

## 1 Introduction

The main motivation to study the problem (P) comes from the seminal paper by Ambrosetti and Prodi [2] that studied the existence and non-existence of solution for the problem

$$\begin{cases} -\Delta u = g(u) + f(x), \text{ in } \Omega, \\ u = 0, \text{ in } \partial\Omega, \end{cases} \quad (1)$$

where  $\Omega \subset \mathbb{R}^N$  with  $N \geq 3$ , is a bounded domain,  $g$  is a  $C^2$ -function with

$$g''(s) > 0, \quad \forall s \in \mathbb{R} \quad \text{and} \quad 0 < \lim_{s \rightarrow -\infty} g'(s) < \lambda_1 < \lim_{s \rightarrow \infty} g'(s) < \lambda_2.$$

In order to prove their results, Ambrosetti and Prodi used a global result of inversion to proper functions to show the existence of a closed manifold  $M$  dividing the space  $C^{0,\alpha}(\Omega)$  in two connected components  $O_1$  and  $O_2$  such that:

- (i) If  $f$  belongs to  $O_1$ , the problem (1) has no solution;
- (ii) If  $f$  belongs to  $M$ , the problem (1) has exactly one solution;
- (iii) If  $f$  belongs to  $O_2$ , the problem (1) has exactly two solution.

In [3], Berger and Podolak proposed the decomposition of function  $f$  in the form  $f = t\phi + f_1$ , where  $\phi$  is eigenfunction associated to first eigenvalue of  $-\Delta$

$$\begin{cases} -\Delta u = g(u) + t\phi + f_1, \text{ in } \Omega, \\ u = 0, \text{ in } \partial\Omega, \end{cases} \quad (2)$$

then using the Liapunov-Schmidt method they showed the existence of  $t_0 \in \mathbb{R}$  such that (2) has at least two solutions if  $t < t_0$ , at least one solution if  $t = t_0$  and no solutions if  $t > t_0$ .

Now, before stating our main results, we need to fix the assumptions on the functions  $P$  and  $g$ . In the sequel,  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^1$ -function that satisfies the following inequalities

$$\limsup_{s \rightarrow -\infty} \frac{g(s)}{s} < \lambda_1 < \liminf_{s \rightarrow \infty} \frac{g(s)}{s} \quad (G_1)$$

Related to the function  $P : \mathbb{R}^N \rightarrow \mathbb{R}^+$ , we consider that it is a continuous function satisfying:

$$|\cdot|^2 P(\cdot) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \quad (P_1)$$

$$\int_{\mathbb{R}^N} \frac{P(y)}{|x-y|^{N-2}} dy \leq \frac{C}{|x|^{N-2}}, \forall x \in \mathbb{R}^N \setminus \{0\}, \text{ for some } C > 0 \quad (P_2)$$

We denote by  $\mathcal{N}$  the eigenspace associated with the first eigenvalue  $\lambda_1$ . By [1], it is well known that  $\dim \mathcal{N} = 1$ , then we can assume that  $\mathcal{N} = \text{Span}\{\phi\}$ , where  $\phi$  is one positive eigenfunction associated with  $\lambda_1$  with  $\int_{\mathbb{R}^N} P(x)|\phi|^2 dx = 1$ . Hence, we can write  $f = t\phi + f_1$ , where  $f_1 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  with

$$\int_{\mathbb{R}^N} P(x)f_1 \phi dx = 0 \text{ and } \int_{\mathbb{R}^N} P(x)f \phi dx = t. \quad (3)$$

From this, problem  $(P)$  can be rewritten as follows

$$\begin{cases} -\Delta u = P(x)(g(u) + t\phi(x) + f_1(x)) \text{ in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \lim_{|x| \rightarrow +\infty} u(x) = 0. \end{cases} \quad (\tilde{P})$$

## 2 Main Results

Our first result is the following:

**Theorem 2.1.** *Assume the conditions  $(G_1)$ ,  $(P_1)$  and  $(P_2)$ . Then, for each  $f_1 \in \mathcal{N}^\perp$  there is a number  $\alpha(f_1)$  such that:*

- (i) *The problem  $(\tilde{P})$  has no solution whenever  $t > \alpha(f_1)$ ;*
- (ii) *If  $t < \alpha(f_1)$ , then  $(\tilde{P})$  has at least one solution.*

Our second result is the following:

**Theorem 2.2.** *Assume the conditions  $(G_1)$ ,  $(P_1)$  and  $(P_2)$ . Moreover, assume that  $g$  is an increasing function satisfying*

$$\lim_{s \rightarrow +\infty} \frac{g(s)}{s^\sigma} = 0, \quad (1)$$

where  $\sigma = \frac{N}{N-2}$ . Then, for each  $f_1 \in \mathcal{N}^\perp$  there is a number  $\alpha(f_1)$  such that:

- (i) *If  $t < \alpha(f_1)$ , then  $(\tilde{P})$  has at least two solutions;*
- (ii) *If  $t = \alpha(f_1)$ , then  $(\tilde{P})$  has at least one solution.*

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AN ELLIPTIC SYSTEM WITH MEASURABLE COEFFICIENTS AND SINGULAR  
NONLINEARITIES

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**Abstract**

In this talk we present some new existence results for a system of elliptic equations with a singular nonlinearity. Our approach is based on a comparison principle for weak solutions and the Schauder fixed point Theorem. The difficulties due to the presence of the singularity are tackled using suitable test functions and barriers. Despite our strategy is not variational, as a byproduct of our results, we are able to find saddle points of the functional associated to the system.

## 1 Introduction

In this talk we will focus on the following system with a singular nonlinearity

$$\begin{cases} -\operatorname{div}(A(x)u) + vu^{r-1} = \frac{1}{u^\gamma} & \text{in } \Omega, \\ -\operatorname{div}(M(x)v) = u^r & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \varphi = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where  $\Omega$  is bounded open subset of  $\mathbb{R}^N$ ; the two exponents satisfy  $\gamma > 0$  and  $r > \max\{0, 1 - \gamma\}$  and the measurable matrices  $A(x), M(x)$  are elliptic in the sense that

$$\alpha|\xi|^2 \leq A(x)\xi\xi \leq \beta|\xi|^2 \quad \text{and} \quad \alpha|\xi|^2 \leq M(x)\xi\xi \leq \beta|\xi|^2, \quad (2)$$

for almost every  $x$  in  $\Omega$ , and for every  $\xi$  in  $\mathbb{R}^N$ , with  $0 < \alpha \leq \beta$ .

We stress that when  $\gamma \in (0, 1)$  even the nonlinear term in the left hand side of the first equation in (1) can be singular. Anyway this singularity mild compared with the one on the right hand side (see assumption on  $r$ ).

The literature about singular equation is wide and well establish. Without the intention of being exhaustive we mention the seminal papers [1], [2] and [3]. We stress that in the previously mentioned paper the singularity is of reaction type, meaning something like

$$-\operatorname{div}(A(x)u) = h(u) \quad \text{with} \quad h(u) \approx \frac{1}{|u|^\gamma} \quad \text{near the origin.}$$

If the singularity is of the absorption type, namely something like

$$-\operatorname{div}(A(x)u) + h(u) = 1 \quad \text{with} \quad h(u) \approx \frac{1}{|u|^\gamma} \quad \text{near the origin,}$$

the features of the problem change dramatically. As we already pointed out, since  $r$  is always greater than  $1 - \gamma$ , in broad terms our problem is part of the first setting.

## 2 Main Results

For the sake of brevity here we present the result concerning solutions in the energy space and  $\gamma \leq 1$ . For the more general case we refer to the manuscript [3].

**Definition 2.1.** A couple of functions  $(u, v) \in \left(W_0^{1,2}(\Omega) \cap L(\Omega)^\infty\right)^2$  is a energy solution to system (1) if

$$u, v > 0 \quad a.e. \text{ in } \Omega,$$

$$\frac{\phi}{u^\gamma} \in L(\Omega)^1 \quad \forall \phi \in W_0^{1,2}(\Omega)$$

and if

$$\begin{cases} \int_{\Omega} A(x) \nabla u \nabla \phi + \int_{\Omega} v u^{r-1} \phi = \int_{\Omega} \frac{\phi}{u^\gamma} & \forall \phi \in W_0^{1,2}(\Omega) \\ \int_{\Omega} M(x) \nabla v \nabla \psi = \int_{\Omega} u^r \psi & \forall \psi \in W_0^{1,2}(\Omega). \end{cases} \quad (1)$$

**Theorem 2.1.** Let  $\Omega$  be a bounded open set of  $\mathbb{R}^N$  and assume (2). Given  $\gamma \in (0, 1]$  and  $r > 1 - \gamma$ , there exists  $(u, v) \in \left(W_0^{1,2}(\Omega) \cap L(\Omega)^\infty\right)^2$  energy solution to (1). Moreover such a couple is a saddle point to the functional

$$J(w, z) = \frac{1}{2} \int_{\Omega} A(x) \nabla w \nabla w - \frac{1}{2r} \int_{\Omega} M(x) \nabla z \nabla z + \frac{1}{r} \int_{\Omega} z^+ |w|^r - \frac{1}{1-\gamma} \int_{\Omega} (w^+)^{1-\gamma},$$

namely

$$J(u, z) \leq J(u, v) \leq J(w, v) \quad \text{for any} \quad (w, z) \in \left(W_0^{1,2}(\Omega)\right)^2.$$

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FOURTH-ORDER NONLOCAL TYPE ELLIPTIC PROBLEMS WITH INDEFINITE  
NONLINEARITIES

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### Abstract

In this work we establish the existence of at least one weak solution and one ground state solution for the following class of fourth-order nonlocal elliptic problems

$$\begin{cases} \Delta^2 u - g \left( \int_{\Omega} |\nabla u|^2 dx \right) \Delta u = \mu a(x)|u|^{q-2}u + b(x)|u|^{p-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $N \geq 5$ ,  $\Omega \subset \mathbb{R}^N$  is a smooth bounded domain,  $\Delta^2 = \Delta \circ \Delta$  is the biharmonic operator,  $\mu > 0$ ,  $1 < q < 2 < p < 2N/(N-4)$  and  $g : [0, \infty) \rightarrow [0, \infty)$  satisfies suitable assumptions. We deal with the case where  $a, b : \Omega \rightarrow \mathbb{R}$  can be sign changing functions, which means that the problem is indefinite. Our approach is based on variational methods jointly with a fine analysis on the Nehari manifold, by giving a complete description of the fibering maps, which strongly depend on the sign of the weights.

## 1 Introduction

In this work we study the following class of fourth-order elliptic problems

$$\begin{cases} \Delta^2 u - g \left( \int_{\Omega} |\nabla u|^2 dx \right) \Delta u = \mu a(x)|u|^{q-2}u + b(x)|u|^{p-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \quad (P_\mu)$$

where  $\Delta^2 = \Delta \circ \Delta$  is the biharmonic operator,  $\mu > 0$ ,  $N \geq 5$ ,  $\Omega \subset \mathbb{R}^N$  is a smooth bounded domain and  $1 < q < 2 < p < 2_*$ , where  $2_* := 2N/(N-4)$  is the critical Sobolev exponent. The function  $g$  is a smooth function satisfying some assumptions and  $a, b$  can be sign changing functions. Before introducing our assumptions and main results, we give a brief survey on the related results, which motivate this work.

## 2 Main Results

Throughout this work we suppose that  $a, b : \Omega \rightarrow \mathbb{R}$  are bounded which can be sign changing functions and  $b$  satisfies the following condition:

- (A) There exist  $\Omega_0, \Omega_1 \subset \Omega$  with  $|\Omega_0|, |\Omega_1| > 0$ , such that  $a(x) > 0$  for all  $x \in \Omega_0$  and  $b(x) > 0$ , for all  $x \in \Omega_1$ , where  $|\cdot|$  denotes the Lebesgue measure.

For the function  $g \in C^2([0, +\infty), [0, +\infty))$  we shall consider the following conditions:

- (G<sub>1</sub>) The function  $g$  is nonnegative and nondecreasing.

(G<sub>2</sub>) There exists  $r \geq 2/(p - 2)$  such that

$$g(t) \geq rg'(t)t, \quad \text{for all } t \geq 0.$$

(G<sub>3</sub>) There exist  $\sigma \in (2/p, 1)$  and  $m \in (2/p, 2/q)$  such that

$$\sigma g(t)t \leq G(t) \leq mg(t)t, \quad \text{for all } t \geq 0,$$

where  $G(t) = \int_0^t g(s) ds, t \in \mathbb{R}$ .

(G<sub>4</sub>) There exists  $\rho \in (2/(p - 4), \infty)$  such that

$$g'(t) \geq \rho g''(t)t, \quad \text{for all } t \geq 0.$$

(G<sub>5</sub>) There exist constants  $c_1, c_2 > 0$  and  $k < (p - 2)/2$  such that

$$g(t) \leq c_1 + c_2 t^k, \quad \text{for all } t \geq 0.$$

The first main result of this paper can be stated as follows:

**Theorem 2.1.** *Suppose that  $1 < q < 2 < p < 2_* = 2N/(N-4)$  and (A), (G<sub>1</sub>)–(G<sub>5</sub>) are satisfied. Then there exists  $\mu_* > 0$  such that Problem  $(P_\mu)$  has at least two nontrivial solutions  $u_1, u_2 \in \mathcal{H}$  satisfying  $J_\mu(u_1) < 0 < J_\mu(u_2)$ , whenever  $\mu \in (0, \mu_*)$ . Furthermore,  $u_1$  is a ground state solution, that is,  $u_1$  has the least energy level among all nontrivial solutions of  $(P_\mu)$ .*

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## ON THE FRICTIONAL CONTACT PROBLEM OF $P(X)$ -KIRCHHOFF TYPE

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### Abstract

In this article we consider a class of frictional contact problem of  $p(x)$ -Kirschhoff type. By means of an abstract Lagrange multiplier technique and the Schauder's fixed point theorem we establish the existence of weak solutions.

## 1 Introduction

The purpose of this work is to investigate the existence of weak solutions for the following boundary value problem

$$\begin{aligned} -M\left(\int_{\Omega} \frac{1}{p(x)}|\nabla u|^{p(x)} dx\right) \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right) &= f_1(x, u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma_1 \\ M\left(\int_{\Omega} \frac{1}{p(x)}|\nabla u|^{p(x)} dx\right)|\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} &= f_2 \quad \text{in } \Gamma_2 \\ \left|M\left(\int_{\Omega} \frac{1}{p(x)}|\nabla u|^{p(x)} dx\right)|\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu}\right| &\leq g, \\ M\left(\int_{\Omega} \frac{1}{p(x)}|\nabla u|^{p(x)} dx\right)|\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} &= -g \frac{u}{|u|}, \quad \text{if } u \neq 0 \quad \text{in } \Gamma_3 \end{aligned} \tag{1}$$

where  $\Omega \subseteq \mathbb{R}^2$  is a bounded domain with smooth enough boundary  $\Gamma$ , partitioned in three parts  $\Gamma_1, \Gamma_2, \Gamma_3$  such that  $\operatorname{meas}(\Gamma_i) > 0$ , ( $i = 1, 2, 3$ );  $f_1 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $f_2 : \Gamma_2 \rightarrow \mathbb{R}$ ,  $g : \Gamma_3 \rightarrow \mathbb{R}$  and  $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  are given functions,  $p \in C(\overline{\Omega})$ .

The study of the  $p(x)$ -Kirchhoff type equations with nonlinear boundary condition of different class have attracted expensive interest in recent years (See e.g. [1,3]). Motivated by the ideas in [2] we consider problem (1.1) (which has already been treated for constant exponent, with  $M(s) = 1$ ,  $f_1(x, u) = f_1(x)$ ) with  $M$  a nonconstant continuous function in the setting of the variable exponent spaces.

## 2 Assumptions and Main Result

First, we introduce the space

$$X = \{u \in W^{1,p(x)}(\Omega) : \gamma u = 0 \quad \text{on } \Gamma_1\}$$

herein  $W^{1,p(x)}(\Omega)$  ( $p \in C(\overline{\Omega})$ ,  $2 \leq p(x) < +\infty$ ) is the well known variable exponent Sobolev space.

(A<sub>1</sub>)  $M : [0, +\infty[ \rightarrow [m_0, +\infty[$  is a continuous and decreasing function;  $m_0 > 0$ .

(A<sub>2</sub>)  $f_1 : \Omega \times \mathbb{R} \rightarrow$  is a Caratheodory function satisfying

$$|f_1(x, t)| \leq c_1 + c_2 |t|^{\alpha(x)-1} , \quad \forall (x, t) \in \Omega \times \mathbb{R} , \quad \alpha \in C_+(\bar{\Omega}) , \quad \alpha(x) < p^*(x)$$

(A<sub>3</sub>)  $f_2 \in L^{p'(x)}(\Gamma_2), g \in L^{p'(x)}(\Gamma_3), g(x) \geq 0$  a.e on  $\Gamma_3$

Now we introduce the spaces

$$S = \{u \in W^{1-\frac{1}{p'(x)}, p(x)}(\Gamma) : \exists v \in X \text{ such that } u = \gamma v \text{ on } \Gamma\},$$

$Y = S'$  (the dual of the space S) and the set of Lagrange multipliers

$$\Lambda = \{u \in Y : \langle \mu, z \rangle \leq \int_{\Gamma_3} g(x)|z(x)|d\Gamma , \quad \forall z \in S\}.$$

Next we define a Lagrange multiplier  $\lambda \in Y$

$$\langle \mu, z \rangle = - \int_{\Gamma_3} M\left(\frac{1}{p(x)}|\nabla u|^{p(x)}\right)|\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} z d\Gamma , \quad \forall z \in S.$$

So, our main result can be stated as follows.

**Theorem 2.1.** Suppose (A<sub>1</sub>) - (A<sub>3</sub>) hold. Then problem (1.1) has a solution  $(u, \lambda) \in X \times \Lambda$ .

**Proof** We apply an abstract result on [4] and the Schauder's fixed point theorem.

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# GLOBAL MULTIPLICITY OF SOLUTIONS FOR A MODIFIED ELLIPTIC PROBLEM WITH SINGULAR TERMS

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## Abstract

We establish global multiplicity results of solutions for a singular nonlinear problem. We first prove a comparison principle to prove the existence of a minimal solution by the method of sub and super solutions and then we also obtain the second solution by critical point theory.

## 1 Introduction

In this paper, we are interested in the global multiplicity results and qualitative properties of the solutions for the modified quasilinear equation with singular nonlinearities

$$\begin{cases} -\Delta u - u\Delta u^2 = \lambda a(x)u^{-\alpha} + b(x)u^\beta \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases} \quad (1)$$

where  $\Omega \subset \mathbb{R}^N$  is a smooth bounded domain with  $N \geq 3$ ,  $0 < a \in C(\Omega) \cap L^\infty(\Omega)$ ,  $b \in C(\Omega) \cap L^\infty(\Omega)$  with  $b^-/a \in L^\infty(\Omega)$ ,  $0 < \alpha < 1 < \beta \leq 22^* - 1$  and  $\lambda > 0$  is a real parameter.

The study of the modified quasilinear Schrödinger equation in the whole space had received a lot of attention in the last decades. The existence of a positive ground state solution of equation (1) has been proved in [1] by introducing parameter  $\lambda$  in front of the nonlinear term. In [2], by changing of variables, the authors studied the quasilinear problem was transformed to a semilinear one and the existence of a positive solution was proved by the Mountain-Pass lemma in an Orlicz working space. Different from the changing variable methods, in [3] the authors introduced a new perturbation techniques to study a class of subcritical quasilinear problems including the Modified Schrödinger equation (1).

However, to the best of our knowledge, the problem of global multiplicity of solutions for the modified quasilinear equation on bounded domain with sub critical and critical growth is still open. In the present paper we are going to study the existence and global multiplicity of the solutions for the singular modified elliptic equation driven by operator  $-\Delta u - u\Delta u^2$  and the critical term. We will observe that how the parameter  $\lambda$  and the nonconstant functions  $a(x), b(x)$  will affect the existence and multiplicity of the solutions.

## 2 Main Results

**Theorem 2.1** (Singular/superlinear-Global existence). *Assume that  $a > 0$ ,  $b^+ \neq 0$ , and  $0 < \alpha < 1 < \beta \leq 22^* - 1$ . Then there exists a  $0 < \lambda_* < \infty$  such that problem (1) admits at least one solution  $u_\lambda \in H_0^1(\Omega)$  for  $0 < \lambda \leq \lambda_*$ , and no  $H_0^1(\Omega)$ -solution for  $\lambda > \lambda_*$ . Besides this, we have:*

- (1) *any solution of problem (1) belongs to  $L^\infty(\Omega)$ ,*
- (2) *that there exists  $\tau_0 > 0$  such that  $u_\lambda$  is the unique solution of the problem (1) with  $L^\infty(\Omega)$ -norm smaller or equal than  $\tau_0$ ,*

- (3)  $\|u_\lambda\| = o(\lambda^\theta)$  as  $\lambda \rightarrow 0$ , for any  $0 < \theta < 1/(1 + \alpha)$ ,
- (4)  $u_\lambda$  is a minimal solution if  $b^- \equiv 0$  holds. In particular,  $\|u_\lambda\| < \|u_\mu\|$  if  $0 < \lambda < \mu \leq \lambda_*$  holds.

**Proof** The first, we will apply a suitable change of variable to transform the quasilinear equation (1) into a semilinear equation which remains singular at zero and behaves superlinearly at infinity:

$$\begin{cases} -\Delta\omega = [\lambda a(x)h(\omega)^{-\alpha} + b(x)h(\omega)^\beta]h'(\omega) \text{ in } \Omega, \\ \omega > 0 \text{ in } \Omega, \omega = 0 \text{ on } \partial\Omega, \end{cases} \quad (1)$$

where  $h : \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $h'(t) = (1 + 2|h(t)|^2)^{-1/2}$ ,  $t > 0$  and  $h(-t) = -h(t)$  for  $t \leq 0$ .

The second, we will use the method of sub and super solutions and minimization to prove that there exists a  $0 < \lambda_* < \infty$  such that problem (1) admits at least one solution  $u_\lambda = h(\omega_\lambda) \in H_0^1(\Omega)$  for  $0 < \lambda \leq \lambda_*$  and no solution for  $\lambda > \lambda_*$ . Besides this,  $u_\lambda$  satisfies (1), (2), (3) and (4) listed above.

**Theorem 2.2** (Singular/superlinear-Global multiplicity). *Assume that  $a > 0$ ,  $b^+ \neq 0$ ,  $0 < \alpha < 1$  and one of the following assumptions:*

- (i)  $1 < \beta < 22^* - 1$ , (ii)  $b(x) \equiv 1$  and  $\beta = 22^* - 1$ .

*Then the problem (1) admits a second solution  $v_\lambda$  in  $H_0^1(\Omega) \cap L^\infty(\Omega)$ , with  $u_\lambda \leq v_\lambda$ , for any  $0 < \lambda < \lambda_*$  given.*

**Proof** The second solution of (1) will be found by making the use of Ekeland's variational Principle on the following set

$$T = \{\omega \in H_0^1(\Omega), \omega \geq \omega_\lambda \text{ a.e. in } \Omega\}$$

where  $\omega_\lambda$  is the first solution of (1). By the proof of Theorem 2.1, there exists  $0 < l_0 \leq \|\omega_\lambda\|$  such that  $I_\lambda(\omega) \geq I_\lambda(\omega_\lambda)$ ,  $\forall \omega$  with  $\|\omega - \omega_\lambda\| \leq l_0$ . Then one of the following cases holds:

- (P1)  $\inf\{I_\lambda(\omega), \omega \in T, \|\omega - \omega_\lambda\| = l\} = I_\lambda(\omega_\lambda)$ ,  $\forall l \in (0, l_0)$ ;
- (P2) There exists  $l_1 \in (0, l_0)$  such that  $\inf\{I_\lambda(\omega), \omega \in T, \|\omega - \omega_\lambda\| = l_1\} > I_\lambda(\omega_\lambda)$ ,

where  $I_\lambda(\omega)$  is the energy functional associated to (1).

If (P1) is true, we prove that there exists a solution  $\eta_\lambda$  of (1) such that  $\eta_\lambda \leq \omega_\lambda$  in  $\Omega$  and  $\|\omega_\lambda - \eta_\lambda\| = l$  for any  $l \in (0, l_0)$  and each  $\lambda \in (0, \lambda_*)$ .

If (P2) is true, we prove that there exists a solution  $\eta_\lambda$  of (1) such that  $\eta_\lambda \leq \omega_\lambda$  in  $\Omega$  and  $\|\omega_\lambda - \eta_\lambda\| = l_1$  for each  $\lambda \in (0, \lambda_*)$ .

Finally, we take  $v_\lambda = h(\eta)$  and we have that  $v_\lambda \neq u_\lambda$  is a second solution for (1).

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THE IVP FOR THE EVOLUTION EQUATION OF WAVE FRONTS IN CHEMICAL REACTIONS  
 IN LOW-REGULARITY SOBOLEV SPACES

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**Abstract**

In this work, we study the initial-value problem for an equation of evolution of wave fronts in chemical reactions. We show that the associated initial value problem is locally and globally well-posed in Sobolev spaces  $H^s(\mathbb{R})$ , where  $s > 1/2$ . The well-posedness in critical space  $\dot{H}^{1/2}(\mathbb{R})$ , for small initial data is obtained. We also show that our result is sharp, in the sense that the flow-map data-solution is not  $C^2$  at origin, for  $s < 1/2$ . Furthermore, we study the behavior of the solutions when  $\mu \downarrow 0$ .

## 1 Introduction

This work is concerned with the initial-value problem (IVP), for the evolution equation of wave fronts in chemical reactions (WFCR)

$$\begin{cases} u_t - \partial_x^2 u - \mu(1 - \partial_x^2)^{-1/2}u - \frac{1}{2}(\partial_x u)^2 = 0, & x \in \mathbb{R}, t \geq 0, \\ u(x, 0) = \phi(x), \end{cases} \quad (1)$$

where above  $\mu > 0$  is a constant,  $u$  is a real-valued function and the operator  $(1 - \partial_x^2)^{-1/2}$  is defined via your Fourier transform by

$$((1 - \partial_x^2)^{-1/2}f)^\vee = (1 + \xi^2)^{-1/2}\hat{f}(\xi).$$

The IVP (1) describe vertical propagation of chemical waves fronts in the presence instability due to density gradients.

## 2 Main Results

In the following, we show our main results, see [3].

**Theorem 2.1.** (*Local well-posedness*). *Let  $\mu > 0$  and  $s > 1/2$ , then for all  $\phi \in H^s(\mathbb{R})$ , there exists  $T = T(\|\phi\|_{H^s})$ , a space*

$$\mathcal{X}_T^s \hookrightarrow C([0, T]; H^s(\mathbb{R}))$$

*and a unique solution  $u$  of (1) in  $\mathcal{X}_T^s$ . In addition, the flow map data-solution*

$$S : H^s(\mathbb{R}) \rightarrow \mathcal{X}_T^s \cap C([0, T]; H^s), \phi \mapsto u$$

*is smooth and*

$$u \in C((0, T]; H^\infty(\mathbb{R})).$$

**Theorem 2.2.** *Let  $\mu > 0$  and  $0 < T \leq 1$ . If  $\phi \in H^{1/2}(\mathbb{R})$  is such that  $\|\phi\|_{H^{1/2}} < (4kC_\mu)^{-1}$ , then there exists a space*

$$\mathcal{X}_T^{1/2} \hookrightarrow C([0, T]; H^{1/2}(\mathbb{R})),$$

and a unique solution  $u$  of (1) in  $\mathcal{X}_T^{1/2}$ . In addition, the flow map data-solution

$$S : H^{1/2}(\mathbb{R}) \rightarrow \mathcal{X}_T^{1/2} \cap C([0, T]; H^{1/2}), \phi \mapsto u$$

is smooth and

$$u \in C((0, T]; H^\infty(\mathbb{R})).$$

For the next result,  $\dot{H}^s$  denotes the homogeneous Sobolev space. The constants  $k$  and  $C_\mu$ , in the next results, depend on  $s$ ,  $T$  and  $\mu$ .

**Theorem 2.3.** *If the initial data is such that  $\|\phi\|_{\dot{H}^{1/2}} < (4kC_\mu)^{-1}$ , then the IVP (1) is locally well-posed in  $\dot{H}^{1/2}$ .*

**Theorem 2.4.** *(Global well-posedness). Let  $\mu > 0$  and  $s > 1/2$ , then the initial value problem (1) is globally well-posed in  $H^s(\mathbb{R})$ .*

**Theorem 2.5.** *(Ill-posedness). Let  $s < 1/2$ , if there exists some  $T > 0$ , such that the problem (1) is locally well-posed in  $H^s(\mathbb{R})$ , then the flow-map data solution*

$$S : H^s(\mathbb{R}) \rightarrow C([0, T]; H^s(\mathbb{R})), \phi \mapsto u,$$

is not  $C^2$  at zero.

To obtain the above results, we use techniques present in [2].

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## EXISTÊNCIA DE SOLUÇÕES PERIÓDICAS EM ESCOAMENTOS DE FERROFLUIDOS

JÁUBER C. OLIVEIRA<sup>1</sup><sup>1</sup>Departamento de Matemática, UFSC, SC, Brasil, j.c.oliveira@ufsc.br**Abstract**

Neste trabalho apresentamos resultados sobre a existência de soluções fortes periódicas no tempo para um sistema de equações diferenciais parciais associadas a um modelo (de Shliomis) bidimensional e tridimensional para o escoamento de fluidos ferro-magnéticos. O regime periódico no tempo é induzido por um campo magnético externo. No caso tridimensional, supomos que o campo magnético externo é suficientemente pequeno em determinada norma.

**1 Introdução**

Escoamentos de fluidos magnéticos ([2]) aparecem em várias aplicações industriais ([1]). Este estudo foi motivado pelo interesse em aplicações em que deseja-se induzir escoamentos de fluídos magnéticos em regime periódico no tempo por meio de um campo magnético externo.

O modelo considerado é o modelo de Shliomis, representado pelo seguinte sistema de equações diferenciais parciais.

$$\nabla \cdot \mathbf{u} = 0 \text{ em } \Omega \times (0, \infty), \quad (1)$$

$$\rho (\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u}) - \eta \Delta \mathbf{u} + \nabla p = \mu_0 (\mathbf{m} \cdot \nabla) \mathbf{h} + \frac{\mu_0}{2} \nabla \times (\mathbf{m} \wedge \mathbf{h}) \text{ em } \Omega \times (0, +\infty), \quad (2)$$

$$\rho (\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u}) - \eta \Delta \mathbf{u} + \nabla p = \mu_0 (\mathbf{m} \cdot \nabla) \mathbf{h} + \frac{\mu_0}{2} \nabla \times (\mathbf{m} \wedge \mathbf{h}) \text{ em } \Omega \times (0, +\infty), \quad (3)$$

$$\mathbf{m}_t + (\mathbf{u} \cdot \nabla) \mathbf{m} - \sigma \Delta \mathbf{m} = \frac{1}{2} \operatorname{rot} \mathbf{u} \wedge \mathbf{m} - \frac{1}{\tau} (\mathbf{m} - \chi_0 \mathbf{h}) - \beta \mathbf{m} \wedge (\mathbf{m} \wedge \mathbf{h}) \text{ em } \Omega \times (0, +\infty), \quad (4)$$

$$\nabla \cdot \mathbf{h} = -\nabla \cdot \mathbf{m} + F, \quad \nabla \times \mathbf{h} = 0 \text{ em } \Omega \times (0, +\infty). \quad (5)$$

Estas equações descrevem o balanço de massa, de momento linear e de magnetização, respectivamente.  $\mathbf{u}$  denota a velocidade do ferro-fluído,  $p$  denota a pressão dinâmica,  $\mathbf{h}$  representa o campo magnético e  $\mathbf{m}$  é a magnetização.  $\rho, \eta, \mu_0, \tau, \chi_0, \beta$  são constantes positivas. O modelo descreve o escoamento de um fluido magnético sujeito a um campo magnético externo  $H_{ext}$ , com  $F = -\nabla \cdot H_{ext}$ . Se o termo regularizante  $-\sigma \Delta \mathbf{m}$  é desconsiderado (desprezando-se o momento magnético de rotação [3]), até mesmo a existência de soluções fracas não é conhecida. As condições de contorno e de periodicidade são as seguintes:

$$\mathbf{u} = 0, \quad \mathbf{m} \cdot \nu = 0, \quad \nabla \times \mathbf{m} \wedge \nu = 0 \quad \text{on } \partial\Omega_T, \quad (6)$$

$$\mathbf{u}(0) = \mathbf{u}(T), \quad \mathbf{m}(0) = \mathbf{m}(T), \quad \mathbf{h}(0) = \mathbf{h}(T) \text{ em } \Omega. \quad (7)$$

Os seguintes trabalhos anteriores abordam questões de existência de soluções para este modelo: [3],[5],[1],[7].

## 2 Resultados Principais

Sejam

$$\mathcal{V}(\Omega) := \{\varphi \in \mathbb{D}(\Omega) : \operatorname{div}\varphi = 0\}, \quad H_{\operatorname{div}}(\Omega) := \{\mathbf{u} \in L^2(\Omega) : \operatorname{div}\mathbf{u} \in L^2(\Omega)\}$$

com norma

$$\|\mathbf{u}\|_{H_{\operatorname{div}}} := (\|\mathbf{u}\|^2 + \|\operatorname{div} \mathbf{u}\|^2)^{1/2}.$$

$V(\Omega)$  é o fecho de  $\mathcal{V}(\Omega)$  em  $H_0^1(\Omega)$ .  $H_{\operatorname{div},0}(\Omega)$  é o fecho de  $\mathbb{D}(\Omega)$  em  $H_{\operatorname{div}}(\Omega)$ .

Seja  $A$  o operador de Stokes,  $A : V(\Omega) \rightarrow V(\Omega)^*$  definido por  $\langle A\mathbf{v}, \mathbf{u} \rangle_{V^*,V} = \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} \, dx$ ,  $\forall \mathbf{u}, \mathbf{v} \in V(\Omega)$  com domínio  $D(A) := \{\mathbf{u} \in V(\Omega) : A\mathbf{u} \in H(\Omega)\}$  e  $(\mathbf{u}|\mathbf{v})_{D(A)} := (\mathbf{u}|\mathbf{v}) + (A\mathbf{u}|A\mathbf{v})$ ,  $\forall \mathbf{u}, \mathbf{v} \in D(A)$ .

Temos a seguinte caracterização para o espaço  $H_{\nu,0}(\Omega) = H^1(\Omega) \cap \mathcal{N}(\gamma_{\nu})$ , onde  $\gamma_{\nu}(\mathbf{u}) = \mathbf{u} \cdot \nu$  é o operador traço, contínuo de  $H_{\operatorname{div}}(\Omega)$  em  $H^{-1/2}(\partial\Omega)$ . Seja  $\mathcal{L}(\ ) = -\Delta(\ )$  com domínio  $D(\mathcal{L}) = H^2(\Omega) \cap H_{\operatorname{div},0}(\Omega)$ .

Apresentamos neste trabalho os seguintes resultados de regularidade das soluções fracas. O existência de soluções fracas está entre os resultados desta investigação. Apresentamos somente resultados de regularidade destas soluções.

**Teorema 2.1.** (*Regularidade de soluções fracas - caso 2d*)

Seja  $T > 0$  o período da função  $F \in H^1(0, T; L^2(\Omega))$  tal que  $(F|1) = 0$  em  $[0, T]$ . Seja  $\Omega \subset \mathbb{R}^2$  um conjunto aberto limitado com fronteira regular (pelo menos de classe  $C^3$ ). Então, as soluções fracas  $T$ -periódicas  $(\mathbf{u}, \mathbf{m}, \mathbf{h})$  têm a seguinte regularidade adicional:

$$\begin{aligned} \mathbf{u} &\in L^{\infty}(0, T; V(\Omega)) \cap L^2(0, T; D(A)), \quad \mathbf{m} \in L^{\infty}(0, T; H_{\nu,0}(\Omega)) \cap L^2(0, T; D(\mathcal{L})), \text{ e} \\ \mathbf{h} &\in L^{\infty}(0, T; H_{\nu,0}(\Omega)). \end{aligned}$$

Além disso, se  $F \in L^2(0, T; H^1(\Omega))$ , então  $\mathbf{h} \in L^2(0, T; H^2(\Omega))$ .

**Teorema 2.2.** (*Regularidade de soluções fracas - caso 3d*)

Seja  $F \in H^1(0, T; H^1(\Omega))$  tal que  $(F|1) = 0$  em  $[0, T]$ .  $\Omega \subset \mathbb{R}^3$  é um subconjunto aberto, limitado, simplesmente conexo com fronteira suave (pelo menos de classe  $C^3$ ). Existe uma constante  $c_0$  tal que se  $\|F\|_{H^1(0, T; H^1(\Omega))} \leq c$ , então as soluções fracas  $T$ -periódicas  $(\mathbf{u}, \mathbf{m}, \mathbf{h})$  têm a seguinte regularidade adicional:

$$\begin{aligned} \mathbf{u} &\in L^{\infty}(0, T; V(\Omega)) \cap L^2(0, T; D(A)), \quad \mathbf{m} \in L^{\infty}(0, T; H_{\nu,0}(\Omega)) \cap L^2(0, T; D(\mathcal{L})) \\ \mathbf{h} &\in L^{\infty}(0, T; H_{\nu,0}(\Omega)) \cap L^2(0, T; H^2(\Omega)). \end{aligned}$$

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**EXACT BOUNDARY CONTROLLABILITY FOR THE WAVE EQUATION IN MOVING  
BOUNDARY DOMAINS WITH A STAR-SHAPED HOLE**

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**Abstract**

We consider an exact boundary control problem for the wave equation in a moving bounded domain which has a star-shaped hole. The boundary domain is composed by two disjoint parts, one is the boundary of the hole, which is fixed, and the other one is the external boundary which is moving. The initial data has finite energy and the control obtained is square integrable and is obtained by means of the conormal derivative. We use the method of controllability presented by Russell in [2], and assume that the control acts only in the moving part of the boundary.

## 1 Introduction

In this work we study an exact boundary control problem for the standard wave equation on a domain with moving boundary which has a single fixed hole. The boundary of such domains is composed by two disjoint parts: one it is the boundary on hole which is fixed, and the other one is the external boundary which is moving. We shall consider the control acting only on the moving boundary part. A illustrative example would be a flexible body that is crossed by a cylindrical pillar and is fixed to it. Without any variation in the temperature of the environment the body has no dilation and thus its external boundary remains static. However, if there is a variation in the temperature, the body would have a dilation or a contraction, causing the mobility of the its external boundary. In this work when we deal with a domain with a hole, and we refer by external boundary as being the part of the boundary of the domain that does not coincide with that one of the hole.

To establish these concepts in more detail, we consider  $B \subset \mathbb{R}^n$ ,  $n \geq 2$ , a convex compact set with the origin in its interior with smooth boundary  $\Gamma_0$ . We set  $\Omega_\infty = \mathbb{R}^n - B$ . Let  $\Xi \subset \mathbb{R}^n$  be a simply connected bounded domain with piecewise smooth boundary  $\Gamma_1$ , with no cusps, such that  $B \subset \Xi$ . We assume that  $\text{dist}(\Gamma_0, \Gamma_1) \geq \epsilon > 0$  and set  $\Omega = \Xi - B$ . Hence, the boundary of  $\Omega$  is  $\partial\Omega = \Gamma_0 \cup \Gamma_1$ . Note that  $\Omega$  is a holed domain whose hole has the shape of  $B$ . We also consider the moving boundary domain  $\Xi_t \subset \mathbb{R}^n$  where  $\Xi_t = \{x \in \mathbb{R}^n : x = \alpha(t)y, y \in \Xi\}$ ,  $t \in [0, +\infty)$  whose boundary is denoted by  $\Gamma_t$  and  $\Xi_0 = \Xi$ . Here  $\alpha : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$  is a piecewise bounded smooth function, where

$$\Xi_t \times \mathbb{R} \subset \cup_{\bar{x} \in \Xi} \{(x, t) \in \mathbb{R}^N \times \mathbb{R} : |x - \bar{x}|^2 \leq t^2\}, \quad (1)$$

$\Xi_t \subset B(0, r)$  for all  $t \in [0, +\infty)$ . Defining  $\tilde{\Omega} = B(0, r) - B$  and  $\Omega_t = \Xi_t - B$  we can see that  $\Omega_t \subset \tilde{\Omega}$  for all  $t \in [0, +\infty)$ . The boundary of  $\Omega_t$  is  $\partial\Omega_t = \Gamma_0 \cup \Gamma_t$ . Now, for  $T > 0$ , let us consider the non-cylindrical domain of  $\mathbb{R}^{n+1}$ ,  $Q_T = \cup_{0 < t < T} \Omega_t \times \{t\}$  whose the lateral boundary is  $\Sigma_T \cup \Sigma_0$ , where  $\Sigma_T = \cup_{0 < t < T} \Gamma_t \times \{t\}$  and  $\Sigma_0 = \Gamma_0 \times [0, T]$ .

We denote by  $(\nu_x, \nu_t)$  the outward unit normal vector defined almost all on  $\Sigma_T \cup \Sigma_0$ . Note that  $Q_T$  is a holed non-cylindrical domain in  $\mathbb{R}^{n+1}$  whose the lateral boundary is composed by two disjoint parts  $\Sigma_T$  and  $\Sigma_0$ . Here, we requires that  $B$  be star-shaped with respect the origin, that is,  $\{\nu_x \cdot x\} \leq 0$  for  $x \in \partial B$ . The assumption (1) assures that the surface  $\Sigma_T$  is time-like. This is known to be sufficient to guarantee the well-posedness of the initial and boundary value problem studied here. The purpose of this work is to study the exact boundary controllability problem

**Theorem 1.1.** Let  $\Omega$  be as defined above. Given  $(f, g) \in \mathcal{H}^1(\Omega) \times L^2(\Omega)$ , there exist  $T > 0$  sufficiently large and a control function  $h(\cdot, t) \in L^2(\Sigma_T)$  such that the solution  $u \in H^1(Q_T)$  of the problem

$$\begin{aligned} u_{tt} - \Delta u &= 0 && \text{in } Q_T \\ u(\cdot, 0) = f, \quad u_t(\cdot, 0) = g, & \quad \text{in } \Omega \\ u(\cdot, t) = 0, & \quad \text{on } \Gamma_0 \times [0, T] \\ \nu_t u_t - \nabla u \cdot \nu_x &= h(\cdot, t), \quad \text{on } \Sigma_T. \end{aligned} \tag{2}$$

satisfy the final condition

$$u(\cdot, T) = 0 = u_t(\cdot, T) \quad \text{in } \Omega_T. \tag{3}$$

## 2 Idea of the proof of the Theorem 1

The Theorem 1 is proved completely in the paper [1]. Here we show the sketch of its proof. Firstly we take the solution  $\tilde{u}$  of the initial-boundary value problem

$$\begin{aligned} \tilde{u}_{tt} - \Delta \tilde{u} &= 0 && \text{in } \Omega_\infty \times (0, +\infty) \\ \tilde{u}(\cdot, 0) = \tilde{f}, \quad \tilde{u}_t(\cdot, 0) = \tilde{g}, & \quad \text{in } \Omega_\infty \\ \tilde{u}(\cdot, t) = 0, & \quad \text{in } \Gamma_0 \times (0, +\infty). \end{aligned} \tag{4}$$

Being  $\tilde{f}$  and  $\tilde{g}$  extension of  $f$  and  $g$  respectively to  $\Omega_\infty$ . After, for a  $T > 0$  great sufficiently we take the state  $(\tilde{u}(\cdot, T), \tilde{u}_t(\cdot, T)) \in \mathcal{H}^1(\tilde{\Omega}) \times L^2(\tilde{\Omega})$  and we solve the exact boundary control problem

$$\begin{aligned} v_{tt} - \Delta v &= 0 && \text{in } \tilde{\Omega} \times [0, T] \\ v(\cdot, T) = \tilde{u}(\cdot, T), \quad v_t(\cdot, T) = \tilde{u}_t(\cdot, T), & \quad \text{in } \tilde{\Omega} \\ v(\cdot, t) = 0, & \quad \text{on } \Gamma_0 \times [0, T] \\ \nu_t v_t - \nabla v \cdot \nu_x &= h(\cdot, t), \quad \text{on } \tilde{\Gamma} \times [0, T], \end{aligned} \tag{5}$$

which satisfies, at the instant  $t = 0$ , the condition  $v(\cdot, 0) = 0 = v_t(\cdot, 0)$  in  $\tilde{\Omega}$ .

Considering  $\Omega_t$  as defined in the Section 1, note that  $\Omega_t \subset \tilde{\Omega}$  for all  $t > 0$ , so follows that for each  $T > 0$  we have  $Q_T = \cup_{0 < t < T} \Omega_t \times \{t\} \subset \tilde{\Omega} \times [0, T]$ . Defining  $u = \tilde{u} - v$  we can see that the restriction of  $u$  to  $Q_T$  satisfies (2) and the condition (3). The control function  $h$  is obtained by means of a trace theorem established in [3].

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## MAXIMAL ATTRACTORS FOR SEMIGROUPS

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### Abstract

The theory of compact global attractors for semigroups relies on the existence of a bounded absorbing set. Here, suppressing this condition, we present a general theory of maximal attractors for semigroups. Such attractors are, in general, unbounded. We present sufficient conditions to ensure their existence. As an example we present a semilinear parabolic equation.

This is a joint work with Juliana Fernandes (UFRJ), and the results presented are extracted from our submitted paper [1].

### 1 Introduction and Main Results

For the last several decades, semigroups with bounded absorbing sets have been extensively studied by many authors. However, if unbounded solutions exist, the system has no such bounded absorbing set. A simple example is the semigroup generated by a linear map in  $\mathbb{R}^2$  with one eigenvalue outside and the other inside the unitary circle, in which case the solutions *converge* to the line defined by the eigenvector associated with the eigenvalue outside the unitary circle. Despite the fact that unbounded attractors are quite common objects in evolution equations, they are harder to study, due to the lack of compactness, and much less is known in their regard. In this work, we explore the asymptotic behavior of semigroups without assuming the existence of a bounded absorbing set.

We explain now our main results. In a metric space  $X$ , we consider a semigroup  $T = \{T(t) : t \geq 0\}$ . A **maximal attractor** for  $T$  is a closed subset  $\mathcal{U}$  of  $X$  that satisfies:

- (i)  $\mathcal{U}$  is **invariant**, that is,  $T(t)\mathcal{U} = \mathcal{U}$  for all  $t \geq 0$ ;
- (ii)  $\mathcal{U}$  **attracts** bounded subsets of  $X$ , that is, for each bounded subset  $B$  of  $X$  we have  $\text{dist}_H(T(t)B, \mathcal{U}) \xrightarrow{t \rightarrow \infty} 0$ , where  $\text{dist}_H(C, D) = \sup_{c \in C} \inf_{d \in D} d(c, d)$  denotes the Hausdorff semidistance between two nonempty subsets  $C, D$  of  $X$ ;
- (iii) there is no proper closed subset  $\mathcal{V}$  of  $\mathcal{U}$  that satisfies both (i) and (ii).

The task to find a maximal attractor for a semigroup is arduous, and not possible in general. However, following the ideas of Chepyzhov and Goritskiĭ [2], we find conditions under which the **set of bounded in the past global solutions** of a semigroup  $T$ , given by  $\mathcal{J} = \{\xi(0) : \xi \text{ is a bounded in the past global solution of } T\}$ , is a maximal attractor for  $T$ . The main result of our work is as follows:

**Theorem 1.1.** *Let  $(X, d)$  be a complete metric space and  $T$  be a semigroup in  $X$  with the set of bounded in the past global solutions  $\mathcal{J}$  nonempty. Assume that:*

- (a)  $T$  is **asymptotically compact**, that is, given a bounded subset  $B$  of  $X$  there exists  $t_0 = t_0(B) \geq 0$  such that for each  $t \geq t_0$  there exist a compact subset  $K = K(B, t)$  of  $X$  and  $\epsilon = \epsilon(B, t) \geq 0$  with  $\text{dist}_H(T(t)B, K) < \epsilon$ , with  $\epsilon \xrightarrow{t \rightarrow \infty} 0$ ;
- (b)  $T$  has a **strongly absorbing set**  $G$ , which means that:

- (H<sub>1</sub>) *G is positively invariant, that is,  $T(t)G \subset G$  for all  $t \geq 0$ ;*
- (H<sub>2</sub>) *for each bounded subset B of X, there exists  $t_0 = t_0(B) \geq 0$  such that  $T(t)B \subset G$  for all  $t \geq t_0$ ;*
- (H<sub>3</sub>) *there exists a sequence of bounded subsets  $(H_n)_{n \in \mathbb{N}}$  of G with the following properties:*
  - \*  $H_n \subset H_{n+1}$  for all  $n \in \mathbb{N}$ ;*
  - \*  $G \setminus H_n$  is positively invariant for T for each  $n \in \mathbb{N}$ ;*
  - \* if  $B \subset G$  is bounded, then  $B \subset H_n$  for some  $n \in \mathbb{N}$ .*
- (H<sub>4</sub>)  $\text{dist}_H(T(t)G, \mathcal{J}) \xrightarrow{t \rightarrow \infty} 0$ .

Then  $\mathcal{J}$  is the unique maximal attractor for T, and  $\mathcal{J}$  is **bounded-compact**, that is, the intersection of  $\mathcal{J}$  with each closed bounded subset of X is compact.

To illustrate the theory we use Theorem 1.1 to obtain that  $\mathcal{J}$  is the bounded-compact maximal attractor for the semigroup generated by the following n-dimensional semilinear parabolic equation:

$$\begin{cases} u_t = \Delta u + bu + f(u), & \text{in } (0, \infty) \times \Omega \\ u = 0, & \text{in } (0, \infty) \times \partial\Omega, \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \quad (1)$$

where  $\Omega \subset \mathbb{R}^n$  is a bounded domain (open and connected) with sufficiently smooth boundary,  $b > 0$ ,  $u_0 \in L^2(\Omega)$  and  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^2$  function satisfying

$$|f(r)| \leq c \quad \text{and} \quad |f'(r)| \leq c \quad \text{for all } r \in \mathbb{R}, \quad (2)$$

for some given constant  $c \geq 0$ .

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## CONTROLLABILITY UNDER POSITIVE CONSTRAINTS FOR QUASILINEAR PARABOLIC PDES

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### Abstract

In this work, we deals with the analysis of the internal control with constraint of positive kind of a parabolic PDE with nonlinear diffusion when the time horizon is large enough. The minimal controllability time will be strictly positive.

We prove a global steady state constrained controllability result for a quasilinear parabolic with nonlinearity in the diffusion term. Then, under suitable dissipative assumption in the system and local controllability results, we conclude the result to any initial datum and any target trajectory.

### 1 Introduction

Let  $\Omega \subset \mathbb{R}^N$  ( $N \geq 1$  is an integer) be a non-empty bounded connected open set, with regular boundary  $\partial\Omega$ . We fix  $T > 0$  and set  $Q := \Omega \times (0, T)$  and  $\Sigma := \partial\Omega \times (0, T)$ .

Let  $\omega, \omega_1 \subset \Omega$  be non-empty open sets, such that  $\omega_1 \subset \subset \omega$ . We deal with the exact controllability to trajectories for the quasilinear system

$$\begin{cases} y_t - \nabla \cdot (a(y)\nabla y) = v\varrho_\omega & \text{in } Q, \\ y(x, t) = 0 & \text{on } \Sigma, \\ y(x, 0) = y_0(x) & \text{in } \Omega, \end{cases} \quad (1)$$

where  $y$  is the associated state,  $v$  is the control and  $\varrho_\omega \in C_0^\infty(\overline{\Omega})$ , such that  $\varrho_\omega = 0$  in  $\Omega \setminus \omega$  and  $\varrho_\omega = 1$  in  $\omega_1$ .

Here, it will be assumed that the real-valued function  $a = a(r)$  satisfies

$$a \in C^2(\mathbb{R}), \quad 0 < a_0 \leq a(r) \quad \text{and} \quad |a'(r)| + |a''(r)| \leq M, \quad \forall r \in \mathbb{R}. \quad (2)$$

**Definition 1.1.** Let  $\bar{v} \in C^{1/2}(\overline{\Omega})$ , a function  $\bar{y} \in C^{2+1/2}(\overline{\Omega})$  is said to be a **steady state** for (1) if it is a solution to

$$-\nabla \cdot (a(\bar{y})\nabla \bar{y}) = \bar{v}\varrho_\omega \quad \text{in } \Omega, \quad \bar{y} = 0 \quad \text{in } \partial\Omega. \quad (3)$$

The function  $\bar{v} \in C^{1/2}(\overline{\Omega})$  is called the **steady control**.

**Remark 1.1.** The application  $\Lambda : \bar{v} \mapsto \bar{y}$  shown in (3) is continuous, since  $a(\cdot)$  satisfies (2).

We will denote by  $\mathcal{S} := \Lambda(C^{1/2}(\overline{\Omega}))$  the set of all the steady-states with steady controls in  $C^{1/2}(\overline{\Omega})$ .

**Definition 1.2.** Fixed  $y_0, y_1 \in \mathcal{S}$  and fixed  $\bar{v}^0, \bar{v}^1$  such that  $\Lambda(\bar{v}^0) = y_0$  and  $\Lambda(\bar{v}^1) = y_1$ , we define a **path-connected steady states** that drive  $y_0$  to  $y_1$  as a continuous path

$$\begin{aligned} \gamma : [0, 1] &\xrightarrow{\lambda} C^{1/2}(\Omega) \xrightarrow{\Lambda} \mathcal{S} \\ s &\longmapsto \lambda(s) \longmapsto \gamma(s) = \Lambda(\lambda(s)), \end{aligned}$$

where  $\lambda(s)$  is a continuous path of steady controls that drive  $\bar{v}^0$  to  $\bar{v}^1$  ( $\lambda(0) = \bar{v}^0$  and  $\lambda(1) = \bar{v}^1$ ).

For each  $s \in [0, 1]$ , we denote  $\bar{y}^s := \gamma(s)$  the steady state and  $\bar{v}^s := \lambda(s)$  the steady control of continuous path  $\gamma$ .

**Definition 1.3.** Let us define a **target trajectory**  $\bar{y} = \bar{y}(x, t)$  for (1) as solution to

$$\begin{cases} \bar{y}_t - \nabla \cdot (a(\bar{y}) \nabla \bar{y}) = \bar{v} \varrho \omega & \text{in } Q, \\ \bar{y}(x, t) = 0 & \text{on } \Sigma, \\ \bar{y}(x, 0) = \bar{y}_0(x) & \text{in } \Omega, \end{cases} \quad (4)$$

with  $\bar{y}_0 \in C^{2+1/2}(\bar{\Omega})$  and  $\bar{v} \in C^{1/2, 1/4}(\bar{Q})$  such that

$$M_a \|\nabla \bar{y}\|_{L^\infty(\Omega \times (0, T))} \leq \frac{a_0}{2 C(\Omega)}, \quad (5)$$

where  $C(\Omega)$  is the Poincaré inequality constant, so  $\|u\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla u\|_{L^2(\Omega)}$  and the constant  $M_a$  is defined by  $M_a := \sup_{r \in \mathbb{R}} |a'(r)|$ .

## 2 Main Results

Now, let us state the main result in this section is the following

**Theorem 2.1.** Let  $y_0, y_1 \in \mathcal{S}$  fixed and let  $\gamma(s) := \bar{y}^s$  be path-connected steady states that drive  $y_0$  to  $y_1$  with steady control  $\bar{v}^s$ . Let us assume there exists a constant  $\eta > 0$  such that

$$\bar{v}^s \geq \eta, \quad \forall s \in [0, 1]. \quad (1)$$

Then there exists  $T_0 > 0$  such that, for every  $T \geq T_0$  there exists a control  $v \in L^\infty(\Omega \times (0, T))$  such that, the system (1) admits a unique solution  $y$  satisfying  $y(\cdot, T) = y_1(\cdot)$  in  $\Omega$  and  $v \geq 0$  in  $\Omega \times (0, T)$ .

Now, we will extend Theorem 2.1 in the following Theorem:

**Theorem 2.2.** Suppose there exists a target trajectory  $\bar{y}$  satisfying the condition (5) with initial datum  $\bar{y}_0$  and control  $\bar{v}$ . Let us assume there exist a constant  $\eta > 0$  such that

$$\bar{v} \geq \eta \quad \text{in } \Omega \times \mathbb{R}^+. \quad (2)$$

For any  $y_0 \in C^{2+1/2}(\bar{\Omega})$  initial datum, there exists  $T_0 > 0$  such that for every  $T \geq T_0$ , we can find a control  $v \in L^\infty(\Omega \times (0, T))$  such that the unique solution  $y$  to (1) satisfies  $y(T) = \bar{y}(T)$  in  $\Omega$  and  $v \geq 0$  in  $\Omega \times (0, T)$ . Furthermore, if  $y_0 \neq \bar{y}_0$  then the minimal controllability time  $T_{\min}$  is strictly positive, where

$$T_{\min} := \inf \{T > 0; \exists v \in L^\infty(\Omega \times (0, T))^+, \text{ such that } y(T) = \bar{y}(T) \text{ in } \Omega\}. \quad (3)$$

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**ON A VARIATIONAL INEQUALITY FOR A BEAM EQUATION WITH INTERNAL DAMPING  
AND SOURCE TERMS**

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**Abstract**

In this paper we investigate the unilateral problem for a extensible beam equation with internal damping and source terms

$$u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + u_t = |u|^{r-1}u$$

where  $r > 1$  is a constant,  $M(s)$  is a continuous function on  $[0, +\infty)$ . The global solutions are constructed by using the Faedo-Galerkin approximations, taking into account that the initial data is in appropriate set of stability created from the Nehari manifold.

## 1 Introduction

In [7] the authors establish existence of global solution to the problem

$$u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + u_t = |u|^{r-1}u \quad (1)$$

$$u(., 0) = u_0, \quad u_t(., 0) = u_1 \text{ in } \Omega, \quad (2)$$

$$u(., t) = \frac{\partial u}{\partial \eta}(., t) \text{ in } \partial\Omega, t \geq 0, \quad (3)$$

where  $\Omega$  is a bounded domain of  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$ ,  $r > 1$  is a constant and  $M(s)$  is a continuous function on  $[0, +\infty)$ ,  $u = 0$  is the homogeneous Dirichlet boundary condition and the normal derivative  $\frac{\partial u}{\partial \eta} = 0$  is the homogeneous Neumann boundary condition, where  $\eta$  unit outward normal on  $\partial\Omega$ .

A nonlinear perturbation of problem (1) is given by

$$u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + u_t - |u|^{r-1}u \geq 0. \quad (4)$$

In the present work we investigated the unilateral problem associated with this perturbation, that is, a variational inequality given for (4) (see [5]). Making use of the penalty method, the potential well theory and Galerkin's approximations, we establish existence and uniqueness of global solutions.

Unilateral problem is very interesting too, because in general, dynamic contact problems are characterized by nonlinear hyperbolic variational inequalities. For contact problem on elasticity and finite element method see Kikuchi-Oden [4] and reference there in. For contact problems on viscoelastic materials see [6]. For contact problems on Klein-Gordon operator see [8]. For contact problems on Oldroyd Model of Viscoelastic fluids see [3]. For contact problems on Navier-Stokes Operator with variable viscosity see [1]. For contact problems on viscoelastic plate equation see [1].

## 2 Main Results

**Theorem 2.1.** Consider the spaces

$$H_{\Gamma}^4(\Omega) = \{u \in H^4(\Omega) | u = \Delta u = 0 \text{ on } \Gamma\} \text{ and } H_{\Gamma}^3(\Omega) = \{u \in H^3(\Omega) | u = \Delta u = 0 \text{ on } \Gamma\}.$$

If  $u_0 \in W_1 \cap H_{\Gamma}^4(\Omega)$ ,  $J(u_0) < d$ ,  $u_1 \in H_0^1(\Omega) \cap H^2(\Omega)$ ,  $1 < r \leq 5$  and the hypothesis  $(H_1)$  and  $(H_2)$  holds, then there exists a function  $u : [0, T] \rightarrow L^2(\Omega)$  in the class

$$u \in L^{\infty}(0, T; (H_0^1(\Omega) \cap H^2(\Omega)) \cap H_{\Gamma}^3(\Omega)) \cap L^{\infty}(0, T; L^{r+1}(\Omega)) \quad (1)$$

$$u_t \in L^{\infty}(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \quad (2)$$

$$u_{tt} \in L^{\infty}(0, T; L^2(\Omega)), \quad u_t(t) \in K \text{ a.e. in } [0, T], \quad (3)$$

satisfying

$$\int_Q (u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + u_t - |u|^{r-1}u)(v - u_t) \geq 0, \forall v \in L^2(0, T; H_0^1(\Omega)), v(t) \in K \text{ a.e. in } t \quad (4)$$

$$u(0) = u_0, \quad u_t(0) = u_1$$

where  $J : H_0^1(\Omega) \cap H^2(\Omega) \rightarrow \mathbb{R}$  defined by  $J(u) = \frac{1}{2}|\Delta u|^2 + \frac{m_0}{2}|\nabla u|^2 - \frac{1}{r+1}|u|^{r+1}$ , and

$$(H_1) \quad M \in C^1([0, \infty)) \text{ with } M(\lambda) \geq m_0 > 0, \quad \forall \lambda \geq 0, \quad (H_2) \quad (r-1)n \leq rn \leq q = \frac{2n}{n-2}, n > 2.$$

**Proof** The proof of Theorem 2.1 is made by the penalization method. It consists in considering a perturbation of the problem (1) adding a singular term called penalty, depending on a parameter  $\epsilon > 0$ . We solve the mixed problem in  $Q$  for the penalization operator and the estimates obtained for the local solution of the penalized equation, allow to pass to limits, when  $\epsilon$  goes to zero, in order to obtain a function  $u$  which is the solution of our problem.

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**THE NONLINEAR QUADRATIC INTERACTIONS OF THE SCHRÖDINGER TYPE ON THE HALF-LINE**

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**Abstract**

In this work we study the initial boundary value problem associated with the coupled Schrödinger equations with quadratic nonlinearities, that appears in nonlinear optics, on the half-line. We obtain local well-posedness for data in Sobolev spaces with low regularity, by using a forcing problem on the full line with a presence of a forcing term in order to apply the Fourier restriction method of Bourgain. The crucial point in this work is the new bilinear estimates on the classical Bourgain spaces  $X^{s,b}$  with  $b < \frac{1}{2}$ , jointly with bilinear estimates in adapted Bourgain spaces that will be used to treat the traces of nonlinear part of the solution. Here the understanding of the dispersion relation is the key point in these estimates, where the set of regularity depends strongly of the constant  $a$  measures the scaling-diffraction magnitude indices.

This work was submitted for publication and can be accessed in <https://arxiv.org/abs/2104.05137>.

## 1 Introduction

This work is dedicated to the study the initial boundary value problem associated to system nonlinear quadratic of the Schrödinger on the half-line, more precisely

$$\begin{cases} i\partial_t u(x, t) + \partial_x^2 u(x, t) + \bar{u}(x, t)v(x, t) = 0, & x \in (0, +\infty), t \in (0, T), \\ i\partial_t v(x, t) + a\partial_x^2 v(x, t) + u^2(x, t) = 0, & x \in (0, +\infty), t \in (0, T), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in (0, +\infty), \\ u(0, t) = f(t), \quad v(0, t) = g(t), & t \in (0, T), \end{cases} \quad (1)$$

where  $u$  and  $v$  are complex valued functions, where  $a > 0$ . The model (1) is given by the nonlinear coupling of two dispersive equations of Schrödinger type through the quadratic terms  $N_1(u, v) = \bar{u} \cdot v$  and  $N_2(u, v) = u^2$ .

An important point in this model is the fact that the functional mass is not conserved, since some bad terms of boundary appear in the mass functional. More precisely, define the functional of mass for the system (1) by

$$\mathcal{M}(t) = \|u(t)\|_{L_x^2(\mathbb{R}^+)}^2 + \|v(t)\|_{L_x^2(\mathbb{R}^+)}^2.$$

Formally, by multiplying the first equation of the system (1) by  $\bar{u}$  and the second equation by  $\bar{v}$ , integrating by parts, taking the imaginary part and using  $\text{Im}(\bar{u}^2 v) = -\text{Im}(u^2 \bar{v})$ , we get

$$\mathcal{M}(t) = \mathcal{M}(0) + \text{Im} \int_0^t \bar{u}(0, s) \partial_x u(0, s) ds + a \text{Im} \int_0^t \bar{v}(0, s) \partial_x v(0, s) ds. \quad (2)$$

This identity suggesters on the case of homogeneous boundary conditions a global result on the space  $L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$ .

Physically, according to the article [4], the complex functions  $u$  and  $v$  represent amplitude packets of the first and second harmonic of an optical wave. In the mathematical context on the paper [1] the first author obtained local well posedness for the model posed on real line by assuming low regularity assumptions.

## 2 Main Results

Our main local well-posedness result is the following statement.

**Theorem 2.1.** *Let the Sobolev index pair  $(\kappa, s)$  verifying  $s \neq \frac{1}{2}$  and  $\kappa \neq \frac{1}{2}$  and*

*(i)  $|\kappa| - 1/2 \leq s < \min\{\kappa + 1/2, 2\kappa + 1/2, 1\}$  and  $\kappa < 1$  for  $a > \frac{1}{2}$  (first non resonant case);*

*(ii)  $0 \leq \kappa = s < 1$  for  $a = \frac{1}{2}$  (resonant case);*

*(iii)  $\max\{-\frac{1}{2}, |\kappa| - 1\} \leq s < \min\{\kappa + 1, 2\kappa + 1, 1\}$  and  $\kappa < 1$  for  $0 < a < \frac{1}{2}$  (second non resonant case). For any  $a > 0$  and  $(u_0, v_0) \in H^\kappa(\mathbb{R}^+) \times H^s(\mathbb{R}^+)$  and  $(f, g) \in H^{\frac{2\kappa+1}{4}}(\mathbb{R}^+) \times H^{\frac{2s+1}{4}}(\mathbb{R}^+)$ , verifying the additional compatibility conditions*

$$\begin{cases} u(0) = f(0), & \text{for } \kappa > \frac{1}{2}; \\ v(0) = g(0), & \text{for } s > \frac{1}{2}. \end{cases} \quad (3)$$

*Then there exist a positive time  $T = T\left(\|u_0\|_{H^\kappa(\mathbb{R}^+)}, \|v_0\|_{H^s(\mathbb{R}^+)}, \|f\|_{H^{\frac{2\kappa+1}{4}}(\mathbb{R}^+)}, \|g\|_{H^{\frac{2s+1}{4}}(\mathbb{R}^+)}, a\right)$  and a distributional solution  $(u(t), v(t))$  for the initial boundary value problem (1) on the classes*

$$u \in C([0, T]; H^\kappa(\mathbb{R}^+)) \quad \text{and} \quad v \in C([0, T]; H^s(\mathbb{R}^+)). \quad (4)$$

*Moreover, the map  $(u_0, v_0) \mapsto (u(t), v(t))$  is locally Lipschitz from  $H^\kappa(\mathbb{R}^+) \times H^s(\mathbb{R}^+)$  into  $C([0, T]; H^\kappa(\mathbb{R}^+) \times H^s(\mathbb{R}^+))$ .*

The approach used to prove this result is based on the arguments introduced in [3] and [2]. The main idea to solve the IBVP (1) is the construction of an auxiliary forced IVP in the line  $\mathbb{R}$ , analogous to (1); more precisely:

$$\begin{cases} i\partial_t u(x, t) + \partial_x^2 u(x, t) + \bar{u}(x, t)v(x, t) = \mathcal{T}_1(x)h_1(t), & (x, t) \in \mathbb{R} \times (0, T) \\ i\partial_t v(x, t) + a\partial_x^2 v(x, t) + u^2(x, t) = \mathcal{T}_2(x)h_2(t), & (x, t) \in \mathbb{R} \times (0, T) \\ u(x, 0) = \tilde{u}_0(x), \quad v(x, 0) = \tilde{v}_0(x), & x \in \mathbb{R} \end{cases} \quad (5)$$

where  $\mathcal{T}_1, \mathcal{T}_2$  are appropriate distributions supported in  $\mathbb{R}^-$ ,  $\tilde{u}_0, \tilde{v}_0$  are nice extensions of  $u_0$  and  $v_0$  in  $\mathbb{R}$  and the boundary forcing functions  $h_1, h_2$  are selected to ensure that

$$\tilde{u}(0, t) = f(t) \quad \text{and} \quad \tilde{v}(0, t) = g(t)$$

for all  $t \in (0, T)$ .

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## VIBRATIONS OF A BAR SUBMITTED TO AN IMPACT

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### Abstract

In this paper is investigated the existence of solutions of a mathematical model that describes the vibrations of a bar by an impact in one of its ends.

## 1 Introduction

Consider an elastic homogeneous cylindrical bar of lenght  $L$  where the cross sections of the bar are small when comparing with its lenght. In the rest position the bar coincides whith the interval  $[0,L]$  of the axis Ox. At the end  $x = 0$ , the bar is clamped and the end  $x = L$  is free. At the initial time  $t = 0$ , the free end is hit by a mass  $M$ , which is moving with velocity  $\alpha_0$  in the direction of the axis of the bar. Then the mass remains glued at the end  $x = L$ . Under the impact, the cross sections of the bar begin to vibrate longitudinally. Assume that these vibrations are small.

The above physical problem is modeled by the following mathematical model:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{E}{\rho} \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \quad 0 < x < L, t > 0; \quad (1)$$

$$u(0,t) = 0, \quad M \frac{\partial^2 u(L,t)}{\partial t^2} + AE \frac{\partial u(L,t)}{\partial x} = 0, \quad t > 0; \quad (2)$$

$$u(x,0) = 0, \quad 0 \leq x \leq L; \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad 0 \leq x < L, \quad \frac{\partial u(L,0)}{\partial t} = -\alpha_0. \quad (3)$$

where  $u(x,t)$  denotes the displacement of the cross section  $x$  of the bar at time  $t$ . Here  $E$  is the Young's modulus of the material of the bar,  $\rho$  its constant density and  $A$  the area of the uniform cross sections.

The above mathematical model was introduced by Koslyakov et al.[1]

The objective of this paper is to investigate the existence of solutions of Problem (1.1)-(1.3).

## 2 Main Results

Denote by  $(u, v)$  and  $|u|$  the usual scalar product and norm of the space  $L^2(0, L)$  By V is represented the Hilbert space

$$V = \{u \in H^1(0, L); u(0) = 0\}$$

equipped with the scalar product

$$((u, v)) = \int_0^L \frac{du}{dx} \frac{dv}{dx} dx$$

and norm  $\|u\| = ((u, u))^{1/2}$ . By  $\delta_L$  is denoted the fucntional

$$\langle \delta_L, v \rangle = v(L), \quad v \in C^0([0, L]; \mathbb{R}) = X$$

then  $\delta_L \in X'$ .

Let  $T > 0$  be an arbitrary fixed real number. Consider the problem

$$\theta''(x, t) - \theta_{xx}(x, t) = f(x, t), \quad 0 < x < L, \quad 0 < t < T; \quad (1)$$

$$\theta(0, t) = 0, \quad \theta''(L, t) + \theta_x(L, t) = 0, \quad 0 < t < T; \quad (2)$$

$$\theta(x, T) = 0, \quad \theta'(x, T) = 0 \quad (3)$$

where  $\theta' = \frac{\partial \theta}{\partial t}$  and  $\theta_x = \frac{\partial \theta}{\partial x}$ . For each  $f \in L^1(0, T; L^2(0, L))$  is determined the weak solution  $\theta$  of the Problem (2,1)-(2.3). One has

$$\frac{1}{2}|\theta'(t)|^2 + \frac{1}{2}\|\theta(t)\|^2 + \frac{1}{2}[\theta'(L, t)]^2 \leq \int_0^T |f(t)|\|\theta'(t)\|dt, \quad \forall 0 \leq t \leq T.$$

**Definition 2.1.** A function  $u \in L^\infty(0, T; L^2(0, L))$  is named a solution defined by transposition of Problem (1.1)-(1.3) if

$$\int_0^T \int_0^L u(x, t)f(x, t)dxdt = -2\alpha_0 < \delta_L, \theta(., 0) >, \quad \forall f \in L^1(0, T; L^2(0, L))$$

where  $\theta$  is the weak solution of (2.1)-(2.3) with  $f$  (see [2] and [3])

**Theorem 2.1.** There exists a unique solution  $u$  defined by transposition of Problem (1.1)-(1.3). Furthermore  $u$  satisfies

$$\begin{aligned} u'' - u_{xx} &= 0 \text{ in } L^\infty(0, T; L^2(0, L)); \\ u(0, .) &= 0, \quad u''(L, .) + u_x(L, .) = 0 \text{ in } L^\infty(0, T); \\ u(0) &= 0, \quad u'(0) = -\alpha_0 \delta_L. \end{aligned}$$

The theorem is obtained by applying the Galerkin method, results of the Trace Theorem and the interpolation of Hilbert spaces.

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## ABOUT POLYNOMIAL STABILITY FOR THE POROUS-ELASTIC SYSTEM WITH FOURIER'S LAW

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### **Abstract**

In this work, we consider the porous-elastic equations mixing Kelvin-Voigt dissipation mechanisms and the thermal effect given by Fourier's law. We prove that the system lack the exponential decay property for a particular equality between damping parameters. In that direction, we prove the polynomial decay and the optimal decay rate.

## 1 Introduction

Based on Quintanilla and Ueda [1], we consider the one-dimensional porous elastic system with Fourier's law is given by

$$\rho u_{tt} - \alpha u_{xx} - \beta \phi_x - \gamma u_{xxt} - \varepsilon \phi_{xt} + \xi \theta_x = 0 \quad \text{in } (0, l) \times (0, \infty), \quad (1)$$

$$\kappa \phi_{tt} - \delta \phi_{xx} + \beta u_x + \eta \phi + \tau \phi_t + \varepsilon u_{xt} - \hbar \theta = 0 \quad \text{in } (0, l) \times (0, \infty), \quad (2)$$

$$\rho_2 \theta_t - K \theta_{xx} + \xi u_{xt} + \hbar \phi_t = 0 \quad \text{in } (0, l) \times (0, \infty), \quad (3)$$

where  $\rho, \alpha, \gamma, \tau, \kappa, K, \delta, \xi, \hbar$  and  $\eta$  are constitutive coefficients,  $\beta \neq 0$  satisfies  $\alpha\eta - \beta^2 > 0$ . Moreover, the functions  $u, \phi$  and  $\theta$  represent the displacement of a solid elastic material, the volume of porous fraction and the temperature, respectively. In addition, we consider that the coefficients  $\gamma > 0, \tau > 0$  and  $\varepsilon \neq 0$  satisfies the relationship

$$\gamma\tau - \varepsilon^2 = 0. \quad (4)$$

Here, we assume the Dirichlet-Neumann-Neumann boundary conditions

$$u(0, t) = u(l, t) = \phi_x(0, t) = \phi_x(l, t) = \theta_x(0, t) = \theta_x(l, t) = 0, \quad t > 0, \quad (5)$$

and the initial conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \phi(x, 0) = \phi_0(x), \quad \phi_t(x, 0) = \phi_1(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in (0, l). \quad (6)$$

## 2 Main Results

**Theorem 2.1.** *Let us suppose that  $\gamma\tau - \varepsilon^2 = 0$ . Then the semigroup  $S(t) = e^{\mathcal{A}t}$  associated with the system (1)–(6) is not exponentially stable.*

**Proof** To prove this result we will argue by contradiction, that is, we will show that there exists a sequence of number  $(\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{R}$  with  $|\lambda_n| \rightarrow \infty$  and  $(U_n)_{n \in \mathbb{N}} \subset D(\mathcal{A})$  for  $(F_n)_{n \in \mathbb{N}} \subset \mathcal{H}$ , with  $\|F_n\|_{\mathcal{H}} < \infty$  such that

$$(i\lambda_n I - \mathcal{A})U_n = F_n, \quad (1)$$

where  $F_n$  is bounded in  $\mathcal{H}$ , but  $\|U_n\|_{\mathcal{H}}$  tends to infinity. To show the lack of exponential stability, we consider

$$\begin{pmatrix} (\alpha + i\lambda\gamma)\omega_n^2 - \lambda^2\rho & (\beta + i\lambda\varepsilon)\omega_n & -\xi\omega_n \\ (\beta + i\lambda\varepsilon)\omega_n & (\eta + i\lambda\tau) - (\lambda^2\kappa - \omega_n^2\delta) & -\hbar \\ i\lambda\xi\omega_n & i\lambda\hbar & i\lambda\rho + K\omega_n^2 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \quad (2)$$

Solving Eq. (2) we have

$$B_n \sim \frac{i\lambda_n\gamma K\omega_n^4 + [K(\alpha - \delta\rho/\kappa) - \delta\gamma\rho/\kappa]\omega_n^4 + \mathcal{O}(n^3)}{(\varepsilon^2 - \tau\gamma)\frac{K\delta}{\kappa}\omega_n^6 - i\lambda K\gamma(w_0 - \Gamma)\omega_n^4 + \mathcal{O}(n^4)}, \quad (3)$$

where  $\Gamma := (\eta\gamma - 2\varepsilon\beta)/\gamma + (\varepsilon^2 - \tau\gamma)\delta\rho/K\gamma\kappa + (\alpha - \delta\rho/\kappa)\tau/\gamma$ . Since  $\gamma\tau - \varepsilon^2 = 0$  and choosing  $w_0 := \Gamma$  we have

$$B_n \sim \mathcal{O}(n). \quad (4)$$

Therefore,

$$\|U_n\|_{\mathcal{H}}^2 \geq \kappa \int_0^L |\phi_1^n|^2 dx = \kappa\lambda_n^2 |B_n|^2 \int_0^L |\cos(\omega_n x)|^2 dx \sim O(n^4) \implies \lim_{n \rightarrow \infty} \|U_n\|_{\mathcal{H}}^2 \geq \kappa\|\phi_1^n\|^2 = \infty. \quad (5)$$

**Theorem 2.2** (Polynomial decay). *The semigroup associated with the system (1)–(6) satisfies*

$$\|S(t)U_0\|_{\mathcal{H}} \leq \frac{C}{t^{1/2}} \|U_0\|_{D(\mathcal{A})}, \quad \forall t > 0, \quad U_0 \in D(\mathcal{A}). \quad (6)$$

Moreover, this rate is optimal.

**Proof** To show the polynomial stability, we use Borichev and Tomilov's Theorem [2]. Then using technical lemmas we get

$$\|U\|_{\mathcal{H}}^2 \leq \lambda^2 C \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + C \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + C \|F\|_{\mathcal{H}}^2. \quad (7)$$

Consequently, we have

$$\frac{1}{\lambda^2} \|(i\lambda I - \mathcal{A})^{-1}F\|_{\mathcal{H}} \leq C \|F\|_{\mathcal{H}}, \quad (8)$$

and therefore, by Borichev and Tomilov result, we prove the polynomial decay.

Now let us suppose that the rate of decay can be improved from  $t^{-1/2}$  to  $t^{-1/(2-\epsilon)}$  for some  $\epsilon > 0$ , then we will have that

$$\frac{1}{|\lambda|^{2-\epsilon}} \|(i\lambda I - \mathcal{A})^{-1}F\|_{\mathcal{H}}, \quad (9)$$

must be bounded. But this is not possible because of the lack of stability. The proof is now complete.  $\square$

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# EXISTENCE AND EXPONENTIAL DECAY FOR WAVE EQUATION IN WHOLE HYPERBOLIC SPACE

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## Abstract

In this work we study the exponential decay of the energy associated to an initial value problem involving the wave equation on the hyperbolic space  $\mathbb{B}^N$ . The main tools are Faedo-Galerkin method, multipliers techniques, and an appropriate Hardy inequality.

## 1 Introduction

In this work we prove the existence of solution and the exponential decay of the energy associated to the following problem

$$u_{tt} - \Delta_{\mathbb{B}^N} u + f(u) + a(x)u_t = 0 \text{ in } \mathbb{B}^N \times (0, \infty), \quad (1)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \text{ for } x \in \mathbb{B}^N, \quad (2)$$

where  $a$ ,  $f$ ,  $u_0$  and  $u_1$  are known functions and  $\Delta_{\mathbb{B}^N}$  is the Laplace-Beltrami operator in the disc model of the Hyperbolic  $\mathbb{B}^N$ . The space  $\mathbb{B}^N$  is the unit disc  $\{x \in \mathbb{R}^N : |x| < 1\}$  of  $\mathbb{R}^N$  endowed with the Riemannian metric  $g$  given by  $g_{ij} = p^2 \delta_{ij}$ , where  $p(x) = \frac{2}{1-|x|^2}$  and  $\delta_{ij} = 1$ , if  $i = j$  and  $\delta_{ij} = 0$ , if  $i \neq j$ . The hyperbolic gradient  $\nabla_{\mathbb{B}^N}$  and the hyperbolic Laplacian  $\Delta_{\mathbb{B}^N}$  are given by

$$\nabla_{\mathbb{B}^N} u = \frac{\nabla u}{p} \quad \text{and} \quad \Delta_{\mathbb{B}^N} u = p^{-N} \operatorname{div}(p^{N-2} \nabla u) = p^{-2} \Delta + \frac{(N-2)}{p} x \cdot \nabla, \quad (3)$$

where  $\cdot$  is the standard scalar product in  $\mathbb{R}^N$ ; and  $\nabla$  and  $\Delta$  are the usual gradient and Laplacian of  $\mathbb{R}^N$ .

Since the pioneer work of Zuazua [5], where the author, based on the multiplier techniques and on the unique continuation results, showed the exponential decay for the semilinear wave equation with localized damping in an unbounded domains, many authors have been studied this class of problems.

In this work, we extend the work of Zuazua [5] to the hyperbolic space  $\mathbb{B}^N$  which is a non-compact manifold, with curvature  $-1$  and without boundary. The main idea of our paper is to consider the damping acting away of the origin, as in [5]. But, now in the context of the hyperbolic space.

The main novelty of our work is to present a new technique which combines the multipliers one with the use of a Hardy inequality. This techniques was used in the context of elliptic equations in [1, 2, 3], but in evolution problem it is a novelty.

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## REALIZABILITY OF THE RAPID DISTORTION THEORY SPECTRUM

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In this work we show that the Rapid Distortion Theory (RDT) model for the spectral tensor of the homogeneous turbulence problem in the whole three-dimensional domain preserves the symmetry, positive semidefiniteness and integrability properties required in Cramér's characterization of the spectral tensor of a continuous homogeneous random process. The correlation tensor recovered from the spectral tensor model is statistically valid and satisfies realizability conditions. The RDT spectral tensor model is a system of transport equations plus an algebraic restriction due to incompressibility, therefore, we deal with the existence, uniqueness and persistence of solutions in a specific set of functions by using DiPerna-Lions renormalization techniques.

**1 Introduction**

We consider an incompressible fluid in the whole three-dimensional domain  $\mathbb{R}^3$  with constant density  $\rho$ , kinematic viscosity  $\nu$  and in constant-shear flow. Starting from the equation of continuity and the Navier-Stokes equations in the continuous homogeneous random process framework known as homogeneous turbulence, it is possible to derive the equations for the evolution of the spectral tensor  $\Phi(\mathbf{k}, t)$ , where  $\mathbf{k}$  denotes the wavenumber vector and  $t$  is the temporal variable. The spectral tensor is the spatial Fourier transform of the velocity correlation tensor  $\mathbf{R}$  which is defined componentwise as  $R_{ij}(\mathbf{r}, t) = \langle u_i(\mathbf{x}, t) u_j(\mathbf{x} + \mathbf{r}, t) \rangle$ ,  $(\mathbf{r}, t) \in \mathbb{R}^3 \times \mathbb{R}$ ,  $t \geq 0$ , where  $u_i(\mathbf{x}, t) = U_i(\mathbf{x}, t) - \langle U_i(\mathbf{x}, t) \rangle$  denotes the fluctuations of each component of the random velocity field  $U_i$ ,  $i = 1, 2, 3$ , and  $\langle \cdot \rangle$  is the notation for the expected value. Note that  $\mathbf{R}$  and any statistical moment of order  $n \geq 2$  are invariant under spatial translations because of the homogeneous turbulence assumption.

The evolution equations for the spectral tensor are part of an infinite hierarchy of coupled nonlinear integro-differential equations such that the equations for the statistical moments of order  $n$  involve the moments of order  $n + 1$ . That is why a closure scheme is introduced, the simplest of which consists of discarding the higher-order moments terms in the equations for the highest-order moments considered, as it is the case of the RDT model in which third-order moments are discarded. As a result, the following first-order homogeneous linear system arises

$$\frac{\partial \Phi_{ij}^R}{\partial t} = C_{nm} \left[ k_m \frac{\partial \Phi_{ij}^R}{\partial k_n} + \frac{2k_m}{k^2} (k_i \Phi_{nj}^R + k_j \Phi_{in}^R) - (\delta_{im} \Phi_{nj}^R + \delta_{jm} \Phi_{in}^R) \right] - 2\nu k^2 \Phi_{ij}^R, \quad (1)$$

where summation convention is adopted for repeated Latin indices, the superscript  $R$  indicates that  $\Phi_{ij}^R$  is in general different from  $\Phi_{ij}$  and a constant mean velocity gradient

$$C_{nm} = \frac{\partial \langle U_m \rangle}{\partial x_n} = \delta_{n3} \delta_{m1} C_{31}, \quad C_{31} \neq 0, \quad (2)$$

is assumed, which corresponds to a constant shear rate of the mean velocity  $\langle U_1 \rangle$  in the  $x_3$  direction. Besides, the continuity equation expressed in the form of the incompressibility condition in the physical domain returns

$$\Phi_{ij}^R k_j = 0, \quad (3)$$

adding an algebraic equation to be satisfied by  $\Phi^R$ . The approximation for  $\Phi$  provided by the RDT spectral tensor model has been validated in practice for rapidly straining turbulent flow, as described in [1]. It is then pertinent to analyze if the model preserves the statistical properties that the spectral tensor of a continuous homogeneous random processes must satisfy, as considered in the next section.

## 2 Main Results

Cramér's theorem provides a characterization of the spectral tensor of a continuous homogeneous random process as a  $\mathbf{k}$ -absolutely integrable Hermitian matrix representing a positive semidefinite quadratic form. Simplifying to the real valued case, we define *an admissible initial condition* as a real absolutely integrable symmetric positive semidefinite matrix  $\Phi^0(\mathbf{k})$  such that  $\Phi^0(\mathbf{k})\mathbf{k} = \mathbf{0}$  almost everywhere, which serves as initial condition for system (1)–(3). In this setting, the following theorem establishes the fulfillment of Cramér's characterization together with the existence and uniqueness of solutions for the model.

**Theorem 2.1.** *If  $\Phi^0$  is an admissible initial condition then there exists a matrix  $\Phi^R$  such that*

1.  $\Phi^R$  is symmetric and positive semidefinite,
2.  $\Phi^R(\mathbf{k}, t)\mathbf{k} = \mathbf{0}$ , for all  $t \geq 0$  and almost everywhere in  $\mathbf{k}$ ,
3. the components of  $\Phi^R$  are continuous functions with respect to  $t$ , for  $t \geq 0$ , with values in  $L^1(\mathbb{R}^3)$ ,
4.  $\Phi^R$  is the unique weak solution of the system of equations (1), with initial condition  $\Phi^0$ , that satisfies property 3
5. If  $\Phi^0$  is also a continuously differentiable function on  $\mathbb{R}^3 - \{\mathbf{0}\}$  then  $\Phi^R$  is also continuously differentiable on  $(\mathbf{k}, t) \in \{\mathbb{R}^3 - \{\mathbf{0}\}\} \times \mathbb{R}$ ,  $t \geq 0$ .

A detailed proof can be found in [2]. It relies on the use of the Kelvin-Townsend system of differential equations which solutions serve as factors for the complete solution. The structure of the system that allows this factorization approach is intrinsic to the original equations. The results are also valid for the complex Hermitian case. A statistically valid correlation tensor is recovered from the spectral tensor model via inverse Fourier transform. Therefore, it satisfies physically meaningful probabilistic inequalities, including, at zero spatial separation, realizability conditions for the Reynolds stress tensor  $\mathbf{R}(\mathbf{0}, t)$ .

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## STABILITY OF PERIODIC SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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### Abstract

We establish the existence of periodic solutions for the Navier-Stokes equations, assuming that the external force is periodic and  $C^1$  in time, and small enough in the norm of the considered space. We also prove uniqueness and stability of the solutions in various norms. The proof of existence is based on a set of estimates for the family of finite-dimensional approximate solutions.

## 1 Introduction

### 1.1 Problem Statement

Given a periodic force in time  $\mathbf{f} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^n$ ,  $\Omega \subset \mathbb{R}^n$ ,  $n = 2$  or  $3$ ,  $\mathbf{f}(x, t) = \mathbf{f}(x, t + \tau)$ , we search for a periodic solution in time

$$\begin{aligned} \mathbf{u} : \Omega \times \mathbb{R} &\rightarrow \mathbb{R}^n \quad \mathbf{u}(x, t) = \mathbf{u}(x, t + \tau) \\ p : \Omega \times \mathbb{R} &\rightarrow \mathbb{R} \quad p(x, t) = p(x, t + \tau) \end{aligned}$$

of the Navier-Stokes equations

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \mu \Delta \mathbf{u} + \nabla p = \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0, \end{cases} \quad (1)$$

subject to the following boundary condition:

$$\mathbf{u}(x, t) = 0 \text{ on } \partial\Omega \times (0, T). \quad (2)$$

Also, we consider the initial-value boundary problem associated to (1)–(2), i.e. (1)–(2) together with

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x) \quad \text{in } \Omega. \quad (3)$$

### 1.2 Preliminaries

We use the usual function spaces for the Navier-Stokes equations, see Lions [1]. We will denote by  $\|\cdot\|$  the usual norm in  $L^2(\Omega)$  and associated product spaces.

The following results on existence and uniqueness can be found for instance in [6, 3, 5]:

**Theorem 1.1.** *Let  $\mathbf{f} \in C^1(\tau; \mathbf{L}^2(\Omega))$ . There exists  $M_1 > 0$  such that, if*

$$\sup_t \|\mathbf{f}(t)\| \leq M_1, \quad (4)$$

*the corresponding system (1)–(2) has a strong  $\tau$ -periodic solution*

$$\mathbf{u}_p \in H^2(\tau; \mathbf{H}) \cap H^1(\tau; D(A)) \cap L^\infty(\tau; D(A)) \cap W^{1,\infty}(\tau; \mathbf{V}). \quad (5)$$

Moreover, if  $\|\mathbf{f}\|_{C^1(\tau; \mathbf{L}^2(\Omega))}$  is small enough, the solution is unique and any solution  $\mathbf{u}$  to (1)–(3) defined for all  $t \in (0, +\infty)$  with values in  $D(A)$  and such that

$$\sup_{0 \leq t \leq T} \|\mathbf{u}(t)\| + \int_0^T \|\nabla \mathbf{u}(t)\|^2 dt < +\infty \quad \forall T > 0 \quad (6)$$

satisfies  $\|\mathbf{u}(t) - \mathbf{u}_p(t)\| \rightarrow 0$  exponentially as  $t \rightarrow +\infty$ .

## 2 Main Results

**Theorem 2.1.** If  $\mathbf{f}$  and  $\mathbf{u}$  are as in the second part of Theorem 2.3, one has

$$\|\mathbf{u}(t) - \mathbf{u}_p(t)\|_{\mathbf{H}^2} \rightarrow 0 \text{ exponentially as } t \rightarrow +\infty. \quad (1)$$

The proof can be achieved by establishing appropriate estimates of  $\|\mathbf{u} - \mathbf{u}_p\|$  in some norms.

**Theorem 2.2.** If  $\mathbf{f}$  and  $\mathbf{u}$  are as in the second part of Theorem 2.3,  $\partial\Omega \in C^\infty$ ,  $D_t^l \mathbf{f} \in L^\infty(0, \infty; \mathbf{H}^k(\Omega))$  for all  $l, k \geq 0$  and  $\mathbf{u}_0 \in D(A)$ , we have

$$\|D_t^n \mathbf{u}(t) - D_t^n \mathbf{u}_p(t)\|_{\mathbf{H}^k} \rightarrow 0 \text{ exponentially as } t \rightarrow +\infty. \quad (2)$$

The proof can be obtained by induction, following some arguments already used in [2] and [1] together with the arguments in the proof of Theorem 2.1.

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## A DAMPED NONLINEAR HYPERBOLIC EQUATION WITH NONLINEAR STRAIN TERM

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**Abstract**

In this work, we investigate an initial boundary value problem related to the nonlinear hyperbolic equation  $u_{tt} + u_{xxxx} + \alpha u_{xxxxt} = f(u_x)_x$ , for  $f(s) = |s|^\rho + |s|^\sigma$ ,  $1 < \rho, \sigma, \alpha > 0$ . Under suitable conditions, we prove the existence of global solutions and the exponential decay of energy.

**1 Introduction**

In this research, we consider the initial boundary value problem of a nonlinear hyperbolic equation with Kelvin-Voigt type damping term

$$\begin{aligned} u_{tt} + u_{xxxx} + \alpha u_{xxxxt} &= f(u_x)_x, \quad x \in \Omega, \quad t > 0, \\ u(0, t) &= u(1, t) = 0, \quad u_{xx}(0, t) = u_{xx}(1, t) = 0, \quad t \geq 0, \\ u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \end{aligned} \tag{1}$$

where  $\Omega = (0, 1)$ ,  $\alpha$  is a positive constant and  $f(s) = |s|^\rho + |s|^\sigma$  is the strain term.

There have been many impressive works on the well-posedness and energy decay of solutions of the nonlinear beam equations of the type (1), with  $f(s)$  satisfying  $f(s)s \geq 0$  and  $|f(s)| \leq a|s|^q$ ,  $a > 0$ , see [1] and the references therein. The authors in these works introduced the potential well method to obtain their results.

However, to the best of our knowledge, there is little research about equation (1) with nonlinearity  $f(s) = |s|^\rho + |s|^\sigma$ ,  $\rho, \sigma > 1$ , which also implies that (1) has no positive definite energy.

**2 Main Results**

We define the space

$$H = \{u \in H^3(\Omega) \cap H_0^1(\Omega) : u_{xx} \in H_0^1(\Omega)\}$$

endowed with the norm

$$\|u\|_H^2 = \|u_{xx}\|^2 + \|u_{xxx}\|^2,$$

and introduce the following notations

$$\begin{aligned} B_1 &= \sup_{\substack{u \in H^2(\Omega) \cap H_0^1(\Omega) \\ u \neq 0}} \frac{\|u_x\|_{\rho+1}}{\|u_{xx}\|}, & B_2 &= \sup_{\substack{u \in H^2(\Omega) \cap H_0^1(\Omega) \\ u \neq 0}} \frac{\|u_x\|_{\sigma+1}}{\|u_{xx}\|}, \\ \gamma_1 &= \frac{1}{\rho+1} B_1^{\rho+1}, & \gamma_2 &= \frac{1}{\sigma+1} B_2^{\sigma+1}. \end{aligned}$$

Let  $\lambda_1 = \min \left\{ \left[ \frac{1}{4(\rho+1)\gamma_1} \right]^{1/(\rho-1)}, \left[ \frac{1}{6(\sigma+1)\gamma_2} \right]^{1/(\sigma-1)} \right\}$  and the energy associated with problem (1) by

$$E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|u_{xx}\|^2 - \frac{1}{\rho+1} \int_0^1 |u_x|^\rho u_x dx - \frac{1}{\sigma+1} \int_0^1 |u_x|^\sigma u_x dx.$$

Now, we have the following two theorems

**Theorem 2.1.** Assume  $u_0 \in H$ ,  $u_1 \in L^2(\Omega)$ . Then problem (1) has a unique weak solution  $u$  for  $T$  small enough.

*Proof.* We prove the local existence of weak solutions by using the Faedo-Galerkin method.  $\square$

**Theorem 2.2** (The main result.). Assume that the assumptions of Theorem 2.1 hold and that

$$0 < \|u_{0xx}\| < \lambda_1 \text{ and } (4E(0))^{1/2} < \lambda_1,$$

where

$$E(0) = \frac{1}{2} \|u_1\|^2 + \frac{1}{2} \|u_{0xx}\|^2 - \frac{1}{\rho+1} \int_\Omega |u_{0x}|^\rho u_{0x} dx - \frac{1}{\sigma+1} \int_\Omega |u_{0x}|^\sigma u_{0x} dx.$$

Then problem admits a global weak solution in time. This solution satisfies

$$E(t) \leq L_0 e^{-\gamma t}, \forall t \geq 0, \text{ for some } L_0, \gamma > 0.$$

*Proof.* We apply an argument due to Tartar [4] (See also Milla Miranda et al.[2]) to get the global existence. The exponential decay is obtained via an integral inequality introduced by Komornik.  $\square$

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## COMPORTAMENTO ASSINTÓTICO PARA AS EQUAÇÕES MAGNETO-MICROPOLARES

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**Abstract**

Estudamos o comportamento assintótico das soluções globais fracas para as equações dos fluidos magneto-micropolares nos espaços de Sobolev  $H^m(\mathbb{R}^n)$ , com  $m \in \mathbb{N} \cup \{0\}$  e  $n \in \{2, 3\}$ . Além disso, mostramos que a velocidade micro-rotacional decai mais rápido do que a velocidade linear do fluido. Também discutimos alguns resultados de decaimento para a pressão total do fluido e para as derivadas da solução na região do espaço-tempo.

## 1 Introdução

Consideramos o PVI

$$\left\{ \begin{array}{l} \mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} - (\mu + \chi) \Delta \mathbf{u} + \nabla \left( \Pi + \frac{|\mathbf{b}|^2}{2} \right) = (\mathbf{b} \cdot \nabla) \mathbf{b} + \chi \operatorname{rot} \mathbf{w}, \\ \mathbf{w}_t + (\mathbf{u} \cdot \nabla) \mathbf{w} - \gamma \Delta \mathbf{w} - \kappa \nabla(\operatorname{div} \mathbf{w}) + 2\chi \mathbf{w} = \chi \operatorname{rot} \mathbf{u}, \\ \mathbf{b}_t + (\mathbf{u} \cdot \nabla) \mathbf{b} - \nu \Delta \mathbf{b} = (\mathbf{b} \cdot \nabla) \mathbf{u}, \\ \operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{b} = 0, \\ (\mathbf{u}, \mathbf{w}, \mathbf{b})|_{t=0} = (\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0), \end{array} \right. \quad (1)$$

em  $\mathbb{R}^n \times (0, \infty)$ , onde  $\mathbf{u}_0$ ,  $\mathbf{w}_0$  e  $\mathbf{b}_0$  são funções dadas e  $n = 2$  ou  $3$ .

No sistema (1), as incógnitas são as funções  $\mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^n$ ,  $\Pi(\mathbf{x}, t) \in \mathbb{R}$ ,  $\mathbf{w}(\mathbf{x}, t) \in \mathbb{R}^n$  e  $\mathbf{b}(\mathbf{x}, t) \in \mathbb{R}^n$ , as quais representam, respectivamente, o campo velocidade incompressível (velocidade linear), a pressão hidrostática, a velocidade micro-rotacional e o campo magnético em um ponto  $\mathbf{x} \in \mathbb{R}^n$  no tempo  $t > 0$ . A função  $|\mathbf{b}|^2/2$  é a pressão magnética. Assim, denotamos por  $p := \Pi + |\mathbf{b}|^2/2$  a pressão total do fluido. Este sistema descreve o movimento de um fluido incompressível micropolar viscoso na presença de um campo magnético (veja [1] e [2]). As constantes positivas  $\mu$ ,  $\chi$ ,  $\gamma$ ,  $\kappa$  e  $\nu$  estão associadas a propriedades específicas do fluido; mais especificamente,  $\mu$  é a viscosidade cinemática (usual),  $\chi$  é a viscosidade do vórtice,  $\gamma$  e  $\kappa$  são as viscosidades de rotação e, por último,  $1/\nu$  é o número magnético de Reynolds. Os dados iniciais para os campos velocidade e magnético, dados por  $\mathbf{u}_0$  e  $\mathbf{b}_0$ , são assumidos livres de divergente, i.e.,  $\operatorname{div} \mathbf{u}_0 = \operatorname{div} \mathbf{b}_0 = 0$ . Vale ressaltar que o sistema (1) se reduz às equações de Navier-Stokes, quando  $\mathbf{w} = \mathbf{b} = \mathbf{0}$ ; ao sistema MHD, quando  $\mathbf{w} = \mathbf{0}$ ; e ao sistema micropolar, quando  $\mathbf{b} = \mathbf{0}$ .

## 2 Resultados Principais

Por simplicidade, assumimos  $\mu = \chi = 1/2$  e  $\gamma = \kappa = \nu = 1$ .

**Teorema 2.1.** *Seja  $(\mathbf{u}, p, \mathbf{w}, \mathbf{b})$  uma solução global do sistema (1). Se  $\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0 \in \mathbf{H}^m(\mathbb{R}^n) \cap \mathbf{L}^1(\mathbb{R}^n)$ , com  $\operatorname{div} \mathbf{u}_0 = \operatorname{div} \mathbf{b}_0 = 0$ , e  $m \in \mathbb{N} \cup \{0\}$  e  $n \in \{2, 3\}$ , então*

$$\|D^m \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2(\mathbb{R}^n)} + \|D^m \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2(\mathbb{R}^n)} + \|D^m \mathbf{b}(\cdot, t)\|_{\mathbf{L}^2(\mathbb{R}^n)} \leq C(t+1)^{-\frac{m}{2} - \frac{n}{4}}, \quad (1a)$$

para todo  $t$  suficientemente grande. Ademais, temos a seguinte taxa de decaimento melhorada para a micro-rotação:

$$\|D^m \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2(\mathbb{R}^n)} \leq C(t+1)^{-\frac{m}{2} - \frac{n}{4} - \frac{1}{2}}, \quad \forall t \gg 1. \quad (1b)$$

Também comparamos a evolução das soluções  $\mathbf{z}(\cdot, t) := (\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)$  do sistema (1) com as soluções  $\bar{\mathbf{z}}(\cdot, t) := (\bar{\mathbf{u}}, \bar{\mathbf{w}}, \bar{\mathbf{b}})(\cdot, t)$  do sistema linear associado. Em [3], M. Wiegner forneceu tal estimativa para as equações de Navier-Stokes (comparando com a equação do calor com os mesmos dados iniciais). Embora tenhamos outro sistema linear associado, o resultado permanece válido e, para o campo micro-rotacional, fornecemos uma taxa de decaimento extra. Nossa segundo resultado principal é o seguinte

**Teorema 2.2.** *Seja  $(\mathbf{u}, p, \mathbf{w}, \mathbf{b})$  uma solução global do sistema (1). Se  $\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0 \in \mathbf{L}^1(\mathbb{R}^n) \cap \mathbf{H}^1(\mathbb{R}^n)$ , com  $\operatorname{div} \mathbf{u}_0 = \operatorname{div} \mathbf{b}_0 = 0$ , então existe uma constante  $C \in \mathbb{R}^+$  tal que*

$$\|\mathbf{z}(\cdot, t) - \bar{\mathbf{z}}(\cdot, t)\|_{\mathbf{L}^2(\mathbb{R}^n)} \leq C(t+1)^{-\frac{n}{4} - \frac{1}{2}}, \quad \forall t \geq 0. \quad (2a)$$

Além disso, melhoramos a taxa de decaimento para o campo micro-rotacional da seguinte forma:

$$\|\mathbf{w}(\cdot, t) - \bar{\mathbf{w}}(\cdot, t)\|_{\mathbf{L}^2(\mathbb{R}^n)} \leq C(t+1)^{-\frac{n}{4} - 1}, \quad \forall t \geq 0. \quad (2b)$$

**Observação 1.** Note que o resultado acima nos diz que as soluções do sistema magneto-micropolar são assintoticamente equivalentes às soluções do problema linear associado com os mesmos dados iniciais.

Por fim, para  $\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0 \in \mathbf{L}^1(\mathbb{R}^n) \cap \mathbf{H}^{m+1}(\mathbb{R}^n)$ , com  $\operatorname{div} \mathbf{u}_0 = \operatorname{div} \mathbf{b}_0 = 0$ , também obtivemos a seguinte taxa de decaimento para a pressão total do fluido

$$\|D^m p(\cdot, t)\|_{\mathbf{L}^2(\mathbb{R}^n)} \leq C(t+1)^{-\frac{m}{2} - \frac{3n}{4}}, \quad \forall t \gg 1,$$

e, supondo que  $\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0 \in \mathbf{L}^1(\mathbb{R}^n) \cap \mathbf{H}^M(\mathbb{R}^n)$ , com  $\operatorname{div} \mathbf{u}_0 = \operatorname{div} \mathbf{b}_0 = 0$ , também mostramos que existe uma constante  $C > 0$  tal que

$$\begin{aligned} \|D^m \partial_t^k \mathbf{u}(\cdot, t)\|_{\mathbf{L}^2(\mathbb{R}^n)} &\leq C(t+1)^{-\frac{m}{2} - k - \frac{n}{4}}, \quad \forall t \gg 1, \\ \|D^m \partial_t^k \mathbf{w}(\cdot, t)\|_{\mathbf{L}^2(\mathbb{R}^n)} &\leq C(t+1)^{-\frac{m}{2} - k - \frac{n}{4} - \frac{1}{2}}, \quad \forall t \gg 1, \\ \|D^m \partial_t^k \mathbf{b}(\cdot, t)\|_{\mathbf{L}^2(\mathbb{R}^n)} &\leq C(t+1)^{-\frac{m}{2} - k - \frac{n}{4}}, \quad \forall t \gg 1, \end{aligned}$$

para todo  $M \geq m + 2k$ ,  $m, k \in \mathbb{N} \cup \{0\}$  e  $n \in \{2, 3\}$ .

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# RESULTADOS DE EXISTÊNCIA GLOBAL PARA SOLUÇÕES DE EQUAÇÕES DE ADVECÇÃO-DIFUSÃO

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## Abstract

Neste trabalho usamos uma técnica baseada em métodos de energia para analisar a existência global da solução do problema evolutivo  $u_t + (b(x, t)u^{k+1})_x = \mu(t)u_{xx}$  com condição inicial  $u(\cdot, 0) = u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ . Encontramos condições que garantam a existência global da solução.

## 1 Introdução

Neste trabalho, apresentamos um estudo detalhado sobre o comportamento assintótico de soluções limitadas não negativas do problema evolutivo do tipo

$$\begin{aligned} u_t + (b(x, t)u^{k+1})_x &= \mu(t)u_{xx} \quad \forall x \in \mathbb{R}, t > 0, \\ u(\cdot, 0) &= u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \end{aligned} \tag{1}$$

para campos de advecção arbitrários continuamente diferenciáveis  $b$  satisfazendo

$$b(\cdot, t) \in L^\infty_{loc}([0, \infty), L^\infty(\mathbb{R})) \quad \forall x \in \mathbb{R}, t \geq 0; \tag{2}$$

$$\left| \frac{\partial b(x, t)}{\partial x} \right| < B(t) \quad \forall x \in \mathbb{R}, t \geq 0, \tag{3}$$

onde  $B \in C^0([0, \infty))$ ,  $\mu(t) \in C^0([0, \infty))$  positiva e  $0 \leq k < 2$  constante.

Aqui, por uma solução limitada do problema (1) em algum intervalo de tempo  $[0, T_*]$ , entendemos como qualquer função  $u \in C^0([0, T_*], L^1(\mathbb{R})) \cap L^\infty_{loc}([0, T_*], L^\infty(\mathbb{R}))$  satisfazendo a equação do problema (1). Para resultados de existência local (no tempo) pode-se consultar [2] and [6], Ch. 7.

Nosso objetivo é investigar para quais valores de  $k$  podemos garantir que a solução do problema exista globalmente e por este motivo é tão importante conhecer o comportamento das normas mais altas  $L^q$ , em especial da norma  $L^\infty$  no intervalo de existência da solução, a fim de que possamos estender este intervalo a intervalos de existência mais amplos.

Note que para  $k = 1$  em (1), temos a equação de Burgers com um termo advectivo arbitrário  $b(x, t)$ , apesar do vasto estudo a respeito desta equação, um dos modelos mais simples que combina os efeitos do operador não linear advectivo com o operador difusivo, a dependência explícita de  $x$  em  $b$  dificulta bastante a análise da existência global das soluções deste problema. Para ilustrar esta dificuldade, reescrevemos a equação geral do problema (1) da seguinte forma

$$u_t + (k+1)b(x, t)u^k u_x = -\frac{\partial b(x, t)}{\partial x}u^{k+1} + \mu(t)u_{xx}. \tag{4}$$

Como  $b$  depende de  $x$ , na região onde  $\frac{\partial b(x, t)}{\partial x} < 0$ , o termo  $-\frac{\partial b(x, t)}{\partial x} u^{k+1}$  estimula a solução  $u$  a crescer em magnitude. Porém, a solução de (1) conserva massa, então a medida que  $u$  cresce, o perfil da solução fica mais afinado, tornando-se mais suscetível aos efeitos do operador difusivo. Desta forma, o resultado da competição entre os termos difusivo e advectivo do lado esquerdo da equação (4) torna-se difícil de ser previsto.

Quando  $b$  não depende explicitamente de  $x$ , ou mais geralmente, quando  $\partial b(x, t)/\partial x \geq 0$  for all  $x \in \mathbb{R}$ , já se sabe da existência global para as soluções no caso  $k = 0$  em (1), e neste caso, além das soluções serem definidas para todo tempo, elas também decaem a zero quando  $t \rightarrow \infty$ , ver os seguintes trabalhos nesta direção [4, 1, 7]. Para o caso de  $k = 0$  e  $b(x, t)$  arbitrária, ver [3, 5]. Finalmente, no nosso trabalho, para  $b(x, t)$  arbitrária e  $k \in [0, 2)$  garantimos a existência global.

## 2 Resultados Principais

Obtemos uma estimativa para a norma do sup da solução do problema (1):

**Teorema 2.1.** *Let  $\bar{q} \geq 1$ ,  $0 \leq t_0 < t < T_*$  and  $0 \leq k < 2$  in (1). Then*

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \bar{q}^{\frac{1}{2\bar{q}-k}} \max \left\{ \|u(\cdot, t_0)\|_{L^\infty(\mathbb{R})}; B_\mu(t_0; t)^{\frac{1}{2\bar{q}-k}} U_{\bar{q}}(t_0; t)^{\frac{2\bar{q}}{2\bar{q}-k}} \right\},$$

where  $U_{\bar{q}}(t_0; t)$ ,  $B_\mu(t_0; t)$  is defined by

$$U_q(t_0; t) := \sup \{ \|u(\cdot, \tau)\|_{L^q(\mathbb{R})}; t_0 \leq \tau \leq t \}, \text{ and } B_\mu(t_0; t) := \sup \left\{ \frac{B(\tau)}{\mu(\tau)}; t_0 \leq \tau \leq t \right\}, \text{ respectivamente.}$$

Em particular, se tomarmos  $t_0 = 0$  e  $\bar{q} = 1$  no Teorema 2.1, sabendo que a solução do problema (1) conserva massa, garantimos a sua existência global, isto é,  $T_* = \infty$ .

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LONG-TIME DYNAMICS FOR A FRACTIONAL PIEZOELECTRIC SYSTEM WITH MAGNETIC  
EFFECTS AND FOURIER'S LAW

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**Abstract**

In this work, we use a variational approach to model vibrations on a piezoelectric beam with fractional damping depending on a parameter  $\nu \in (0, 1/2)$ . Magnetic and thermal effects are taken into account via the Maxwell's equations and Fourier law, respectively. Existence and uniqueness of solutions of the system is proved by the semigroup theory. The existence of smooth global attractors with finite fractal dimension and the existence of exponential attractors for the associated dynamical system are proved. Finally, the upper-semicontinuity of global attractors as  $\nu \rightarrow 0^+$  is shown.

## 1 Introduction

In this work, we consider the longitudinal vibrations on a piezoelectric beam system with thermal and magnetic effects and with friction damping

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} + \delta \theta_x + f_1(v, p) = h_1 & \text{in } (0, L) \times (0, T), \\ \mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} + A^\nu p_t + f_2(v, p) = h_2 & \text{in } (0, L) \times (0, T), \\ c\theta_t - \kappa \theta_{xx} + \delta v_{xt} = 0 & \text{in } (0, L) \times (0, T), \end{cases} \quad (1)$$

where the physical constants  $\rho, \alpha, \beta, \gamma, \delta, \kappa, \mu$  and  $c$  are positive constants,  $f_1, f_2$  are nonlinear source terms and  $h_1, h_2$  are external forces. Moreover, we consider the relationship  $\alpha = \alpha_1 + \gamma^2 \beta$  with  $\alpha_1 > 0$ . Moreover,  $A : D(A) \subset L^2(0, L) \rightarrow L^2(0, L)$  is the one-dimensional Laplacian operator defined by

$$A = -\partial_{xx}, \quad \text{with domain } D(A) = \{v \in H^2(0, L) \cap H_*^1(0, L) : v_x(L) = 0\} \quad (2)$$

where  $H_*^1(0, L) := \{u \in H^1(0, L); u(0) = 0\}$  and  $A^\nu : D(A^\nu) \subset L^2(0, L) \rightarrow L^2(0, L)$  is the fractional power associated with operator  $A$  of order  $\nu \in (0, 1/2)$ .

The system  $(P_j)$  is supplemented by the clamped-free boundary and initial conditions

$$\begin{cases} v(0, t) = \alpha v_x(L, t) - \gamma \beta p_x(L, t) = 0, & t > 0, \\ p(0, t) = p_x(L, t) - \gamma v_x(L, t) = 0, & t > 0, \\ \theta(0, t) = \theta(L, t) = 0, & t > 0, \\ v(x, 0) = v_0, \quad v_t(x, 0) = v_1, \quad 0 < x < L, \\ p(x, 0) = p_0, \quad p_t(x, 0) = p_1, \quad 0 < x < L, \\ \theta(x, 0) = \theta_0(x), \quad 0 < x < L. \end{cases} \quad (3)$$

We assume that

- (i) The external forces  $h_1, h_2 \in L^2(0, L)$ ;
- (ii) There exists a function  $F \in C^2(\mathbb{R}^2)$  such that

$$\nabla F = (f_1, f_2); \quad (4)$$

- (iii) There exist  $q \geq 1$  and  $C > 0$  such that

$$|\nabla f_j(v, p)| \leq C (|v|^{q-1} + |p|^{q-1} + 1), \quad j = 1, 2; \quad (5)$$

- (iv) There exist constants  $\eta \geq 0$ ,  $m_F > 0$  such that

$$F(v, p) \geq -\eta (|v|^2 + |p|^2) - m_F, \quad \nabla F(v, p) \cdot (v, p) - F(v, p) \geq -\eta (|v|^2 + |p|^2) - m_F. \quad (6)$$

## 2 Main Results

First, we observe that the system (P<sub>j</sub>)-(3) defines a dynamical system  $(\mathcal{H}, S(t))$ . To this system we study the existence of global attractors and their properties.

**Theorem 2.1.** *Suppose that assumptions (4)-(6) hold. Then,*

- (i) *The dynamical system  $(\mathcal{H}, S(t))$  possesses a unique compact global attractor  $\mathcal{A} \subset \mathcal{H}$ ;*
- (ii) *The global attractor  $\mathcal{A}$  has finite fractal and Hausdorff dimension;*
- (iii) *The complete trajectories  $(v, p, v_t, p_t, \theta)$  in  $\mathcal{A}$  has further regularity*

$$\begin{aligned} & \| (v, p) \|_{(H^2(0, L) \cap H_*^1(0, L))^2}^2 + \| \theta \|_{H^2(0, L) \cap H_0^1(0, L)} + \| (v_t, p_t) \|_{(H_*^1(0, L))^2}^2 \\ & + \| (v_{tt}, p_{tt}) \|_{(L^2(0, L))^2}^2 + \| \theta_t \|_2^2 \leq C, \end{aligned} \quad (1)$$

for some constant  $C > 0$ ;

- (iv) *The dynamical system  $(\mathcal{H}, S(t))$  has a generalized exponential attractor  $\mathcal{A}_{\text{exp}}$  with finite fractal dimension in the extended space*

$$\mathcal{H}_{-1} := (L^2(0, L))^2 \times (H_*^{-1}(0, L))^2 \times H^{-1}(0, L), \quad (2)$$

where  $H_*^{-1}(0, L)$  is the dual space of  $H_*^1(0, L)$  pivoted with respect to  $L^2(0, L)$ . In addition, from interpolation theorem, there exists a generalized exponential attractor whose fractal dimension is finite in a smaller extended space  $\mathcal{H}_{-\sigma}$ , where

$$\mathcal{H} = \mathcal{H}_0 \subset \mathcal{H}_{-\sigma} \subset \mathcal{H}_{-1}, \quad 0 < \sigma \leq 1. \quad (3)$$

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## GLOBAL SOLUTIONS TO THE NON-LOCAL NAVIER-STOKES EQUATIONS

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### Abstract

We study the global well-posedness for a non-local-in-time Navier-Stokes equation. Our results recover in particular other existing well-posedness results for the Navier-Stokes equations and their time-fractional version. We show the appropriate manner to apply Kato's strategy and this context, with initial conditions in the divergence-free Lebesgue space  $L_d^\sigma(\mathbb{R}^d)$ .

### 1 Introduction

Consider the fractional-in-time Navier-Stokes equation

$$\begin{aligned} \partial_t^\alpha u - \Delta u + (u \cdot \nabla) u + \nabla p &= f, & t > 0, x \in \Omega \subset \mathbb{R}^d, \\ \nabla \cdot u &= 0, & t > 0, x \in \Omega \subset \mathbb{R}^d, \\ u(0, x) &= u_0(x), & x \in \Omega \subset \mathbb{R}^d, \end{aligned}$$

where  $\partial_t^\alpha u$  denotes the fractional derivative of  $u$  in the Caputo's sense with order  $\alpha \in (0, 1)$ . If the product  $(k * v)$  denotes the convolution on the positive halffline  $\mathbb{R}_+ := [0, \infty)$  with respect to time variable, then we have  $\partial_t^\alpha u = g_{1-\alpha} * u_t$ , for an absolutely continuous function  $u$ , where  $g_\beta$  is the standard notation for the function  $g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ ,  $t > 0$ ,  $\beta > 0$ . Toward the possibility of considering more general nonlocal-in-time effects, we will replace  $g_\alpha$  by  $k$ , and we assume as a general hypothesis that  $k$  is a kernel of type  $(\mathcal{PC})$ , by which we mean that the following condition is satisfied:

$(\mathcal{PC})$   $k \in L_{1,loc}(\mathbb{R}_+)$  is nonnegative and nonincreasing, and there exists a kernel  $\ell \in L_{1,loc}(\mathbb{R}_+)$  such that  $k * \ell = 1$  on  $(0, \infty)$ .

We also write  $(k, \ell) \in \mathcal{PC}$ . We point out that the kernels of type  $(\mathcal{PC})$  are called *Sonine kernels* and they have been successfully used to study integral equations of first kind in the spaces of Hölder continuous, Lebesgue and Sobolev functions, see [1].

Therefore, we consider the following problem for the following nonlocal-in-time Navier-Stoke-type equation

$$\partial_t(k * (u - u_0)) - \Delta u + (u \cdot \nabla) u + \nabla p = f, \quad t > 0, x \in \mathbb{R}^d, \quad (1)$$

$$\nabla \cdot u = 0, \quad t > 0, x \in \mathbb{R}^d, \quad (2)$$

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^d, \quad (3)$$

where  $u(t, x)$  represents the velocity field and  $p(t, x)$  is the associated pressure of the fluid. The function  $u_0(x) = u(0, x)$  is the initial velocity and  $f(t, x)$  represents an external force. The problem (1)-(3), can be written in an abstract form as

$$\partial_t(k * (u - u_0)) + \mathcal{A}_p u = F(u, u) + Pf, \quad t > 0, \quad (4)$$

where  $\mathcal{A}_p u := P(-\Delta)u$ ,  $P : L_p(\mathbb{R}^d) \rightarrow L_p^\sigma(\mathbb{R}^d)$  is well-known as Helmholtz-Leray's projection, and the nonlinear term  $F(u, v) := -P(u \cdot \nabla)v$ . Equation (4) can be written as a Volterra equation of the form

$$u + (\ell * \mathcal{A}_r u)(t) = u_0 + (\ell * [F(u, u) + Pf])(t), \quad t > 0, \quad (5)$$

by condition  $(k, \ell) \in \mathcal{PC}$ .

## 2 Main Results

We investigate the existence and uniqueness of global mild solutions for equation (5). Before we state the main result, we introduce space where the mild solution will dwell. Let  $d \in \mathbb{N}$ . For any  $2 \leq d < q < \infty$ , consider the space  $X_q$  of the functions  $v$  satisfying  $v \in C_b([0, \infty); L_d^\sigma(\mathbb{R}^d))$ ,  $(1 * \ell)^{\frac{1}{2} - \frac{d}{2q}}v \in C_b((0, \infty); L_q^\sigma(\mathbb{R}^d))$  and  $(1 * \ell)^{\frac{1}{2}}\nabla v \in C_b((0, \infty); L_d^\sigma(\mathbb{R}^d))$ , which is a Banach space with norm

$$\|v\|_{X_q} := \max\{\sup_{t>0} \|v(t)\|_{L_d^\sigma(\mathbb{R}^d)}, \sup_{t>0} [(1 * \ell)(t)]^{\frac{1}{2} - \frac{d}{2q}} \|v(t)\|_{L_q^\sigma(\mathbb{R}^d)}, \sup_{t>0} [(1 * \ell)(t)]^{\frac{1}{2}} \|\nabla v(t)\|_{L_d^\sigma(\mathbb{R}^d)}\}.$$

The existence of the mild solutions solution for (5) will be a consequence of the following fixed point lemma (see [2, Lemma 1.5]).

**Lemma 2.1.** *Let  $X$  be an abstract Banach Space and  $L : X \times X \rightarrow X$  a bilinear operator. Assume that there exists  $\eta > 0$  such that, given  $x_1, x_2 \in X$ , we have  $\|L(x_1, x_2)\| \leq \eta \|x_1\| \|x_2\|$ . Then for any  $y \in X$ , such that  $4\eta \|y\| < 1$ , the equation  $x = y + L(x, x)$  has a solution  $x$  in  $X$ . Moreover, this solution  $x$  is the only one such that*

$$\|x\| \leq \frac{1 - \sqrt{1 - 4\eta \|y\|}}{2\eta}. \quad (1)$$

**Theorem 2.1.** *Let  $d \in \mathbb{N}$ ,  $2 \leq d < q < \infty$ ,  $\eta$  an appropriate constant and  $f \in C_b([0, \infty); L_{\frac{qd}{q+d}}(\mathbb{R}^d))$  be such that  $\alpha := \{\sup_{t>0} [(1 * \ell)(t)]^{1 - \frac{d}{2q}} \|f(t)\|_{L_{\frac{qd}{q+d}}(\mathbb{R}^d)}\} < \infty$ . For  $u_0 \in L_d^\sigma(\mathbb{R}^d)$  and  $\alpha > 0$  sufficiently small, there exists  $0 < \lambda < \frac{1 - 4\alpha\vartheta C\eta}{4\eta}$ , where  $\vartheta$  and  $C$  are positive real constants, such that if  $\|u_0\|_{L_d(\mathbb{R}^d)} \leq \min\{1, C^{-1}\}\lambda$ , then the problem (5) has a global mild solution  $u \in X_q$  that is the unique one satisfying (1). In particular,*

$$\|u(t, \cdot)\|_{L_q(\mathbb{R}^d)} \leq \frac{1 - \sqrt{1 - 4\eta(\lambda + \alpha\vartheta C)}}{2\eta} [(1 * \ell)(t)]^{-\frac{1}{2} + \frac{d}{2q}} \text{ and } \|\nabla u(t, \cdot)\|_{L_d(\mathbb{R}^d)} \leq \frac{1 - \sqrt{1 - 4\eta(\lambda + \alpha\vartheta C)}}{2\eta} [(1 * \ell)(t)]^{-\frac{1}{2}}.$$

If, in addition,  $f \equiv 0$ , we have

$$[(1 * \ell)(t)]^{\frac{1}{2} - \frac{d}{2q}} \|u(t, \cdot)\|_{L_q(\mathbb{R}^d)} \rightarrow 0 \text{ and } [(1 * \ell)(t)]^{\frac{1}{2}} \|\nabla u(t, \cdot)\|_{L_d(\mathbb{R}^d)} \rightarrow 0,$$

as  $t \rightarrow 0^+$ . Furthermore, let  $u, v \in X_q$  be two solutions given by the existence part corresponding to the initial data  $u_0$  and  $v_0$ , respectively. Then,

$$\|u - v\|_{X_q} \leq \frac{C}{\sqrt{1 - 4\eta(\lambda + \alpha\vartheta C)}} \|u_0 - v_0\|_{L_d(\mathbb{R}^d)}.$$

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EXISTÊNCIA GLOBAL E NÃO GLOBAL DE SOLUÇÕES PARA UMA EQUAÇÃO DO CALOR  
COM COEFICIENTES DEGENERADOS

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### Abstract

Neste trabalho estabelecemos condições para a existência global e não global de soluções não negativas da seguinte equação do calor  $u_t - \operatorname{div}(\omega(x)\nabla u) = h(t)f(u) + l(t)g(u)$  em  $\mathbb{R}^N \times (0, T)$ , com condição inicial  $0 \leq u(0) = u_0 \in C_0(\mathbb{R}^N)$ , onde  $\omega(x)$  é um peso adequado na classe Muckenhoupt  $A_{1+\frac{2}{N}}$  que pode ter uma linha de singularidades e  $(h, f, l, g) \in (C[0, \infty))^4$ . Quando  $h(t) \sim t^r$  ( $r > -1$ ),  $l(t) \sim t^s$  ( $s > -1$ ),  $f(u) = (1+u)[\ln(1+u)]^p$  e  $g(u) = u^q$  obtemos o expoente de Fujita e o segundo expoente crítico no sentido de Lee e Ni [5]. Nossos resultados ampliam os obtidos por Fujishima et al. [2].

## 1 Introdução

Consideramos a seguinte equação do calor

$$\begin{cases} u_t - \operatorname{div}(\omega(x)\nabla u) = h(t)f(u) + l(t)g(u) & \text{em } \mathbb{R}^N \times (0, T), \\ u(0) = u_0 \geq 0 & \text{em } \mathbb{R}^N, \end{cases} \quad (1)$$

onde  $u_0 \in C_0(\mathbb{R}^N)$ ,  $h, l \in C[0, \infty)$ ,  $\omega(x)$  é tal que

(A)  $\omega(x) = |x_1|^a$  com  $a \in [0, 1)$  se  $N = 1, 2$  e  $a \in [0, 2/N)$  se  $N \geq 3$ ,

(B)  $\omega(x) = |x|^b$  com  $b \in [0, 1)$ ,  $(x = (x_1, \dots, x_N))$ ,

e  $f, g \in C[0, \infty)$  são funções localmente Lipschitz não negativas. Para a existência não global consideramos

(F<sub>1</sub>)  $\int_w^\infty \frac{d\sigma}{f(\sigma)} < \infty$  para todo  $w > 0$  e  $f(S(t)v_0) \leq S(t)f(v_0)$  para todo  $0 \leq v_0 \in C_0(\mathbb{R}^N)$  e  $t > 0$ .

(G<sub>1</sub>)  $\int_w^\infty \frac{d\sigma}{g(\sigma)} < \infty$  para todo  $w > 0$  e  $g(S(t)v_0) \leq S(t)g(v_0)$  para todo  $0 \leq v_0 \in C_0(\mathbb{R}^N)$  e  $t > 0$ .

Onde  $S(t)u_0(x) := \int_{\mathbb{R}^N} \Gamma(x, y, t)u_0(y)dy$  para  $t > 0$ , e  $\Gamma(x, y, t)$  é a solução fundamental do problema homogêneo  $u_t - \operatorname{div}(\omega(x)\nabla u) = 0$ . Quando  $\omega(x) = |x_1|^a$  o problema (1) está relacionado com o laplaciano fracionário por meio da extensão de Caffarelli-Silvestre, veja [1] e [2]. A equação do calor (1) aparece em modelos que descrevem processos de propagação do calor em meios não homogêneos, veja [3].

## 2 Resultados Principais

No presente trabalho estabelecemos condições para a existência global e não global de soluções não negativas de (1). O primeiro trabalho nesta direção foi obtido por Fujishima, Kawakami e Sire em [2], nesse trabalho os autores obtêm resultados do tipo Fujita para o problema (1), quando  $h = 1$ ,  $l = 0$  e  $f(u) = u^p$  ( $p > 1$ ). Nosso principal resultado é o seguinte

**Teorema 2.1.** *Assuma à condição (A) ou (B) e suponha que  $(f, g) \in (C[0, \infty))^2$  são funções não negativas localmente lipchitz contínuas tal que  $f(0) = g(0) = 0$ .*

(i) Se  $f, g, f(s)/s, g(s)/s$  são não decrescentes num intervalo  $(0, m]$  e existe  $v_0 \neq 0$ ,  $0 \leq v_0 \in C_0(\mathbb{R}^N)$ ,  $\|v_0\|_\infty \leq m$  tal que  $\int_0^\infty h(\sigma) \frac{f(\|S(\sigma)v_0\|_\infty)}{\|S(\sigma)v_0\|_\infty} d\sigma + \int_0^\infty l(\sigma) \frac{g(\|S(\sigma)v_0\|_\infty)}{\|S(\sigma)v_0\|_\infty} d\sigma < 1$ , então existe uma constante  $\delta > 0$  tal que para  $\delta v_0 = u_0$  a solução de (1) é global.

(ii) Seja  $0 \leq u_0 \in C_0(\mathbb{R}^N)$ ,  $u_0 \neq 0$  e suponha que alguma das seguintes condições sejam satisfeitas

(a)  $(F_1)$  é verdade e  $f$  é não decrescente tal que  $f(s) > 0$  para todo  $s > 0$  e existe  $\tau > 0$  tal que  $\int_{\|S(\tau)u_0\|}^\infty \frac{d\sigma}{f(\sigma)} \leq \int_0^\tau h(\sigma) d\sigma$ .

(b)  $(G_1)$  é verdade e  $g$  é não decrescente tal que  $g(s) > 0$  para todo  $s > 0$  e existe  $\tau > 0$  tal que  $\int_{\|S(\tau)u_0\|}^\infty \frac{d\sigma}{g(\sigma)} \leq \int_0^\tau l(\sigma) d\sigma$ .

Então a solução de (1) com condição inicial  $u_0$  não é global.

*Proof.* Primeiro obtemos às soluções  $u \in C((0, T), C_0(\mathbb{R}^N))$  que satisfazem a formulação integral  $u(x, t) = \int_{\mathbb{R}^N} \Gamma(x, y, t) u_0(y) dy + \int_0^t \int_{\mathbb{R}^N} \Gamma(x, y, t - \sigma) h(\sigma) f(u(y, \sigma)) d\sigma$ . Daí, a prova é obtida adaptando às ideias de [4] junto com às estimativas obtidas em [2].  $\square$

Agora, para  $\rho > 0$  considere  $I^\rho = \{\psi \in C_0(\mathbb{R}^N), \psi \geq 0 \text{ e } \limsup_{|x| \rightarrow \infty} |x|^\rho \psi(x) < \infty\}$  e  $I_\rho = \{\psi \in C_0(\mathbb{R}^N), \psi \geq 0 \text{ e } \liminf_{|x| \rightarrow \infty} |x|^\rho \psi(x) > 0\}$ . Do teorema (2.1) e os métodos empregados em [5] e [4] obtemos os seguintes resultados concernentes ao expoente crítico de Fujita e ao segundo expoente crítico no sentido de [5].

**Corolário 2.1.** Assuma a condição (A) ou (B). Suponha  $f(t) = (1+t)(\ln(1+t))^p$  ( $p > 1$ ),  $g(t) = t^q$  ( $q > 1$ ),  $(h, l) \in (C[0, \infty))^2$  tal que  $h(t) \sim t^r$  ( $r > -1$ ) e  $l(t) \sim t^s$  ( $s > -1$ ) para  $t$  suficientemente grande.

(i) Se  $1 < p \leq 1 + \frac{(2-\alpha)(r+1)}{N}$  ou  $1 < q \leq 1 + \frac{(2-\alpha)(s+1)}{N}$ , então o problema (1) não têm soluções globais não triviais.

(ii) Suponha  $1 + \frac{(2-\alpha)(r+1)}{N} < p$  e  $1 + \frac{(2-\alpha)(s+1)}{N} < q$ .

(a) Se  $0 < \rho < \frac{(2-\alpha)(r+1)}{p-1}$  ou  $0 < \rho < \frac{(2-\alpha)(s+1)}{q-1}$ , então o problema (1) não têm soluções globais não triviais para qualquer condição inicial  $\psi \in I_\rho$ .

(b) Se  $\frac{(2-\alpha)(r+1)}{p-1} < \rho$  e  $\frac{(2-\alpha)(s+1)}{q-1} < \rho$ , então para qualquer condição inicial  $\psi \in I^\rho$  existe  $\lambda > 0$  tal que o problema (1) com condição inicial  $\lambda\psi$  tem uma solução global não trivial.

Onde  $\alpha = a$  quando a condição (A) é verdade e  $\alpha = b$  quando (B) é verdade.

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## EXISTÊNCIA DE ESCOAMENTOS DE FLUÍDOS MAGNÉTICOS PERIÓDICOS NO TEMPO

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Neste trabalho estabelecemos a existência de soluções fortes  $T$ -periódicas para um sistema de equações diferenciais parciais associadas a um modelo (de Rosensweig) para o escoamento de fluídos magnéticos em domínios limitados bidimensionais e tridimensionais sob a ação de um campo magnético externo.

**1 Introdução**

Investigamos a existência de movimentos periódicos no tempo em escoamentos de fluidos magnéticos. Tais fluidos são encontrados em várias aplicações industriais como em mancais hidrostáticos e hidrodinâmicos, em sistemas de amortecimento, em atuadores e máquinas elétricas, dentre outros. Várias aplicações estão descritas em [1]. O sistema de equações diferenciais parciais do modelo é o seguinte:

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) - (\eta + \xi) \Delta \mathbf{u} + \nabla p = \mu_0 (\mathbf{m} \cdot \nabla) \cdot \mathbf{h} + 2\xi \nabla \times \mathbf{w} \text{ em } (0, \infty) \times \Omega, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \text{ em } (0, \infty) \times \Omega, , \quad (2)$$

$$\frac{\partial \mathbf{m}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{m} = \mathbf{w} \wedge \mathbf{m} - \frac{1}{\tau} (\mathbf{m} - \chi_0 \mathbf{h}) \quad \text{em } (0, \infty) \times \Omega \quad (3)$$

$$\rho k \left( \frac{\partial \mathbf{w}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{w} \right) - \sigma' \Delta \mathbf{w} - (\eta' + \lambda') \nabla \nabla \cdot \mathbf{w} = \mu_0 \mathbf{m} \wedge \mathbf{h} \quad (4)$$

$$+ 2\xi (\nabla \times \mathbf{u} - 2\mathbf{w}) \quad \text{em } (0, \infty) \times \Omega, \quad (5)$$

$$\nabla \cdot (\mathbf{h} + \mathbf{m}) = F, \quad \nabla \times \mathbf{h} = 0, \quad \text{em } (0, \infty) \times \Omega. \quad (6)$$

Estas equações descrevem (respectivamente) o balanço de momento linear, massa, magnetização, momento angular, seguido das denominadas equações magneto-estáticas para o campo magnético  $\mathbf{h}$ .  $\mathbf{u}$  denota a velocidade do ferro-fluído,  $p$  representa a pressão dinâmica,  $\mathbf{m}$  denota a magnetização, e  $\mathbf{w}$  é a velocidade de rotação.  $\rho, \eta, \xi, \mu_0, \tau, \chi_0, k, \sigma', \eta', \lambda'$  são constantes positivas. O campo magnético externo é denotado por  $H_{ext}$  e  $F = -\nabla \cdot H_{ext}$ . Se o termo regularizante  $-\sigma \Delta \mathbf{m}$  é desconsiderado (desprezando-se o momento magnético de rotação [5]), até mesmo a existência de soluções fracas não é conhecida. As condições de contorno e de periodicidade são as seguintes:

$$\mathbf{u} = 0, \quad \mathbf{w} = 0, \quad \mathbf{m} \cdot \nu = 0, \quad \nabla \times \mathbf{m} \wedge \nu = 0 \text{ sobre } \partial \Omega_T, \quad (7)$$

com a condição de  $T$ -periodicidade para  $\mathbf{u}, \mathbf{w}, \mathbf{m}$  e  $\mathbf{h}$ . Os seguintes trabalhos anteriores abordam questões de existência de soluções para este modelo: [3],[4],[6].

## 2 Resultados Principais

Sejam

$$\mathcal{V}(\Omega) := \{\varphi \in \mathbb{D}(\Omega) : \operatorname{div}\varphi = 0\}, \quad H_{\operatorname{div}}(\Omega) := \{\mathbf{u} \in L^2(\Omega) : \operatorname{div}\mathbf{u} \in L^2(\Omega)\}$$

com norma  $\|\mathbf{u}\|_{H_{\operatorname{div}}} := (\|\mathbf{u}\|^2 + \|\operatorname{div} \mathbf{u}\|^2)^{1/2}$ .

$V(\Omega)$  é o fecho de  $\mathcal{V}(\Omega)$  em  $H_0^1(\Omega)$ .  $H_{\operatorname{div},0}(\Omega)$  é o fecho de  $\mathbb{D}(\Omega)$  em  $H_{\operatorname{div}}(\Omega)$ .

Seja  $A$  o operador de Stokes,  $A : V(\Omega) \rightarrow V(\Omega)^*$  definido por  $\langle A\mathbf{v}, \mathbf{u} \rangle_{V^*,V} = \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} \, dx$ ,  $\forall \mathbf{u}, \mathbf{v} \in V(\Omega)$  com domínio  $D(A) := \{\mathbf{u} \in V(\Omega) : A\mathbf{u} \in H(\Omega)\}$  e  $(\mathbf{u}|\mathbf{v})_{D(A)} := (\mathbf{u}|\mathbf{v}) + (A\mathbf{u}|A\mathbf{v})$ ,  $\forall \mathbf{u}, \mathbf{v} \in D(A)$ .

Temos a seguinte caracterização para o espaço  $H_{\nu}(\Omega) = H^1(\Omega) \cap \mathcal{N}(\gamma_{\nu})$ , onde  $\gamma_{\nu}(\mathbf{u}) = \mathbf{u} \cdot \nu$  é o operador traço, contínuo de  $H_{\operatorname{div}}(\Omega)$  em  $H^{-1/2}(\partial\Omega)$ . Sejam  $\mathcal{L}(\ ) = -\Delta(\ )$  com domínio  $D(\mathcal{L}) = H^2(\Omega) \cap H_{\operatorname{div},0}(\Omega)$ , e  $\tilde{\mathcal{L}} = -\eta' \Delta - (\eta' + \lambda') \nabla \operatorname{div}$  com domínio

$$D(\tilde{\mathcal{L}}) = H^2(\Omega) \cap H_0^1(\Omega).$$

Apresentamos neste trabalho os seguintes resultados de regularidade das soluções fracas. O existência de soluções fracas está entre os resultados desta investigação. Apresentamos somente resultados de regularidade destas soluções.

**Teorema 2.1.** (*Regularidade - caso 2d*)

Seja  $F \in H^1(0, T; H^1(\Omega))$  tal que  $(F|1) = 0$  sobre  $[0, T]$ . Seja  $\Omega \subset \mathbb{R}^3$  um subconjunto aberto limitado, simplesmente conexo com fronteira suave (pelo menos de classe  $C^3$ ). Então, as soluções fracas  $T$ -periódicas do sistema de equações do modelo de Rosensweig tem a seguinte regularidade adicional:

$$\begin{aligned} \mathbf{u} &\in L^{\infty}(0, T; V(\Omega)) \cap L^2(0, T; D(A)), \quad \mathbf{w} \in L^{\infty}(0, T; H_0^1(\Omega)) \cap L^2(0, T; D(\tilde{\mathcal{L}})) \\ \mathbf{m} &\in L^{\infty}(0, T; H_{\nu}(\Omega)) \cap L^2(0, T; D(\mathcal{L})), \quad \mathbf{h} \in L^{\infty}(0, T; H_{\nu}(\Omega)) \cap L^2(0, T; H^2(\Omega)). \end{aligned}$$

**Teorema 2.2.** (*Regularidade - caso 3d*)

Seja  $F \in H^1(0, T; H^1(\Omega))$  tal que  $(F|1) = 0$  sobre  $[0, T]$ . Seja  $\Omega \subset \mathbb{R}^3$  um subconjunto aberto limitado, simplesmente conexo com fronteira suave (pelo menos de classe  $C^3$ ). Existe uma constante positiva  $c$  tal que se  $\|F\|_{H^1(0, T; H^1)} \leq c$ , então as soluções fracas  $T$ -periódicas do sistema de Rosensweig tem a seguinte regularidade adicional:

$$\begin{aligned} \mathbf{u} &\in L^{\infty}(0, T; V(\Omega)) \cap L^2(0, T; D(A)), \quad \mathbf{w} \in L^{\infty}(0, T; H_0^1(\Omega)) \cap L^2(0, T; D(\tilde{\mathcal{L}})) \\ \mathbf{m} &\in L^{\infty}(0, T; H_{\nu}(\Omega)) \cap L^2(0, T; D(\mathcal{L})), \quad \mathbf{h} \in L^{\infty}(0, T; H_{\nu}(\Omega)) \cap L^2(0, T; H^2(\Omega)). \end{aligned}$$

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## ASYMPTOTIC BEHAVIOR OF THE COUPLED KLEIN-GORDON-SCHRÖDINGER SYSTEMS ON COMPACT MANIFOLDS

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### Abstract

This paper is concerned with a 2-dimensional Klein-Gordon-Schrödinger system subject to two types of locally distributed damping on a compact Riemannian manifold  $\mathcal{M}$  without boundary. Making use of unique continuation property, the observability inequalities, and the smoothing effect due to Aloui, we obtain exponential stability results.

### 1 Introduction

In this paper, we consider the Cauchy problem of the following Klein-Gordon-Schrödinger equations through Yukawa interaction,

$$\begin{cases} i\psi_t + \Delta\psi + i\alpha\mathcal{B}_j(x, \psi) = \phi\psi\chi_\omega & \text{in } \mathcal{M} \times (0, \infty), \ j = 1, 2 \\ \phi_{tt} - \Delta\phi + a(x)\phi_t = |\psi|^2\chi_\omega & \text{in } \mathcal{M} \times (0, \infty) \\ \psi(0) = \psi_0, \ \phi(0) = \phi_0, \ \phi_t(0) = \phi_1 & \text{in } \mathcal{M} \end{cases} \quad (P_j)$$

where  $(\mathcal{M}, g)$  is a bidimensional compact Riemannian manifold without boundary and  $g$  represents your metric,  $\mathcal{B}_j(x, \psi)$  is non-linear locally distributed damping,  $\omega$  is a region on  $\mathcal{M}$  where the dissipative effect is effective, and  $\chi_\omega$  is the characteristic function on  $\omega$ . Moreover the constant  $\alpha$  will be characterized later. We study two types of damping, defined by  $\mathcal{B}_1(x, \psi) = b(x)(1 - \Delta)^{1/2}b(x)\psi$  and  $\mathcal{B}_2(x, \psi) = b(x)(|\psi|^2 + 1)\psi$

We assume that  $a(\cdot), b(\cdot)$  are non-negative functions satisfying

$$\begin{cases} a, b \in W^{1,\infty}(\mathcal{M}) \cap C^\infty(\mathcal{M}) \\ a(x) \geq a_0 > 0 \text{ in } \omega, \quad \text{and} \quad b(x) \geq b_0 > 0 \text{ in } \omega, \end{cases}$$

where  $\omega$  is an open subset of  $\mathcal{M}$  such that  $\text{meas}(\omega) > 0$  satisfying geometric control condition.

An interesting result of exponential decay considering Klein-Gordon-Schrödinger system with localized damping in both equations are due the authors in [1]. Uniform decay rates were have obtained combining multipliers method, integral inequalities of energy, and regularizing effect due to Aloui. Recently in [2], the authors generalize the previous results considering the weaker damped structure  $iab(x)(|\psi|^2 + 1)\psi$  instead of  $iab(x)(-\Delta)^{1/2}b(x)\psi$  assumed in [1], making use of the observability inequality in both equations, the linear wave and the Schrödinger equation, furthermore, combined with other tools have proven exponential decay. The purpose of the present article is to extend substantially all previous results given by [1] and [2] in the geometric sense and exhibit an important multiplier function. Here we study the problem  $(P_j)$ ,  $j = 1, 2$ , on a compact Riemannian manifold with arbitrary metric.

## 2 Main Results

We shall use standard Sobolev spaces on Riemannian manifolds. In the present paper, we consider some crucial assumptions about the dissipative region  $\omega$ , in order to establish the geometric control condition.

**Assumption 1:** We assume that  $a, b \in W^{1,\infty}(\mathcal{M}) \cap C^\infty(\mathcal{M})$  are nonnegative functions such that  $a(x) \geq a_0 > 0$  and  $b(x) \geq b_0 > 0$  in  $\omega$ . In addition, if  $a(x) \geq a_0 > 0$  in  $\mathcal{M}$ , then we consider  $\chi_\omega \equiv 1$  in  $\mathcal{M}$ . If  $b(x) \geq b_0 > 0$  in  $\mathcal{M}$ , then we consider  $\chi_\omega \equiv 1$  in  $\mathcal{M}$ .

**Definition 2.1.** (*Geometric Control Condition*):  $\omega$  geometrically controls  $\mathcal{M}$ , i.e there exists  $T_0 > 0$ , such that every geodesic of  $\mathcal{M}$  travelling with speed 1 and issued at  $t = 0$ , enters the set  $\omega$  in a time  $t < T_0$ .

**Assumption 2:** We assume that  $\omega$  is an open subset of  $\mathcal{M}$  such that  $\text{meas}(\omega) > 0$  and satisfying the geometric control condition

In what follows, we consider the energy associated with problems  $(P_j)$ ,  $j = 1, 2$ , by

$$E(t) := \frac{1}{2} \int_{\mathcal{M}} (|\psi(x, t)|^2 + |\nabla \phi(x, t)|^2 + |\phi_t(x, t)|^2) d\mathcal{M}. \quad (1)$$

**Theorem 2.1** (Well-Posedness). *Suppose Assumption 1 holds. In addition, assume that  $5(2a_0b_0)^{-1} \leq \alpha$  in the problem  $(P_2)$ . Then, given  $(\psi_0, \phi_0, \phi_1) \in \{\mathcal{V} \cap H^2(\mathcal{M})\}^2 \times \mathcal{V}$  problems  $(P_1)$  and  $(P_2)$  has a unique regular solution satisfying*

$$\begin{aligned} \psi &\in L^\infty(0, \infty; \mathcal{V} \cap H^2(\mathcal{M})), \psi' \in L^\infty(0, \infty; L^2(\mathcal{M})), \\ \phi &\in L^\infty(0, \infty; \mathcal{V} \cap H^2(\mathcal{M})), \phi' \in L^\infty(0, \infty; \mathcal{V}), \text{ and } \phi'' \in L^\infty(0, \infty; L^2(\mathcal{M})). \end{aligned} \quad (2)$$

Considering the phase space  $\mathcal{H} := \{\mathcal{V} \cap H^2(\mathcal{M})\}^2 \times \mathcal{V}$ , in the next theorem, below, we provide a local uniform decay of the energy. Indeed, we shall consider the initial data taken in bounded sets of  $\mathcal{H}$ , namely,  $\|(\psi_0, \phi_0, \phi_1)\|_{\mathcal{H}} \leq L$ , where  $L$  is a positive constant. This is strongly necessary due to the non linear character of system  $(P_1)$  and since the energy  $E(t)$  is not naturally a non increasing function of the parameter  $t$ . Thus, the constants,  $C$  and  $\gamma$  which appear below, will depend on  $L > 0$ . We shall denote  $d = d(c, \|b\|_\infty, L)$ , to be fixed, where  $c$  comes from the embedding  $D[(1 - \Delta)^{\frac{1}{4}}] \equiv H^{\frac{1}{2}}(\mathcal{M}) \hookrightarrow L^4(\mathcal{M})$ . So, under the above considerations, we can establish the main result concerning uniform stabilization from the problems  $(P_1)$  and  $(P_2)$ .

**Theorem 2.2.** *Suppose that the hypotheses of Theorem 2.1 holds. In addition,  $\alpha > \frac{a_0^{-1}b_0^{-4}d}{2}$  or  $d$  is sufficiently small. Then, there exist  $C, \gamma$  positive constants such that following decay rate holds  $E(t) \leq Ce^{-\gamma t}E(0)$ , for all  $t \geq 0$ , for every regular solution of problem  $(P_1)$  satisfying (2), provided the initial data are taken in bounded sets of  $\mathcal{H}$ .*

**Theorem 2.3.** *Suppose that the hypotheses of Theorem 2.1, and Assumption 2 hold. Then, there exist  $C, \gamma$  positive constants such that the following decay rate holds  $E(t) \leq Ce^{-\gamma t}E(0)$ , for all  $t \geq 0$ , for every regular solution of problem  $(P_2)$  satisfying (2), provided the initial data are taken in bounded sets of  $\mathcal{H}$ .*

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## GLOBAL REGULARITY FOR A 1D SUPERCRITICAL TRANSPORT EQUATION

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In this manuscript we investigate a nonlocal transport 1D-model with supercritical dissipation  $\gamma \in (0, 1)$  in which the velocity is coupled via the Hilbert transform. We show global existence of non-negative  $H^{3/2}$ -strong solutions in a supercritical subrange (close to 1) that depends on the initial data norm.

**1 Introduction**

We consider the initial value problem for the 1D transport equation with nonlocal velocity

$$\begin{cases} \partial_t \theta - \mathcal{H}\theta \theta_x + \Lambda^\gamma \theta = 0 & \text{in } \mathbb{T} \times (0, \infty) \\ \theta(x, 0) = \theta_0(x) & \text{in } \mathbb{T}, \end{cases} \quad (1)$$

where  $0 < \gamma \leq 2$ ,  $\mathbb{T}$  is the 1D torus,  $\Lambda = (-\Delta)^{1/2}$  and  $\mathcal{H}$  denotes the Hilbert transform. This model arises as a lower dimensional model for the well known 2D dissipative quasi-geostrophic equation and in connection with vortex-sheet problems.

The IVP (1) has three basic cases: subcritical  $1 < \gamma \leq 2$ , critical  $\gamma = 1$  and supercritical  $0 < \gamma < 1$ . The global smoothness problem in the critical and subcritical cases have already been solved (see [1, 3]).

The global regularity problem for solutions of (1) in the supercritical case remains an open problem. In the part  $0 < \gamma < \frac{1}{2}$  of the supercritical range, Li and Rodrigo [6] proved blow-up of solutions in finite time for non-positive, smooth, even and compactly supported initial data satisfying  $\theta_0(0) = 0$  and a suitable weighted integrability condition. In [5], still in the same range, Kiselev showed blow-up of solutions in finite time for even, positive, bounded and smooth initial data  $\theta_0$  satisfying  $\max_{x \in \mathbb{R}} \theta_0(x) = \theta_0(0)$  and suitable integrability conditions.

In the range  $\frac{1}{2} \leq \gamma < 1$ , the formation of singularity in finite time or global smoothness is an open problem (stated by [5, p. 251]), even for sign restriction on the initial data, i.e.,  $\theta_0 \geq 0$  or  $\theta_0 \leq 0$ . In [2], for  $0 < \gamma < 1$ , Do obtained eventual regularization of solutions for non-negative initial data.

In this work we focus on supercritical values of  $\gamma$  contained in the range  $\frac{1}{2} \leq \gamma < 1$  (close to 1) and prove existence of global classical solutions for (1). More precisely, we show existence of  $H^{\frac{3}{2}}$ -strong solution for arbitrary non-negative initial data  $\theta_0 \in H^{\frac{3}{2}}$  and  $\gamma_1 \leq \gamma < 1$ , where  $\gamma_1$  depends on  $H^{\frac{3}{2}}$ -norm of  $\theta_0$ .

**2 Main Results**

**Theorem 2.1.** *Suppose that  $\gamma \in [1/2, 1)$  and  $\theta_0 \in L^\infty(\mathbb{T})$  is non-negative. Let  $\alpha \in (1 - \gamma, 1)$  and define*

$$T^* = C\alpha^{\frac{1}{1-\gamma}} \|\theta_0\|_{L^\infty(\mathbb{T})}^{\frac{\gamma}{1-\gamma}}, \quad (1)$$

where  $C = \gamma^{-1} k_1 k_2^{\frac{\gamma}{1-\gamma}} > 0$  with  $k_1$  and  $k_2$  being independent of  $\alpha, \gamma$  and  $\theta_0$ . Let  $\theta$  be a solution of (1) in  $C([0, T]; H^{\frac{3}{2}}(\mathbb{T}))$  with existence time  $0 < T < \infty$ . If  $T^* < T$ , then  $\theta \in C^\infty(\mathbb{T} \times (T^*, T])$ .

**Proof.** See [4].  $\square$

**Theorem 2.2.** Let  $\theta_0 \in H^{\frac{3}{2}}(\mathbb{T})$  be an arbitrary non-negative initial data. Then, there exists  $\gamma_1 = \gamma_1(\|\theta_0\|_{H^{\frac{3}{2}}}) \in (1/2, 1)$  such that for each  $\gamma \in [\gamma_1, 1)$  the IVP (1) has a unique global (classical)  $H^{\frac{3}{2}}$ -solution.

**Proof.** The Theorem 2.1 provides an explicit control on the regularization time  $T^*$ ,

$$T^* = C\alpha^{\frac{1}{1-\gamma}} \|\theta_0\|_{L^\infty(\mathbb{T})}^{\frac{\gamma}{1-\gamma}}.$$

Afterwards, we obtain an explicit lower bound of the local existence time with  $H^{\frac{3}{2}}$ -initial data. More precisely, there exists a constant  $C > 0$  such that  $H^{\frac{3}{2}}$ -solution does not blow up until

$$T = C \left( \|\theta_0\|_{L^2(\mathbb{T})}^{\frac{2\gamma(9+2\gamma)}{3(9+4\gamma)}} \|\theta_0\|_{H^{\frac{3}{2}}(\mathbb{T})}^{2 - \frac{4\gamma(6+\gamma)}{3(9+4\gamma)}} \right)^{-1}. \quad (2)$$

By comparison of the local existence time  $T$  (2) with the eventual regularization time  $T^*$  (1) we can choose  $\alpha = \min\{2(1-\gamma), 1/2\}$  such that there exists  $\gamma_1 \in (1/2, 1)$  such that  $T^* < T$  for all  $\gamma \in [\gamma_1, 1)$ .  $\square$

The equation (1) has a scaling property: if  $\theta$  is a solution, then so is  $\theta_\lambda(t, x) = \lambda^{\gamma-1}\theta(\lambda^\gamma t, \lambda x)$ , for any  $\lambda > 0$ . Thus, the quantity  $\|\cdot\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T})}^{1-\frac{2\gamma}{3}} \|\cdot\|_{L^2(\mathbb{T})}^{\frac{2\gamma}{3}}$  is scaling invariant. Now, for each  $\gamma \in (0, 1)$ , define

$$R_\gamma = \sup\{R > 0 \text{ such that, for any } \theta_0 \in H^{\frac{3}{2}}(\mathbb{T}) \text{ with } \|\theta_0\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T})}^{1-\frac{2\gamma}{3}} \|\theta_0\|_{L^2(\mathbb{T})}^{\frac{2\gamma}{3}} \leq R, \text{ the unique } H^{\frac{3}{2}} \text{-solution of (1) with initial data } \theta_0 \text{ does not blow up in finite time.}\}$$

By small data results for  $\gamma \in (0, 1)$  we know that  $R_\gamma > 0$ , while from the global regularity results in the critical case, we have that  $R_1 = \infty$ . The Theorem 2.2 state shows that

$$R_\gamma \rightarrow \infty \text{ as } \gamma \rightarrow 1^-.$$

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A KATO TYPE EXPONENT FOR A CLASS OF SEMILINEAR EVOLUTION EQUATIONS WITH  
TIME-DEPENDENT DAMPING

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### Abstract

In this presentation, we derive suitable optimal  $L^p - L^q$  decay estimates,  $1 \leq p \leq 2 \leq q \leq \infty$ , for the solutions to the  $\sigma$ -evolution equation,  $\sigma > 1$ , with scale-invariant time-dependent damping and power nonlinearity  $|u|^p$ ,

$$u_{tt} + (-\Delta)^\sigma u + \frac{\mu}{1+t} u_t = |u|^p, \quad t \geq 0, \quad x \in \mathbb{R}^n,$$

where  $\mu > 0$ ,  $p > 1$ . The critical exponent  $p = p_c$  for the global (in time) existence of small data solutions to the Cauchy problem is related to the long time behavior of solutions, which changes accordingly with  $\mu$ . Under the assumption of small initial data in  $L^1 \cap L^2$ , we find the critical exponent

$$p_c = 1 + \frac{2\sigma}{[n - \sigma + \sigma\mu]_+},$$

for  $\mu \in (0, 1)$ . This critical exponent it is a shift of a Kato type exponent.

## 1 Introduction

In this presentation we study the global (in time) existence of small data solutions to the Cauchy problem for the semilinear damped  $\sigma$ -evolution equations with scale-invariant time-dependent damping

$$u_{tt} + (-\Delta)^\sigma u + \frac{\mu}{1+t} u_t = |u|^p, \quad u(0, x) = 0, \quad u_t(0, x) = u_1(x), \quad t \geq 0, \quad x \in \mathbb{R}^n. \quad (1)$$

where  $\mu \in (0, 1)$ ,  $\sigma > 1$  and  $f(u) = |u|^p$  for some  $p > 1$ .

Naturally the size of the parameter  $\mu$  is relevant to describe the asymptotic behavior of solutions. When  $\mu \in (0, 1)$ , this model is related to the semilinear free  $\sigma$ -evolution equations. For this reason let us introduce some previous results to the Cauchy problem

$$u_{tt} + (-\Delta)^\sigma u = |u|^p, \quad u(0, x) = 0, \quad u_t(0, x) = u_1(x). \quad (2)$$

For  $\sigma = 1$  this problem was considered by several authors. If  $1 < p < p_K(n) = \frac{n+1}{[n-1]_+}$ , Kato [5] proved the nonexistence of global generalized solutions to (2), for small initial data with compact support. On the other hand, John [4] showed that  $p = 1 + \sqrt{2}$  is the critical exponent for the global existence of classical solutions with small initial data in space dimension  $n = 3$ . A bit later some authors, for instance, Glassey ([2], [3]), Sideris [7], Lindblad and Sogge [6], proved that the critical exponent is  $p_S(n)$  for  $n \geq 2$ , which is the positive root of

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

Then, for  $\sigma > 1$  and for space dimensions  $1 \leq n \leq 2\sigma$ , in [1] it was obtained the critical exponent to (2),  $p_K(n) = \frac{n+\sigma}{[n-\sigma]_+}$ , which is of Kato type.

The main goals in this presentation are to derive  $L^p - L^q$  estimates and energy estimates for solutions to the linear Cauchy problem associated to (1) and to obtain the critical exponent for the global (in time) existence of small initial data solutions to (1) for  $\mu \in (0, 1)$ . We show that the critical exponent is a shift of Kato type exponent  $p_K(n + \sigma\mu) \doteq \frac{n+\sigma+\sigma\mu}{[n-\sigma+\sigma\mu]_+}$ .

## 2 Main Results

This two results shows that the critical exponent for the Cauchy problem (1), when  $\mu \in (0, 1)$ , is given by a shift of Kato type exponent  $p_K(n + \sigma\mu) \doteq \frac{n+\sigma+\sigma\mu}{[n-\sigma+\sigma\mu]_+}$ .

Let us consider  $\mu_{\sharp} = \infty$  if  $\mu \leq 2 - \frac{2n}{\sigma}$  or  $\mu_{\sharp} = \frac{1}{2\sigma}(\sigma - n + \sqrt{9\sigma^2 - 10n\sigma + n^2})$  if  $\mu > 2 - \frac{2n}{\sigma}$ .

**Theorem 2.1.** *Let  $\sigma > 1$ ,  $1 \leq n < \sigma$ ,  $1 - \frac{n}{\sigma} < \mu < \min\{\mu_{\sharp}; 1\}$  and  $\mu \neq 2 - \frac{2n}{\sigma}$ . If*

$$1 + \frac{2\sigma}{n - \sigma + \sigma\mu} \doteq p_K(n + \sigma\mu) < p \leq 1 + \frac{2\sigma - \sigma\mu}{[2n - 2\sigma + \sigma\mu]_+} \doteq q_1, \quad (3)$$

then there exists  $\epsilon > 0$  such that for any initial data

$$u_1 \in \mathcal{A} = L^2(\mathbf{R}^n) \cap L^1(\mathbf{R}^n), \quad \|u_1\|_{\mathcal{A}} \leq \epsilon,$$

there exists a unique energy solution  $u \in C([0, \infty), H^\sigma(\mathbf{R}^n)) \cap C^1([0, \infty), L^2(\mathbf{R}^n)) \cap L^\infty([0, \infty) \times \mathbf{R}^n)$  to (1). Moreover, for  $2 \leq q \leq q_1$  the solution satisfies the following estimates

$$\|u(t, \cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{q})+1-\mu} \|u_1\|_{\mathcal{A}}, \quad \|u(t, \cdot)\|_{L^\infty} \lesssim (1+t)^{-\min\{\frac{n}{\sigma}+\mu-1, \frac{\mu}{2}\}} \|u_1\|_{\mathcal{A}},$$

and

$$\|u(t, \cdot)\|_{\dot{H}^\sigma} + \|\partial_t u(t, \cdot)\|_{L^2} \lesssim (1+t)^{-\frac{\mu}{2}} \|u_1\|_{\mathcal{A}}, \quad \forall t \geq 0.$$

For the sake of simplicity, in the next result we restrict our analysis for integer  $\sigma$ .

**Proposition 2.1.** *Let  $\sigma \in \mathbb{N}$ ,  $0 < \mu \leq 1$  and*

$$1 < p \leq p_K(n + \sigma\mu) \doteq \frac{n + \sigma + \sigma\mu}{[n - \sigma + \sigma\mu]_+}.$$

If  $u_1 \in L^1(\mathbf{R}^n)$  such that

$$\int_{\mathbf{R}^n} u_1(x) dx > 0, \quad (4)$$

then there exists no global (in time) weak solution  $u \in L^p_{loc}([0, \infty) \times \mathbf{R}^n)$  to (1).

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EXISTENCE AND CONTINUOUS DEPENDENCE OF THE LOCAL SOLUTION OF NON  
HOMOGENEOUS KDV-K-S EQUATION

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**Abstract**

In this work, we prove that initial value problem associated to the nonhomogeneous KdV-Kuramoto-Sivashinsky (KdV-K-S) equation in periodic Sobolev spaces has a local solution in  $[0, T]$  with  $T > 0$ , and the solution has continuous dependence with respect to the initial data and the nonhomogeneous part of the problem. We do this in an intuitive way using Fourier theory and introducing a  $C_o$ -semigroup inspired by the work of Iorio [1] and Santiago and Rojas [3]. Also, we prove the uniqueness solution of the homogeneous and nonhomogeneous KdV-K-S equation, using its dissipative property, inspired by the work of Iorio [1] and Santiago and Rojas [4].

## 1 Introduction

First, we want to comment that from Theorem 3.1 in [2], we have that the KdV-K-S homogeneous problem is globally well posed and, in addition to the inequality (3.2) in [2], we have the continuous dependence of the solution of homogeneous problem.

In this work, in Theorem 2.1 we will prove the existence and uniqueness of the local solution for the non homogeneous problem and from inequality (4) we will get the continuous dependence of the solution with respect to the initial data and respect to the non homogeneous part.

Thus, in both homogeneous and non homogeneous cases, the estimatives are made from the explicit form of the solution, that is, by applying the Fourier transform to the respective equation.

Another result in this work is about the dissipative property of the homogeneous problem and some estimates of it, using differential calculus in  $H_{per}^s$ . This is included in Theorem 2.2 which we will develop. So, using Theorem 2.2, we deduce the results of continuous dependence and uniqueness of solution for both homogeneous and non homogeneous problems, respectively.

Finally, we give some conclusions and generalizations.

## 2 Main Results

We prove that the non homogeneous problem ( $P_c^F$ ) is locally well posed.

**Theorem 2.1.** *Let  $\phi \in H_{per}^s$ ,  $s \in R$ ,  $\mu > 0$ ,  $F \in C([0, T], H_{per}^s)$ , where  $T > 0$ , and  $\{S(t)\}_{t \geq 0}$  the semigroup of class  $C_o$  of contraction in  $H_{per}^s$  for homogeneous case ( $F = 0$ ), introduced in the Theorem 3.2 from [2], then*

*1. The function:*

$$u^F(t) := S(t)\phi + \underbrace{\int_0^t S(t-\tau)F(\tau)d\tau}_{u_p(t)=}, \quad t \in [0, T] \quad (1)$$

*belongs to  $C([0, T], H_{per}^s) \cap C^1([0, T], H_{per}^{s-4})$  and*

2.  $u^F(t)$  is the unique solution of

$$(P_c^F) \quad \left| \begin{array}{l} u_t + u_{xxx} + \mu(u_{xxxx} + u_{xx}) = F(t) \in H_{per}^{s-4} \\ u(0) = \phi \end{array} \right. \quad (2)$$

with the derivative given by

$$\lim_{h \rightarrow 0} \left\| \frac{u(t+h) - u(t)}{h} + u_{xxx} + \mu(u_{xxxx} + u_{xx}) - F(t) \right\|_{s-4} = 0. \quad (3)$$

3. Let  $\psi_j \in H_{per}^s$ ,  $F_j \in C([0, T], H_{per}^s)$ ,  $j = 1, 2$ . The map  $\psi \rightarrow u$  is continuous in the following sense. Let  $u_1$  and  $u_2$  the corresponding solutions to initial data  $\psi_1$  and  $\psi_2$ , and with non homogeneity  $F_1$  and  $F_2$  respectively. Then

$$\|u_1 - u_2\|_{\infty, s} \leq \|\psi_1 - \psi_2\|_s + T\|F_1 - F_2\|_{\infty, s}, \quad (4)$$

$$\begin{aligned} \|\partial_t u_1(t) - \partial_t u_2(t)\|_{s-4} &\leq (1+2\mu)\|u_1 - u_2\|_{\infty, s} + \|F_1 - F_2\|_{\infty, s-4} \\ &\leq (1+2\mu)\|\psi_1 - \psi_2\|_s + [(1+2\mu)T + 1]\|F_1 - F_2\|_{\infty, s} \end{aligned} \quad (5)$$

where we have used the notation:

$$\|h\|_{\infty, r} = \sup_{t \in [0, T]} \|h(t)\|_r, \quad h \in C([0, T], H_{per}^r). \quad (6)$$

Now, we study the dissipative property of the homogeneous problem.

Let  $\mu > 0$ ,  $s \in R$  and the homogeneous problem

$$(P_c) \quad \left| \begin{array}{l} w \in C([0, \infty), H_{per}^s) \cap C^1([0, \infty), H_{per}^{s-4}) \\ \partial_t w + \partial_x^3 w + \mu(\partial_x^4 w + \partial_x^2 w) = 0 \in H_{per}^{s-4} \\ w(0) = \phi \in H_{per}^s. \end{array} \right.$$

**Theorem 2.2.** Let  $w$  the solution of  $(P_c)$  with initial data  $\phi \in H_{per}^s$ , then we obtain the following results:

1.  $\partial_t \|w(t)\|_{s-4}^2 = -2\mu < \partial_x^2 w(t) + \partial_x^4 w(t), w(t) >_{s-4} \leq 0$ .
2.  $\|w(t)\|_{s-4} \leq \|\phi\|_{s-4} \leq \|\phi\|_s$ ,  $t \geq 0$ .

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## GROTHENDIECK-TYPE SUBSETS OF BANACH LATTICES

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In the setting of Banach lattices the weak (resp. positive) Grothendieck spaces have been defined. We localize such notions by defining new classes of sets that we study and compare with some quite related different classes. This allows us to introduce and compare the corresponding linear operators. This talk corresponds the results in Sections 2 and 3 of the preprint [1].

## 1 Introduction

Recall that a Banach space  $X$  has the Grothendieck property if every weak\* null sequence in  $E'$  is weakly null. In the class of Banach lattices, by considering disjoint and positive sequences, two further Grothendieck properties have been considered. Following [2] (resp. [3]), a Banach lattice  $E$  has the *weak Grothendieck property* (resp. *positive Grothendieck property*) if every disjoint weak\* null sequence in  $E'$  is weakly null (resp. every positive weak\* null sequence in  $E'$  is weakly null). Of course the Grothendieck property implies both the positive and the weak Grothendieck properties. The lattice  $c$  of all real convergent sequences has the positive Grothendieck property but it fails to have the weak Grothendieck property [2, p. 10]. On the other hand,  $\ell_1$  is a Banach lattice with the weak Grothendieck property without the positive Grothendieck property [3, p. 6].

Recall that a subset  $A$  of a Banach space  $X$  is a *Grothendieck set* if  $T(A)$  is relatively weakly compact in  $c_0$  for each bounded linear operator  $T : X \rightarrow c_0$ . Keeping this  $c_0$ -valued operators point of view, we introduce and study a new class of sets in Banach lattices- that we name *almost Grothendieck* (see Definition 2.1)- and which characterizes the weak Grothendieck property. In an analogous way, the notion of *positive Grothendieck* set is defined.

## 2 Main Results

Every bounded linear operator  $T : E \rightarrow c_0$  is uniquely determined by a weak\* null sequence  $(x'_n) \subset E'$  such that  $T(x) = (x'_n(x))$  for all  $x \in E$  where  $x'_n$  is the  $n^{th}$  component of  $T$ . When this sequence is disjoint in the dual Banach lattice  $E'$ , we say that  $T$  is a *disjoint operator*.

**Definition 2.1.** *We say that  $A \subset E$  is an almost Grothendieck set if  $T(A)$  is relatively weakly compact in  $c_0$  for every disjoint operator  $T : E \rightarrow c_0$ .*

It is obvious that every Grothendieck set in a Banach lattice is almost Grothendieck. Obviously, each almost Grothendieck subset of  $c_0$  is relatively weakly compact. In particular, we can localize the weak Grothendieck property as follows:

**Proposition 2.1.** *For a Banach lattice  $E$ , the following are equivalent:*

1.  *$E$  has the weak Grothendieck property.*
2. *Every disjoint operator  $T : E \rightarrow c_0$  is weakly compact.*

3.  $B_E$  is an almost Grothendieck set.

By Proposition 2.1, we get that the unit ball of every L-space is an almost Grothendieck set, e.g.  $B_{\ell_1}$  and  $B_{L_1[0,1]}$  are almost Grothendieck sets that are not Grothendieck sets.

The question whether the solid hull of an almost Grothendieck set is still almost Grothendieck belongs to a type of questions usual in Banach lattice theory. In particular, we have the following results:

**Theorem 2.1.** *Let  $E$  be a Banach lattice with the property (d) and let  $A \subset E$ . Then  $|A| = \{|x| : x \in A\}$  is almost Grothendieck if and only if  $\text{sol}(A)$  is also almost Grothendieck.*

It follows immediately from Theorem 2.1 that if  $A \subset E^+$  is an almost Grothendieck set in a Banach lattice with the property (d), then  $\text{sol}(A)$  is also an almost Grothendieck set. By using this observation, we could establish conditions so that the solid hull of an almost Grothendieck set is also an almost Grothendieck set.

Recall that a bounded operator  $T : X \rightarrow Y$  is said to be *Grothendieck* if  $T(B_X)$  is a Grothendieck set in  $Y$ . Or, equivalently,  $T'y'_n \xrightarrow{\omega} 0$  in  $X'$  for every weak\* null sequence  $(y'_n) \subset Y'$ . In a natural way, we introduce the class of almost Grothendieck operators.

**Definition 2.2.** *A bounded operator  $T : X \rightarrow F$  is said to be almost Grothendieck if  $T'y'_n \xrightarrow{\omega} 0$  in  $X'$  for every disjoint weak\* null sequence  $(y'_n) \subset F'$ .*

It is immediate that every Grothendieck operator  $T : X \rightarrow F$  from a Banach space into a Banach lattice is almost Grothendieck. The converse does not hold though. For example, the identity map  $I_{\ell_1} : \ell_1 \rightarrow \ell_1$  is almost Grothendieck but not Grothendieck.

An characterization of almost Grothendieck operators concerning almost Grothendieck sets was proved as follows: A bounded linear operator  $T : X \rightarrow F$  is almost Grothendieck if and only if  $T(B_X)$  is an almost Grothendieck subset of  $F$ . As a consequence, we have that a Banach lattice  $F$  has the weak Grothendieck property if and only if every weakly compact operator from any Banach space  $X$  into  $F$  is almost Grothendieck.

Moreover, the following result gives necessary and sufficient conditions so that every almost Grothendieck set is relatively weakly compact.

**Theorem 2.2.** *For a Banach lattice  $E$ , every almost Grothendieck subset of  $E$  is relatively weakly compact if and only if every almost Grothendieck operator  $T : X \rightarrow E$  is weakly compact, for all Banach spaces  $X$ .*

In the class of the positive linear operators in Banach lattices, there is a dominated type problem. For instance, let  $S, T : E \rightarrow F$  be positive operators such that  $S \leq T$ . The question is, if  $T$  has some property (\*), does  $S$  also have it? We present condition under the Banach lattice  $F$  in order to get a positive answer when  $T$  is an almost Grothendieck operator.

By considering positive operators  $T : E \rightarrow c_0$  instead of disjoint operators in Definition 2.1, we define the positive Grothendieck sets. A study concerning this class of sets, the positive Grothendieck property and a class of related operators was made in an analogous way.

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LOWER BOUNDS FOR THE CONSTANTS IN THE REAL MULTIPOLYNOMIAL  
BOHNENBLUST-HILLE INEQUALITY

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**Abstract**

We extend to multipolynomials the method established by [3] and [1] for finding lower bounds for the constants in the multilinear and polynomial Bohnenblust–Hille inequalities for the case of real scalars.

## 1 Introduction

Given positive integers  $m$  and  $n_1, \dots, n_m$ , we say that a mapping  $P : E^m \rightarrow \mathbb{R}$  is an  $(n_1, \dots, n_m)$ -homogeneous polynomial if, for each  $i$  with  $1 \leq i \leq m$ , the mapping

$$P(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_m) : E \rightarrow \mathbb{R}$$

is an  $n_i$ -homogeneous polynomial for all fixed  $x_j \in E$  with  $j \neq i$ . Continuous multipolynomials are all those bounded over products of the unit ball  $B_E$  of  $E$ . In that case,

$$\|P\| := \sup \{|P(x_1, \dots, x_m)| : x_1, \dots, x_m \in B_E\}$$

defines a norm on the space of all continuous  $(n_1, \dots, n_m)$ -homogeneous polynomials from  $E^m$  into  $\mathbb{R}$ . As for the basics of the theory of multipolynomials between Banach spaces, we refer to [2, 4]. Similar to the polynomial case (see [1, p. 392]), one may show that every continuous  $(n_1, \dots, n_m)$ -homogeneous polynomial  $P : c_0 \times \dots \times c_0 \rightarrow \mathbb{R}$  can be written as

$$P(x_1, \dots, x_m) = \sum c_\alpha x_1^{\alpha_1} \cdots x_m^{\alpha_m}$$

for all  $x_1, \dots, x_m \in c_0$ , where  $c_\alpha \in \mathbb{R}$  and where the summation is taken over all matrices  $\alpha \in \mathbb{M}_{m \times \infty}(\mathbb{N}_0)$  such that  $|\alpha_i| = n_i$ , for each  $i$  with  $1 \leq i \leq m$ . The multipolynomial Bohnenblust–Hille inequality [4] for real scalars asserts that for all positive integers  $m$  and  $n_1, \dots, n_m$  there exists a constant  $C_M \geq 1$  such that

$$\left( \sum_{|\alpha_1|=n_1, \dots, |\alpha_m|=n_m} |c_\alpha|^{\frac{2M}{M+1}} \right)^{\frac{M+1}{2M}} \leq C_M \|P\|$$

for all continuous  $(n_1, \dots, n_m)$ -homogeneous polynomials  $P : c_0 \times \dots \times c_0 \rightarrow \mathbb{R}$ . In [3, Sec. 5], the best lower bound for the constants in the Bohnenblust–Hille inequality for  $m$ -linear forms is given by

$$C_m \geq 2^{\frac{m-1}{m}} \tag{1}$$

for every  $m \geq 2$ . As for the constants in the Bohnenblust–Hille inequality for  $m$ -homogeneous polynomials, the best-obtained estimate in [1, Theorem 2.2] is given by

$$D_{\mathbb{R},m} \geq \begin{cases} \left(3^{\frac{m}{2}}\right)^{\frac{m+1}{2m}} & \text{if } m \text{ is even} \\ \left(\frac{5}{4}\right)^{\frac{m}{2}} & \text{if } m \text{ is odd} \end{cases}$$

and

$$D_{\mathbb{R},m} \geq \frac{\left(4 \cdot 3^{\frac{m-1}{2}}\right)^{\frac{m+1}{2m}}}{2 \cdot \left(\frac{5}{4}\right)^{\frac{m-1}{2}}} \quad \text{if } m \neq 1 \text{ is odd.}$$

In any case, we have

$$D_{\mathbb{R},m} > (1.17)^m, \quad (2)$$

which holds, therefore, for every positive integer  $m > 1$ . In this work, we adapt the techniques due to [3] and [1] aiming to yield non-trivial lower bounds for  $C_M$ .

## 2 Main Results

In order to determine proper lower bounds for  $C_M$ , let us set a couple of notations. Let  $f$  and  $g$  denote the real-valued functions defined by means of the equations

$$f(n_i) = \begin{cases} 1 & , \text{ if } n_i = 1 \\ 3^{\frac{n_i}{2}} & , \text{ if } n_i \text{ is even} \\ 4 \cdot 3^{\frac{n_i-1}{2}} & , \text{ if } n_i \neq 1 \text{ is odd} \end{cases}$$

and

$$g(n_i) = \begin{cases} 1 & , \text{ if } n_i = 1 \\ \left(\frac{5}{4}\right)^{\frac{n_i}{2}} & , \text{ if } n_i \text{ is even} \\ 2 \cdot \left(\frac{5}{4}\right)^{\frac{n_i-1}{2}} & , \text{ if } n_i \neq 1 \text{ is odd} \end{cases}$$

for each  $i$  with  $1 \leq i \leq m$ .

### Theorem 2.1.

$$C_M \geq \frac{(4^{m-1} f(n_1) \cdots f(n_m))^{\frac{M+1}{2M}}}{2^{m-1} g(n_1) \cdots g(n_m)}$$

for all positive integers  $m$  and  $n_1, \dots, n_m$ .

We conclude this study by noting that the classical multilinear and polynomial estimates can be derived from this result. Indeed, it reduces to the best estimate (1) for the  $m$ -linear constants  $C_m$  when  $m > 1$  and  $n_1 = \dots = n_m = 1$ . An application of the theorem by assuming  $m = 1$  and then  $n_1 = m$ , on the other hand, yields the more accurate lower bound (2) for  $D_{\mathbb{R},m}$ .

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## TIGHTNESS IN BANACH SPACES WITH TRANSFINITE BASIS

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In this work we extend the definition of a tight space for Banach spaces with transfinite basis. We show some basic properties of tight transfinite basis and we prove that a Banach space with a tight transfinite basis fails to have minimal subspaces. Related open questions are discussed.

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## 1 Introduction

As part of the program of classification of Banach spaces up to subspaces initiated by W. T. Gowers [7], V. Ferenczi and Ch. Rosendal [1] introduced the notion of tightness and proved several dichotomies between different notions of tightness and minimality. Let  $X$  be a Banach space with normalized Schauder basis  $(x_n)_n$ . In [1] is defined a Banach space  $Y$  to be tight in the basis  $(x_n)_n$  if, and only if, there is a sequence of intervals  $I_0 < I_1 < \dots < I_k < \dots$  such that for every infinite subset  $A$  of  $\mathbb{N}$ , we have that  $Y$  is not isomorphically embedded in the closed span  $[x_n : n \notin \cup_{i \in A} I_i]$ .  $(x_n)_n$  is a tight basis for  $X$  if every Banach space  $Y$  is tight in  $(x_n)_n$ . A Banach space  $X$  is tight if it has a tight basis.

In [2] it was proved that a Banach space  $Y$  is tight in the basis  $(x_n)_n$  if, and only if, the set

$$\{u \subseteq \omega : Y \text{ is isomorphically embedded in } [x_n : n \in u]\} \quad (1)$$

is comeager in the Cantor space  $2^\omega$ , after the natural identification of  $\mathbb{P}(\omega)$  with  $2^\omega$ .

H. Rosenthal defined a separable Banach space to be minimal if it can be isomorphically embedded into any of its closed subspaces.

Tightness is hereditary by taking block subspaces meanwhile any subspace of a minimal space is always minimal. In [1] was proved that any shrinking basic sequence of a tight space is a tight basis and that in a reflexive Banach space every basic sequence is tight. Also, minimality and tightness are incompatible properties:

**Proposition 1.1** ([1]). *A tight Banach space with basis does not have minimal subspaces.*

The classical spaces  $\ell_p$ ,  $c_0$ , Schlumprecht space  $\mathcal{S}$  [1] are minimal, meanwhile Tsirelson's space  $\mathcal{T}$ , the  $p$ -convexification  $\mathcal{T}^p$  of the Tsirelson's space [5] and Gowers-Maurey unconditional space  $G_u$  [3] are examples of tight spaces (see [1] and [3]).

## 2 Main Results

We extend the notion of tightness from Banach spaces with Schauder basis to Banach spaces with transfinite basis as follows.

**Definition 2.1.** *Let  $\alpha$  be an infinite ordinal. Let  $X$  be a Banach space with transfinite basis  $(x_\gamma)_{\gamma < \alpha}$ . We say that a Banach space  $Y$  is tight in  $X$  if, and only if,*

$$E_Y := \{u \subseteq \alpha : Y \hookrightarrow [x_\gamma : \gamma \in u]\}$$

is meager in  $2^\alpha$ , after the natural identification of  $\mathbb{P}(\alpha)$  with  $2^\alpha$ . The basis  $(x_\gamma)_{\gamma < \alpha}$  is a tight transfinite basis for  $X$  if, and only if, any  $Y$  Banach space is tight in  $X$ .  $X$  is tight if it admits a tight transfinite basis.

It can be proved that the property of tightness for Banach spaces with transfinite basis is hereditary by taking transfinite block subspaces. Also, the following characterization is valid.

**Proposition 2.1.** *Let  $\alpha$  be an infinite ordinal and  $A \subseteq \mathbb{P}(\alpha)$ . The following assertions are equivalent:*

- (i)  *$A$  is comeager in  $2^\alpha$ ,*
- (ii) *there are a sequence  $(I_n)_{n < \omega}$  of non-empty finite pairwise disjoint subsets of  $\alpha$ , and subsets  $a_n \subseteq I_n$ , such that for any  $u \in 2^\alpha$ , if  $|\{n : I_n \cap u = a_n\}| = \aleph_0$ , then  $u \in A$ .*

Also, we prove that

**Proposition 2.2.** *If  $(x_\gamma)_{\gamma < \alpha}$  is a tight shrinking transfinite basic sequence, and  $(\gamma_n)_n$  is an increasing sequence of ordinals in  $\alpha$ , then every basic sequence in  $[x_{\gamma_n}]_n$  is tight.*

In particular, the thesis of the last proposition holds if  $X$  is a reflexive Banach space with transfinite basis  $(x_\gamma)_{\gamma < \alpha}$ . For spaces with transfinite basis, tightness and minimality are also incompatible properties.

**Theorem 2.1.** *Let  $X$  be a Banach space with a tight transfinite basis, then  $X$  does not have any separable minimal subspace.*

We discuss the existence of examples of transfinite tight Banach spaces and give some open questions.

This work is part of the doctorate thesis under the supervision of the professor Valentin Ferenczi.

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RESULTS ON THE FRÉCHET SPACE  $\mathcal{H}_L(B_E)$ LUIZA A. MORAES<sup>1</sup> & ALEX F. PEREIRA<sup>2</sup><sup>1</sup>Instituto de Matemática, UFRJ, RJ, Brasil, luiza@im.ufrj.br,<sup>2</sup>Instituto de Matemática e Estatística, UFF, RJ, Brasil, alexpereira@id.uff.br**Abstract**

For a complex Banach algebra  $E$ , in this work we study topological properties of the space  $\mathcal{H}_L(B_E)$  of the mappings  $f : B_E \rightarrow E$  that are analytic in the sense of Lorch endowed with the topology  $\tau_b$  where  $B_E$  denote the open unit ball in  $E$ . Also, we show that  $\mathcal{H}_L(B_E)$  is homeomorphic to some sequence space in  $E$ .

## 1 Introduction

For a commutative Banach algebra  $E$  let  $B_E$  be the open unit ball of  $E$ . We denote by  $\Gamma(B_E)$  the set of the sequences  $(a_n)_n$  in  $E$  that satisfy  $\limsup \|a_n\|^{1/n} \leq 1$ . We consider  $\Gamma(B_E)$  endowed with the topology  $\tau$  generated by the family of the seminorms  $\|a\|_r = \sup \|a_n\| r^n$  for all  $a = (a_n)_n \in \Gamma(B_E)$  and for all  $0 < r < 1$ . It is easy to see that  $(\Gamma(B_E), \tau)$  is a locally convex space with the usual operations.

We say that  $f : B_E \rightarrow E$  is Lorch analytic in  $B_E$  if and only if there exist unique sequence  $(a_n)_n \in \Gamma(B_E)$  such that  $f(w) = \sum_{n=0}^{\infty} a_n w^n$  for all  $w \in B_E$ . We denote by  $\mathcal{H}_L(B_E)$  the space of Lorch analytic mappings from  $B_E$  into  $E$ . For more information about Lorch analytic mappings we refer to [2]. It is clear that  $\mathcal{H}_L(B_E) \subset \mathcal{H}_b(B_E, E)$  where  $\mathcal{H}_b(B_E, E)$  denotes the space of holomorphic mappings from  $B_E$  into  $E$  which are bounded on the bounded subsets of  $B_E$ . For background on holomorphic mappings between Banach spaces see [1, 7]. Note that we can consider in  $\mathcal{H}_L(B_E)$  the topology  $\tau_b$  of the uniform convergence on the bounded subsets of  $B_E$ .

## 2 Main Results

For  $n \in \mathbb{N}_0$  we denote by  $\mathcal{P}_L(^n E)$  the space of the  $n$ -homogeneous polynomials from  $E$  into  $E$  which are Lorch analytic in  $B_E$  with the usual topology. The proofs of the propositions below can be found in [3].

**Proposition 2.1.** *The following statements are true:*

- (a)  $\{\mathcal{P}_L(^n E)\}_{n \in \mathbb{N}_0}$  is an 1-Schauder decomposition of  $(\mathcal{H}_L(B_E), \tau_b)$ .
- (b)  $\{\mathcal{P}_L(^n E)\}_{n \in \mathbb{N}_0}$  is an  $\mathcal{S}$ -absolute decomposition of  $(\mathcal{H}_L(B_E), \tau_b)$ .
- (c)  $\{\mathcal{P}_L(^n E)\}_{n \in \mathbb{N}_0}$  is shrinking.
- (d)  $\{\mathcal{P}_L(^n E)\}_{n \in \mathbb{N}_0}$  is boundedly complete.

**Proposition 2.2.**  *$E$  has the Schur property if and only if  $(\mathcal{H}_L(B_E), \tau_b)$  has the Schur property.*

**Proposition 2.3.**  *$E$  is separable if and only if  $(\mathcal{H}_L(B_E), \tau_b)$  is separable.*

**Proposition 2.4.**  *$E$  is reflexive if and only if  $(\mathcal{H}_L(B_E), \tau_b)$  is reflexive.*

**Proposition 2.5.**  *$(\mathcal{H}_L(B_E), \tau_b)$  is a Fréchet space.*

It is clear by the definition of  $\Gamma(B_E)$  that we have a natural linear bijection between  $\Gamma(B_E)$  and  $\mathcal{H}_L(B_E)$ .

**Theorem 2.1.**  *$(\Gamma(B_E), \tau)$  e  $(\mathcal{H}_L(B_E), \tau_b)$  are isomorphics. In particular,  $(\Gamma(B_E), \tau)$  is a Fréchet space.*

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# SOBRE UMA REFORMULAÇÃO DA HIPÓTESE DE RIEMANN NO ESPAÇO DE HARDY DO CÍRCULO UNITÁRIO

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## Abstract

Este trabalho busca avançar o esforço de pesquisa iniciado pela versão recente do critério de Nyman-Beurling-Báez-Duarte para a hipótese de Riemann (RH, da sigla em inglês) no espaço de Hardy-Hilbert do disco unitário,  $H^2$ . Questões de densidade e ortogonalidade diretamente atreladas a este critério são abordadas, o que leva a versões fracas de RH. Entre as ferramentas utilizadas, se destacam vários espaços de Hilbert de funções holomorfas no disco unitário. Trabalho em colaboração com J. C. Manzur.

## 1 Introdução

A hipótese de Riemann é a afirmação de que a função definida por  $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$  para  $\operatorname{Re} s > 1$  e estendida analiticamente a  $\mathbb{C} \setminus \{1\}$  não possui zeros com parte real maior do que  $1/2$ . Nyman [4] obteve uma reformulação de RH em termos de densidade e aproximação em  $L^1(0, 1)$ , resultado generalizado para  $L^p(0, 1)$  por Beurling [2] e refinado por Báez-Duarte [1]. Detalhes podem ser encontrados no artigo expositório [2]. Em [5], é encontrada a seguinte versão unitariamente equivalente do critério de Báez-Duarte. O contexto é uma classe de funções holomorfas no disco unitário  $\mathbb{D}$ , a saber o espaço de Hardy-Hilbert

$$H^2 = \left\{ f : \mathbb{D} \rightarrow \mathbb{C} : f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n, \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty \right\},$$

que é um espaço de Hilbert com a norma  $\|f\| = \left( \sum_{n=0}^{\infty} |\hat{f}(n)|^2 \right)^{1/2}$  e contém o espaço  $H^\infty$  das funções holomorfas limitadas em  $\mathbb{D}$ . Toda  $f \in H^2$  possui limites radiais em quase todo ponto do círculo unitário  $\mathbb{T}$ , o que identifica  $H^2$  com um subespaço fechado de  $L^2(\mathbb{T})$ .

**Teorema 1.1.** *Seja  $\mathcal{N}$  o espaço vetorial gerado por  $\{h_k : k \geq 2\}$ , onde*

$$h_k(z) = \frac{1}{1-z} \log \left( \frac{1+z+\dots+z^{k-1}}{k} \right), \quad z \in \mathbb{D}, \quad k \geq 2.$$

*Então a hipótese de Riemann é verdadeira se e somente se  $\mathcal{N}$  é denso em  $H^2$ , o que ocorre se e somente se a constante 1 está no fecho de  $\mathcal{N}$ .*

Este trabalho fortalece resultados de [5] dando respostas parciais aos problemas de encontrar (i) topologias em  $H^2$  com respeito às quais  $\mathcal{N}$  é denso; (ii) subespaços vetoriais  $V \subset H^2$  tais que  $\mathcal{N}^\perp \cap V = \{0\}$ . Tais respostas parciais podem ser interpretadas como versões fracas da hipótese de Riemann, ou seja, afirmações que são implicadas por RH mas demonstradas verdadeiras incondicionalmente.

## 2 Resultados Principais

Uma função  $f \in H^2$  é exterior se  $\{z^n f : n \geq 0\}$  gera um subespaço denso em  $H^2$ . Funções exteriores não se anulam no disco. A classe de Smirnov é

$$\mathcal{N}^+ = \{g/h : g, h \in H^\infty, h \text{ é exterior}\}.$$

Em [8] é estudada a topologia induzida em  $N^+$  pela métrica

$$d(f, g) = \frac{1}{2\pi} \int_0^{2\pi} \log(1 + |f(e^{i\theta}) - g(e^{i\theta})|) d\theta,$$

que é mais fraca que a topologia da norma e mais forte que a da convergência uniforme em compactos.

**Teorema 2.1.** *Com respeito à métrica  $d$ ,  $\mathcal{N}$  é denso em  $N^+$ , e portanto em  $H^2$ .*

Dado  $\xi \in \mathbb{T}$ , o espaço local de Dirichlet em  $\xi$  é

$$\mathcal{D}_\xi = \left\{ f : \mathbb{D} \rightarrow \mathbb{C} : f \text{ é holomorfa, } \int_{\mathbb{D}} |f'(z)|^2 \frac{1 - |z|^2}{|\xi - z|^2} dA(z) < \infty \right\},$$

onde  $dA$  é a medida de área. Temos que  $\mathcal{D}_\xi$  coincide com  $\{(\xi - z)f + c : f \in H^2, c \in \mathbb{C}\}$  (ver [4]) e contém todas as funções holomorfas em vizinhanças do fecho de  $\mathbb{D}$ . Usando resultados de [7] relacionando operadores de multiplicação ilimitados em  $H^2$  e espaços de Branges-Rovnyak, é possível provar o seguinte.

**Teorema 2.2.** *Para todo  $\xi \in \mathbb{T}$ ,  $\mathcal{N}^\perp \cap \mathcal{D}_\xi = \{0\}$ .*

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**TEOREMAS DO TIPO BANACH-STONE PARA ÁLGEBRAS DE GERMES HOLOMORFOS EM  
ESPAÇOS DE BANACH**

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**Abstract**

Neste trabalho, estudamos resultados do tipo Banach-Stone para álgebras de germes holomorfos em espaços de Banach. Mostramos que se  $K$  e  $L$  são subconjuntos compactos, equilibrados e determinantes em espaços de Banach separáveis com propriedade de aproximação, então as álgebras de germes holomorfos  $\mathcal{H}(K)$  e  $\mathcal{H}(L)$  são topologicamente isomórfas se, e somente se, as envoltórias polinomialmente convexas  $\hat{\mathcal{K}}_{\mathcal{P}}$  e  $\hat{\mathcal{L}}_{\mathcal{P}}$  são biholomorficamente equivalentes.

## 1 Introdução

Se  $K$  e  $L$  são espaços topológicos Hausdorff compactos, o Teorema de Banach-Stone clássico afirma que os espaços  $\mathcal{C}(K)$  e  $\mathcal{C}(L)$  são isométricos se, e somente se,  $K$  e  $L$  são homeomorfos. Mais especificamente, se  $T : \mathcal{C}(K) \rightarrow \mathcal{C}(L)$  é uma isometria, então existem um homeomorfismo  $\varphi : L \rightarrow K$  e uma função contínua  $\alpha : L \rightarrow \mathbb{K}$  com  $|\alpha(y)| = 1$ , para todo  $y \in L$  tais que:  $T(f)(y) = \alpha(y) \cdot (f \circ \varphi)(y) = \alpha(y) \cdot f(\varphi(y))$ , para todo  $y \in L$  e para toda  $f \in \mathcal{C}(K)$ .

A versão desse teorema para isomorfismos algébricos foi provada por Gelfand & Kolmogoroff em 1939 e afirma que  $\mathcal{C}(K)$  e  $\mathcal{C}(L)$  são isomórfas como álgebras se, e somente se,  $K$  e  $L$  são homeomorfos. Além disso, todo isomorfismo algébrico  $T : \mathcal{C}(K) \rightarrow \mathcal{C}(L)$  é da forma  $T(f) = f \circ \varphi$ , onde  $\varphi : L \rightarrow K$  é um homeomorfismo.

Neste trabalho, investigamos resultados deste tipo em álgebras de germes holomorfos em espaços de Banach. A seguir, daremos algumas definições.

Sejam  $E$  um espaço de Banach complexo,  $U$  um subconjunto aberto de  $E$ . Denotamos por:  $\mathcal{H}(U)$  o espaço vetorial de todas as funções holomorfas  $f : U \rightarrow \mathbb{C}$  e  $\tau_{\omega}$  a topologia de Nachbin (portada por compactos) em  $\mathcal{H}(U)$ . Desta forma,  $(\mathcal{H}(U), \tau_{\omega})$  é uma álgebra localmente m-convexa.

Se  $K$  é um subconjunto compacto de  $E$ , denotamos por  $h(K) = \cup_{U \supset K} \mathcal{H}(U)$ , onde  $U$  percorre todos os abertos de  $E$  que contém  $K$ . Diremos que duas funções  $f_1, f_2 \in h(K)$  são **equivalentes** ( $f_1 \sim f_2$ ) se elas coincidirem em alguma vizinhança aberta de  $K$ . Denotamos por  $\mathcal{H}(K)$  ao conjunto de todas as classes de equivalências de funções que são holomorfas em alguma vizinhança de  $K$ . Cada elemento de  $\mathcal{H}(K)$  é chamado de **germe holomorfo em  $K$** . As aplicações canônicas  $I_U : \mathcal{H}(U) \hookrightarrow \mathcal{H}(K)$ , com  $U \supset K$ , induzem uma estrutura de espaço vetorial em  $\mathcal{H}(K)$ . O espaço vetorial  $\mathcal{H}(K)$  é então munido da topologia induativa com respeito às aplicações lineares canônicas  $I_U : (\mathcal{H}(U), \tau_{\omega}) \rightarrow \mathcal{H}(K)$ , com  $U \supset K$ . Desta forma, dizemos que  $\mathcal{H}(K)$  é o limite induutivo dos espaços  $(\mathcal{H}(U), \tau_{\omega})$ , com  $U \supset K$ . Tem-se que  $(\mathcal{H}(K), \tau_{\omega})$  é uma álgebra localmente m-convexa.

A **envoltória polinomialmente convexa de  $K$**  é definida por

$$\hat{\mathcal{K}}_{\mathcal{P}(E)} = \{x \in E : |P(x)| \leq \sup_K |P|, \text{ para todo } P \in \mathcal{P}(E)\}.$$

Um compacto  $K$  é **polinomialmente convexo** se  $\hat{\mathcal{K}}_{\mathcal{P}(E)} = K$ . Dizemos que um subconjunto compacto  $K$  de um espaço de Banach  $E$  é **determinante** se  $f \in \mathcal{H}(U)$  é tal que  $f|_K = 0$  então existe uma vizinhança  $V \supset K$ ,  $K \subset V \subset U$  tal que  $f|_V = 0$ . Um espaço de Banach  $E$  possui um compacto determinante se, e somente se,  $E$  é separável.

Sejam  $E$  e  $F$  espaços de Banach, e sejam  $K \subset E$  e  $L \subset F$  subconjuntos compactos. Dizemos que  $K$  e  $L$  são **biholomorficamente equivalentes** se existem abertos  $U$  e  $V$  com  $K \subset U \subset E$  e  $L \subset V \subset F$ , e uma aplicação  $\varphi : V \rightarrow U$  biholomorfa com  $\varphi(L) = K$ .

## 2 Resultados Principais

Nosso principal resultado é o seguinte teorema.

**Teorema 2.1.** *Sejam  $E$  e  $F$  espaços de Banach, ambos separáveis e com a propriedade de aproximação, e sejam  $K \subset E$  e  $L \subset F$  subconjuntos compactos, equilibrados e determinantes. Então as seguintes afirmações são equivalentes:*

- (1)  $\mathcal{H}(K)$  e  $\mathcal{H}(L)$  são topologicamente isomórfas como álgebras.
- (2)  $\widehat{K}_{\mathcal{P}(E)}$  e  $\widehat{L}_{\mathcal{P}(F)}$  são biholomorficamente equivalentes.

Esse teorema está relacionado com um resultado semelhante de [3], para compactos equilibrados em espaços de Banach do tipo Tsirelson. O Teorema 2.1 melhora o resultado de [3] para uma classe muito mais ampla de espaços de Banach, porém a classe de compactos é reduzida aos compactos equilibrados e determinantes.

A demonstração do Teorema 2.1 está baseada em resultados de [5], técnicas de [1, 2, 3], além da seguinte proposição:

**Proposition 2.1** (J. Mujica, D.M.V., 2017). *Sejam  $E$  e  $F$  espaços de Banach, ambos com a propriedade de aproximação, e sejam  $K \subset E$  e  $L \subset F$  subconjuntos compactos e polinomialmente convexos. Se  $\mathcal{H}(K)$  e  $\mathcal{H}(L)$  são topologicamente isomórfas como álgebras, então  $K$  e  $L$  são homeomorfos.*

Como consequência do Teorema 2.1, temos o seguinte corolário:

**Corollary 2.1.** *Sejam  $E$  e  $F$  espaços de Banach, ambos separáveis e com a propriedade de aproximação, e sejam  $K \subset E$  e  $L \subset F$  subconjuntos compactos, equilibrados, determinantes e polinomialmente convexos. Então  $\mathcal{H}(K)$  e  $\mathcal{H}(L)$  são topologicamente isomórfas como álgebras se, e somente se,  $K$  e  $L$  são biholomorficamente equivalentes.*

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## OPERADORES MULTILINEARES SOMANTES E CLASSES DE SEQUÊNCIAS

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Construímos neste trabalho um aparato geral que recupera como casos particulares todas as classes de operadores absolutamente somantes definidos ou caracterizados por transformação de sequências vetoriais já estudadas na literatura.

## 1 Introdução

O estudo de classes de operadores multilineares que generalizam os ideais dos operadores lineares  $(p; q)$ -somantes tem se desenvolvido nas últimas décadas seguindo várias linhas, entre elas: (i) estudar as classes de operadores que podem ser definidas ou caracterizadas por meio de transformação de sequências vetoriais; (ii) considerar as várias formas de se percorrer o conjunto de índices na definição dos operadores, como por exemplo, somar apenas na diagonal de  $\mathbb{N}^d$  (operadores absolutamente somantes) ou somar em todos os índices de  $\mathbb{N}^d$  (operadores múltiplo somantes); (iii) considerar conjuntos de índices intermediários à diagonal e a  $\mathbb{N}^d$  e, por fim, (iv) considerar somas iteradas sobre conjuntos de índices de  $\mathbb{N}^d$  (caso anisotrópico).

O objetivo deste trabalho é introduzir um conceito que unifica todas essas linhas que foram estudadas separadamente. Cada um dos casos até agora já estudados será um caso particular do conceito aqui introduzido.

Usaremos a noção de classe de sequências vetoriais, introduzido em [4]. Assim, dados uma classe de sequências  $X$  e um espaço de Banach  $E$ ,  $X(E)$  será um espaço de sequências a valores em  $E$ , de acordo com [4]. As letras  $X_1, \dots, X_d, Y_1, \dots, Y_k$  denotarão classe de sequências vetoriais, onde  $d$  e  $k$  são números naturais fixos com  $1 \leq k \leq d$ .

## 2 Resultados Principais

Usaremos as letras  $E_1, \dots, E_d, F$  para denotar espaços de Banach, denotaremos por  $\mathcal{I} = \{I_1, \dots, I_k\}$  uma partição do conjunto  $\{1, \dots, d\}$ , ou seja, uma classe de subconjuntos de  $\{1, \dots, d\}$  dois a dois disjuntos cuja união é igual a  $\{1, \dots, d\}$ , e por  $x * e_j$  entenderemos a  $d$ -upla  $(0, \dots, 0, x, 0, \dots, 0)$  com  $x$  na coordenada  $j$  e 0 nas demais, seja quando  $x$  pertencer a um espaço de Banach seja quando  $x$  for um número natural.

**Definition 2.1.** Fixadas uma partição  $\mathcal{I} = \{I_1, \dots, I_k\}$  e  $d$  sequências de números naturais  $(j_n^r)_{n=1}^\infty$ ,  $r = 1, \dots, d$ , tais que a correspondência  $(n_1, \dots, n_k) \in \mathbb{N}^k \mapsto \sum_{s=1}^k \sum_{r \in I_s} j_{n_s}^r * e_r$  é injetiva, o *bloco de  $\mathbb{N}^d$  associado à partição  $\mathcal{I}$  e às sequências*  $(j_n^r)_{n=1}^\infty$ ,  $r = 1, \dots, d$ , é definido por  $B_{\mathcal{I}} = \left\{ \sum_{s=1}^k \sum_{r \in I_s} j_{n_s}^r * e_r \in \mathbb{N}^d : n_1, \dots, n_k \in \mathbb{N} \right\}$ .

A notação  $Y_1(Y_2(F))$  refere-se a todas as sequências em  $Y_2(F)$  que pertencem a classe  $Y_1$ . Assim, iteradamente, construímos o espaço  $Y_1(\cdots Y_k(F) \cdots)$ .

**Definition 2.2.** Um operador  $d$ -linear  $A: E_1 \times \cdots \times E_d \rightarrow F$  é dito *parcialmente  $(B_{\mathcal{I}}; X_1, \dots, X_d; Y_1, \dots, Y_k)$ -somante* se  $\left( \cdots \left( A \left( \sum_{s=1}^k \sum_{r \in I_s} x_{j_{n_s}}^r * e_r \right) \right)_{n_k=1}^\infty \right)_{n_1=1}^\infty \in Y_1(\cdots Y_k(F) \cdots)$  para quaisquer sequências  $(x_j^r)_{j=1}^\infty \in X_r(E_r)$ ,  $r = 1, \dots, d$ . Por  $\mathcal{L}_{X_1, \dots, X_d; Y_1, \dots, Y_k}^{B_{\mathcal{I}}}(E_1, \dots, E_d; F)$  denotamos a classe formada por todos esses operadores.

**Proposition 2.1.** Um operador  $d$ -linear  $A: E_1 \times \dots \times E_d \rightarrow F$  é parcialmente  $(B_{\mathcal{I}}; X_1, \dots, X_d; Y_1, \dots, Y_k)$ -somante se, e somente se, o operador induzido  $\widehat{A}_{B_{\mathcal{I}}}: X_1(E_1) \times \dots \times X_d(E_d) \rightarrow Y_1(\dots Y_k(F) \dots)$ , definido por

$$\widehat{A}_{B_{\mathcal{I}}}((x_j^1)_{j=1}^\infty, \dots, (x_j^d)_{j=1}^\infty) = \left( \dots \left( A \left( \sum_{s=1}^k \sum_{r \in I_s} x_{j_n^r}^r * e_r \right) \right)_{n_k=1}^\infty \dots \right)_{n_1=1}^\infty,$$

está bem definido, é  $d$ -linear e contínuo. Neste caso define-se a norma parcialmente  $(B_{\mathcal{I}}; X_1, \dots, X_d; Y_1, \dots, Y_k)$ -somante de  $A$  por

$$\|A\|_{B_{\mathcal{I}}; X_1, \dots, X_d; Y_1, \dots, Y_k} := \|\widehat{A}_{B_{\mathcal{I}}}\|.$$

**Definition 2.3.** Dizemos que uma  $d+k$ -upla de classes de sequências  $(X_1, \dots, X_d, Y_1, \dots, Y_k)$  é  $B_{\mathcal{I}}$ -compatível se o operador  $I_{\mathbb{K}^d}: \mathbb{K}^d \rightarrow \mathbb{K}$  definido por  $I_{\mathbb{K}^d}(\lambda^1, \dots, \lambda^d) = \prod_{r=1}^d \lambda^r$  é parcialmente  $(B_{\mathcal{I}}; X_1, \dots, X_d; Y_1, \dots, Y_k)$ -somante com  $\|I_{\mathbb{K}^d}\|_{B_{\mathcal{I}}; X_1, \dots, X_d; Y_1, \dots, Y_k} = 1$ .

**Theorema 2.1.** Sejam  $B_{\mathcal{I}} \subseteq \mathbb{N}^d$  um bloco associado à partição  $\mathcal{I}$  e às sequências  $(j_n^r)_{n=1}^\infty$ ,  $r = 1, \dots, d$ , e  $X_1, \dots, X_d, Y_1, \dots, Y_k$  classes de sequências linearmente estáveis que são  $B_{\mathcal{I}}$ -compatíveis. Então  $(\mathcal{L}_{X_1, \dots, X_d; Y_1, \dots, Y_k}^{B_{\mathcal{I}}}, \|\cdot\|_{B_{\mathcal{I}}; X_1, \dots, X_d; Y_1, \dots, Y_k})$  é um ideal de Banach de operadores multilinearares.

Neste ambiente recuperamos todas as classes que foram recuperadas em [5] e também outras classes que lá não eram recuperadas, em particular o caso geral dos operadores estudados em [4]. Vejamos alguns exemplos concretos.

**Exemplo 2.1.** A diagonal  $D(\mathbb{N}^d) := \{(j, \dots, j) : j \in \mathbb{N}\}$  em  $\mathbb{N}^d$  é um bloco associado à partição trivial  $\mathcal{I} = \{\{1, \dots, d\}\}$  e às sequências  $(j_n^r)_{n=1}^\infty = (n)_{n=1}^\infty$ ,  $r = 1, \dots, d$ . Os operadores parcialmente  $(D(\mathbb{N}^d), \ell_{p_1}^w(\cdot), \dots, \ell_{p_d}^w(\cdot); \ell_q(\cdot))$ -somantes são exatamente os operadores absolutamente  $(q; p_1, \dots, p_d)$ -somantes de [1].

**Exemplo 2.2.** Seja  $B_{\mathcal{I}}$  o bloco associado à partição  $\mathcal{I} = \{I_1, \dots, I_k\}$  e às sequências de números naturais  $(j_n^r)_{n=1}^\infty = (n)_{n=1}^\infty$ . Os operadores que são parcialmente  $(B_{\mathcal{I}}; \ell_{p_1}^w(\cdot), \dots, \ell_{p_d}^w(\cdot); \ell_{q_1}(\cdot), \dots, \ell_{q_k}(\cdot))$ -somantes são exatamente os operadores que são  $\mathcal{I}$ -parcialmente múltiplo  $(\mathbf{q}; \mathbf{p})$ -somante com  $(\mathbf{q}; \mathbf{p}) := (p_1, \dots, p_d, q_1, \dots, q_k) \in [1, \infty)^{d+k}$  estudados em [2].

**Exemplo 2.3.** Considere  $\mathbb{N}^d$  como o bloco associado à partição  $\mathcal{I} = \{I_1, \dots, I_d\}$ , com  $I_r = \{r\}$ , e às sequências  $(j_n^r)_{n=1}^\infty = (n)_{n=1}^\infty$ ,  $r = 1, \dots, d$ . Então a classe dos operadores que são parcialmente  $(\mathbb{N}^d; \ell_{p_1}^w(\cdot), \dots, \ell_{p_d}^w(\cdot); \ell_q(\cdot), \dots, \ell_q(\cdot))$ -somantes coincide com a classe de operadores que são múltiplo  $(q; p_1, \dots, p_d)$ -somantes (ver [2, 3]).

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## PROPRIEDADE DE SCHUR POLINOMIAL POSITIVA

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Definimos e demonstramos alguns resultados sobre a *propriedade de Schur polinomial positiva*. Essa propriedade surge como um análogo, no ambiente de reticulados de Banach, à propriedade de Schur polinomial ( $\Lambda$ -espaço) em espaços de Banach.

**1 Introdução**

O estudo da propriedade de Schur polinomial teve início em 1989 com o célebre artigo de Carne, Cole e Gamelin [2], trabalho no qual foi apresentado o conceito de  $\Lambda$ -espaço e foram apresentados os primeiros resultados sobre esse tipo de espaço de Banach. Posteriormente esse conceito foi abordado por outros matemáticos (veja [3, 4, 5]) e os termos *espaço polinomialmente de Schur* e espaço com a *propriedade de Schur polinomial* passaram a ser empregados como sinônimos de  $\Lambda$ -espaço.

**Definição 1.1.** Um espaço de Banach  $X$  tem a *propriedade de Schur polinomial* se toda sequência polinomialmente nula em  $X$  é nula em norma.

é natural ponderar sobre uma versão análoga à propriedade de Schur polinomial em reticulados de Banach que leve em consideração as peculiaridades advindas da estrutura de ordem. Isso nos motiva a introduzir a seguinte definição:

**Definição 1.2.** Um reticulado de Banach  $E$  tem a *propriedade de Schur polinomial positiva* se toda sequência positiva  $(x_j)_{j=1}^\infty$  em  $E$  tal que  $P(x_j) \rightarrow 0$  para todo polinômio homogêneo regular  $P$  em  $E$  é nula em norma. Um reticulado de Banach com a propriedade de Schur polinomial positiva será chamado de *positivamente polinomialmente de Schur (PPS)*.

**2 Resultados Principais**

A seguir listamos alguns exemplos e resultados sobre reticulados de Banach positivamente polinomialmente de Schur, os quais podem ser encontrados em [1].

**Exemplo 2.1.** (a) Todo reticulado de Banach  $E$  com a propriedade de Schur positiva é PPS.

(b)  $L_1[0, 1]$  é um reticulado de Banach PPS que não tem a propriedade de Schur polinomial.

(c) O reticulado de Banach  $\left(\bigoplus_{n \in \mathbb{N}} \ell_\infty^n\right)_1$  é PPS, pois tem a propriedade de Schur positiva, e não é um AL-espaço.

**Proposição 2.1.** Seja  $E$  um reticulado de Banach com a propriedade de Dunford-Pettis e sem a propriedade de Schur positiva. Então  $E$  não é PPS. Em particular, AM-espaços não são PPS.

**Exemplo 2.2.**  $C(K)$ -espaços, em particular  $c_0$ , não são PPS.

A propriedade de Schur polinomial positiva é herdada por subreticulados fechados e preservada por isomorfismo de reticulados, conforme a proposição a seguir.

**Proposição 2.2.** (a) Se  $F$  é um reticulado de Banach positivamente isomorfo a um subespaço de um reticulado de Banach PPS, então  $F$  é PPS.

(b) Se dois reticulados de Banach são isomorfos como reticulados e um deles é PPS, então o outro também será PPS.

(c) Subreticulados fechados de reticulados de Banach PPS são PPS.

A seguir vemos que reticulados de Banach PPS gozam de boas propriedades.

**Proposição 2.3.** Todo reticulado de Banach PPS é um KB-espaco, consequentemente, tem norma ordem-contínua, é fracamente sequencialmente completo e é Dedekind completo.

O próximo exemplo nos mostra que não vale a recíproca da proposição anterior.

**Exemplo 2.3.** O espaço de Tsirelson original  $T^*$  é um KB-espaco que não é PPS.

Notamos que, na realidade, os espaços  $L_p(\mu)$  gozam de uma propriedade mais forte do que ser PPS.

**Definição 2.1.** Dado  $n \in \mathbb{N}$ , um reticulado de Banach  $E$  tem a *propriedade de Schur n-polinomial positiva* se toda sequência  $(x_j)_{j=1}^\infty$  em  $E$  fracamente nula e positiva tal que  $P(x_j) \rightarrow 0$  para todo polinômio  $n$ -homogêneo regular  $P$  em  $E$  é nula em norma. Nesse caso diremos que  $E$  é um reticulado de Banach  $n$ -PPS.

**Teorema 2.1.** Sejam  $1 \leq p < \infty$  e  $\mu$  uma medida qualquer. O reticulado de Banach  $L_p(\mu)$  é  $n$ -PPS para todo  $n \geq p$ .

Esse teorema nos apresenta exemplos de reticulados de Banach PPS que não possuem a propriedade de Schur positiva.

**Corolário 2.1.** Todos os reticulados de Hilbert e  $\ell_p$ ,  $1 \leq p < \infty$ , são PPS.

Para finalizar, observamos que a propriedade de Schur polinomial positiva possui uma interessante relação com as propriedades de Schur positiva e de Dunford-Pettis fraca, conforme teorema a seguir.

**Teorema 2.2.** Um reticulado de Banach tem a propriedade de Schur positiva se, e somente se, ele tem as propriedades de Dunford-Pettis fraca e de Schur polinomial positiva.

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## A SEMIGROUP RELATED TO THE RIEMANN HYPOTHESIS

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### **Abstract**

According to S. Waleed Noor, the cyclic vector of a semigroup of weighted composition operators are intimately related to the Riemann hypothesis. In this work we focus on the analysis of this semigroup. In particular, a new reformulation for the Riemann hypothesis says that the study of invariant subspaces of any element of this semigroup are also related to this conjecture. We also provide a generalization for the Băez-Duarte criterion in  $H^2$  through cyclic vectors.

### 1 Introduction

The Riemann hypothesis is a famous open problem, which says that all the non-trivial zeros of the  $\zeta$ -function lie on the vertical line with real part  $1/2$ . This conjecture is considered to be the most important unsolved problem in mathematics.

In 1950, [2, 4], Nyman and Beurling gave a reformulation for this problem: they proved that the Riemann hypothesis holds if and only if the constant 1 belongs to the closure linear span of  $\{f_\lambda : 0 < \lambda \leq 1\}$  in  $L^2(0, 1)$ , where  $f_\lambda(x) = \{\lambda/x\} - \lambda\{1/x\}$ ; here  $\{x\}$  denotes the fractional part of a real number  $x$ . In 2003, [1], Băez-Duarte showed a stronger version: the family  $\{f_\lambda : 0 < \lambda \leq 1\}$  was replaced by the countable family  $\{f_{1/k} : k \geq 1\}$ . Recently, S. Waleed Noor, [7], gave the  $H^2$  version of the Băez-Duarte reformulation:

**Theorem 1.1.** *For each  $k \geq 2$ , define*

$$h_k(z) = \frac{1}{1-z} \log \left( \frac{1+z+\cdots+z^{k-1}}{k} \right).$$

*Then the Riemann hypothesis holds if and only if the constant 1 belongs to the closure linear span of  $\{h_k : k \geq 2\}$  in  $H^2$ .*

S. Waleed Noor also constructed a semigroup  $\{W_n : n \geq 1\}$  on  $H^2$ , where  $W_n f(z) = (1+z+\cdots+z^{n-1})f(z^n)$ , of weighted composition operators having a closed relation with the Riemann hypothesis. He showed that the constant 1 appearing in Theorem 1.1 may be replaced by any cyclic vector of  $\{W_n : n \geq 1\}$ . So the generalization of Theorem 1.1 was stated as follows.

**Theorem 1.2.** *The following statements are equivalents:*

- 1) *Riemann hypothesis,*
- 2) *the closed linear span of  $\{h_k : k \geq 2\}$  contains a cyclic vector of  $\{W_n : n \geq 1\}$ ,*
- 3) *the closed linear span of  $\{h_k : k \geq 2\}$  is dense in  $H^2$ .*

This semigroup  $\{W_n : n \geq 1\}$  is also related to another important problem: To characterize all the 2-periodic functions  $\phi$  on  $(0, \infty)$  having the property that the span of its dilates  $\{\phi(nx) : n \geq 1\}$  is dense in  $L^2(0, 1)$ . This open problem is known as the Periodic Dilation Completeness problem (PDCP).

N. Nikolski, [3], proved that solving this problem is equivalent to characterizing the cyclic vectors of a semigroup  $\{T_n : n \geq 1\}$  on  $H_0^2 := H^2 \ominus \mathbb{C}$ , defined by  $T_n f(z) = f(z^n)$ . Although the semigroup  $\{T_n : n \geq 1\}$  and  $\{W_n : n \geq 1\}$  are not unitarily equivalent, they are semiconjugate; this is,  $T_n(I - S) = (I - S)W_n$ , where  $S$  is the shift of multiplication by  $z$  in  $H^2$ . This relation allowed S. Waleed Noor to guarantee that cyclic vectors of  $\{W_n : n \geq 1\}$  are properly embedded into the PDCP functions.

The purpose of this work is to investigate this semigroup  $\{W_n : n \geq 1\}$ . In particular, we introduce a new reformulation of the Riemann hypothesis in terms of the invariance of the Hilbert subspace spanned by  $\{h_k : k \geq 2\}$  under  $W_n^*$ , for any  $n \geq 2$ . This result lead us to focus on the study of invariant subspaces of  $\{W_n^* : n \geq 1\}$ . For this reason, a series of questions will be discussed and we shall provide an answer. We also present a generalization for the BÃ¡ez-Duarte criterion in  $H^2$  trough a family of cyclic vector for  $\{W_n : n \geq 1\}$ . Recall that at this point there is only one known cyclic vector: the constant 1 in  $H^2(\mathbb{D})$ .

## 2 Main Results

**Theorem 2.1.** *Let  $\mathcal{N}$  be the linear span of  $\{h_k : k \geq 2\}$ . Then the Riemann hypothesis is true if and only if the closure of  $\mathcal{N}$  is  $W_k^*$ -invariant for any  $k \geq 2$ .*

In order to generalize the BÃ¡ez-Duarte criterion in  $H^2$ , we provide a family of cyclic vectors for  $\{W_n : n \geq 1\}$ . Let  $p_{m,\lambda}(z) := z^m + \dots + z - \lambda$ ,  $m \in \mathbb{N}$  and  $\lambda \in \mathbb{C}$ .

**Theorem 2.2.**  *$p_{m,\lambda}$  is a cyclic vector for  $\{W_n : n \geq 1\}$ , for every  $m \in \mathbb{N}$  and  $\lambda \in \mathbb{C}$  such that  $|\lambda + 1| > \sqrt{m + 1}$ .*

**Corollary 2.1.** *Let  $m \in \mathbb{N}$  and  $\lambda \in \mathbb{C}$  such that  $|\lambda + 1| > \sqrt{m + 1}$ . Then the Riemann hypothesis is true if and only if  $p_{m,\lambda}$  belongs to the closure linear span of  $\{h_k : k \geq 2\}$  in  $H^2$ .*

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# ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO NONLINEAR INTEGRAL EQUATIONS VIA RENORMALIZATION

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## Abstract

In this work we use the renormalization group method to study the long-time asymptotics of solutions to a class of nonlinear integral equations with a generalized heat kernel. We classify the nonlinearities according to its role in the asymptotic behavior and we prove that, if one adds nonlinearities classified as irrelevant in the renormalization group sense, then the behavior of the solution in the limit as  $t$  goes to infinity remains unchanged but if marginal nonlinear terms are added in the equation, then the asymptotic behaviour acquires an extra logarithmic decay factor.

## 1 Introduction

Here we use the renormalization group method as proposed by Bricmont et al. [1] to study nonlinear integral equations with a generalized heat kernel, obtaining global existence and uniqueness of the solution, as well as the asymptotic behavior. The nonlinearities are classified according to their contribution to the asymptotic behavior. We show that the so called *irrelevant* perturbations do not affect the asymptotic profile of the solution, in the sense that the profile is the same as in the linear case, whereas, by adding *marginal* perturbations, an extra logarithmic factor appears on the decay of the solution.

Our proof relies on the Renormalization Group approach which was originally introduced in quantum field theory and statistical mechanics and it was afterwards applied to the asymptotic analysis of deterministic differential equations, both analytically and numerically. It proved to be very useful on the asymptotic analysis in problems involving an infinite number of scales and has been used since then in different applications and approaches. Our results here generalize the problems presented in [2, 3], where the Renormalization Group method was applied to study the asymptotic behavior of the solution to I.V.P.  $u_t = c(t)u_{xx} + \lambda F(u)$ ,  $t > 1$ ,  $x \in \mathbb{R}$ ,  $u(x, 1) = f(x)$  with  $c(t) = t^p + o(t^p)$  and nonlinearities of type  $F(u) = \sum_{j \geq \alpha} a_j u^j$ .

## 2 Main Results

More specifically, we obtain the asymptotic behavior of solutions to equations of type

$$u(x, t) = \int G(x - y, s(t))f(y)dy + \int_1^t \int G(x - y, s(t) - s(\tau))F(u(y, \tau))dyd\tau, \quad (1)$$

with  $x \in \mathbb{R}$  and  $t > 1$ . By imposing conditions on the kernel without specifying  $G = G(x, t)$ , we generalize the study of asymptotics for initial value problems. We consider therefore the integral kernel  $G(x, t)$  satisfying the following general conditions:

(i) There are integers  $q > 1$  and  $M > 0$  such that  $G(\cdot, 1) \in C^{q+1}(\mathbb{R})$  and

$$\sup_{x \in \mathbb{R}} \{(1 + |x|)^{M+2} |G^{(j)}(x, 1)|\} < \infty, \quad j = 0, 1, \dots, q+1,$$

where  $G^{(j)}(x, 1)$  denotes the  $j$ -th derivative  $(\partial_x^j G)(x, 1)$ .

(ii) There is a positive constant  $d$  such that

$$G(x, t) = t^{-\frac{1}{d}} G\left(t^{-\frac{1}{d}} x, 1\right), \quad x \in \mathbb{R}, \quad t > 0;$$

(iii)  $G(x, t) = \int_{\mathbb{R}} G(x - y, t - s) G(y, s) dy$ , for  $x \in \mathbb{R}$  and  $t > s > 0$ .

This outlook was adopted in [4, 5] where it is shown that, under similar conditions on  $G$ , with  $s(t) = t$ , the solution  $u(x, t)$  to (1) behaves for long time as

$$\frac{A}{t^{1/d}} G\left(\frac{x}{t^{1/d}}, 1\right),$$

where  $d > 0$  is such that  $G(x, t) = t^{-\frac{1}{d}} G\left(t^{-\frac{1}{d}} x, 1\right)$ . We recover and extend the above result using a renormalization group approach showing that, if  $c(t)$  is a positive function in  $L^1_{loc}((1, +\infty))$  of type  $t^p + o(t^p)$ , with  $p > 0$  and

$$s(t) = \int_1^t c(\tau) d\tau = \frac{t^{p+1} - 1}{p+1} + r(t), \quad (2)$$

then, for  $F(u) = \sum_{j \geq \alpha} a_j u^j$  with  $\alpha > (p+1+d)/(p+1)$ ,

$$u(x, t) \sim \frac{A}{t^{(p+1)/d}} G\left(\frac{x}{t^{(p+1)/d}}, \frac{1}{p+1}\right) \text{ when } t \rightarrow \infty.$$

Furthermore, if  $F(u) = -\mu u^\alpha + \lambda \sum_{j > \alpha} a_j u^j$  with  $\mu$  small and positive and  $\alpha_c = (d+p+1)/(p+1)$ , then

$$u(x, t) \sim \frac{A}{(t \ln t)^{(p+1)/d}} G\left(\frac{x}{(t \ln t)^{(p+1)/d}}, \frac{1}{p+1}\right) \text{ when } t \rightarrow \infty.$$

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# SHARP ESTIMATES FOR THE COVERING NUMBERS OF THE WEIERSTRASS FRACTAL KERNEL

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## Abstract

In this job we use the infamous continuous and nowhere differentiable Weierstrass function as a prototype to define a ‘Weierstrass fractal kernel’. We investigate properties of reproducing kernel Hilbert space (RKHS) associated to this kernel by presenting an explicit characterization of this space. In particular, we show that this space has a dense subset composed of continuous but nowhere differentiable functions. Moreover, we present sharp estimates for the covering numbers of the unit ball of this space as a subset of the continuous functions.

## 1 Introduction

In 1872 K. Weierstrass presented a particular class of trigonometric series as a collection of continuous but nowhere differentiable functions (CNDF). These functions are defined in terms of the following Fourier series expansion

$$w_{a,b}(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \quad x \in \mathbb{R}. \quad (1)$$

For  $0 < a < 1$  it is clear that it defines a continuous bounded function. Under this assumption, Weierstrass proved that  $w_{a,b}$  is nowhere differentiable provided that  $ab \geq 1 + \frac{3\pi}{2}$ , with  $b$  an odd integer ([2, 3]). G.H. Hardy relaxed this condition in [1] by showing that for  $ab \geq 1$ , the Weierstrass function is nowhere differentiable.

Here, as usual,  $C([a, b])$  stands for the normed real vector space of real-valued continuous functions on  $[a, b]$  with the supremum norm.

Let  $I = [-1, 1]$ . We consider here  $W : I \times I \longrightarrow \mathbb{R}$ , the ‘Weierstrass Fractal kernel’, defined for  $0 < a < 1$ ,  $b$  an integer such that  $ab \geq 1$ , and given by

$$W(x, y) := w_{a,b}(x - y), \quad x, y \in I, \quad (2)$$

where  $w_{a,b}$  is the Weierstrass function (1). This is a continuous, nowhere differentiable, symmetric and positive definite kernel. The theory of RKHS tell us that there exists only one RKHS  $\mathcal{H}_W := \mathcal{H}_W(I)$  having the Weierstrass Fractal kernel as reproducing kernel. We present a complete characterization of the space  $\mathcal{H}_W$ , given in terms of Fourier series expansions. It is expected that the functions in a RKHS inherit some of the properties of the generating kernel such as smoothness.

If  $A$  is a subset of a metric space  $M$  and  $\epsilon > 0$ , the covering number

$$\mathcal{C}(\epsilon, A) := \mathcal{C}(\epsilon, A, M)$$

is the minimal number of balls in  $M$  of radius  $\epsilon$  covering the set  $A$ . Clearly,  $\mathcal{C}(\epsilon, A) < \infty$ , whenever  $A$  is a compact subset of  $M$ . For  $X, Y$  Banach spaces and  $T : X \rightarrow Y$  an operator the covering numbers are defined in terms of unit

balls as follows. For  $\epsilon > 0$ , if  $B_X$  and  $B_Y$  are the unit balls in  $X$  and  $Y$ , respectively, then the covering numbers of  $T$  are

$$\mathcal{C}(\epsilon, T) := \mathcal{C}(\epsilon, T(B_X), Y),$$

and given by

$$\mathcal{C}(\epsilon, T) = \min \left\{ n \in \mathbb{N} : \exists y_1, y_2, \dots, y_n \in Y \text{ s.t. } T(B_X) \subset \bigcup_{j=1}^n (y_j + \epsilon B_Y) \right\}.$$

Our main goal is to investigate the covering numbers of the embedding  $I_W : \mathcal{H}_W \rightarrow C(I)$ , where  $\mathcal{H}_W$  is the reproducing kernel Hilbert space associated to  $W$  defined in equation (2).

In this job we present upper and lower estimates for  $I_W : \mathcal{H}_W \rightarrow C(I)$  achieving tight bounds. In short, here, the approach is mainly based on operator norm estimate of  $I_W$  and some others related finite rank operators. In [4] this approach was employed to obtain estimates for the covering numbers of the embedding operator over the RKHS associated to the Gaussian kernel over non-empty interior sets of  $\mathbb{R}^d$ .

Nevertheless, only for a few infinite-dimensional spaces there has been success in determining the precise asymptotics of covering numbers.

## 2 Main Results

The main result of this job reads as follows.

**Theorem 2.1.** *Let  $W$  as in (2). The covering numbers of the embedding  $I_W : \mathcal{H}_W \rightarrow C(I)$  behave asymptotically as follows*

$$\ln(\mathcal{C}(\epsilon, I_W)) \asymp \frac{[\ln(1/\epsilon(1-a)^{1/2})]^2}{\ln(1/a)}, \quad \text{as } \epsilon \rightarrow 0^+.$$

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## DIRICHLET SERIES WITH MAXIMAL BOHR'S STRIP

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We study linear and algebraic structures in sets of Dirichlet series with maximal Bohr's strip. More precisely, we consider a set  $\mathcal{M}$  of Dirichlet series which are uniformly continuous on the right half plane and whose strip of uniform but not absolute convergence has maximal width, i.e.,  $1/2$ . We show that  $\mathcal{M}$  contains an isometric copy of  $\ell_1$  (except zero) and is strongly  $\aleph_0$ -algebrable. Also, we prove that there is a dense  $G_\delta$  set such that any of its elements generates a free algebra contained in  $\mathcal{M} \cup \{0\}$ .

**1 Introduction**

Mathematics is plenty of examples that seem to challenge the intuition. For instance, discontinuous additive functions, Weierstrass' Monsters, Peano curves, non-extendable holomorphic functions, and so on and so forth. The counter-intuitiveness of these examples may lead us to believe they must be rare, but usually this is not the case. Moreover, recent investigations are presenting a very interesting picture. Many of these peculiar examples/objects are not only far away from being rare: in many situations, the set formed by these objects can even contain big linear or algebraic structures. As a seminal example, Gurariy in [5] constructed infinite dimensional subspaces of  $C([0, 1])$  all whose nonzero elements are nowhere differentiable functions. Since then, a whole theory was built in this direction, especially in the last years. Many of these advances are documented in the recent monograph [1].

Our main goal in the present work was to search for linear and algebraic structures in the set of Dirichlet series with maximal Bohr's strips (see below for the definition). A *Dirichlet series* is a series of the form  $\sum_{n=1}^{\infty} a_n n^{-s}$ , where the coefficients  $a_n$  are complex numbers and  $s$  is a complex variable. The natural domains of convergence of Dirichlet series are half-planes. Given a Dirichlet series  $D = \sum a_n n^{-s}$  we can consider three natural abscissas which define the biggest half-planes on which  $D$  converges, converges uniformly and converges absolutely:

$$\sigma_c(D) \leq \sigma_u(D) \leq \sigma_a(D).$$

It is not hard to see that

$$\sup_{D \text{ Dir. ser.}} \sigma_a(D) - \sigma_c(D) = 1.$$

Harald Bohr was among the first to consider the problem of finding the maximal width of the strip on which a Dirichlet series can converge uniformly but not absolutely (this strip is usually called *Bohr's strip*). Thus, the so called *Bohr's absolute convergence problem* was to determine the number

$$S := \sup_{D \text{ Dir. ser.}} \sigma_a(D) - \sigma_u(D).$$

Bohr first showed in 1913 that  $S \leq 1/2$ , and later in 1931 Bohnenblust and Hille proved that actually  $S = 1/2$ . For a modern reference about the solution of this problem we refer to [4, Ch. 1-4].

For  $a \in \mathbb{R}$  we let  $\mathbb{C}_a$  denote the set of all complex numbers  $z$  such that  $\operatorname{Re} z > a$ . Bohr's fundamental theorem (see [4, Theorem 1.13]) ensures that every bounded holomorphic function  $f : \mathbb{C}_0 \rightarrow \mathbb{C}$  which may be represented as

a Dirichlet series in some half-plane converges uniformly on  $\mathbb{C}_\delta$  for each  $\delta > 0$ . Let  $\mathcal{H}_\infty$  denote the space of all such functions. It is well-known that  $\mathcal{H}_\infty$  is actually a Banach algebra when equipped with the supremum norm. As a consequence of Bohr's results, the absolute convergence problem and its solution can be written as

$$S = \sup_{D \in \mathcal{H}_\infty} \sigma_a(D) = \frac{1}{2}. \quad (1)$$

As expected, finding explicit Dirichlet series  $D \in \mathcal{H}_\infty$  such that  $\sigma_a(D) = \frac{1}{2}$  is not an easy task. However, in this work we show that we have plenty of them and that the set of such series contains large linear and algebraic structures. Moreover, we can get all this in a much smaller subalgebra of  $\mathcal{H}_\infty$  which we now define.

A *Dirichlet polynomial* is a Dirichlet series of the form  $\sum_{n=1}^N a_n n^{-s}$ . Let  $\mathcal{A}(\mathbb{C}_0)$  denote the subalgebra of  $\mathcal{H}_\infty$  of all Dirichlet series which are uniform limits on  $\mathbb{C}_0$  of a sequence of Dirichlet polynomials. Our goal is to study the following set of Dirichlet series:

$$\mathcal{M} := \left\{ D \in \mathcal{A}(\mathbb{C}_0) : \sigma_a(D) = \frac{1}{2} \right\}.$$

Note that, by (1), the Dirichlet series belonging to  $\mathcal{M}$  are those with maximal Bohr's strip.

## 2 Main Results

Let  $E$  be a topological vector space and  $\kappa$  be a cardinal number. A subset  $Z \subset E$  is said to be  *$\kappa$ -spaceable* if  $Z \cup \{0\}$  contains a closed vector subspace of  $E$  with dimension  $\kappa$ . Moreover, we say that a subset  $Z \subset E$  is *maximal spaceable* if  $Z$  is  $\dim(E)$ -spaceable. Our first main result is the following.

**Theorem 2.1.** *The set  $\mathcal{M} \cup \{0\}$  contains an isometric copy of  $\ell_1$ . In particular, it is maximal spaceable.*

Let us recall the precise definition of algebrability. Let  $X$  be an arbitrary set,  $\mathcal{A}$  be an algebra of functions  $f: X \rightarrow \mathbb{C}$  and  $\kappa$  be a cardinal number. A subset  $Z \subset \mathcal{A}$  is said to be *strongly  $\kappa$ -algebrable* if there is a sub-algebra  $\mathcal{B}$  of  $\mathcal{A}$  which is generated by an algebraically independent set of generators with cardinality  $\kappa$  and such that  $\mathcal{B} \subset Z \cup \{0\}$ . Let  $\aleph_0$  denote the cardinality of the natural numbers. With this, we can state our second main result.

**Theorem 2.2.** *The set  $\mathcal{M}$  is strongly  $\aleph_0$ -algebrable. Also, there is a dense  $G_\delta$  subset of  $\mathcal{A}(\mathbb{C}_0)$  such that any of its elements generates a free algebra contained in  $\mathcal{M} \cup \{0\}$ .*

It is worth noting that some ideas for the proofs of our results come from [2, 3].

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## EXTENSÕES DE ARENS DE MULTIMORFISMOS EM ESPAÇOS DE RIESZ E RETICULADOS DE BANACH

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**Abstract**

Provamos que todas as extensões de Arens de multimorfismos de Riesz de posto finito tomado valores em espaços de Riesz Arquimedianos coincidem e são multimorfismo de Riesz. Consequências para extensões de Aron-Berner de multimorfismos de Riesz em reticulados de Banach são obtidas.

**1 Introdução**

Para um espaço de Riesz  $E$ , por  $E^\sim$  denotamos o espaço de todos os funcionais lineares regulares em  $E$  e por  $E_n^\sim$  o espaço de todos os funcionais lineares regulares ordem contínuos. Sejam  $E_1, \dots, E_m, F$  espaços de Riesz e  $A: E_1 \times \dots \times E_m \rightarrow F$  um operador  $m$ -linear regular. Em [1] foi mostrado que se  $E_1, E_2, F$  são Arquimedianos e  $A: E_1 \times E_2 \rightarrow F$  é um bimorfismo de Riesz, então a extensão de Arens  $A^{***}: (E_1^\sim)^\sim \times (E_2^\sim)^\sim \rightarrow (F^\sim)^\sim$  é também um bimorfismo de Riesz. Para reticulados de Banach tal resultado foi mostrado em [2].

Seguindo nessa linha, neste trabalho provamos que: (i) para cada permutação  $\sigma \in S_m$ , se  $A$  é positivo/regular, então todas as extensões de Arens  $AR_m^\sigma(A): E_1^{\sim\sim} \times \dots \times E_m^{\sim\sim} \rightarrow F^{\sim\sim}$  são positivas/regulares; (ii) quando  $F$  é Arquimédiano, todas as extensões de Arens  $AR_m^\sigma(A)$  coincidem e são multimorfismos de Riesz sempre que  $A$  é multimorfismo de Riesz de posto finito; (iii) para o caso vetorial mostramos que para todo  $A$  multimorfismo de Riesz,  $|AR_m^\sigma(A)(x_1'', \dots, x_m'')|(y') = AR_m^\sigma(A)(|x_1''|, \dots, |x_m''|)(y')$  para todos  $x_1'' \in E_1^{\sim\sim}, \dots, x_m'' \in E_m^{\sim\sim}$  e todo  $y' \in F^\sim$  homomorfismo de Riesz. Consequências desses resultados para extensões de Aron-Berner em reticulados de Banach são provadas.

**2 Resultados Principais**

Por  $\mathcal{L}_r(E_1, \dots, E_m; F)$  denotamos o espaço vetorial de todos os operadores  $m$ -lineares regulares. Considere a seguinte notação: dados espaços de Riesz  $E_1, \dots, E_m$ , uma permutação  $\sigma \in S_m$  e  $k \in \{1, \dots, m\}$ , denotamos

$$E_1, \dots,_{\sigma(1)} E, \dots,_{\sigma(k-1)} E, \dots, E_m = \begin{cases} E_1, \dots, E_m & \text{se } k = 1, \\ E_1, \dots, E_m & \text{nessa mesma ordem onde } E_{\sigma(1)}, \dots, \\ & E_{\sigma(k-1)} \text{ são retirados para } k = 2, \dots, m. \end{cases}$$

Por exemplo,  $(E_1, {}_2E, E_3) = (E_1, E_3)$ . O mesmo define-se para  $(x_1, \dots, {}_{\sigma(1)} x, \dots, {}_{\sigma(k-1)} x, \dots, x_m)$ . E para  $k = 1, \dots, m-1$ , denotamos  $(E_1, \dots, {}_{\sigma(1)} E, \dots, {}_{\sigma(k)} E, \dots, E_m) = (E_1, \dots, E_m)$  nessa mesma ordem, onde os espaços de Riesz  $E_{\sigma(1)}, \dots, E_{\sigma(k)}$  são retirados. O mesmo define-se para  $(x_1, \dots, {}_{\sigma(1)} x, \dots, {}_{\sigma(k)} x, \dots, x_m)$ . Se  $k = m$ , denotamos  $\mathcal{L}_r(E_1, \dots, {}_{\sigma(1)} E, \dots, {}_{\sigma(m)} E; \mathbb{R}) = \mathbb{R}$ .

Fixe  $k \in \{1, \dots, m\}$  e sejam  $\sigma \in S_m$  uma permutação e  $A \in \mathcal{L}_r(E_1, \dots, {}_{\sigma(1)} E, \dots, {}_{\sigma(k-1)} E, \dots, E_m)$ . Para cada  $x_r \in E_r, r \in \{1, \dots, n\} \setminus \{\sigma(1), \dots, \sigma(k)\}$  defina,  $A(x_1, \dots, {}_{\sigma(1)} x, \dots, {}_{\sigma(k)} x; \bullet; \dots, x_m): E_{\sigma(k)} \rightarrow \mathbb{R}$ ,

$$A(x_1, \dots, {}_{\sigma(1)} x, \dots, {}_{\sigma(k)} x; \bullet; \dots, x_m)(x_{\sigma(k)}) = A(x_1, \dots, {}_{\sigma(1)} x, \dots, {}_{\sigma(k-1)} x, \dots, x_m),$$

onde o ponto  $\bullet$  está na  $\sigma(k)$ -ésima coordenada. Cada funcional definido acima é linear e regular.

**Definição 2.1.** Sejam  $E_1, \dots, E_m, F$  espaços de Riesz. Um operador  $m$ -linear  $A: E_1 \times \dots \times E_m \rightarrow F$  é dito:

- (i) Positivo (em notação  $A \geq 0$  ou  $A \leq 0$ ) se  $A(x_1, \dots, x_m) \in F^+$ , para todos  $x_1 \in E_1^+, \dots, x_m \in E_m^+$ .
- (ii) Regular se  $A$  pode ser escrito como a diferença de dois operadores positivos.
- (iii) Um multimorfismo de Riesz se  $|A(x_1, \dots, x_m)| = A(|x_1|, \dots, |x_m|)$  para todos  $x_1 \in E_1, \dots, x_m \in E_m$ .

**Proposição 2.1.** Para  $\sigma \in S_m$ ,  $k \in \{1, \dots, m\}$  e  $x''_{\sigma(k)} \in E_{\sigma(k)}^{\sim\sim}$ , considere os operadores

$$\overline{x''_{\sigma(k)}}^\sigma : \mathcal{L}_r(E_1, \dots, \sigma(1)E, \dots, \sigma(k-1)E, \dots, E_m) \longrightarrow \mathcal{L}_r(E_1, \dots, \sigma(1)E, \dots, \sigma(k)E, \dots, E_m),$$

$$\overline{x''_{\sigma(k)}}^\sigma(A)(x_1, \dots, \sigma(1)x, \dots, \sigma(k)x, \dots, x_m) = x''_{\sigma(k)}(A(x_1, \dots, \sigma(1)x, \dots, \sigma(k)x; \bullet; \dots, x_m)).$$

Então  $\overline{x''_{\sigma(k)}}^\sigma$  é linear e regular. Além disso, se  $0 \leq x''_{\sigma(k)} \in E_{\sigma(k)}^{\sim\sim}$ , então os operadores  $\overline{x''_{\sigma(k)}}^\sigma$  são positivos.

**Teorema 2.1.** Para  $\sigma \in S_m$  e  $A \in \mathcal{L}_r(E_1, \dots, E_m; F)$ , considere  $AR_m^\sigma(A): E_1^{\sim\sim} \times \dots \times E_m^{\sim\sim} \rightarrow F^{\sim\sim}$  dado por

$$AR_m^\sigma(A)(x'_1, \dots, x'_m)(y') = (\overline{x''_{\sigma(m)}}^\sigma \circ \dots \circ \overline{x''_{\sigma(1)}}^\sigma)(y' \circ A) \text{ para todo } y' \in F^\sim.$$

- (a) Cada operador  $AR_m^\sigma(A)$  é regular e estende  $A$  no sentido que  $AR_m^\sigma(A) \circ (J_{E_1}, \dots, J_{E_m}) = J_F \circ A$ .
- (b) Se  $A \in \mathcal{L}_r(E_1, \dots, E_m; F)$  é positivo, então cada  $AR_m^\sigma(A)$  é positivo.

**Teorema 2.2.** Se  $F$  é um espaço de Riesz Arquimediano e  $A: E_1 \times \dots \times E_m \rightarrow F$  é um multimorfismo de Riesz de posto finito, então todas as extensões de Arens de  $A$ ,  $AR_m^\sigma(A): E_1^{\sim\sim} \times \dots \times E_m^{\sim\sim} \rightarrow F^{\sim\sim}$ ,  $\sigma \in S_m$ , coincidem e são multimorfismos de Riesz.

**Proposição 2.2.** Se  $A \in \mathcal{L}_r(E_1, \dots, E_m; F)$  é um multimorfismo de Riesz e  $\sigma \in S_m$ . Então para cada homomorfismo de Riesz  $y' \in F^\sim$ ,  $J_{F^\sim}(y') \circ AR_m^\sigma(A)$  é um multimorfismo de Riesz e  $|AR_m^\sigma(A)(x'_1, \dots, x'_m)|(y') = AR_m^\sigma(A)(|x'_1|, \dots, |x'_m|)(y')$ , para todo  $x'_1 \in E_1^{\sim\sim}, \dots, x'_m \in E_m^{\sim\sim}$ .

**Proposição 2.3.** Se  $A \in \mathcal{L}_r(E_1, \dots, E_m; F)$  é um multimorfismo de Riesz e  $\sigma \in S_m$ . Então:

- (a)  $y^{***} \circ AB_m^\sigma(A)$  é um multimorfismo de Riesz para cada homomorfismo de Riesz  $w^*$ -contínuo  $y^{***} \in F^{***}$ .
- (b)  $|AB_m^\sigma(A)(x_1^{**}, \dots, x_m^{**})|(y^*) = AB_m^\sigma(A)(|x_1^{**}|, \dots, |x_m^{**}|)(y^*)$ , para todo  $x_1^{**} \in E_1^{**}, \dots, x_m^{**} \in E_m^{**}$  e qualquer  $y^* \in \overline{\text{span}}\{\varphi \in F^* : \varphi \text{ é um homomorfismo de Riesz}\}$ .

**Corolário 2.1.** Seja  $F$  um reticulado de Banach tal que  $F^*$  tem uma base de Schauder formada por homomorfismos de Riesz. Então todas as extensões de Aron-Berner de qualquer multimorfismo de Riesz tomando valores em  $F$  são multimorfismo de Riesz.

**Exemplo 2.1.** O corolário anterior se aplica para os seguintes espaços:  $c_0, \ell_p, 1 < p < \infty$ ,  $F$  um espaço de Banach com base de Schauder 1-incondicional que não contém uma cópia de  $\ell_1$ ,  $F$  um espaço de Banach reflexivo com base de Schauder 1-incondicional, o espaço original de Tsirelson's  $T^*$  e seu dual  $T$ , o espaço de Schreier's  $S$ , o predual  $d_*(w, 1)$  do espaço de sequências de Lorenz  $d(w, 1)$ , cada reticulado de Banach  $F$  que é uma faixa projetada em qualquer dos reticulados de Banach listados acima.

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## UNIVERSAL TOEPLITZ OPERATORS ON THE HARDY SPACE OVER THE POLYDISK

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### Abstract

The Invariant Subspace Problem (ISP) for Hilbert spaces asks if every bounded linear operator has a non-trivial closed invariant subspace. Due to the existence of universal operators (in the sense Rota), the ISP may be solved by describing the invariant subspaces of these operators alone. We characterize all analytic Toeplitz operators  $T_\phi$  on the Hardy space  $H^2(\mathbb{D}^n)$  over the polydisk  $\mathbb{D}^n$  for  $n > 1$  whose adjoints satisfy the Caradus criterion for universality, that is, when  $T_\phi^*$  is surjective and has infinite dimensional kernel. In particular, if  $\phi$  is a non-constant inner function on  $\mathbb{D}^n$ , or a polynomial in the ring  $\mathbb{C}[z_1, \dots, z_n]$  that has zeros in  $\mathbb{D}^n$  but is zero-free on  $\mathbb{T}^n$ , then  $T_\phi^*$  is universal for  $H^2(\mathbb{D}^n)$ . The analogs of these results for  $n = 1$  are not true.

### 1 Introduction

One of the most important open problems in operator theory is the ISP, which asks: Given a complex separable Hilbert space  $\mathcal{H}$  and a bounded linear operator  $T$  on  $\mathcal{H}$ , does  $T$  have a non-trivial invariant subspace? An invariant subspace of  $T$  is a closed subspace  $E \subset \mathcal{H}$  such that  $TE \subset E$ . The recent monograph by Chalendar and Partington [2] is a reference for some modern approaches to the ISP. In 1960, Rota [7] demonstrated the existence of operators that have an invariant subspace structure so rich that they could model every Hilbert space operator.

**Definition 1.1.** Let  $\mathcal{B}$  be a Banach space and  $U$  a bounded linear operator on  $\mathcal{B}$ . Then  $U$  is said to be universal for  $\mathcal{B}$ , if for any bounded linear operator  $T$  on  $\mathcal{B}$  there exists a constant  $\alpha \neq 0$  and an invariant subspace  $\mathcal{M}$  for  $U$  such that the restriction  $U|_{\mathcal{M}}$  is similar to  $\alpha T$ .

If  $U$  is universal for a separable, infinite dimensional Hilbert space  $\mathcal{H}$ , then the ISP is equivalent to the assertion that every minimal invariant subspace for  $U$  is one dimensional. The main tool thus far for identifying universal operators has been the following criterion of Caradus [1].

**Theorem 1.1.** Let  $\mathcal{H}$  be a separable infinite dimensional Hilbert space and  $U$  a bounded linear operator on  $\mathcal{H}$ . If  $\ker(U)$  is infinite dimensional and  $U$  is surjective, then  $U$  is universal for  $\mathcal{H}$ .

Let  $\mathbb{D}$  be the unit disk in the complex plane  $\mathbb{C}$  and  $\mathbb{T}$  be the boundary of  $\mathbb{D}$ . The polydisk  $\mathbb{D}^n$  and torus  $\mathbb{T}^n$  are the cartesian products of  $n$  copies of  $\mathbb{D}$  and  $\mathbb{T}$ , respectively. We let  $L^p(\mathbb{T}^n) = L^p(\mathbb{T}^n, \sigma)$  denote the usual Lebesgue space on  $\mathbb{T}^n$ , where  $\sigma = \sigma_n$  is the normalized Haar measure on  $\mathbb{T}^n$ , and  $L^\infty(\mathbb{T}^n)$  the essentially bounded functions with respect to  $\sigma$ . The Hardy space  $H^2(\mathbb{D}^n)$  is the Hilbert space of holomorphic functions  $f$  on  $\mathbb{D}^n$  satisfying

$$\|f\|^2 := \sup_{0 < r < 1} \int_{\mathbb{T}^n} |f(r\zeta)|^2 d\sigma(\zeta) < \infty.$$

Denote by  $H^\infty(\mathbb{D}^n)$  the space of bounded analytic functions on  $\mathbb{D}^n$ . It is well-known that both  $H^2(\mathbb{D}^n)$  and  $H^\infty(\mathbb{D}^n)$  can be viewed as subspaces of  $L^2(\mathbb{T}^n)$  and  $L^\infty(\mathbb{T}^n)$  respectively by identifying  $f$  with its boundary function  $f(\zeta) := \lim_{r \rightarrow 1} f(r\zeta)$  for almost every  $\zeta \in \mathbb{T}^n$ . If  $|f| = 1$  almost everywhere on  $\mathbb{T}^n$ , then  $f$  is called inner function.

Let  $P$  denote the orthogonal projection of  $L^2(\mathbb{T}^n)$  onto  $H^2(\mathbb{D}^n)$ . The Toeplitz operator  $T_\phi$  with symbol  $\phi$  in  $L^\infty(\mathbb{T}^n)$  is defined by

$$T_\phi f = P(\phi f)$$

for  $f \in H^2(\mathbb{D}^n)$ . Just like on the disk, we have that  $T_\phi$  is a bounded linear operator on  $H^2(\mathbb{D}^n)$  and  $T_\phi^* = T_{\bar{\phi}}$ . Moreover, if  $\phi \in H^\infty(\mathbb{D}^n)$ , then  $T_\phi f = \phi f$  for all  $\phi \in H^2(\mathbb{D}^n)$  and  $T_\phi$  is called an analytic Toeplitz operator.

The best known examples of universal operators are all adjoints of analytic Toeplitz operators on  $H^2(\mathbb{D})$ , or are equivalent to one of them. For example  $T_\phi^*$  when  $\phi$  is a singular inner function or infinite Blaschke product. In the last few years, Cowen and Gallardo-Gutiérrez [3, 4, 5, 6] have undertaken a thorough analysis of adjoints of analytic Toeplitz operators that are universal for  $H^2(\mathbb{D})$ . The objective of this presentation is to consider analytic Toeplitz operators  $T_\phi$  whose adjoints are universal on  $H^2(\mathbb{D}^n)$  for  $n > 1$ .

## 2 Main Results

**Theorem 2.1.** *Let  $\phi \in H^\infty(\mathbb{D}^n)$  for  $n > 1$ . Then  $T_\phi^*$  satisfies the Caradus criterion for universality if and only if  $\phi$  is invertible in  $L^\infty(\mathbb{T}^n)$  but non-invertible in  $H^\infty(\mathbb{D}^n)$ .*

**Corollary 2.1.** *Let  $T_\phi$  be a left-invertible analytic Toeplitz operator on  $H^2(\mathbb{D}^n)$  for some  $n > 1$ . Then either  $T_\phi$  is invertible or  $T_\phi^*$  is universal.*

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# CICLICIDADE E HIPERCICLICIDADE DE OPERADORES DE COMPOSIÇÃO NO ESPAÇO DE HARDY DO SEMI-PLANO DIREITO

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## Abstract

Seja  $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$  o semi-plano direito. Neste trabalho, estudamos os operadores de composição  $C_\Phi f = f \circ \Phi$  induzidos no espaço de Hardy do semi-plano direito  $H^2(\mathbb{C}_+)$  por funções holomorfas  $\Phi : \mathbb{C}_+ \rightarrow \mathbb{C}_+$ . Aqui caracterizamos completamente os operadores de composição cíclicos e hipercíclicos em  $H^2(\mathbb{C}_+)$  que são induzidos por funções da forma  $\Phi(z) = az + b$ , onde  $a > 0$  e  $\operatorname{Re}(b) \geq 0$ .

## 1 Introdução

Seja  $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$  o semi-plano direito. O espaço de Hardy do semi-plano direito, denotado por  $H^2(\mathbb{C}_+)$ , é o espaço de Hilbert de todas as funções holomorfas  $f : \mathbb{C}_+ \rightarrow \mathbb{C}$  para o qual

$$\|f\| := \left( \sup_{x>0} \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x+iy)|^2 dy \right)^{1/2} \quad (1)$$

é finito. A quantidade (1) descreve a norma de Hilbert de  $H^2(\mathbb{C}_+)$ .

Se  $\Phi : \mathbb{C}_+ \rightarrow \mathbb{C}_+$  é uma função holomorfa, então o *operador de composição*  $C_\Phi$  com símbolo  $\Phi$  é definido por

$$C_\Phi f = f \circ \Phi, \quad f \in H^2(\mathbb{C}_+).$$

A ênfase na teoria de operadores de composição está na comparação das propriedades de  $C_\Phi$  com as do símbolo  $\Phi$ . Por exemplo, Elliott e Jury mostraram que  $C_\Phi$  é limitado em  $H^2(\mathbb{C}_+)$  se, e somente se,  $\Phi$  tem derivada angular finita em  $\infty$  (veja [1, Theorem 3.1]). Relembre que uma transformação fracionária linear de  $\mathbb{C}_+$  é uma função  $\Phi : \mathbb{C}_+ \rightarrow \mathbb{C}_+$  da forma

$$\Phi(z) = \frac{az+b}{cz+d}, \quad z \in \mathbb{C}_+.$$

Devido ao critério de limitação para  $C_\Phi$  segue que as transformações fracionárias lineares de  $\mathbb{C}_+$  que induzem operadores de composição limitados em  $H^2(\mathbb{C}_+)$  tem a forma

$$\Phi(z) = az + b, \quad z \in \mathbb{C}_+ \quad (2)$$

onde  $a > 0$  e  $\operatorname{Re}(b) \geq 0$ .

Sejam  $X$  um espaço normado e  $\mathcal{L}(X)$  o espaço de todos os operadores lineares limitados  $T : X \rightarrow X$ . Um operador  $T \in \mathcal{L}(X)$  é *cíclico* se existe um vetor  $x \in X$  tal que o espaço gerador por  $\{T^n x\}_{n \in \mathbb{N}}$  é denso em  $X$ . Se  $\{T^n x\}_{n \in \mathbb{N}}$  é denso em  $X$ , então  $T$  é dito ser *hipercíclico*. Nestes casos,  $x$  é chamado de *vetor cíclico* e *hipercíclico*, respectivamente. Aqui caracterizamos quais dos símbolos em (2) induzem operadores de composição cíclicos ou hipercíclicos.

## 2 Resultados Principais

Os principais resultados deste trabalho podem ser sumarizados na seguinte tabela:

As demonstrações dos resultados apresentados na tabela podem ser encontrados em [2].

Símbolo $\Phi(z) = az + b$	Ciclicidade de $C_\Phi$	Hiperciclicidade de $C_\Phi$
$\operatorname{Re}(b) = 0$	Não	Não
$a = 1$ and $\operatorname{Re}(b) > 0$	Sim	Não
$a < 1$ and $\operatorname{Re}(b) > 0$	Não	Não
$a > 1$ and $\operatorname{Re}(b) > 0$	Sim	Não

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## ANÉIS LINEARMENTE TOPOLOGIZADOS ESTRITAMENTE MINIMAIS

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Nesta nota introduzimos a noção de anel linearmente topologizado estritamente minimal, provamos que todo anel de valorização discreta é estritamente minimal e fornecemos condições necessárias e suficientes para que um anel linearmente topologizado de Hausdorff seja estritamente minimal.

## 1 Introdução

Nesta nota *anel* significará *anel comutativo com elemento unidade diferente de 0* e *módulo* significará *módulo unitário*.

**Definição 1.1.** Um anel linearmente topologizado [1] (§7) de Hausdorff  $(R, \tau_R)$  (munido de sua estrutura canônica de  $R$ -módulo) é dito **estritamente minimal** se toda topologia de Hausdorff em  $R$  que o torne um  $(R, \tau_R)$ -módulo linearmente topologizado coincidir com  $\tau_R$ .

O resultado a seguir fornece exemplos importantes de anéis linearmente topologizados estritamente minimais.

**Proposição 1.1.** Sejam  $R$  um anel de valorização discreta e  $\tau_R$  sua topologia [4] (Capítulo I). Então  $(R, \tau_R)$  é estritamente minimal.

**Prova:** Com efeito, se  $\tau$  é uma topologia de Hausdorff em  $R$  tal que  $(R, \tau)$  é um  $(R, \tau_R)$ -módulo linearmente topologizado, da continuidade da aplicação

$$(\lambda, \mu) \in (R \times R, \tau_R \times \tau) \longmapsto \lambda\mu \in (R, \tau) \quad (1)$$

segue a continuidade da aplicação

$$\lambda \in (R, \tau_R) \longmapsto \lambda \in (R, \tau); \quad (2)$$

logo,  $\tau$  é menos fina do que  $\tau_R$ .

Reciprocamente, mostremos que  $\tau_R$  é menos fina do que  $\tau$ . De fato, sejam  $\pi R$  o ideal maximal de  $R$  e  $m$  um inteiro  $\geq 1$  arbitrário. Como  $\pi^m \neq 0$  e  $\tau$  é uma topologia de Hausdorff, existe uma  $\tau$ -vizinhança  $U$  de 0 em  $R$  que é um ideal de  $R$  tal que  $\pi^m \notin U$ . Afirmamos que  $U \subset \pi^m R$ , o que assegurará que  $\tau_R$  é menos fina do que  $\tau$ . Realmente, seja  $v$  uma valorização discreta no corpo de frações  $K$  de  $R$  tal que  $R = \{\lambda \in K; v(\lambda) \geq 0\}$  [4] (p. 17) e admitamos a existência de  $\xi \in U$  tal que  $\xi \notin \pi^m R$ . Então  $\xi \neq 0$  e  $v(\xi) \in \{0, 1, \dots, m-1\}$ . Como  $0 = v(1) = v(\xi\xi^{-1}) = v(\xi) + v(\xi^{-1})$ ,  $v(\xi^{-1}) = -v(\xi) \in \{-(m-1), \dots, -1, 0\}$ . Logo,  $\xi^{-1}\pi^m \in R$ , pois  $v(\xi^{-1}\pi^m) = v(\xi^{-1}) + v(\pi^m) = v(\xi^{-1}) + m > 0$ . Consequentemente,  $\pi^m = \xi(\xi^{-1}\pi^m) \in UR \subset U$ , o que não ocorre. Portanto,  $U \subset \pi^m R$ .  $\square$

## 2 Resultado Principal

Argumentando como em [2, 3], podemos estabelecer o

**Teorema 2.1.** *Para um anel linearmente topologizado de Hausdorff  $(R, \tau_R)$ , as seguintes condições são equivalentes:*

- (a)  $(R, \tau_R)$  é estritamente minimal;
- (b) para todo  $(R, \tau_R)$ -módulo linearmente topologizado de Hausdorff  $F$ , onde  $F$  é um  $R$ -módulo livre com uma base de 1 elemento, todo isomorfismo de  $R$ -módulos de  $R$  em  $F$  é um homeomorfismo de  $(R, \tau_R)$  em  $F$ ;
- (c) todo  $R$ -módulo livre com uma base de 1 elemento admite uma única topologia que o torna um  $(R, \tau_R)$ -módulo linearmente topologizado de Hausdorff;
- (d) para todo  $(R, \tau_R)$ -módulo linearmente topologizado  $E$  e para todo  $(R, \tau_R)$ -módulo linearmente topologizado de Hausdorff  $F$ , onde  $F$  é um  $R$ -módulo livre com uma base de 1 elemento, toda aplicação  $R$ -linear sobrejetora de  $E$  em  $F$  com núcleo fechado é contínua.
- (e) para todo  $(R, \tau_R)$ -módulo linearmente topologizado  $E$  e para todo  $(R, \tau_R)$ -módulo linearmente topologizado de Hausdorff  $F$ , onde  $F$  é um  $R$ -módulo livre com uma base de 1 elemento, toda aplicação  $R$ -linear de  $E$  em  $F$  com gráfico fechado é contínua.

Como consequência da Proposição 1.1 e do Teorema 2.1 resulta que as condições (b), (c), (d) e (e) são válidas se  $(R, \tau_R)$  é um anel de valorização discreta arbitrário.

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THE RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL AS A SEMIGROUP IN  
 BOCHNER-LEBESGUE SPACES

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**Abstract**

The Riemann-Liouville fractional integral is a classic tool from fractional calculus and the literature about its properties is very huge. In this short communication, we would like to present this fractional integral as a semigroup in  $\mathcal{L}(L^p(t_0, t_1; X))$ , with respect to the order of integration, when  $t_0, t_1 \in \mathbb{R}$ , with  $t_0 < t_1$ , and  $X$  is a Banach space. Then we prove that its infinitesimal generator is an unbounded linear operator, which allows me to conclude that the fractional integral is not an uniformly continuous semigroup.

## 1 Introduction

Let us begin by recalling the notions of fractional integral and Bochner-Lebesgue spaces  $L^p(t_0, t_1; X)$ , when we have  $t_0, t_1 \in \mathbb{R}$ , with  $t_0 < t_1$  and  $X$  a Banach space.

**Definition 1.1.** Consider  $1 \leq p \leq \infty$ . We use the symbol  $L^p(t_0, t_1; X)$  to represent the set of all Bochner measurable functions  $f : I \rightarrow X$  for which  $\|f\|_X \in L^p(t_0, t_1; \mathbb{R})$ , where  $L^p(t_0, t_1; \mathbb{R})$  stands for the classical Lebesgue space. Moreover,  $L^p(t_0, t_1; X)$  is a Banach space when considered with the norm

$$\|f\|_{L^p(t_0, t_1; X)} := \begin{cases} \left[ \int_{t_0}^{t_1} \|f(s)\|_X^p ds \right]^{1/p}, & \text{if } p \in [1, \infty), \\ \text{ess sup}_{s \in [t_0, t_1]} \|f(s)\|_X, & \text{if } p = \infty. \end{cases}$$

**Definition 1.2.** For  $\alpha \in (0, \infty)$  and  $f : [t_0, t_1] \rightarrow X$ , the Riemann-Liouville (RL for short) fractional integral of order  $\alpha$  at  $t_0$  of a function  $f$  is defined by

$$J_{t_0, t}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s) ds, \quad (1)$$

for every  $t \in [t_0, t_1]$  such that integral (1) exists. Above  $\Gamma(z)$  denotes the classical Euler's gamma function.

With these definitions, and by considering Riesz-Thorin interpolation theorem, we are able to prove that:

**Theorem 1.1.** Let  $\alpha > 0$ ,  $1 \leq p \leq \infty$  and  $f \in L^p(t_0, t_1; X)$ . Then  $J_{t_0, t}^\alpha f(t)$  is Bochner integrable and belongs to  $L^p(t_0, t_1; X)$ . Furthermore, it holds that

$$\left[ \int_{t_0}^{t_1} \|J_{t_0, t}^\alpha f(t)\|_X^p dt \right]^{1/p} \leq \left[ \frac{(t_1 - t_0)^\alpha}{\Gamma(\alpha + 1)} \right] \|f\|_{L^p(t_0, t_1; X)}. \quad (2)$$

In other words,  $J_{t_0, t}^\alpha$  is a bounded linear operator from  $L^p(t_0, t_1; X)$  into itself.

## 2 Main Results

The results presented above, together with the Abstract Semigroup Theory, are enough for us to present the following results:

**Theorem 2.1.** *Let  $1 \leq p \leq \infty$ . Then the family  $\{J_{t_0,t}^\alpha : \alpha \geq 0\}$  defines a  $C_0$ -semigroup in  $L^p(t_0, t_1; X)$ .*

**Theorem 2.2.** *Let  $1 \leq p \leq \infty$  and assume that  $A : D(A) \subset L^p(t_0, t_1; X) \rightarrow L^p(t_0, t_1; X)$  is the infinitesimal generator of the  $C_0$ -semigroup  $\{J_{t_0,t}^\alpha : \alpha \geq 0\}$  in  $L^p(t_0, t_1; X)$ . Then  $f \in D(A)$  if, and only if,*

$$\int_{t_0}^t \ln(t-s)f(s) ds$$

*is absolutely continuous from  $[t_0, t_1]$  into  $X$  and its derivative belongs to  $L^p(t_0, t_1; X)$ . Moreover, we have*

$$Af(t) = -\psi(1)f(t) + \frac{d}{dt} \left[ \int_{t_0}^t \ln(t-s)f(s) ds \right], \quad (1)$$

*for almost every  $t \in [t_0, t_1]$ , where  $\psi(t)$  denotes the digamma function.*

Finally we present the main result of this short communication.

**Theorem 2.3.** *Assume that  $1 \leq p \leq \infty$ . If  $A : D(A) \subset L^p(t_0, t_1; X) \rightarrow L^p(t_0, t_1; X)$  is the infinitesimal generator of the  $C_0$ -semigroup  $\{J_{t_0,t}^\alpha : \alpha \geq 0\} \subset \mathcal{L}(L^p(t_0, t_1; X))$ , then  $A : D(A) \subset L^p(t_0, t_1; X) \rightarrow L^p(t_0, t_1; X)$  is an unbounded operator.*

*Proof.* If  $p = \infty$ ,  $x \in X$ , with  $\|x\|_X = 1$ , and we define  $\phi \in L^\infty(t_0, t_1, X)$  by  $\phi(t) = x$ , then  $\phi \notin D(A)$ , when  $D(A)$  is viewed as a domain in  $L^\infty(t_0, t_1; X)$ . This implies that  $D(A) \subsetneq L^\infty(t_0, t_1; X)$ , i.e.,  $A \notin \mathcal{L}(L^\infty(t_0, t_1; X))$ .

If  $1 \leq p < \infty$  and we consider  $x \in X$ , with  $\|x\|_X = 1$ ,  $n \in \mathbb{N}^*$  and the sequence  $\phi_n(t) = (t - t_0)^n x$ , then

$$\lim_{n \rightarrow \infty} \frac{\|A\phi_n\|_{L^p(t_0, t_1; X)}}{\|\phi_n\|_{L^p(t_0, t_1; X)}} = \infty,$$

and therefore  $A$  is an unbounded operator.  $\square$

## 3 Acknowledgement

It is worth emphasising that these results can be found in [3, 4], which are recently submitted works that were done together with Prof. Renato Fehlberg Junior.

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## A HIPER-TRANSFORMADA DE BOREL POLINOMIAL

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Desenvolvemos neste trabalho uma técnica para representar funcionais lineares em espaços de polinômios homogêneos contínuos entre espaços de Banach, a qual denominamos de hiper-transformada de Borel polinomial. Como exemplo de aplicação desta técnica, representamos os funcionais lineares em espaços de polinômios compactos como operadores lineares integrais.

**1 Introdução**

A transformada de Borel polinomial clássica é uma ferramenta muito utilizada para representação de funcionais lineares em espaços de polinômios homogêneos e funções holomorfas. Dados  $n \in \mathbb{N}$  e espaços de Banach  $E$  e  $F$ , denotaremos por  $\mathcal{P}(^n E; F)$  o espaço de Banach dos polinômios  $n$ -homogêneos contínuos de  $E$  em  $F$  com a norma usual. Escrevemos  $\mathcal{L}(E; F)$  no caso linear e  $\mathcal{P}(^n E)$  no caso em que  $F$  é o corpo dos escalares. Se  $\mathcal{Q}(^n E; F)$  é um subespaço de  $\mathcal{P}(^n E; F)$  munido de uma norma completa  $\|\cdot\|_{\mathcal{Q}}$ , a transformada de Borel polinomial é o operador

$$\beta_n : (\mathcal{Q}(^n E; F), \|\cdot\|_{\mathcal{Q}})^* \longrightarrow \mathcal{P}(^n E^*; F^*) , \quad \beta_n(\phi)(x^*)(y) = \phi([x^*]^n \otimes y) , \quad x^* \in E^*, \quad y \in F,$$

onde  $([x^*]^n \otimes y)(x) = x^*(x)^n y$  é um polinômio  $n$ -homogêneo e contínuo.

Embora frutífera, essa técnica possui a seguinte limitação: para  $\beta_n$  ser injetiva, combinações lineares de polinômios do tipo  $[x^*]^n \otimes y$ , chamados de polinômios de tipo finito, devem formar um subespaço denso de  $\mathcal{Q}(^n E; F)$ . Em razão dessa restrição, desenvolvemos uma nova técnica de representação que, a grosso modo, contempla subespaços maiores de  $\mathcal{P}(^n E; F)$ . Chamamos esta variante da transformada de Borel polinomial clássica de *hiper-transformada de Borel polinomial*, denotada por  $\mathfrak{B}_n$ , a qual é a contrapartida polinomial da hiper-transformada de Borel desenvolvida em [1] para representar funcionais lineares em espaços de operadores multilineares. No caso linear temos  $\beta_1 = \mathfrak{B}_1 =: \mathfrak{B}$ .

Sejam  $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$  um ideal de Banach de operadores lineares no sentido de [3] e  $(\mathcal{I} \circ \mathcal{P}(^n E; F), \|\cdot\|_{\mathcal{I} \circ \mathcal{P}})$  o respectivo ideal composição de polinômios homogêneos no sentido de [2]. Tendo em vista as equivalências em [1, Theorem 2.7], é esperado que as seguintes condições sejam equivalentes para quaisquer espaços de Banach  $E$  e  $F$ :

- (i) A transformada de Borel linear  $\mathfrak{B}$  representa funcionais lineares em  $(\mathcal{I}(E; F), \|\cdot\|_{\mathcal{I}})$ .
- (ii) A hiper-transformada de Borel polinomial  $\mathfrak{B}_n$  representa funcionais lineares em  $(\mathcal{I} \circ \mathcal{P}(^n E; F), \|\cdot\|_{\mathcal{I} \circ \mathcal{P}})$  para todo  $n \in \mathbb{N}$ .
- (iii) A hiper-transformada de Borel polinomial  $\mathfrak{B}_n$  representa funcionais lineares em  $(\mathcal{I} \circ \mathcal{P}(^n E; F), \|\cdot\|_{\mathcal{I} \circ \mathcal{P}})$  para algum  $n \in \mathbb{N}$ .

Como é usual, alguns procedimentos multilineares funcionam bem para polinômios, enquanto que outros não. Experimentaremos as duas situações neste trabalho. Por um lado, veremos que (i) e (ii) são equivalentes e, obviamente, implicam (iii); mas não sabemos se as três condições são equivalentes. E temos razões para conjecturar que não é verdade em geral.

## 2 Resultados Principais

Dados  $P \in \mathcal{P}(^n E)$  e  $y \in F$ , denotaremos por  $P \otimes y$  o polinômio em  $\mathcal{P}(^n E; F)$  definido por

$$(P \otimes y)(x) = P(x)y, \quad x \in E.$$

Combinações lineares de polinômios deste tipo são chamados de polinômios de posto finito. A hiper-transformada de Borel polinomial é o operador  $\mathfrak{B}_n$  definido no próximo resultado.

**Teorema 2.1.** *Seja  $\mathcal{Q}(^n E; F)$  um subespaço vetorial de  $\mathcal{P}(^n E; F)$  munido com uma norma completa  $\|\cdot\|_{\mathcal{Q}}$  contendo os polinômios de posto finito e tal que  $\|P \otimes y\|_{\mathcal{Q}} \leq \|P\| \cdot \|y\|$  para todos  $P \in \mathcal{P}(^n E; F)$  e  $y \in F$ . Então:*

(a) *A aplicação*

$$\mathfrak{B}_n: (\mathcal{Q}(^n E; F), \|\cdot\|_{\mathcal{Q}})^* \longrightarrow \mathcal{L}(\mathcal{P}(^n E); F^*), \quad \mathfrak{B}_n(\phi)(P)(y) = \phi(P \otimes y),$$

*é um operador linear bem definido e  $\|\mathfrak{B}_n\| \leq 1$ .*

(b)  *$\mathfrak{B}_n$  é injetivo se, e somente se, o subespaço dos polinômios de posto finito é  $\|\cdot\|_{\mathcal{Q}}$ -denso em  $\mathcal{Q}(^n E; F)$ .*

Semi-ideais à esquerda de operadores lineares foram introduzidos em [1]. Por propriedade geométrica de espaços de Banach entendemos uma propriedade que é invariante por isomorfismos isométricos.

**Teorema 2.2.** *Sejam  $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$  um ideal de Banach de operadores lineares,  $\alpha$  um semi-ideal de operadores à esquerda e  $P_1$  e  $P_2$  propriedades geométricas de espaços de Banach. Suponha que para todos espaços de Banach  $E$  e  $F$  tais que  $E^*$  tem  $P_1$  e  $F$  tem  $P_2$ , a transformada de Borel linear  $\mathfrak{B}: (\mathcal{I}(E, F), \|\cdot\|_{\mathcal{I}})^* \longrightarrow \mathcal{L}_{\alpha}(E^*; F^*)$  seja um isomorfismo isométrico. Então, para todo  $n \in \mathbb{N}$  e todos espaços de Banach  $E$  e  $F$  tais que  $\mathcal{P}(^n E)$  tem  $P_1$  e  $F$  tem  $P_2$ , a hiper-transformada de Borel polinomial*

$$\mathfrak{B}_n: (\mathcal{I} \circ \mathcal{P}(^n E; F), \|\cdot\|_{\mathcal{I} \circ \mathcal{P}})^* \longrightarrow \mathcal{L}_{\alpha}(\mathcal{P}(^n E); F^*), \quad \mathfrak{B}_n(\phi)(P)(y) = \phi(P \otimes y),$$

*é um isomorfismo isométrico.*

Conforme dissemos, conjecturamos que a recíproca do teorema acima não é verdadeira. De toda forma, temos a seguinte recíproca parcial:

**Teorema 2.3.** *Sejam  $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$  um ideal de Banach,  $\alpha$  um semi-ideal de operadores à esquerda e  $E$  e  $F$  espaços de Banach. Suponha que exista  $n \in \mathbb{N}$  tal que a hiper-transformada de Borel polinomial  $\mathfrak{B}_n: (\mathcal{I} \circ \mathcal{P}(^n E; F), \|\cdot\|_{\mathcal{I} \circ \mathcal{P}})^* \longrightarrow \mathcal{L}_{\alpha}(\mathcal{P}(^n E); F^*)$  seja um isomorfismo isométrico sobre sua imagem. Então, a transformada de Borel linear  $\mathfrak{B}: (\mathcal{I}(E; F), \|\cdot\|_{\mathcal{I}})^* \longrightarrow \mathcal{L}_{\alpha}(E^*; F^*)$  também é um isomorfismo isométrico sobre sua imagem.*

Como exemplo de aplicação do Teorema 2.2, no próximo resultado usamos a hiper-transformada de Borel polinomial para representar funcionais lineares no dual hiper-ideal fechado  $\mathcal{P}_{\mathcal{K}}$  dos polinômios compactos (polinômios que transformam conjuntos limitados em conjuntos relativamente compactos). Veremos que, na presença da propriedade da aproximação, os funcionais em  $\mathcal{P}_{\mathcal{K}}$  podem ser representados por operadores lineares integrais. O ideal de Banach dos operadores lineares integrais será denotado por  $\mathcal{J}$ .

**Teorema 2.4.** *Se  $\mathcal{P}(^n E)$  ou  $F$  tem a propriedade da aproximação, então a hiper-transformada de Borel polinomial  $\mathfrak{B}_n: [\mathcal{P}_{\mathcal{K}}(^n E; F)]^* \longrightarrow \mathcal{J}(\mathcal{P}(^n E); F^*)$  é um isomorfismo isométrico.*

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## SOME PROPERTIES OF ALMOST SUMMING OPERATORS

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In this work we will present the scope of three important results in the linear theory of absolute summing operators. The first one was obtained by Bu and Kranz in [3] and it asserts that a continuous linear operator between Banach spaces takes almost unconditionally summable sequences into Cohen strongly  $q$ -summable sequences for any  $q \geq 2$ , whenever its adjoint is  $p$ -summing for some  $p \geq 1$ . The second of them states that  $p$ -summing operators with hilbertian domain are Cohen strongly  $q$ -summing operators ( $1 < p, q < \infty$ ), this result is due to Bu [2]. The third one is due to Kwapien [7] and it characterizes spaces isomorphic to a Hilbert space using 2-summing operators. We will show that these results are maintained replacing the hypothesis of the operator to be  $p$ -summing by almost summing.

**1 Introduction**

If  $1 \leq p < \infty$ , we say that a linear operator  $u : X \rightarrow Y$  is *absolutely  $p$ -summing* (or  *$p$ -summing*) if  $(u(x_i))_{i=1}^\infty \in \ell_p(Y)$  whenever  $(x_i)_{i=1}^\infty \in \ell_p^w(X)$ . The class of absolutely  $p$ -summing linear operators from  $X$  to  $Y$  will be represented by  $\Pi_p(X, Y)$  (see [6]). In [5] Cohen introduced a class of operators which characterizes the  $p^*$ -summing adjoint operators. If  $1 < p \leq \infty$ , we say that a linear operator  $u$  from  $X$  to  $Y$  is *Cohen strongly  $p$ -summing* (or *strongly  $p$ -summing*) if  $(u(x_i))_{i=1}^\infty \in \ell_p\langle Y \rangle$  whenever  $(x_i)_{i=1}^\infty \in \ell_p(X)$ . The class of Cohen strongly  $p$ -summing linear operators from  $X$  to  $Y$  will be denoted by  $\mathcal{D}_p(X, Y)$ . According to [1], a linear operator  $u \in \mathcal{L}(X; Y)$  is called to be *almost  $p$ -summing*,  $1 \leq p < \infty$ , if there is a constant  $C \geq 0$  such that

$$\left( \int_0^1 \left\| \sum_{i=1}^m r_i(t)u(x_i) \right\|^2 dt \right)^{\frac{1}{2}} \leq C \cdot \| (x_i)_{i=1}^m \|_{w,p}$$

for every  $m \in \mathbb{N}$  and  $x_1, \dots, x_m \in X$ , whose  $r_i$  are the Rademacher functions. The class of all almost summing operators from  $X$  to  $Y$  is denoted by  $\Pi_{al.s.p}(X, Y)$ . When  $p = 2$ , these operators are simply called *almost summing* and we write  $\Pi_{al.s}$  instead of  $\Pi_{al.s.2}$  (see [6, Chapter 12]). By [6, Proposition 12.5],  $\bigcup_{1 \leq p < \infty} \Pi_p(X, Y) \subseteq \Pi_{al.s}(X, Y)$ .

Using strong tools as Pietsch domination theorem and Khinchin and Kahane inequalities, the main result obtained by Bu and Kranz in [3] was:

**Theorem 1.1.** [3, Theorem 1] *Let  $X$  and  $Y$  be Banach spaces and  $u$  be a continuous linear operator from  $X$  to  $Y$ . If  $u^*$  is  $p$ -summing for some  $p \geq 1$ , then for any  $q \geq 2$ ,  $u$  takes almost unconditionally summable sequences in  $X$  into members of  $\ell_q\langle Y \rangle$ .*

Let  $X$  be a Hilbert space. Cohen in [4] has shown that

$$\Pi_2(X, Y) \subseteq \mathcal{D}_2(X, Y) \text{ for all Banach space } Y. \quad (1)$$

In [2], Bu showed that (1) is valid with no restrictions of the parameters  $p, q \in (1, \infty)$  instead of  $p = q = 2$ . Cohen [4] also asked if (1) characterizes spaces isomorphic to a Hilbert space. Kwapien [7] proved that this question has a positive answer. These important results are as follows:

**Theorem 1.2.** [2, Main Theorem] Let  $1 < p, q < \infty$ , and let  $X$  be a Hilbert space and  $Y$  be a Banach space. Then

$$\Pi_p(X, Y) \subseteq \mathcal{D}_q(X, Y).$$

**Theorem 1.3.** [7] The following properties of Banach space  $X$  are equivalent.

- (i) The space  $X$  is isomorphic to a Hilbert space.
- (ii) For every Banach space  $Y$ ,  $\Pi_2(X, Y) \subseteq \mathcal{D}_2(X, Y)$ .

The work is organized as follows: We will present our first result which is an improvement on the Bu and Kranz [3] result through a simpler argument than the original. Afterward, we will extend the statement of the main result of Bu in [2, Main Theorem]. Finally, we will show a Kwapień type theorem using almost summing operators to characterize spaces isomorphic to a Hilbert space.

## 2 Main Results

**Theorem 2.1.** (Extension of the Bu-Kranz Theorem) Let  $X$  and  $Y$  be Banach spaces and  $u$  be a continuous linear operator from  $X$  to  $Y$ . If  $u^*$  is almost  $p^*$ -summing for some  $p \geq 1$ , then  $u$  takes almost unconditionally summable sequences in  $X$  into members of  $\ell_p(Y)$ .

**Theorem 2.2.** (Extension Bu's Theorem) Let  $2 \leq p < \infty$ ,  $1 < q \leq \infty$ ,  $X$  and  $Y$  be Banach spaces such that  $X$  is an  $\mathcal{L}_{p^*}$ -space. Then

$$\Pi_{al.s.p}(X, Y) \subseteq \mathcal{D}_q(X, Y).$$

**Theorem 2.3.** (Extension Kwapień's Theorem) The following properties of Banach space  $X$  are equivalent.

- (i) The space  $X$  is isomorphic to a Hilbert space.
- (ii) For every  $1 < q \leq \infty$  and every Banach space  $Y$ ,  $\Pi_{al.s}(X, Y) \subseteq \mathcal{D}_q(X, Y)$ .
- (iii) For every Banach space  $Y$ ,  $\Pi_{al.s}(X, Y) \subseteq \mathcal{D}_2(X, Y)$ .

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**ON THE BISHOP-PHELPS-BOLLOBÁS THEOREM FOR BILINEAR FORMS FOR FUNCTION  
MODULE SPACES**

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**Abstract**

In this talk we study a version of the Bishop-Phelps-Bollobás theorem called Bishop-Phelps-Bollobás property for bilinear forms. Under appropriate conditions for a function module space  $X$  we prove that the pair  $(X, X)$  satisfies the BPBp for bilinear forms.

## 1 Introduction

Let  $E$  and  $F$  be Banach spaces. The Bishop-Phelps-Bollobás property for operators (BPBp for operators) has been defined in [1], is a version of the Bishop-Phelps-Bollobás Theorem and is related to the density of the set of norm attaining operators in the space of all bounded linear operators between  $E$  and  $F$ . Over the years, other versions of this theorem have appeared. In [3] the authors defined another version of this theorem called Bishop-Phelps-Bollobás property for bilinear forms (BPBp for bilinear forms) and proved that this property fails for bilinear forms on  $\ell_1 \times \ell_1$ . In [2] Acosta, Becerra-Guerrero, García and Maestre presented classes of spaces satisfying this property, such as, when the domain space  $E$  is an uniformly convex Banach space, then for every Banach space  $F$ , the pair  $(E, F)$  satisfies the BPBp for bilinear forms. It is known that the BPBp for bilinear forms on  $E \times F$  implies the BPBp for operators, and the converse is no longer true. Considering a function module space  $X$ , Grando and Lourenço [4], presented conditions for  $X$  such that the pair  $(\ell_1, X)$  satisfies the BPBp for operators. In this note, will be present some conditions to the function module space  $X$  such that the BPBp for bilinear forms is satisfied for the pair  $(X, X)$ .

## 2 Main Results

**Definition 2.1.** Let  $E$  and  $F$  be Banach spaces. We say that the pair  $(E, F)$  has the Bishop-Phelps-Bollobás property for operators (shortly BPBp for operators) if given  $\varepsilon > 0$ , there is  $\eta(\varepsilon) > 0$  such that whenever  $T \in S_{\mathcal{L}(E, F)}$  and  $x_0 \in S_E$  satisfy that  $\|Tx_0\| > 1 - \eta(\varepsilon)$ , then there exist a point  $u_0 \in S_E$  and an operator  $S \in S_{\mathcal{L}(E, F)}$  satisfying the following conditions

$$\|Su_0\| = 1, \quad \|u_0 - x_0\| < \varepsilon, \quad \text{and} \quad \|S - T\| < \varepsilon.$$

**Definition 2.2.** Let  $E$  and  $F$  be Banach spaces. We say that the pair  $(E, F)$  has the Bishop-Phelps-Bollobás property for bilinear forms (shortly BPBp for bilinear forms) if given  $\varepsilon > 0$ , there are  $\eta(\varepsilon) > 0$  and  $\beta(\varepsilon) > 0$  with  $\lim_{t \rightarrow 0} \beta(t) = 0$  such that for any  $A \in S_{\mathcal{L}^2(E \times F)}$  and  $(x_0, y_0) \in S_E \times S_F$  is such that that  $|A(x_0, y_0)| > 1 - \eta(\varepsilon)$ , then are  $B \in S_{\mathcal{L}^2(E \times F)}$  and  $(u_0, v_0) \in S_E \times S_F$  satisfying the following conditions

$$|B(u_0, v_0)| = 1, \quad \|u_0 - x_0\| < \beta(\varepsilon), \quad \|v_0 - y_0\| < \beta(\varepsilon) \quad \text{and} \quad \|B - A\| < \varepsilon.$$

**Definition 2.3.** Function Module is (the third coordinate of) a triple  $(K, (X_t)_{t \in K}, X)$ , where  $K$  is a nonempty compact Hausdorff topological space,  $(X_t)_{t \in K}$  a family of Banach spaces, and  $X$  a closed  $C(K)$ -submodule of the  $C(K)$ -module  $\prod_{t \in K}^\infty X_t$  (the  $\ell_\infty$ -sum of the spaces  $X_t$ ) such that the following conditions are satisfied:

1. For every  $x \in X$ , the function  $t \rightarrow \|x(t)\|$  from  $K$  to  $\mathbb{R}$  is upper semi-continuous.
2. For every  $t \in K$ , we have  $X_t = \{x(t) : x \in X\}$ .
3. The set  $\{t \in K : X_t \neq 0\}$  is dense in  $K$ .

**Theorem 2.1.** Let  $(K, (X_t)_{t \in K}, X)$  be a function module space. Suppose that for every  $x_t \in X_t$  there exists  $f \in X$  such that  $f(t) = x_t$  and  $\|f \leq x_t\|$  then the pair  $(X, X)$  satisfies the BPBp for bilinear forms.

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## EM DIREÇÃO A UM TEOREMA ESPECTRAL PARA SEMIGRUPOS CONVOLUTOS

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### Abstract

Seja  $k$  uma função localmente integrável em  $[0, \infty[$ . Semigrupos  $k$ -convolutos são operadores que incluem semigrupos e semigrupos integrados como casos particulares, e é parte da solução fraca de certas equações diferenciais funcionais de primeira ordem. Neste trabalho, o objetivo principal é obter uma versão do Teorema Espectral para o espectro de pontos, o espectro aproximado e o espectro residual, para um semigrupo  $k$ -convoluto.

### 1 Introdução

Seja  $A : D(A) \subseteq X \rightarrow X$  um operador linear e fechado, definido em um espaço de Banach  $X$ , cujo domínio não é necessariamente denso. Nosso interesse está focado no espectro de  $A$ , denotado por  $\sigma(A)$ . Em particular, se  $A$  é gerador de uma família resolvente  $R(t)$  e dado  $\lambda \in \sigma(A)$ , o objetivo é determinar o que podemos dizer sobre os elementos de  $\sigma(R(t))$ , ou seja, quais são as condições que permitem obter

$$\sigma(R(t)) = \{r(\lambda) : \lambda \in \sigma(A)\}, \quad (1)$$

para uma determinada função  $r(\cdot)$  que depende de  $R(t)$ . É conhecido da teoria que a igualdade (4) é satisfeita para os casos de semigrupo e semigrupo integrado, nas referências [3] e [2] respectivamente.

Para estabelecer os principais resultados deste trabalho, consideramos diferentes partes do espectro de um operador  $A$ , chamadas de espectro de pontos, aproximado e residual, definidos respectivamente por

$$\begin{aligned} \sigma_p(A) &= \{\lambda \in \mathbb{C} : \ker(\lambda I - A) \neq \{0\}\}, \\ \sigma_a(A) &= \{\lambda \in \mathbb{C} : (\lambda I - A) \text{ não é injetivo, ou } \overline{\text{ran}(\lambda I - A)} \text{ não é fechado}\}, \\ \sigma_r(A) &= \{\lambda \in \mathbb{C} : \ker(\lambda I - A) = \{0\} \text{ ou } \overline{\text{ran}(\lambda I - A)} \neq X\}. \end{aligned}$$

A partição do espectro fornecida acima é aplicada no seguinte contexto. Seja  $A$  um operador fechado e  $k \in L^1_{loc}(\mathbb{R}_+)$  tal que  $k(0) = 0$ , consideramos aqui a seguinte versão do problema abstrato de primeira ordem:

$$\begin{cases} u'(t) = Au(t) + k(t)x, \\ u(0) = 0, \end{cases}$$

onde  $x \in X$ , e cuja solução é dada pelo chamado semigrupo  $k$ -convoluto gerado por  $A$ , apresentado nas referências [1] e [3]. Este semigrupo é uma família fortemente contínua  $\{R(t)\}_{t \geq 0} \subset B(X)$  que satisfaz as seguintes propriedades:

1.  $R(t)x \in D(A)$  e  $R(t)Ax = AR(t)x$  para todo  $x \in D(A)$  e  $t \geq 0$ .
2.  $\int_0^t R(s)x ds \in D(A)$  para todo  $x \in X$  e  $t \geq 0$ , e  $R(t)x = \int_0^t k(s)x ds + A \int_0^t R(s)x ds$ .

## 2 Resultados Principais

**Teorema 2.1.** Seja  $\{R(t)\}_{t \geq 0}$  um semigrupo  $k$ -convoluto com gerador  $A$  em um espaço de Banach  $X$ . Então, temos

$$\sigma(R(t)) \cup \{0\} \supseteq \left\{ \int_0^t k(t-s)e^{\lambda s} ds : \lambda \in \sigma(A) \right\} \cup \{0\},$$

e as seguintes inclusões são certas:

$$\begin{aligned} \sigma_p(R(t)) \cup \{0\} &\supseteq \left\{ \int_0^t k(t-s)e^{\lambda s} ds : \lambda \in \sigma_p(A) \right\} \cup \{0\}, \\ \sigma_a(R(t)) \cup \{0\} &\supseteq \left\{ \int_0^t k(t-s)e^{\lambda s} ds : \lambda \in \sigma_a(A) \right\} \cup \{0\}. \end{aligned}$$

Além disso, se  $A$  é densamente definido, então

$$\sigma_r(R(t)) \cup \{0\} \supseteq \left\{ \int_0^t k(t-s)e^{\lambda s} ds : \lambda \in \sigma_r(A) \right\} \cup \{0\}.$$

**Prova:** Para mostrar a validade das inclusões acima, veja [5], Teoremas 5.3, 5.5, 5.6 e 5.7.

**Teorema 2.2.** Seja  $\{R(t)\}_{t \geq 0}$  um semigrupo  $k$ -convoluto com gerador  $A$ , então

$$\sigma_p(R(t)) \cup \{0\} = \left\{ \int_0^t k(t-s)e^{\lambda s} ds : \lambda \in \sigma_p(A) \right\} \cup \{0\}.$$

**Teorema 2.3.** Seja  $\{R(t)\}_{t \geq 0}$  um semigrupo  $k$ -convoluto gerado por um operador  $A$  que é também gerador de um  $C_0$ -semigrupo, então

$$\begin{aligned} \sigma_a(R(t)) \cup \{0\} &= \left\{ \int_0^t k(t-s)e^{\lambda s} ds : \lambda \in \sigma_a(A) \right\} \cup \{0\}, \\ \sigma_r(R(t)) \cup \{0\} &= \left\{ \int_0^t k(t-s)e^{\lambda s} ds : \lambda \in \sigma_r(A) \right\} \cup \{0\}. \end{aligned}$$

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## SOLUTIONS FOR FUNCTIONAL VOLTERRA-STIELTJES INTEGRAL EQUATIONS

ANNA CAROLINA LAFETÁ<sup>1</sup><sup>1</sup>Departamento de Matemática, UnB, DF, Brasil, lafeta.carol@gmail.com**Abstract**

In this work, we introduce a class of equations called functional Volterra–Stieltjes integral equations. This type of equations encompasses many other kinds of equations such as functional Volterra equations, functional Volterra equations with impulses, functional Volterra delta integral equations on time scales, functional fractional differential equations with and without impulses, among others. We present here results concerning local existence, uniqueness and prolongation of solutions.

**1 Introduction**

This presentation is based on the work [3]. Here, we are interested in a more general formulation of functional Volterra integral equations involving the so called Stieltjes integral given by

$$\begin{cases} x(t) = \phi(0) + \int_{\tau_0}^t a(t,s)f(x_s,s)dg(s), & t \geq \tau_0, \\ x_{\tau_0} = \phi, \end{cases} \quad (1)$$

where the integral in the right-hand side is understood in the sense of Henstock–Kurzweil–Stieltjes,  $\tau_0 \geq t_0$ ,  $\phi \in G([-r, 0], \mathbb{R}^n)$  and we assume the following conditions on the functions  $f$ ,  $a$  and  $g$ :

- (A1) The function  $g: [t_0, d] \rightarrow \mathbb{R}$  is nondecreasing and left-continuous on  $(t_0, d)$ .
- (A2) The function  $a: [t_0, d]^2 \rightarrow \mathbb{R}$  is nondecreasing with respect to the first variable and regulated with respect to the second variable.

- (A3) The Henstock–Kurzweil–Stieltjes integral

$$\int_{\tau_1}^{\tau_2} a(t,s)f(x_s,s)dg(s)$$

exists for each compact interval  $[\tau_0, \tau_0 + \sigma] \subset [t_0, d]$ , all  $x \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R}^n)$ ,  $t \in [t_0, d]$  and all  $\tau_0 \leq \tau_1 \leq \tau_2 \leq \tau_0 + \sigma$ .

- (A4) There exists a locally Henstock–Kurzweil–Stieltjes integrable function  $M: [t_0, d] \rightarrow \mathbb{R}^+$  with respect to  $g$  such that for each compact interval  $[\tau_0, \tau_0 + \sigma] \subset [t_0, d]$ , we have

$$\left\| \int_{\tau_1}^{\tau_2} (c_1 a(\tau_2, s) + c_2 a(\tau_1, s)) f(x_s, s) dg(s) \right\| \leq \int_{\tau_1}^{\tau_2} |c_1 a(\tau_2, s) + c_2 a(\tau_1, s)| M(s) dg(s),$$

for all  $x \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R}^n)$ , all  $c_1, c_2 \in \mathbb{R}$  and all  $\tau_0 \leq \tau_1 \leq \tau_2 \leq \tau_0 + \sigma$ .

- (A5) There exists a locally regulated function  $L: [t_0, d] \rightarrow \mathbb{R}^+$  such that for each compact interval  $[\tau_0, \tau_0 + \sigma] \subset [t_0, d]$ , we have

$$\left\| \int_{\tau_1}^{\tau_2} a(\tau_2, s)[f(x_s, s) - f(z_s, s)] dg(s) \right\| \leq \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| L(s) \|x_s - z_s\|_{\infty} dg(s),$$

for all  $x, z \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R}^n)$ , and all  $\tau_0 \leq \tau_1 \leq \tau_2 \leq \tau_0 + \sigma$ .

This type of equation also encompasses impulsive Volterra–Stieltjes integral equations and Volterra functional  $\Delta$ -integral equations.

## 2 Main Results

Our main results are the following.

**Theorem 2.1.** *Assume  $f: G([-r, 0], \mathbb{R}^n) \times [t_0, d] \rightarrow \mathbb{R}^n$  satisfies conditions (A3), (A4) and (A5),  $a: [t_0, d]^2 \rightarrow \mathbb{R}$  satisfies condition (A2) and  $g: [t_0, d] \rightarrow \mathbb{R}$  satisfies condition (A1). Then for all  $\tau_0 \in [t_0, d]$  and all  $\phi \in G([-r, 0], \mathbb{R}^n)$ , there exists a  $\sigma > 0$  and a unique solution  $x: [\tau_0 - r, \tau_0 + \sigma] \rightarrow \mathbb{R}^n$  of the initial value problem:*

$$\begin{cases} x(t) = \phi(0) + \int_{\tau_0}^t a(t, s)f(x_s, s)dg(s) \\ x_{\tau_0} = \phi. \end{cases} \quad (2)$$

In the next theorem, let conditions (B1)–(B5) be the same as conditions (A1)–(A5) but with  $d = +\infty$ .

**Theorem 2.2.** *Suppose  $f: G([-r, 0], \mathbb{R}^n) \times [t_0, +\infty) \rightarrow \mathbb{R}^n$  satisfies conditions (B3), (B4) and (B5),  $a: [t_0, +\infty)^2 \rightarrow \mathbb{R}$  satisfies condition (B2) and  $g: [t_0, +\infty) \rightarrow \mathbb{R}$  satisfies condition (B1). Then, for every  $\tau_0 \geq t_0$  and  $\phi \in G([-r, 0], \mathbb{R}^n)$ , there exists a unique maximal solution  $x: I \rightarrow \mathbb{R}^n$  of the equation (1), where  $I$  is a nondegenerate interval with  $\tau_0 - r = \min I$ . Also,  $I = [\tau_0 - r, \omega]$ , with  $\omega \leq +\infty$ .*

Moreover, besides presenting results that guarantee existence and uniqueness of local and maximal solutions, we also present the correspondences between equation (1) and impulsive Volterra–Stieltjes integral equations and Volterra functional  $\Delta$ -integral equations.

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# A CONJECTURA DE BESSE, ESPAÇO VÁCUO ESTÁTICO E ESPAÇO $\sigma_2$ -SINGULAR

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## Abstract

Chamamos métricas CPE (*Critical Point Equation*) os pontos críticos do funcional da curvatura escalar total restrito ao espaço de métricas com curvatura escalar constante de volume unitário. Neste trabalho, daremos uma condição necessária e suficiente para que uma métrica crítica seja Einstein em termos de espaços  $\sigma_2$ -singulares. Tal resultado melhora nosso entendimento sobre métricas CPE e a conjectura de Besse com um novo ponto de vista geométrico. Além disso, provamos que a condição CPE pode ser trocada pela condição de espaço vazio estático para caracterizar as variedades de Einstein fechadas em termos de espaços  $\sigma_2$ -singulares.

## 1 Introdução

Uma variedade Riemanniana  $(M^n, g)$  é dita ser Einstein se o tensor de Ricci é múltiplo da métrica  $g$ , i.e.,  $\text{Ric}_g = \lambda g$ , onde  $\lambda : M \rightarrow \mathbb{R}$ , em particular se  $(M^n, g)$  é conexa, então  $\lambda$  é constante. Em outras palavras,  $(M^n, g)$  é Einstein se o traço do tensor

$$\dot{\text{Ric}}_g = \text{Ric}_g - \frac{R_g}{n}g$$

é identicamente zero, onde  $\text{Ric}_g$  e  $R_g$  são as curvaturas de Ricci e escalar, respectivamente.

Sejam  $(M^n, g)$  uma variedade conexa, fechada de dimensão  $n \geq 3$ ,  $\mathcal{M}$  o espaço das métricas Riemannianas e  $S_2(M)$  o espaço dos 2-tensores simétricos em  $M$ . Fischer e Marsden, ver [3], consideraram a aplicação da curvatura escalar  $\mathcal{R} : \mathcal{M} \rightarrow C^\infty(M)$  que associa a cada métrica  $g \in \mathcal{M}$  sua curvatura escalar. Sejam  $\gamma_g$  a linearização da aplicação  $\mathcal{R}$  e  $\gamma_g^*$  a sua adjunta  $L^2$ -formal, então eles usaram que

$$\gamma_g(h) = -\Delta_g \text{tr}_g h + \delta_g^2 h - \langle \text{Ric}_g, h \rangle$$

e

$$\gamma_g^* f = \nabla^2 f - (\Delta f)g - f \text{Ric}_g,$$

onde  $\delta_g = -\text{div}_g$ ,  $h \in S_2(M)$ ,  $\nabla_g^2$  é a Hessiana e  $\Delta_g \text{tr}_g h$  é o Laplaciano do traço de  $h$ , no estudo da sobrejetividade da aplicação da curvatura escalar  $R_g$ , e ainda consideraram a equação de vácuo estática  $\gamma_g^*(f) = 0$ .

Nas últimas décadas, várias pesquisas tem sido feitas nestes espaços. O problema de classificação é uma questão fundamental, assim como os resultados de rigidez. O funcional de Einstein-Hilbert  $\mathcal{S} : \mathcal{M} \rightarrow \mathcal{R}$  é definido por:

$$\mathcal{S}(g) = \int_M R_g dv_g. \tag{1}$$

Em 1987 Besse conjecturou, ver [2], que os pontos críticos do funcional da curvatura escalar total (1), restrito a  $\mathcal{M}_1 = \{g \in M; R_g \in C \text{ e } \text{vol}_g(M) = 1\}$ , onde  $C = \{g \in M; R_g \text{ é constante}\} \neq \emptyset$ , precisam ser Einstein. Mais precisamente, a equação de Euler-Lagrange da ação Hilbert-Einstein restrita a  $\mathcal{M}_1$  pode ser escrita como a seguinte equação do ponto crítico (CPE)

$$\gamma_g^* f = \nabla_g^2 f - (\Delta_g f)g - f \text{Ric}_g = \dot{\text{Ric}}_g.$$

## 2 Resultados Principais

Nesta seção, serão apresentados alguns resultados.

**Teorema 2.1.** *Seja  $(M^n, g, f)$ ,  $n \geq 3$ , uma métrica CPE com função potencial não constante  $f$ .  $(M^n, g)$  é Einstein se, e somente se,  $f \in \text{Ker} \Lambda_g^*$ , onde  $\Lambda_g : S_2(M) \rightarrow C^\infty(M)$  é a linearização da  $\sigma_2$ -curvatura e  $\Lambda_g^*$  é a adjunta  $L^2$ -formal do operador, i.e.,  $(M^n, g, f)$  é um espaço  $\sigma_2$ -singular.*

Uma consequência é a seguinte

**Corolário 2.1.** *Seja  $(M^n, g, f)$ ,  $n \geq 3$ , uma métrica CPE com função potencial não constante  $f$ . Se  $f \in \text{Ker} \Lambda_g^*$ , então  $(M^n, g)$  é isométrica a esfera redonda com raio  $r = \left(\frac{n(n-1)}{R_g}\right)^{1/2}$  e  $f$  é uma autofunção do Laplaciano associada ao primeiro autovalor  $\frac{R_g}{n-1}$  em  $\mathbb{S}^n(r)$ . Além disso,  $\dim \text{Ker} \Lambda_g^* = n+1$  e  $\int_M f dv_g = 0$ .*

Além disso, provamos que se  $(M^n, g)$  é uma variedade Riemanniana fechada e  $\text{ker}_g \Lambda_g^* \cap \text{ker} \gamma_g^* \neq \{0\}$ , então  $(M^n, g)$  é uma variedade de Einstein. Portanto, ela é isométrica a esfera redonda  $\mathbb{S}^n$ .

**Teorema 2.2.** *Seja  $(M^n, g, f)$  um espaço vácuo estático, onde  $(M^n, g)$  é uma variedade Riemannina fechada de dimensão  $n \geq 3$ .  $(M^n, g)$  é Einstein se, e somente se, o espaço  $(M^n, g, f)$  é  $\sigma_2$ -singular. Se  $f$  é uma função não constante, então  $(M^n, g)$  é isométrica à esfera redonda  $\mathbb{S}^n$ , em outro caso  $(M^n, g)$  deve ser Ricci plana.*

Vale comentar que a abordagem utilizada para provar estes resultados são: obter a linearização de uma certa aplicação geométrica, depois calcular a adjunta  $L^2$ -formal dessa linearização e entender o núcleo dessa adjunta. Os detalhes podem ser vistos em [1].

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EIGENVALUE PROBLEMS FOR FREDHOLM OPERATORS WITH SET-VALUED  
PERTURBATIONS

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**Abstract**

By means of a suitable degree theory, we prove persistence of eigenvalues and eigenvectors for set-valued perturbations of a Fredholm linear operator. As a consequence, we prove existence of a bifurcation point for a non-linear inclusion problem in abstract Banach spaces. Finally, we provide applications to differential inclusions.

## 1 Introduction

The present paper is devoted to the study of the following eigenvalue problem with a set-valued perturbation:

$$\begin{cases} Lx - \lambda Cx + \varepsilon \phi(x) \ni 0 \\ x \in \partial\Omega. \end{cases} \quad (1)$$

Here  $L : E \rightarrow F$  is a Fredholm linear operator of index 0 between two real Banach spaces  $E$  and  $F$  such that  $\ker L \neq 0$ ,  $C$  is another bounded linear operator,  $\Omega$  is an open subset of  $E$  not necessarily bounded and containing  $0$ ,  $\phi : \overline{\Omega} \rightarrow 2^F$  is a locally compact, upper semi-continuous (u.s.c. for short) set-valued map of  $CJ$ -type, and  $\lambda, \varepsilon \in \mathbb{R}$  are parameters.

Problem (1) can be seen as a set-valued perturbation of a linear eigenvalue problem (which is retrieved for  $\varepsilon = 0$ ):

$$\begin{cases} Lx - \lambda Cx = 0 \\ x \in \partial\Omega. \end{cases} \quad (2)$$

So, it is reasonable to expect that, under suitable assumptions, solutions of (1) appear in a neighborhood of the eigenpairs  $(x, \lambda)$  of (2). In fact, we show that this is the case for the trivial eigenpairs  $(x, 0)$ , provided  $\dim(\ker L)$  is odd, the set  $\overline{\Omega} \cap \ker L$  is compact, and the following transversality condition holds:

$$\operatorname{im} L + C(\ker L) = F. \quad (3)$$

More precisely, we denote  $\mathcal{S}_0 = \partial\Omega \cap \ker L$  the set of trivial solutions of (2). We prove that there exist a rectangle  $\mathcal{R} = [-a, a] \times [-b, b]$  ( $a, b > 0$ ) and  $c > 0$  such that for all  $\varepsilon \in [-a, a]$  the set of real parameters  $\lambda \in [-b, b]$  for which (1) admits a nontrivial solution  $x \in E$  with  $\operatorname{dist}(x, \mathcal{S}_0) < c$  is nonempty and depends on  $\varepsilon$  by means of an u.s.c. set-valued map. Similarly, for all  $\varepsilon \in [-a, a]$  the set of vectors  $x \in E$  with  $\operatorname{dist}(x, \mathcal{S}_0) < c$  that solve (1) for some  $\lambda \in [-b, b]$  is nonempty and depends on  $\varepsilon$  by means of an u.s.c. set-valued map. This is usually referred to as a *persistence result* for eigenpairs. Using such persistence, we prove that  $\mathcal{S}_0$  contains at least one bifurcation point, i.e., a trivial solution  $x_0$  such that any neighborhood of  $x_0$  in  $E$  contains a nontrivial solution.

This type of investigation of nonlinear eigenvalue problems goes back to various papers in the last two decades. Here we extend the study of the problem to the case of a set-valued perturbation. Such an extension requires a more general degree theory for set-valued maps, which extends Brouwer's degree for nonlinear maps on  $C^1$ -manifolds.

Such a degree theory has been introduced in [1] and redefined in [2] by a precise notion of orientation for set-valued perturbations of nonlinear Fredholm maps between Banach spaces.

Our abstract results find a natural application to *differential inclusions*. Here we consider an ordinary differential inclusion with Neumann boundary conditions and an integral constraint:

$$\begin{cases} u'' + u' - \lambda u + \varepsilon \Phi(u) \ni 0 \text{ in } [0, 1] \\ u'(0) = u'(1) = 0 \\ \|u\|_1 = 1. \end{cases}$$

Here  $\Phi(u) : [0, 1] \rightarrow 2^{\mathbb{R}}$  is a set-valued map depending on  $u$ , to be chosen according to several requirements (three different examples will be presented). We shall prove that the transversality condition (3) holds, and hence the above problem admits at least one bifurcation point.

## 2 Main Results

**Theorem 2.1.** *Let  $\dim(\ker L)$  be odd, (3) hold, and  $\Omega_0$  be compact. Then, problem (1) has at least one bifurcation point.*

**Proof** We argue by contradiction: assume that  $\mathcal{S}_0$  contains no bifurcation points, i.e., for all  $x \in \mathcal{S}_0$  there exists an open neighborhood  $\mathcal{U}_x \subset E \times \mathbb{R} \times \mathbb{R}$  of  $(x, 0, 0)$ , such that for all  $(x, \varepsilon, \lambda) \in \mathcal{S} \cap \mathcal{U}_x$  we have  $(\varepsilon, \lambda) = (0, 0)$ . The family  $(\mathcal{U}_x)_{x \in \mathcal{S}_0}$  is an open covering of the compact set  $\mathcal{S}_0 \times \{(0, 0)\}$  in  $E \times \mathbb{R} \times \mathbb{R}$ , so we can find a finite sub-covering, which we relabel as  $(\mathcal{U}_i)_{i=1}^m$ .

Let  $a, b, c > 0$  be such that

$$B_c(\mathcal{S}_0) \times \mathcal{R} \subset \bigcup_{i=1}^m \mathcal{U}_i,$$

where as usual  $\mathcal{R} = [-a, a] \times [-b, b]$ . Thus, we have

$$\mathcal{S} \cap (B_c(\mathcal{S}_0) \times \mathcal{R}) = \mathcal{S}_0 \times \{(0, 0)\}$$

(i.e., there are no solutions in  $B_c(\mathcal{S}_0) \times \mathcal{R}$  except the trivial ones). By reducing  $a, b, c > 0$  if necessary, for all  $\varepsilon \in [-a, a] \setminus \{0\}$  there exist  $x \in \partial\Omega \cap B_c(\mathcal{S}_0)$ ,  $\lambda \in [-b, b]$  such that  $(x, \varepsilon, \lambda) \in \mathcal{S}$ , a contradiction.  $\square$

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## ON A CLASS OF ABEL DIFFERENTIAL EQUATIONS OF THIRD KIND

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Consider a class of Abel equations of third kind  $[d(x) + c(x)y^m]y'_x = a(x) + b(x)y$ . Suppose that each of them has the same constants  $m \in \mathbb{R} - \{-1\}$  and the same real continuous functions  $a(x)$ , then if there exists a certain functional relation among the variable coefficients  $b(x)$ ,  $c(x)$  and  $d(x)$ , we can construct new exact general solutions that are shared by all these equations. This result, which improves and generalizes earlier results from literature, is proved in the present work. Here the notation  $\cdot' = \frac{d}{dx}$  denotes the classical derivative with respect to the independent variable  $x$ .

**1 Introduction**

The Abel nonlinear differential equations have been widely studied, either calculating their solutions (see [1, 2]), or specifying their centers, or characterizing the behaviour of their solutions to obtain qualitative properties like blow up or exponential decay in finite or infinity time (see [3]). Particularly, when it comes to calculating their solutions, many authors search for functional relations between the variable coefficients and integrating factors that allow the construction of exact analytic solutions (see [1, 2]).

In 2020, by means of Poincaré compactification, Regilene Oliveira and Cláudia Valls [3] classified the topological phase portraits of the Abel equation of third kind

$$C(x)y^2y'_x = A(x) + B(x)y \quad (1)$$

(where the functions  $A(x)$ ,  $B(x)$  and  $C(x)$  are polynomials in  $x$ ) to understand the behaviour of their solutions. This problem becomes very hard when the number of parameters in the equation increases and we know that the analysis of particular solutions for the differential equations is very important for understanding the solutions sets of a differential equation and for assisting qualitative and numerical studies. Thus, to collaborate with future qualitative and numerical studies about cases with more parameters, we present a new theorem whose constructive demonstration leads to exact general solutions for the following more general case of equation (1)

$$[d(x) + c(x)y^m]y'_x = a(x) + b(x)y. \quad (2)$$

satisfying  $y = y(x)$ ,  $c(x), d(x) \in \mathcal{C}^1(x_1, x_2)$  and  $a(x), b(x) \in \mathcal{C}(x_1, x_2)$ , where  $x_1, x_2 \in \mathbb{R}$ .

In fact, a new direct analytic method is introduced to obtain these solutions for the general form of equation (2) with  $b(x), c(x), d(x) \neq 0$  and  $a(x)$  can be equals to 0 or not. For this, we propose a new functional relation between variable coefficients of equation (2) and we use an argument of integrating factors.

**2 Main Result**

In this section, we prove the following result:

**Theorem 2.1.** *For the general form of the Abel equation of third kind (2) with  $b(x), c(x), d(x) \neq 0$ , if their variable coefficients satisfy the functional relation*

$$c'_x d = c(b + d'_x) \quad (1)$$

then equation (2) admits the exact implicit general solution

$$y^{m+1} + \frac{(m+1)d(x)}{c(x)} \left\{ y - \frac{1}{\mu(x)} \left[ \int \frac{a(x)}{d(x)} \mu(x) dx + C \right] \right\} = 0, \quad (2)$$

where  $\mu(x) = \exp \left[ - \int \frac{b(x)}{d(x)} dx \right]$  is an integrating factor and  $C$  is an arbitrary constant of integration.

**Proof** The proof is resumed as follows: firstly, equation (2) can be rewritten in the form

$$y'_x + \frac{c(x)}{d(x)} y^m y'_x = \frac{a(x)}{d(x)} + \frac{b(x)}{d(x)} y. \quad (3)$$

By using differentiation rules, we deduce

$$\frac{c(x)}{d(x)} y^m y'_x = \frac{1}{m+1} \left\{ \left[ \frac{c(x)}{d(x)} y^{m+1} \right]' - \left[ \frac{c(x)}{d(x)} \right]'_x y^{m+1} \right\}.$$

If we insert the last equation in equation (3), we have

$$\left[ y + \frac{c(x)}{(m+1)d(x)} y^{m+1} \right]'_x = \frac{a(x)}{d(x)} + \frac{b(x)}{d(x)} \left\{ y + \frac{d(x)}{b(x)} \left[ \frac{c(x)}{(m+1)d(x)} \right]'_x y^{m+1} \right\}.$$

Now, if we consider the functional relation between the variable coefficients

$$\frac{d(x)}{b(x)} \left[ \frac{c(x)}{(m+1)d(x)} \right]'_x = \frac{c(x)}{(m+1)d(x)} \Rightarrow c'_x d = c(b + d'_x),$$

so we can assume

$$\psi = \psi(x) = y + \frac{c(x)}{(m+1)d(x)} y^{m+1} \quad (4)$$

such that we obtain the linear differential equation

$$\psi'_x - \frac{b(x)}{d(x)} \psi = \frac{a(x)}{d(x)}. \quad (5)$$

Multiplying both sides of equation (5) by the integrating factor  $\mu(x)$ , we get

$$(\mu\psi)'_x = \mu(x) \frac{a(x)}{d(x)} \Rightarrow \psi(x) = \frac{1}{\mu(x)} \left[ \int \frac{a(x)}{d(x)} \mu(x) dx + C \right].$$

Therefore, we use relation (4) for returning to the original dependent variable  $y = y(x)$ , so we obtain equation (2). This completes the resuming proof of the Theorem. In other words, the Theorem says that if each element of an equations set (2) satisfies relation (1) and has the same constants  $m \in \mathbb{R} - \{-1\}$  and the same real continuous functions  $a(x)$ , then all these elements (equations) have the same exact general solutions given by equation (2). ■

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## CONVERGENCE OF A LEVEL-SET ALGORITHM FOR SCALAR CONSERVATION LAWS

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### Abstract

In this paper we study the convergence of the level-set algorithm introduced by Aslam for tracking the discontinuities in scalar conservation laws in the case of linear or strictly convex flux function (2001, *J. Comput. Phys.* 167, 413–438). The numerical method is deduced by the level-set representation of the entropy solution: the zero of a level-set function is used as an indicator of the discontinuity curves and two auxiliary states, which are assumed continuous through the discontinuities, are introduced. Following the ideas of (2015 *Numer. Meth. for PDE* 31, 1310–1343), we rewrite the numerical level-set algorithm as a procedure consisting of three big steps: (a) initialization, (b) evolution and (c) reconstruction. In (a) we choose an entropy admissible level-set representation of the initial condition. In (b), for each iteration step, we solve an uncoupled system of three equations and select the entropy admissible level-set representation of the solution profile at the end of the time iteration. In (c) we reconstruct the entropy solution by using the level-set representation. Assuming that in the step (b) we can use a monotone scheme to approximate each equation we prove the convergence of the numerical solution of the level set algorithm to the entropy solution in  $L^p$  for  $p > 1$ . In addition, some numerical examples focused on the elementary wave interaction are presented.

## 1 Introduction

In this work we introduce a convergent numerical method for the Cauchy problem for a scalar conservation law:

$$u_t + (f(u))_x = 0 \text{ for } (x, t) \in Q_T := \mathbb{R} \times \mathbb{R}_+, \text{ with } u(x, 0) = u_0(x) \text{ for } x \in \mathbb{R},$$

where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is the flux function and  $u$  is the conserved variable. We consider the following data assumptions:  $u_0 \in L^\infty(\mathbb{R})$  and  $f(u) = au$  ( $a$  constant) or  $f \in C^2(\mathbb{R}, \mathbb{R})$  and  $f''(u) \geq \alpha > 0$ , for all  $u \in \mathbb{R}$  and some  $\alpha > 0$ . We recall that the weak solutions satisfy the jump-entropy conditions

$$[u]s = [f(u)], \quad s = \frac{dX}{dt}, \quad [u] = u_l - u_r, \quad [f(u)] = f(u_l) - f(u_r), \quad f'(u_l) > s(t) > f'(u_r),$$

through a discontinuity of  $u$  parameterized by  $(X(t), t)$ .

In order to introduce the numerical method, we consider some notation:  $\text{sgn}^+(x) = \mathbb{1}_{\mathbb{R}^+}(x)$  and  $\text{sgn}^-(x) = -\text{sgn}^+(-x)$ , where  $\mathbb{1}_A : X \rightarrow \{0, 1\}$  is defined by  $\mathbb{1}_A(x) = 1$  for  $x \in A$  and  $\mathbb{1}_A(x) = 0$  for  $x \in X - A$ ;  $a^+ = \max\{a, 0\}$  and  $a^- = \min\{a, 0\}$ ;  $\mathcal{P}_j^n = \text{sgn}^+((p_{j+1}^n - p_{j-1}^n)/2\Delta x)$  and  $\mathcal{E}_j^n = [1 - \text{sgn}^-(f'(u_{L,j}^n) - f'(u_{R,j}^n))] / 2$ , for  $(j, n) \in \mathbb{Z} \times \mathbb{N}$ . The numerical method called LS-scheme consist in three big steps:

**(I) Initialization step.** We consider a continuous function  $p_0 : \mathbb{R} \rightarrow \mathbb{R}$  such that it vanishes over the control volumes where the function  $u_0$  is discontinuous, and we can make the calculus of entropy admissible states  $\mathbf{w}^0$  and  $\mathbf{v}^0$  and the initial speed  $\mathbf{s}^0$ ;

**(II) Evolution step.** The evolution considers three intermediate states: (a) The intermediate states  $\mathbf{w}^{n+1/2}$  and  $\mathbf{v}^{n+1/2}$  are calculated by applying a monotone scheme with numerical flux  $g$ , i.e.  $w_j^{n+1/2} = w_j^n - \lambda(g(w_j^n, w_{j+1}^n) - g(w_{j-1}^n, w_j^n))$  and  $v_j^{n+1/2} = v_j^n - \lambda(g(v_j^n, v_{j+1}^n) - g(v_{j-1}^n, v_j^n))$ ; (b) The level set equation state  $\mathbf{p}^{n+1}$  is calculated by:  $p_j^{n+1} = p_j^n - \lambda(s_j^n)^+(p_j^n - p_{j-1}^n) - \lambda(s_j^n)^-(p_{j+1}^n - p_j^n)$ ; and (c) Using the notation  $\mathcal{P}_j^n$ , we introduce the discrete

left-right states and the extended discrete shock speed as follows:  $u_{L,j}^{n+1} = \mathcal{P}_j^{n+1} w_j^{n+1/2} + (1 - \mathcal{P}_j^{n+1}) v_j^{n+1/2}$ ,  $u_{R,j}^{n+1} = (1 - \mathcal{P}_j^{n+1}) w_j^{n+1/2} + \mathcal{P}_j^{n+1} v_j^{n+1/2}$  and  $s_j^{n+1} = \frac{f(u_{L,j}^{n+1}) - f(u_{R,j}^{n+1})}{u_{L,j}^{n+1} - u_{R,j}^{n+1}}$ ; and (d) using the indicator  $\mathcal{E}_j^n$ , we introduce the states  $\mathbf{w}^{n+1}$  and  $\mathbf{v}^{n+1}$  such that  $\mathbf{u}^{n+1}$  is consistent with the entropy condition:  $w_j^{n+1} = w_j^{n+1/2} + (v_j^{n+1/2} - w_j^{n+1/2})(1 - \text{sgn}^+(p_j^n))\mathcal{E}_j^n$  and  $v_j^{n+1} = v_j^{n+1/2} + (w_j^{n+1/2} - v_j^{n+1/2})\text{sgn}^+(p_j^n)\mathcal{E}_j^n$ ; and

**(III) Reconstruction step.** In this step we apply the definition of level set representation to reconstruct  $\mathbf{u}^{n+1}$  from  $\mathbf{p}^{n+1}$ ,  $\mathbf{w}^{n+1}$  and  $\mathbf{v}^{n+1}$ , i.e.  $u_j^{n+1} = \text{sgn}^+(p_j^{n+1}) w_j^{n+1} + (1 - \text{sgn}^+(p_j^{n+1})) v_j^{n+1}$ .

## 2 Main Result

**Theorem 2.1.** Consider the assumptions: (A1)  $f$  satisfies the hypothesis of strict convexity or linearity; (A2)  $u_0$  satisfies the hypothesis of  $L^\infty$  boundness; (A3)  $p_0$ ,  $v_0$  and  $w_0$  satisfy the requirements specified by the initialization step; (A4)  $g$  is a monotone flux; (A5) The functions  $w_\Delta$ ,  $v_\Delta$ ,  $p_\Delta$  and  $u_\Delta$  defined from  $\mathbb{R} \times \mathbb{R}_0^+$  are determined by the LS-scheme; and (A6)  $\lambda$  satisfies the CFL condition  $\lambda \|f'\|_{L^\infty([u_m, u_M])} < 1 - \xi$  with  $\xi \in ]0, 1[$ . Then, the numerical solution  $u_\Delta$  converges to  $u$ , the entropy solution of the Cauchy problem, in the strong topology of  $L_{\text{loc}}^p(\mathbb{R} \times \mathbb{R}_0^+)$  for all  $p > 1$  when  $\Delta x \rightarrow 0$ .

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## LAGRANGIAN-EULERIAN SCHEME FOR GENERAL BALANCE LAWS

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### Abstract

We present an extension of the Lagrangian-Eulerian method for solving general balance laws [2], by taking into account a nonlinear accumulation term and a non-conservative source. This approach generalizes the *no-flow curves* as in [1] and does not require the use of approximate/exact Riemann solvers. The scheme is easy to implement and provides fast, accurate and stable results. We present fully/semi-discrete methods and illustrate the robustness of the approach with numerical examples for the nontrivial Baer-Nunziato system [3].

## 1 No-flow fully/semi-discrete schemes for general balance laws

**The Fully Discrete Lagrangian Eulerian Scheme (FDLE).** Consider the following system

$$[A(u)]_t + [F(u)]_x = G(u) + [S(u)]_x + N(u)[B(u)]_x, \quad x \in \mathbb{R}, t > 0; \quad u(x, 0) = \eta(x), \quad u(x, t) : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^m. \quad (1)$$

After the discretization, we introduce the control volumes  $D_j^n = \{(t, x) / t^n \leq t \leq t^{n+1}, \varphi_j^n(t) \leq x \leq \varphi_{j+1}^n(t)\}$ , where

$$\frac{d\varphi_j^n(t)}{dt} = \frac{F(u)}{A(u)}, \quad t^n < t \leq t^{n+1}; \quad \varphi_j^n(t^n) = x_j. \quad (2)$$

The *no-flow curves* of the control volumes  $D_j^n$  are determined by the IVPs (2), which play an important role in the Lagrangian-Eulerian method. For numerical purposes, the solution of (2) can be approximate with a simple and robust first-order linearization, which gives us  $\varphi_j^n(t) = x_j^n + (t - t^n)f_j^n$ , where  $f_j^n := \frac{F(u)}{A(u)}$ . For the well-posedness of (2), we essentially require  $\frac{F(u)}{A(u)}$  as Lipschitz. In case of blow-up singularities in the term  $\frac{F(u)}{A(u)}$ , for real-world applications [2], we apply a flux-split modeling strategy [1], whenever necessary to naturally handle this situation.

Thus, by writing (1) in its divergence form and integrating over the control volume  $D_j^n$ , we are able to explore the properties of the no-flow curves through the divergent theorem (where  $I_j^n = \int_{D_j^n} \left[ G(u) + \frac{\partial S(u)}{\partial x} + N(u) \frac{\partial B(u)}{\partial x} \right] dx$ ):

$$\iint_{D_j^n} \nabla_{xt} \cdot \begin{bmatrix} F(u) \\ A(u) \end{bmatrix} dx = \int_{\varphi_j^n(t^{n+1})}^{\varphi_{j+1}^n(t^{n+1})} A(u(x, t^{n+1})) dx - \int_{x_j^n}^{x_{j+1}^n} A(u(x, t^n)) dx = I_j^n, \quad \text{where } A_j^n := \frac{1}{\Delta x} \int_{D_j^n} A(u) dx \quad (3)$$

is the cell averages and projecting back the results of (3) to the original discrete lattice in  $D_j^n$ , and approximating the integrals  $I_j^n$  with a simple and robust quadrature, we get the fully-discrete Lagrangian Eulerian scheme:

$$\begin{aligned} A_j^{n+1} &= \frac{A_{j-1}^n + 2A_j^n + A_{j+1}^n}{4} - \frac{\Delta t^n}{4} \left[ \frac{f_j^n + f_{j+1}^n}{\Delta x_j^{n+1}} (A_j^n + A_{j+1}^n) - \frac{f_{j-1}^n + f_j^n}{\Delta x_{j-1}^{n+1}} (A_{j-1}^n + A_j^n) \right] + \\ &+ \frac{1}{\Delta x} \left[ \left( \frac{\Delta x}{2} + f_j^n \Delta t^n \right) \frac{I_{j-1}^n}{\Delta x_{j-1}^{n+1}} + \left( \frac{\Delta x}{2} - f_j^n \Delta t^n \right) \frac{I_j^n}{\Delta x_j^{n+1}} \right], \end{aligned} \quad (4)$$

where  $\Delta x_j^{n+1} = \Delta x + (f_{j+1}^n - f_j^n)\Delta t$ . In order to recover the cell averages approximation of the original variables at each time step  $U_j^n := \frac{1}{\Delta x} \int_{D_j^n} u(x, t) dx$  we still need to solve the typically non-linear system  $A(U_j^n) = A_j^n$ .

**Semi Discrete Lagrangian Eulerian Scheme (SDLE).** Starting from (4) we can write

$$A_j^{n+1} = A_j^n - \frac{\Delta t}{\Delta x} \left[ \mathbb{F}_j^n - \mathbb{F}_{j-1}^n + \left( \frac{\Delta x}{2} + f_j^n \Delta t \right) \frac{\mathcal{I}_{j-1}^n}{\Delta x_{j-1}^{n+1}} + \left( \frac{\Delta x}{2} - f_j^n \Delta t \right) \frac{\mathcal{I}_j^n}{\Delta x_j^{n+1}} \right], \quad (5)$$

where  $\mathbb{F}_k^n = \frac{1}{4} \left[ \frac{\Delta x}{\Delta t} (A_k^n - A_{k+1}^n) + \Delta x \frac{(f_k^n + f_{k+1}^n)}{\Delta x_k^{n+1}} (A_k^n + A_{k+1}^n) \right]$ ;  $\mathcal{I}_k^n = \frac{I_k^n}{\Delta t}$  and  $k \in \{j, j+1\}$ . We stress that  $I_j^n = \mathcal{O}(\Delta t \Delta x)$ , so  $\mathcal{I}_j^n = \mathcal{O}(\Delta x)$ . Applying  $t \rightarrow 0$ , the derivative  $\frac{dA_j(t)}{dt} = \lim_{t \rightarrow 0} \frac{A_j^{n+1} - A_j^n}{\Delta t}$  can be replaced in (5) and due to the no-flow property  $\left[ \frac{\Delta x}{\Delta t} \right] \propto \frac{F(u)}{A(u)}$ , we can remove the blow-up singularity replacing  $\frac{\Delta x}{\Delta t}$  in (5) by a stability condition that depends on  $\frac{F(u)}{A(u)}$ , leading us to the semi-discrete Lagrangian–Eulerian scheme for balance laws:

$$\frac{dA_j(t)}{dt} = -\frac{1}{\Delta x} \left[ \mathcal{F}_j^n - \mathcal{F}_{j-1}^n + \frac{\mathcal{I}_j^n + \mathcal{I}_{j-1}^n}{2} \right], \quad (6)$$

where  $\mathcal{F}_j^n = \frac{1}{4} [b_{j+\frac{1}{2}} (A_j^n - A_{j+1}^n) + (f_j^n + f_{j+1}^n) (A_j^n + A_{j+1}^n)]$  and  $b_{j+\frac{1}{2}} = \max_j |f_j^n + f_{j+1}^n|$ .

**Numerical Experiments.** To illustrate the robustness of the methods **FDLE** and **SDLE**, we applied them to solve the Baer-Nunziato system from [3], which models a two-phase reactive flow in detonation systems:

$$\left\{ \begin{array}{lcl} [\bar{\alpha}]_t & = & -\bar{u}[\bar{\alpha}]_x + \mathcal{F} + \frac{\mathcal{C}}{\rho} \\ [\bar{\alpha} \bar{\rho}]_t + [\bar{\alpha} \bar{\rho} \bar{u}]_x & = & \mathcal{C} \\ [\bar{\alpha} \bar{\rho} \bar{u}]_t + [\bar{\alpha} (\bar{\rho} \bar{u}^2 + \bar{p})]_x & = & p [\bar{\alpha}]_x + \mathcal{M} \\ [\bar{\alpha} \bar{\rho} \bar{E}]_t + [\bar{\alpha} \bar{u} (\bar{\rho} \bar{E} + \bar{p})]_x & = & \bar{u} p [\bar{\alpha}]_x - p \mathcal{F} + \mathcal{E} \\ [\alpha \rho]_t + [\alpha \rho u]_x & = & -\mathcal{C} \\ [\alpha \rho u]_t + [\alpha (\rho u^2 + p)]_x & = & -p [\bar{\alpha}]_x - \mathcal{M} \\ [\alpha \rho E]_t + [\alpha u (\rho E + p)]_x & = & -\bar{u} p [\bar{\alpha}]_x + p \mathcal{F} - \mathcal{E} \end{array} \right. . \quad (7)$$

The variables are  $\alpha, \rho, p, u$ , the volume fraction, density, pressure and velocity of the gas phase as well as  $\bar{\alpha}, \bar{\rho}, \bar{p}, \bar{u}$  are these same quantities for the solid phase. The solutions are shown in Figure 1 and follow the full model and initial data as described in [3], with a very good agreement given by the methods **FDLE** and **SDLE** in coarse grids by comparing our numerical results along with the available exact solution in [3].



Figure 1: Solutions for  $\bar{\alpha}$  with 128 points at time  $t_M = 1$ : **FDLE** (left) and **SDLE** (right).

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## A POSITIVE LAGRANGIAN-EULERIAN SCHEME FOR HYPERBOLIC SYSTEMS

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In this work, we design a class of positivity preserving Semi-Discrete Lagrangian-Eulerian schemes for solving multidimensional initial value problems for scalar models and systems of conservation laws [2], based on the concept of no-flow curves [1]. The new scheme is genuinely multidimensional in the sense that is Riemann solver free which avoid dimensional splitting strategies. The full rigorous numerical analysis is carried out in [2]. In the general context of multidimensional hyperbolic systems of conservation laws, the scheme satisfies the positivity principle in the sense of the paper [3]. We also provide robust numerical examples to verify the theory and illustrate the scientific computing capabilities of the proposed approach in advanced modeling and simulation.

**1 Motivation and the Lagrangian-Eulerian formulation**

To proceed with the construction of the semi-discrete Lagrangian-Eulerian scheme, consider the scalar 1D problem

$$\frac{\partial u}{\partial t} + \frac{\partial H(u)}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad u(x, 0) = u_0(x), \quad u_0(x) \in L^\infty(R) \quad \text{where } H \in C^2(\Omega), H : \Omega \rightarrow \mathbb{R}, \quad (1)$$

and  $u = u(x, t) : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \Omega \subset \mathbb{R}$ . Following [1, 2], we obtain the fully discrete Lagrangian-Eulerian scheme,

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} [F(u_j^n, u_{j+1}^n) - F(u_{j-1}^n, u_j^n)], \quad \text{with a numerical flux function given by } F(u_j^n, u_{j+1}^n) = \quad (2)$$

$$\frac{1}{4} \left[ \frac{\Delta x}{\Delta t} (u_j^n - u_{j+1}^n) + \Delta x \frac{f_j^n + f_{j+1}^n}{\Delta x_j} (u_j^n + u_{j+1}^n) + \frac{\Delta x^2}{4} \frac{f_j^n + f_{j+1}^n}{\Delta x_j} ((u_x)_j^n - (u_x)_{j+1}^n) + \frac{\Delta x^2}{4\Delta t} ((u_x)_j^n + (u_x)_{j+1}^n) \right]. \quad (3)$$

Thanks to the *no-flow property*  $\left[ \frac{\Delta x}{\Delta t} \right] \propto [O(H(u)/u)]$ , (see [1] and  $u$  and  $H(u)$  given by (1)), we can remove the blow-up singularity of the numerical flux  $F(u_j^n, u_{j+1}^n)$  in (2)–(3) by replacing  $\frac{\Delta x}{\Delta t}$  with a stability condition that depends on  $O((H(u)/u))$ , which allows us to have  $\Delta t \rightarrow 0^+$  and produce an accurate approximation of the local speeds. We set  $\frac{\Delta x}{\Delta t} \propto [O((H(u)/u))]$  in (2)–(3) for a suitable function

$$b_{j+\frac{1}{2}} = b_{j+\frac{1}{2}}(f_j, f_{j+1}), \quad f_j \equiv \frac{H(u_j)}{u_j} \approx \frac{H(u)}{u} \quad \text{for each } j \in \mathbb{Z} \text{ per time step } [t^n, t^{n+1}]. \quad (4)$$

Thus, the new class of SDLE schemes for hyperbolic-transport initial value problems (1) is given by

$$\frac{d}{dt} u_j(t) = -\frac{1}{\Delta x} [\mathcal{F}(u_j, u_{j+1}) - \mathcal{F}(u_{j-1}, u_j)], \quad \mathcal{F}(u_j, u_{j+1}) = \frac{1}{4} \left[ b_{j+\frac{1}{2}} (u_{j+\frac{1}{2}}^- - u_{j+\frac{1}{2}}^+) + (f_j + f_{j+1}) (u_{j+\frac{1}{2}}^- + u_{j+\frac{1}{2}}^+) \right],$$

where  $\lim_{\Delta t \rightarrow 0} \mathcal{F}(u_j^n, u_{j+1}^n) \neq \infty$ , with  $u_{j+\frac{1}{2}}^- = u_j + \frac{\Delta x}{4} ((u_x)_j)$  and  $u_{j+\frac{1}{2}}^+ = u_{j+1} - \frac{\Delta x}{4} ((u_x)_{j+1})$ . The formal 2D extension of the semi-discrete scheme is straightforward and given by

$$\frac{d}{dt} u_{j,k}(t) = -\frac{\mathcal{F}_{j+1/2,k} - \mathcal{F}_{j-1/2,k}}{\Delta x} - \frac{\mathcal{G}_{j,k+1/2} - \mathcal{G}_{j,k-1/2}}{\Delta y}, \quad \text{for the following scalar conservation law} \quad (5)$$

$$u_t + H(u)_x + G(u)_y = 0, \quad u(x, y, 0) = u_0(x, y), \quad \text{where } H, G \in C^2, u_0(x, y) \in L_{loc}^\infty(\mathbb{R}^2). \quad (6)$$

The corresponding multidimensional numerical fluxes in the  $x$ - and  $y$ -directions are, respectively, given by

$$\begin{aligned} \mathcal{F}_{j+\frac{1}{2}, k} &= \frac{1}{4} \left[ b_{j+\frac{1}{2}, k}^x \left( u_{j+\frac{1}{2}, k}^- - u_{j+\frac{1}{2}, k}^+ \right) + (f_{j,k} + f_{j+1,k}) \left( u_{j+\frac{1}{2}, k}^- + u_{j+\frac{1}{2}, k}^+ \right) \right] \quad \text{and} \\ \mathcal{G}_{j, k+\frac{1}{2}} &= \frac{1}{4} \left[ b_{j, k+\frac{1}{2}}^y \left( u_{j, k+\frac{1}{2}}^- - u_{j, k+\frac{1}{2}}^+ \right) + (g_{j,k} + g_{j,k+1}) \left( u_{j, k+\frac{1}{2}}^- + u_{j, k+\frac{1}{2}}^+ \right) \right], \end{aligned} \quad (7)$$

where the discretized multi-D (2D) space-time no-flow curves [1], given by  $(u, H(u), G(u))$  as defined in (6)

$$f_{j,k} = \frac{H(u_{jk})}{u_{jk}} \quad \text{and} \quad g_{j,k} = \frac{G(u_{jk})}{u_{jk}}, \quad \text{with} \quad \left[ \frac{\Delta x}{\Delta t} \right] \propto [O(H(u)/u)] \quad \text{and} \quad \left[ \frac{\Delta y}{\Delta t} \right] \propto [O(G(u)/u)]. \quad (8)$$

The intermediate values are given by  $u_{j+1/2,k}^+ := u_{j+1,k}(t) - \frac{\Delta x}{4}(u_x)_{j+1,k}(t)$ ,  $u_{j+1/2,k}^- := u_{j,k}(t) + \frac{\Delta x}{4}(u_x)_{j,k}(t)$ ,  $u_{j,k+1/2}^+ := u_{j,k+1}(t) - \frac{\Delta y}{4}(u_y)_{j,k+1}(t)$ ,  $u_{j,k+1/2}^- := u_{j,k}(t) + \frac{\Delta y}{4}(u_y)_{j,k}(t)$ , where numerical derivatives  $(u_x)_{j,k}(t)$  and  $(u_y)_{j,k}(t)$  were computed via slope limiter approximations, and subject to the new no-flow CFL stability condition

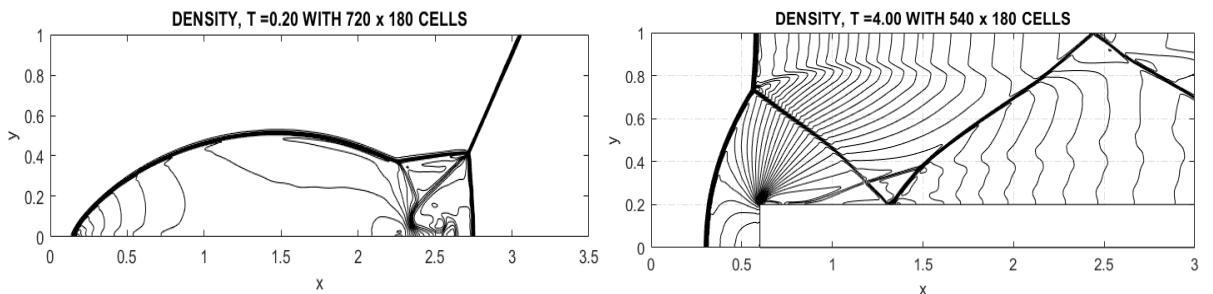
$$\max_{j,k} \left( \frac{\Delta t}{\Delta x} \max_{j,k} |f_{j,k}|, \frac{\Delta t}{\Delta y} \max_{j,k} |g_{j,k}| \right) \leq \frac{1}{4}, \quad \text{without the need to employ the eigenvalues.} \quad (9)$$

Therefore, the extension for systems is straightforward (see [2]), and we will apply this version for systems using the SDLE scheme (5)–(7) to numerically solve a 2D Euler system given by (e.g., [3]),

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, & (\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y = 0, \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y = 0, & E_t + (u(E + p))_x + (v(E + p))_y = 0, \end{cases} \quad (10)$$

where  $\rho$  is the mass density;  $u = u(x, y, t)$  and  $v = v(x, y, t)$ , the  $x$ - and  $y$ -components of the velocity, respectively and  $E = pe + \frac{1}{2}\rho(u^2 + v^2)$ . For a perfect gas,  $p = \rho e(\gamma - 1)$ , where constant  $\gamma$  denotes the ratio of specific heats; and  $e$ , the internal energy of the gas. In all tests, we consider  $\gamma = 1.4$  with the pre-and-post shock initial condition (left, Double Mach Reflection) and the initial condition throughout the channel (right, A Mach 3 wind tunnel with a step), where  $x_s(t) = 10t/\sin(\pi/3) + 1/6 + y/\tan(\pi/3)$  is the shock position for the initial data  $(\rho, p, u, v)_0^T =$

$$\begin{cases} (1.4, 1, 0, 0)^T, & x > x_s(0), \\ (8, 116.5, 4.125\sqrt{3}, -4.125)^T, & x \leq x_s(0), \end{cases} \quad \text{or} \quad \begin{cases} (1.4, 1, 3, 0)^T, & x \leq 0.6 \quad \text{and} \quad y \geq 0, \\ (0, 0, 0, 0)^T, & x > 0.6, \quad y > 0 \quad \text{and} \quad y \leq 0.2, \\ (1.4, 1, 3, 0)^T, & x > 0.6 \quad \text{and} \quad y > 0.2, \end{cases} \quad (11)$$



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A CONVERGENT FINITE DIFFERENCE METHOD FOR A TYPE OF NONLINEAR  
FRACTIONAL ADVECTION-DIFFUSION EQUATION

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### Abstract

In this work, we apply a numerical method based on finite differences to solve a type of nonlinear advection-diffusion fractional differential equation. The fractional operator considered is the fractional Riemann-Liouville derivative or the fractional Riesz derivative of order  $\alpha$ , with  $1 < \alpha \leq 2$ . The nonlinearity is of type  $u^\nu(x, t)$ , with  $\nu > 0$ , on which the spatial fractional derivative acts.

## 1 Introduction

In this work, we apply a finite difference method to solve a nonlinear advection-diffusion fractional equation of the form

$$\frac{\partial}{\partial t} u(x, t) = -a(x) \frac{\partial}{\partial x} u(x, t) + b(x) \frac{\partial^\alpha}{\partial |x|^\alpha} u^\nu(x, t), \quad (1)$$

where  $t > 0$ ,  $x \in I = [L, R] \subseteq \mathbb{R}$ , and  $a(x), b(x) \geq 0$  for all  $x \in I$ . Only positive exponents  $\nu$  are considered. The fractional operator  $\frac{\partial^\alpha}{\partial |x|^\alpha}$  is the fractional Riemann-Liouville derivative or the fractional Riesz derivative of order  $\alpha$  (for details, see Refs. [1, 2, 3]), where  $1 < \alpha \leq 2$ . For simplicity, in the main result, we only consider the case  $a(x) = b(x) = 1$  for all  $x$ .

The method used was introduced in [4], proposed for solving the nonlinear fractional diffusion equation

$$\frac{\partial}{\partial t} u(x, t) = c^2 \frac{\partial^\alpha}{\partial |x|^\alpha} u^\nu(x, t), \quad (2)$$

where the function  $u(x, t) \geq 0$  is unknown,  $c^2$  represents the diffusion coefficient, and the fractional operator is considered with  $1 < \alpha \leq 2$ . The exponent  $\nu \neq 1$  is usually related to unusual relaxation processes and allows us to deal with problems related to anomalous diffusion processes (see, for instance, Ref [1]). The implicit Euler method developed holds for  $\nu > 0$  and it is consistent and unconditionally stable, therefore convergent by Rosinger's Theorem [5], which is a nonlinear extension of the celebrated Lax-Richtmyer equivalence theorem. Such a method was also applied in a recent work [6].

## 2 Main Result

The numerical approach is the standard for finite differences. We consider a mesh with  $0 \leq j \leq M$  and  $0 \leq n \leq N$ . The exact solution  $u(x, t)$  evaluated in the grid point  $(x_j, t^n)$  is denoted by  $u_j^n$ . The boundary values of the domain are  $x_0 = L$  and  $x_M = R$ , and  $t^N = t$  denotes the final time. To denote the numerical solutions, we utilize the notation  $v(x_j, t^n)$  or  $v_j^n$ .

Assuming that  $\nu > 0$  and  $\delta > -1$ , such that  $\nu = 1 + \delta$ , we multiply Eq. (1) by  $(1 + \delta)u^\delta(x, t)$ , yielding

$$\frac{\partial}{\partial t} u^\nu(x, t) = -a(x) \frac{\partial}{\partial x} u^\nu(x, t) + \nu u^\delta(x, t) b(x) \frac{\partial^\alpha}{\partial |x|^\alpha} u^\nu(x, t). \quad (3)$$

In the grid points, we denote the weights  $\nu v^\delta(x_j, t^n)$  only by  $\delta_j^n := \nu (v^\delta)_j^n = \nu v^\delta(x_j, t^n)$ . Our main result is the implicit Euler method (4), given in the next Theorem.

**Theorem 2.1.** *The implicit Euler method*

$$-\lambda\delta_j^{n+1}w_0^\alpha(v^\nu)_{j+1}^{n+1} + (1 - \lambda\delta_j^{n+1}w_1^\alpha + \lambda_1)(v^\nu)_j^{n+1} - (\lambda\delta_j^{n+1}w_2^\alpha + \lambda_1)(v^\nu)_{j-1}^{n+1} - \lambda\delta_j^{n+1}\sum_{i=3}^{j+1} w_i^\alpha(v^\nu)_{(j+1)-i}^{n+1} = (v^\nu)_j^n \quad (4)$$

where  $\lambda = \frac{k}{h^\alpha}$ ,  $\lambda_1 = \frac{k}{h}$  and  $\delta_j^n = \nu (v^\delta)_j^n$ , to solve the Eq. (3) with  $1 < \alpha \leq 2$ ,  $\nu > 0$  and  $a(x) = b(x) = 1$ , on the finite domain  $L \leq x \leq R$ , together with a non-negative bounded initial condition  $u(x, 0) = u_0$  and boundary conditions  $u(x = L, t) = 0 = u(x = R, t)$  for all  $t \geq 0$ , based on the shifted Grünwald approximation (see it in [2] or in [3]), with  $p = 1$  and  $h = (R - L)/M$ , is consistent and unconditionally stable.

**Proof** It is sufficient to adapt the proof of Theorem 3.1 of [4], an extension for the nonlinear case of Theorem 2.7 of Ref. [7].  $\square$

**Corollary 2.1.** *The implicit Euler method (4) is convergent.*

**Proof** Just apply the Theorem presented by Rosinger in Ref. [5].  $\square$

We observe that one cannot apply directly the implicit Euler method (4) since the weights  $\delta_j^{n+1}$  are considered in the  $(n + 1)$ -step. To apply the method shown in Eq. (4), it is necessary to compute the value of the weights  $\delta_j^{n+1}$  in each time-step. We propose an iteration procedure in which each step is evaluated twice: using the known weights to one time-step ago, we evaluate the weights  $\delta_j$  to the next time-step, and then we return and evaluate the unknown function at the next time-step.

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# SOLUTION OF LINEAR RADIATIVE TRANSFER EQUATION IN HOLLOW SPHERE BY DIAMOND DIFFERENCE DISCRETE ORDINATES AND ADOMIAN METHODS

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## Abstract

In this work, a methodology to solve radiative transfer problems in spherical geometry without other forms of heat exchange is presented. The authors used a decomposition method, based on the Adomian routines, together with a diamond difference scheme. The algorithm is simple, highly reproducible and can be easily adapted to further problems or geometries. Also, the authors introduce a brief necessary criterion for convergence and consistency using an algebraic residual term analysis. The numerical results are compared with some classical and recent cases in the literature, along with a simplified version of a complete (fully coupled with heat exchange problem) case.

## 1 Introduction

In this work we present a hybrid methodology with application to a test case with a focus on formalism and a convergence criterion. One of the main objectives here is to take an initial step in this sense and present results that analyze and guarantee the convergence of the method by the quick decay of the algebraic residuals.

We consider the radiative transfer equation in spherical geometry for hollow sphere [1],

$$\frac{\mu}{r^2} \frac{\partial}{\partial r} \left[ r^2 I(r, \mu) \right] + \frac{1}{r} \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) I(r, \mu) \right] + I(r, \mu) = \left( 1 - \omega(r) \right) I_b(T) + \frac{\omega(r)}{2} \int_{-1}^1 p(\mu, \mu') I(r, \mu') d\mu', \quad (1a)$$

where  $r \in [R_1, R_2]$  is the optical space variable and  $\mu \in [-1, 1]$  is the direction cosine.  $R_1$  and  $R_2$  are the radii of the inner and outer spherical surfaces, respectively. Further,  $I(r, \mu)$  is the radiation intensity,  $I_b(T)$  is the black body radiation for temperature  $T$ ,  $\omega$  is the single scattering albedo and  $p(\mu, \mu')$  is the phase function [2]. The boundary conditions of Equation (1a) are

$$I(R_k, (-1)^{k+1} \mu) = \epsilon_k I_{bk}(T) + \rho_k \int_0^1 I(R_k, (-1)^k \mu') \mu' d\mu', \quad 0 < \mu \leq 1 \quad (1b)$$

for  $k \in \{1, 2\}$ , where  $\epsilon_1$  and  $\epsilon_2$  are the emissivities of the inner and outer surfaces, respectively. In the same way,  $\rho_1$  and  $\rho_2$  are the diffusive reflectivities for the inner and outer surfaces, respectively.  $I_{b1}(T)$  and  $I_{b2}(T)$  are the black body radiations for inner and outer surfaces in temperature  $T$ , respectively.

To solve (1) we implement the discrete ordinates method, evaluating the equations in certain  $\mu = \mu_m$ . The derivative with respect to  $\mu$  is approximated using a diamond difference scheme and the integral is evaluated using the Gauss-Legendre quadrature rule. The abscissas of this quadrature rule are the discrete ordinates  $\mu_m$ , and we define  $I(r, \mu_m) = I_m(r)$ . After, we used the decomposition method, which briefly consists in expanding  $I_m$  as an infinite series. For computational purposes and due to the necessity of the application of a numerical integration scheme to solve the recursive equations, we segment the domain in  $N + 1$  nodes  $r_i$ , define  $I_m^i = I_m(r_i)$  and the

decomposed solution writes

$$I_m^t = \sum_{j=0}^n \left( \mathcal{I}_m^t \right)_j. \quad (2)$$

We made a recursive system among the  $(\mathcal{I}_m^t)_j$  using (2) in (1) like

$$\mathbf{A}\mathcal{I}_j = \mathbf{B}\mathcal{I}_{j-1} \quad (3)$$

for  $j = 1, 2, \dots, n$ . By their solution and (2) we reconstruct  $I_m^t$ . Here,  $\mathcal{I}_j$  are  $(\mathcal{I}_m^t)_j$  in vector notation,  $\mathbf{A}$  and  $\mathbf{B}$  are constant two-dimensional arrays. Also, in (1) we considered the terms  $(1 - \omega(r)) I_b(T)$  and  $\epsilon_k I_{bk}(T)$  for  $j = 0$  only.

We demonstrate consistency of this numerical scheme by setting an upper bound to the residual term in (1), substituting (2), using some norm operations and proving it goes to zero as  $n$  increases. In addition, we present a necessary condition to the convergence of (2) using the divergence test.

## 2 Main Results

Several cases of [3] were successfully solved using the presented methodology, with small differences in the numerical results. Using (3) with  $j \rightarrow \infty$  would result in an infinite series in (2), yielding an exact representation of  $I_m^t$ , however using a truncated series, a remaining (residual) term remains from the substitution. Substituting (2) in (1), taking the maximum norm and using the triangle inequality, we obtain

$$\|\boldsymbol{\varepsilon}_n\|_\infty \leq \|\mathbf{C}\|_\infty \|\mathcal{I}_n\|_\infty. \quad (4)$$

where  $\boldsymbol{\varepsilon}_n$  is the vector notation for the residual term using the truncated series (2), and  $\mathbf{C}$  is a constant two-dimensional array. As  $\mathbf{C}$  does not vary with  $n$ ,  $\|\boldsymbol{\varepsilon}_n\|_\infty$  is majored by a constant scale of  $\|\mathcal{I}_n\|_\infty$ . In other words, if  $\|\mathcal{I}_n\|_\infty \rightarrow 0$ , then  $\|\boldsymbol{\varepsilon}_n\|_\infty \rightarrow 0$  and the method is consistent. Now, using norm operations and (3), we see that  $\|\mathcal{I}_n\|_\infty \rightarrow 0$  as  $j = n \rightarrow \infty$  if

$$\|\mathbf{A}\|_\infty > \|\mathbf{B}\|_\infty. \quad (5)$$

This inequality is also a necessary condition for convergence of the series in (2).

The combination of the Adomian decomposition method with the discrete ordinates method yield results that did not differ from the reference by a significant amount, and for the test cases, the processing time did not exceed five seconds in a domestic computer. Despite its simplicity, this research sets the basis for the complex and time demanding research towards the convergence of the Adomian decomposition method, absent in the literature. We are still developing closed and sufficient criteria for other problems involving the application of the Adomian decomposition method in transport problems, and we intend to do it for some usual non-linear terms, like the coupling with the heat diffusion equation.

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ANALYSIS RESULTS ON AN ARBITRARY-ORDER SIR MODEL CONSTRUCTED WITH  
MITTAG-LEFFLER DISTRIBUTION

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**Abstract**

Our recent works discuss the construction of a meaningful arbitrary-order SIR model. We believe that arbitrary-order derivatives may arise from potential laws in the infectivity and removal functions. This work intends to summarize previous results, as well as show new results on a model with Mittag-Leffler distribution. We emphasise our optimization process, the nonlocality of the model and the behavior near the lower terminal.

## 1 Introduction

Arbitrary-Order Calculus, commonly known as Fractional Calculus, is a great tool for describe the dynamic of many processes, mainly because of its “memory effect”. Generally, the models are obtained by replacing a integer derivative with an arbitrary-order one. Compartmental models, for example, have been widely studied with arbitrary orders. We investigate the use of arbitrary orders in SIR-type models, theoretically, analytically and numerically. Recalling that a model is constructed by modelling the physical process, we ask what features are maintained when exchanging the orders. Are consistent models established, regarding the definition of parameters, physical meaning etc.? We need to give attention to how, where, and why the arbitrary-orders interfere in the model.

## 2 Main Results

Arbitrary-Order Calculus, commonly known as Fractional Calculus, is a great tool for describe the dynamic of many processes, mainly because of its “memory effec”. Generally, the models are obtained by replacing an integer derivative with an arbitrary-order one. Compartmental models, for example, have been widely studied with arbitrary orders. We investigate the use of arbitrary orders in SIR-type models, theoretically, analytically and numerically. Recalling that a model is constructed by modelling the physical process, we ask what features are maintained when exchanging the orders. Are consistent models established, regarding the definition of parameters, physical meaning etc.? We need to give attention to how, where, and why the arbitrary orders interfere in the model.

## 3 Main Results

As discussed in [1], so far we have not been able to find a physical-based modelling that simply allows to change the orders of the derivatives. However, arbitrary orders can be obtained through potential laws in the infectivity and removal functions. We present in [2] a physical derivation of an arbitrary-order model, following [3], with the language of the Continuous Time Random Walks (CTRW). The individual's removal time from the infectious compartment follows a Mittag-Leffler distribution related to  $\alpha$ , while the parameter  $\beta$  is related to the infectivity function. The Riemann-Liouville derivative arises from the modelling and the arbitrary-order model with  $1 \geq \beta \geq \alpha > 0$  is given by (1)-(3), where  $\gamma(t)$  is the vital dynamic;  $\omega(t)$ , the extrinsic infectivity;  $N$ , the total population;  $\tau$ , a scale parameter and,  $\theta(t, t')$ , the probability that an infectious since  $t'$  has not died of natural death until  $t$ . If  $\beta = \alpha = 1$  and  $\gamma(t), \omega(t)$  are taken constants, we get the classic SIR model. Note that  $dN(t)/dt = 0$ , so

the population is constant. In [2], we revisited the work and used optimization to apply it to COVID-19 pandemic data.

$$\frac{dS(t)}{dt} = \gamma(t)N - \frac{\omega(t)S(t)\theta(t,0)}{N\tau^\beta} D^{1-\beta} \left( \frac{I(t)}{\theta(t,0)} \right) - \gamma(t)S(t), \quad (1)$$

$$\frac{dI(t)}{dt} = \frac{\omega(t)S(t)\theta(t,0)}{N\tau^\beta} D^{1-\beta} \left( \frac{I(t)}{\theta(t,0)} \right) - \frac{\theta(t,0)}{\tau^\alpha} D^{1-\alpha} \left( \frac{I(t)}{\theta(t,0)} \right) - \gamma(t)I(t), \quad (2)$$

$$\frac{dR(t)}{dt} = \frac{\theta(t,0)}{\tau^\alpha} D^{1-\alpha} \left( \frac{I(t)}{\theta(t,0)} \right) - \gamma(t)R(t), \quad (3)$$

We set up a *L1*-scheme based discretization to implementation on MATLAB and, for optimization, a Feasible Direction Interior Point Algorithm. In [4], we are doing parameter analysis in the model (1)-(3). Also, in [5], we are dealing with equilibrium, reproduction numbers, monotonicity and non-negativity, while in [6] we deal with pandemic data in Brazilian states. Here, we pretend to summarize the equilibrium characterization, present some considerations on the use of FDIPA and also deal with two points that were not discussed: the model is nonlocal and presents a nonintuitive behavior in the lower terminal. In fact, given the asymptotic behavior of the derivatives near the lower terminal, if  $\beta > \alpha$ , one have  $dI/dt < 0$  for  $t$  sufficiently small, as illustrated in Figure 1. About the nonlocality, in the classic SIR model, epidemiological parameters define the epidemic independently of time: for each point, there is a unique trajectory. However, this is not valid for arbitrary-order models. The formulation of the IVP disregards the past, but, once the model is nonlocal, this modifies the trajectory. We illustrate this considering  $N = 1000000$ , initial conditions  $S(0) = N - 1$ ,  $I(0) = 1$  and  $R(0) = 0$  and  $dt = 0.1$ . At time  $t = 90$ , we consider the initial condition given by  $S(90), I(90), R(90)$  and run the model again. In Figure 2, we have the equivalent trajectories for a maximum time  $T = 3000$ . The equilibrium is the same, but the trajectories are not.

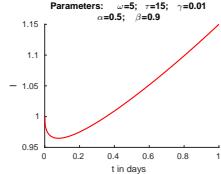


Figure 1: Behaviour Lower Terminal.

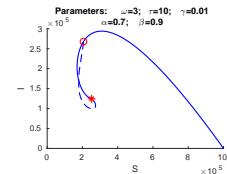


Figure 2: Change on trajectory.

So, one should to start the simulation of an epidemic on its beginning or be able to say about the past, what is a trick question. We aim that the deeper study of the equilibrium points and trajectories are fundamental to predict important features of the epidemic. By other hand, the mathematics of the Fractional Calculus' models is still a black box of surprises that the assembling between analytic and numerical studies can help to investigate.

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## NUMERICAL ANALYSIS FOR A THERMOELASTIC DIFFUSION PROBLEM IN MOVING BOUNDARY

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### Abstract

In this work, error estimates are shown for the semi-discrete and totally discrete problems of a linear thermoelastic diffusion model with a moving boundary, considering the null boundary condition. The resulting linear system is solved through three numerical methods: Coupled, Uncoupled with Predictor-Corrector and Uncoupled. The order of convergence obtained in the  $L^\infty(0, T; L^2(\Omega))$  and  $L^\infty(0, T; H_0^1(\Omega))$  norms is consistent with the theoretical results.

### 1 Introduction

Consider the following thermoelastic diffusion problem:

$$\left| \begin{array}{l} \rho \frac{\partial^2 u}{\partial t^2} - \alpha \frac{\partial^2 u}{\partial x^2} + \gamma_1 \frac{\partial \theta}{\partial x} + \gamma_2 \frac{\partial P}{\partial x} = f_1(x, t), \text{ in } \hat{Q}, \\ c \frac{\partial \theta}{\partial t} + d \frac{\partial P}{\partial t} - k \frac{\partial^2 \theta}{\partial x^2} + \gamma_1 \frac{\partial^2 u}{\partial x \partial t} = f_2(x, t), \text{ in } \hat{Q}, \\ \eta \frac{\partial P}{\partial t} + d \frac{\partial \theta}{\partial t} - \hat{h} \frac{\partial^2 P}{\partial x^2} + \gamma_2 \frac{\partial^2 u}{\partial x \partial t} = f_3(x, t), \text{ in } \hat{Q}, \\ u = \theta = P = 0, \text{ in } \hat{\Sigma}, \\ u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0(x), \quad P(x, 0) = P_0(x), \quad \forall x \in [-K(0), K(0)], \end{array} \right. \quad (1)$$

where the domain  $\hat{Q} \subset \mathbb{R}^2$  is defined by

$$\hat{Q} = \{(x, t) \in \mathbb{R} \times (0, T); x \in I_t\} \text{ and } I_t = \{x \in \mathbb{R}; -K(t) < x < K(t)\},$$

and its lateral boundary is given by  $\hat{\Sigma} = \bigcup_{0 < t < T} \{(-K(t), K(t))\} \times \{t\}$ , where  $K(t)$  is a function  $K : [0, T] \rightarrow \mathbb{R}$ , which defines the moving boundary. The solution  $\{u, \theta, P\}$  of the problem (1) is composed of functions which depend on the spatial and temporal variables  $x$  and  $t$ , respectively, that is,  $u = u(x, t)$ ,  $\theta = \theta(x, t)$ ,  $P = P(x, t)$ . The apostrophe symbol,  $'$ , denotes the partial derivative with respect to  $t$ .

We will show the theoretical and numerical aspects of the (1) problem in order to establish the error estimate of solutions in Sobolev spaces for the discrete problem and semi-discrete problem, as well as perform numerical simulations to analyze the behavior of the solution, order of numerical convergence and errors. For theoretical development, the Faedo-Galerkin method and interpolation theory results are applied to obtain an inequality from the approximate solution to the exact solution. To obtain the numerical solution, the finite element method is applied to the spatial variable and the finite differences method to the temporal variable. The system of ordinary differential equations resulting from the application of these methods is naturally coupled. For the numerical solution of this system, we developed the Coupled, Uncoupled with Predictor-Corrector and Uncoupled methods.

## 2 Main Results

### 2.1 Assumptions

The function  $K(t)$  and the constants satisfy the following assumptions,

$$(H1) \quad K \in C^3([0, T]; \mathbb{R}), \text{ with } K_0 = \min_{0 \leq t \leq T} K(t) > 0.$$

$$(H2) \quad |K''(t)| \leq cK(t), \forall t \in [0, T], \text{ where } c > 0.$$

$$(H3) \quad \text{There is a positive constant } K_1 < 1, \text{ such that } |K'(t)| \leq K_1.$$

$$(H4) \quad \rho, \alpha, \gamma_1, \gamma_2, k \text{ and } \hat{h} \text{ are strictly positive.}$$

$$(H5) \quad \text{The constants } c, d \text{ and } \eta \text{ are positive and satisfy } c\eta - d^2 > 0.$$

Condition (H5) is necessary for stability of the problem (1). The existence and uniqueness of the solution  $\{u, \theta, P\}$  of the problem (1) are known in the literature (see [1]).

### 2.2 Discrete error estimates in the $H_0^1(\Omega)$ and $L^2(\Omega)$ norms

Under the conditions (H1)-(H5) and conveniently chosen initial data, we show that the error estimate in the  $H_0^1(\Omega)$  norm has convergence order  $\mathcal{O}(h + \Delta t^2)$  and the error estimate in the  $L^2(\Omega)$  norm has an order of convergence  $\mathcal{O}(h^2 + \Delta t^2)$  (see [2]). Here,  $\Omega = (-1, 1)$ . Furthermore,  $h$  and  $\Delta t$  are, respectively, the mesh sizes of space and time.

### 2.3 Numerical simulations

Numerical simulations are shown using three numerical methods developed for the resulting system of ordinary differential equations: Coupled, Uncoupled with Predictor-Corrector, and Uncoupled. They all prove the theoretical convergence order, however with very different execution times (see [3, 3]).

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NUMERICAL ANALYSIS AND TRAVELLING WAVE SOLUTIONS FOR AN INTERNAL WAVE  
SYSTEM

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**Abstract**

In this work we focus on approximations of travelling wave solutions for a nonlinear system of Boussinesq type with a nonlocal operator. For numerical purposes, we focus on the case where the solutions are periodic functions in space with period  $2l > 0$ . Three approaches to calculate travelling waves are proposed and compared. For this an efficient and stable scheme for the nonlinear system, based on a von Neumann stability analysis for the linearized problem, is used to capture the evolution of approximate travelling wave solutions. Also, a scheme for the corrugated bottom version of the nonlinear system is proposed and validated.

## 1 Introduction

Asymptotic analysis of the Euler equations is a successful method for the study of internal ocean waves. For the case of intermediate depth for the lower layer and shallow upper layer, a strongly nonlinear model for internal waves was obtained in [1]. It describes the evolution of the interface  $\eta(x, t)$  between the fluids and the upper layer averaged horizontal velocity  $u(x, t)$ , where  $x$  and  $t$  represent the spatial and temporal variables, respectively. Considering a weakly nonlinear wave propagation regime, flat bottom and in nondimensional variables that system reads:

$$\begin{cases} \eta_t - [(1 - \alpha\eta)u]_x = 0, \\ u_t + \alpha u u_x - \eta_x = \sqrt{\beta} \frac{\rho_2}{\rho_1} \mathcal{T}_\delta [u]_{xt} + \frac{\beta}{3} u_{xxt}. \end{cases} \quad (1)$$

The pseudo-differential operator  $\mathcal{T}_\delta$  is the Hilbert transform on the strip and  $\alpha$ ,  $\beta$ ,  $\rho_1$  and  $\rho_2$  are positive constants where  $\alpha = \mathcal{O}(\beta)$ .

## 2 Main Results

For the discretization of the nonlinear system we consider its linearization around the zero equilibrium to implement the method of lines. A fourth order finite difference scheme for spatial derivatives and a spectral approach for the dispersive terms are considered in the semi-discretization and the classical fourth order Runge-Kutta (RK4) scheme is used for time advancing. The stability conditions obtained in a von Neumann analysis are validated in numerical tests and extended to the scheme for the nonlinear system (1) which includes the discretization of the nonlinear terms  $\alpha(\eta u)_x$  and  $\alpha uu_x$  as presented in [2].

Initially, we consider as initial condition for the nonlinear system the traveling wave solutions of the Intermediate Long Wave (ILW) equation and its regularized version (rILW). Both waves perform satisfactorily preserving their shapes in a given time interval as presented in [4]. In addition, three approaches to calculate travelling waves for the nonlinear system (1) are proposed in [3]. Supposing that the system admits a travelling wave solution, we define the variable  $y = x - ct$ , integrate both equations on  $y$  and consider the integration constant to be equal to zero to obtain

$$\begin{cases} -c\eta - (1 - \alpha\eta)u = 0, \\ -cu + \frac{\alpha}{2}u^2 - \eta + c\sqrt{\beta}\frac{\rho_2}{\rho_1}\mathcal{T}_\delta[u_y] + c\frac{\beta}{3}u_{yy} = 0. \end{cases} \quad (2)$$

For the first approach we reduce system (2) to an equation on  $\eta$  using a second order Taylor approximation. For the second approach we obtain an equation on  $u$  from system (2) using algebraic computations. For the third approach we take the complete system (2). In all approaches we consider the wave speed  $c$  as an unknown variable and complete the problem with the conservation law

$$\int_{-l}^l \eta(y) dy = d,$$

from system (1), where  $d$  is a constant.

The discretization is done considering an uniform grid on the interval  $[0, 2l]$  and the resulting system of equations is solved by the Newton's method using the traveling wave of the rILW equation as initial guess. The first approach did not improve the results of the initial guess and the second approach presented profiles that perform worse than the initial ones. On the other hand the third approach improved the results obtained in [4] and proved to be a good method to obtain travelling waves for system (1).

In the last part of the work we consider a more general case of the intermediate wave model (1), where there is an irregular topography on the bottom that can be described by a variable coefficient in a nonlinear system given in the computational domain  $(\xi, t)$  by

$$\begin{cases} \eta_t = \frac{1}{M(\xi)} [(1 - \alpha\eta)u]_\xi, \\ u_t + \frac{\alpha}{M(\xi)} u u_\xi - \frac{1}{M(\xi)} \eta_\xi = \frac{\rho_2}{\rho_1} \frac{\sqrt{\beta}}{M(\xi)} \mathcal{T}[u]_{\xi t} + \frac{\beta}{3M(\xi)} \left( \frac{u_{\xi t}}{M(\xi)} \right)_\xi. \end{cases} \quad (3)$$

This formulation allowed us to propose a numerical method for system (3) based on the one for the nonlinear system (1). The effects of the topography in the solutions and in the stability conditions are illustrated and compared with the solutions for the flat bottom cases.

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## IMPROVED REGULARITY FOR NONLOCAL ELLIPTIC EQUATIONS THROUGH ASYMPTOTIC PROFILES

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We obtain improved regularity for viscosity solutions of a special class of nonlocal  $\mathcal{L}_0(\sigma)$ -elliptic equations. More precisely, we introduce the notion of recession operator to the nonlocal setting, discuss its main features and then apply a compactness method to transfer regularity from asymptotic profiles. The role of this class is confined into some examples and a  $\mathcal{C}^{1,(\sigma-1)^-}$  regularity estimate.

**1 Introduction**

In this work, we are concerned with improved regularity estimates for viscosity solutions  $w$  of

$$\mathcal{I}[w] = f(x) \in L^\infty(B_1), \quad (1)$$

where  $\mathcal{I}$  is a  $\mathcal{L}_0(\sigma)$ -elliptic operator under an *asymptotic regime*. Here,  $\mathcal{L}_0(\sigma)$ -ellipticity means that the inequality

$$\mathcal{M}_{\mathcal{L}_0(\sigma)}^-[u-v](x) \leq \mathcal{I}[u](x) - \mathcal{I}[v](x) \leq \mathcal{M}_{\mathcal{L}_0(\sigma)}^+[u-v](x),$$

holds whenever they are well-defined, where

$$\mathcal{M}_{\mathcal{L}_0(\sigma)}^-[u](x) = \inf_{L \in \mathcal{L}_0(\sigma)} L[u](x), \quad \mathcal{M}_{\mathcal{L}_0(\sigma)}^+[u](x) = \sup_{L \in \mathcal{L}_0(\sigma)} L[u](x),$$

and  $\mathcal{L}_0(\sigma)$  is the class of linear operators of the form

$$L[u](x) = \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x)) \mathcal{K}(y) dy,$$

where  $\mathcal{K}$  is symmetric and satisfies  $\lambda(2-\sigma) \leq |y|^{n+\sigma} \mathcal{K}(y) \leq \Lambda(2-\sigma)$ .

The background for the search of qualitative properties of solutions to nonlocal equations of the type (1) have gained much attention since the seminal work [1] of Caffarelli and Silvestre in 2009. So far, a lot of interesting studies have emerged, for example, [2], from the same authors, in which they establish compactness results for equations like (1) and extend the results from [1] to equations with  $x$ -dependence. They use a scaling argument that relies on the invariance of the class of ellipticity. Therefore, to obtain Hölder regularity for the gradient of the solutions, they need to consider proximity to an operator on the subclass  $\mathcal{L}_1(\sigma) \subset \mathcal{L}_0(\sigma)$ , such that  $\nabla \mathcal{K}$  decay like the gradient of  $|y|^{-n-\sigma-1}$ .

A few years later, in [3], Kriventsov establishes  $\mathcal{C}^{1,\beta}$  for some unknown  $\beta$  to equations like (1) with  $x$ -dependence and merely bounded data using a perturbative argument. One can also find further regularity results when the main operator is concave, or with better boundary regularity datum.

The key novelty of our paper is to bring the notion of recession operator to the nonlocal context (it was already introduced in the local setting, see for example [3]), discuss it properly and adapt the strategy in [2] to transfer regularity from the asymptotic profile of the main operator in (1). Under a special *asymptotic regime*, we are able to improve regularity up to  $\mathcal{C}^{1,(\sigma-1)^-}$ , which is almost optimal. Furthermore, we don't need the presence of concavity or better regularity on the boundary values.

## 2 Main Results

Given a nonlocal operator  $\mathcal{I}$  as in (1), we define the nonlocal recession operator  $\mathcal{I}^*$  as a weakly subsequential limit (in the weak sense of [2]) of the family  $\{\mathcal{I}_\mu\}_{\mu>0}$  as  $\mu \rightarrow 0$ , where  $\mathcal{I}_\mu = \mu\mathcal{I}[\mu^{-1}-]$ . This can be done by stability results from [2], which assures that the nonlocal recession operator  $\mathcal{I}^*$  always exists and is, at least, on the same class of ellipticity of the original operator.

Under uniqueness assumptions of the convergence above, we are able to give some qualitative aspects to the nonlocal recession operator such as

- Homogeneity of degree 1;
- Rate of convergence of the family  $\{\mathcal{I}_\mu\}_{\mu>0}$  as  $\mu \rightarrow 0$ .

Besides that, through an adaptation of the compactness strategy written by Caffarelli and Silvestre in [2], we are able to transfer regularity from the limiting profile to the solutions of the original equation, and so, the following theorem comes out

**Theorem 2.1.** *Let  $\sigma > 1$ ,  $w \in \mathcal{C}(B_1) \cap L^\infty(\mathbb{R}^n)$  be a viscosity solution of (1) with  $\mathcal{I}$  in the following asymptotic regime:*

- (I)  *$\mathcal{I}$  is  $\mathcal{L}_0(\sigma)$ -elliptic and is translation invariant;*
- (II) *The recession operator  $\mathcal{I}^*$  has  $\mathcal{C}^\sigma$  estimates;*
- (III) *For all  $\mu > 0$  and  $0 < \lambda < 1$  there holds*

$$\lambda^\sigma \mu \mathcal{I}[\mu^{-1}v(\lambda^{-1}-)](\lambda-) = \lambda^\sigma \mu \mathcal{I}[\mu^{-1}\lambda^{-\sigma}v](-).$$

*Then  $w \in \mathcal{C}^{1,\alpha}(B_{1/2})$  for every  $\alpha \in (0, 1)$  such that  $1 + \alpha < \sigma$  and hold the estimate*

$$\|w\|_{\mathcal{C}^{1,\alpha}(B_{1/2})} \leq C \left( \|w\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1)} \right),$$

*where  $C$  depends on dimension,  $\alpha, \sigma$  and the constant from the  $\mathcal{C}^\sigma$  estimates of  $\mathcal{I}^*$ .*

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EXISTENCE AND NONEXISTENCE OF SOLUTION FOR A CLASS OF QUASILINEAR  
SCHRÖDINGER EQUATIONS WITH CRITICAL GROWTH

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**Abstract**

In this work, we study the existence and nonexistence of solution for the following class of quasilinear Schrödinger equations:

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = f(x, u) + h(x)g(u) \quad \text{in } \mathbb{R}^N,$$

where  $N \geq 3$ ,  $g : \mathbb{R} \rightarrow \mathbb{R}_+$  is a continuously differentiable function,  $V(x)$  is a potential that can change sign, the function  $h(x)$  belongs to  $L^{2N/(N+2)}(\mathbb{R}^N)$  and the nonlinearity  $f(x, s)$  is possibly discontinuous and may exhibit critical growth. In order to obtain the nonexistence result, we deduce a Pohozaev identity and the existence of solution is proved by means of a fixed point theorem.

## 1 Introduction

We consider the following class of quasilinear elliptic equations:

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = f(x, u) + h(x)g(u) \quad \text{in } \mathbb{R}^N, \quad (1)$$

where  $N \geq 3$ ,  $g : \mathbb{R} \rightarrow \mathbb{R}_+$  is a  $C^1$ -class function,  $V : \mathbb{R}^N \rightarrow \mathbb{R}$  is a potential that can change sign,  $f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$  is a measurable function, which may have critical growth and  $h \in L^{2N/(N+2)}(\mathbb{R}^N)$ ,  $h \neq 0$ . This work is based on the article [4].

The study of equation (P) is related with the existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$i\partial_t w = -\Delta w + W(x)w - \tilde{p}(x, |w|^2)w - \Delta[\rho(|w|^2)]\rho'(|w|^2)w, \quad (2)$$

where  $w : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$  is the unknown,  $W : \mathbb{R}^N \rightarrow \mathbb{R}$  is a given potential,  $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}$  and  $\tilde{p} : \mathbb{R}^N \times \mathbb{R}_+ \rightarrow \mathbb{R}$  are real functions satisfying appropriate conditions. Equation (2) is called in the current literature as *Generalized Quasilinear Schrödinger Equation* and it has been accepted as model in many physical phenomena depending on the function  $\rho$ . If we take  $g^2(u) = 1 + \frac{[(\rho(u^2))']^2}{2}$ , then (2) turns into quasilinear elliptic equation (P) (see [5]). Furthermore, depending on the form of the function  $g$ , equation (P) can take several forms already well known in the literature, such as

$$-\Delta u + V(x)u = p(x, u) \quad \text{in } \mathbb{R}^N,$$

$$-\Delta u + V(x)u - \Delta(u^2)u = p(x, u) \quad \text{in } \mathbb{R}^N,$$

$$-\Delta u + V(x)u - \gamma\Delta(|u|^{2\gamma})|u|^{2\gamma-2}u = p(x, u) \quad \text{in } \mathbb{R}^N,$$

or

$$-\Delta u + V(x)u - \Delta[(1+u^2)^{1/2}] \frac{u}{2(1+u^2)^{1/2}} = p(x, u) \quad \text{in } \mathbb{R}^N.$$

Motivated by these physical and mathematical aspects, equation (P) has attracted a lot of attention of many researchers and some existence and multiplicity results have been obtained, see [1, 2, 3, 5] and references therein. In this work, we intend to prove a Pohozaev identity for the equation

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = p(u) \quad \text{in } \mathbb{R}^N \quad (3)$$

and, as a consequence of the identity, to exhibit the critical exponent for this type of equation. Moreover, under convenient conditions on  $g(s)$ ,  $V(x)$ ,  $f(x, s)$ ,  $h(x)$  and by applying a fixed point theorem, we show that equation (P) admits at least one weak solution.

## 2 Main Results

**Theorem 2.1** (Pohozaev identity). *Suppose that  $u \in C^2(\mathbb{R}^N)$  is a classical solution for problem (3), with  $g \in C^1(\mathbb{R})$ ,  $V \in C^1(\mathbb{R}^N, \mathbb{R})$  and  $p \in C(\mathbb{R})$ . Moreover, assume that*

$$\int_{\mathbb{R}^N} [g^2(u)|\nabla u|^2 + (|x \cdot \nabla V(x)| + |V(x)|)u^2 + |P(u)|] dx < \infty, \quad (1)$$

where  $P(s) = \int_0^s p(\tau)d\tau$ . Then,  $u$  satisfies the identity

$$\begin{aligned} \frac{N-2}{2} \int_{\mathbb{R}^N} g^2(u)|\nabla u|^2 dx + \frac{N}{2} \int_{\mathbb{R}^N} V(x)u^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} [x \cdot \nabla V(x)]u^2 dx \\ = N \int_{\mathbb{R}^N} P(u) dx. \end{aligned} \quad (2)$$

**Proof** To show this Pohozaev identity see [4].

**Theorem 2.2** (Result of Existence). *Assume appropriate conditions on the functions  $g$ ,  $V$  and  $f$ . Furthermore, assuming that  $h \in L^{2N/(N+2)}(\mathbb{R}^N)$ , there exists  $\delta_0 > 0$  such that if  $\|h\|_{2N/(N+2)} \leq \delta_0$  then equation (P) has at least a weak solution.*

**Proof** To know the proper assumptions about  $g$ ,  $V$  and  $f$  and to prove this result of existence, see [4].

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# ON THE FRACTIONAL $P$ -LAPLACIAN CHOQUARD LOGARITHMIC EQUATION WITH EXPONENTIAL CRITICAL GROWTH: EXISTENCE AND MULTIPLICITY

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## Abstract

In this work we present two main results concerning the existence and multiplicity of non-trivial solutions for the Choquard logarithmic equation  $(-\Delta)_p^s u + |u|^{p-2}u + (\ln |\cdot| * |u|^p)|u|^{p-2}u = f(u)$  in  $\mathbb{R}^N$ , where  $N = sp$ ,  $s \in (0, 1)$ ,  $p > 2$ ,  $a > 0$ ,  $\lambda > 0$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous nonlinearity with exponential critical growth. Using variational techniques we guarantee the existence of a non-trivial solution at the mountain pass level and a non-trivial ground state solution under critical growth. Moreover, via genus theory, considering  $f$  with subcritical growth we prove the existence of infinitely many solutions.

## 1 Introduction

In the present work we are concerned with the existence and multiplicity of solutions to the following Choquard logarithmic equation

$$(-\Delta)_p^s u + |u|^{p-2}u + (\ln |\cdot| * |u|^p)|u|^{p-2}u = f(u) \quad \text{in } \mathbb{R}^N, \quad (1)$$

where  $N = sp$ ,  $s \in (0, 1)$ ,  $p > 2$ ,  $a = 1$ ,  $\lambda = 1$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, with primitive  $F(t) = \int_0^t f(\tau)d\tau$ , and  $(-\Delta)_p^s$  is the fractional  $p$ -Laplacian operator.

The results presented here are based in [1] and inspired by [2], where the authors studied existence and multiplicity results for the planar Schrödinger-Poisson system with polynomial nonlinearity, using variational techniques and, when needed, considering subgroups of the rotational group  $O(2) \subset \mathbb{R}^2$ . For a more detailed literature overview we refer to [1] and the references therein.

Based on works that deal with exponential growth nonlinearities, we ask the following conditions over  $f$ .

(f<sub>1</sub>)  $f \in C(\mathbb{R}, \mathbb{R})$ ,  $f(0) = 0$ , has critical exponential growth and  $\lim_{|t| \rightarrow 0} \frac{f(t)}{|t|^{p-2}t} = 0$ .

(f<sub>2</sub>) there exists  $\theta > 2p$  such that  $f(t)t \geq \theta F(t) > 0$ , for all  $t > 0$ .

(f<sub>3</sub>) there exist  $q > 2p$  and  $C_q > \frac{[2(q-p)]^{\frac{q-p}{p}}}{q^{\frac{q}{p}}} \frac{S_q^q}{\rho_0^{q-p}}$  such that  $F(t) \geq C_q|t|^q$ , for all  $t \in \mathbb{R}$ , where  $S_q$  is a suitable constant obtained from the Sobolev embeddings and  $\rho_0 > 0$  is a sufficiently small value.

Since the above conditions are sufficient to obtain the existence result, in order to get multiplicity we need a stronger geometry for the energy functional. In this sense, we will need to consider the following.

(f<sub>1</sub>')  $f \in C(\mathbb{R}, \mathbb{R})$ ,  $f$  is odd, has subcritical exponential growth and  $\lim_{|t| \rightarrow 0} \frac{f(t)}{|t|^{p-2}t} = 0$ .

(f<sub>3</sub>') there exists  $q > 2p$  and  $M_1 > 0$  such that  $F(t) \geq M_1|t|^q$ ,  $\forall t \in \mathbb{R}$ ,

(f<sub>4</sub>) the function  $t \mapsto \frac{f(t)}{t^{2p-1}}$  is increasing in  $(0, +\infty)$ .

The results presented here can be adapted to a general version of (P) considering a  $\mathbb{Z}^N$ -invariant (or asymptotically  $\mathbb{Z}^N$ -invariant) continuous potential  $a : \mathbb{R}^N \rightarrow \mathbb{R}$ .

## 2 Main Results

In the following, we only sketchy the proof. Detailed arguments can be found in [1].

**Theorem 2.1.** *Assume  $(f_1) - (f_3)$ ,  $q > 2p$  and  $C_q > 0$  sufficiently large. Then,*

(i) *Problem (P) has a non-trivial solution  $u \in X$  such that*

$$I(u) = c_{mp} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where  $\Gamma = \{\gamma \in C([0,1], X) ; \gamma(0), I(\gamma(1)) < 0\}$  and  $X \subset W^{s,p}(\mathbb{R}^N)$  is a Banach space.

(ii) *Problem (P) has a non-trivial ground state solution  $u \in X$ , that is,  $u$  satisfies*

$$I(u) = c_g = \inf\{I(v) ; v \in X \text{ is a solution of (P)}\}.$$

**Proof** From conditions  $(f_1) - (f_3)$  it is possible to prove that  $I$  has the mountain pass geometry and, consequently, there exists a Cerami sequence in the level  $c_{mp}$ . Let  $(u_n) \subset X$  be such sequence. Then, as  $I(u_n) \rightarrow c_{mp} > 0$ , one can prove that there exists a sequence  $(y_n) \subset \mathbb{Z}^N$  such that, up to a subsequence,  $\tilde{u}_n = u_n(\cdot - y_n) \rightarrow u$  in  $X$  for a non-trivial critical value of  $I$ . Moreover, considering the set  $\mathcal{K} = \{v \in X \setminus \{0\} ; I'(v) = 0\}$ , that is not empty by item (i). Hence, since  $c_g \in [-\infty, c_{mp}]$ , one can prove in an analogously way that a minimizing sequence for  $\mathcal{K}$  converges to a critical point of  $I$ ,  $u \in \mathcal{K}$ , satisfying  $I(u) = c_g$ .

□

**Theorem 2.2.** *Suppose  $(f'_1), (f'_2), (f'_3)$  and  $(f_4)$ . Then, problem (P) has infinitely many solutions.*

**Proof** From the hypothesis, we can prove that  $\varphi_u(t) = I(tu)$ , for all  $u \in X$  and  $t \in (0, +\infty)$  has the desired geometry that allow us to guarantee that the values  $c_k = \inf\{c \geq 0 ; \gamma_D(I^c) \geq k\}$ , for all  $k \in \mathbb{N}$ , where  $D = I^0$ ,  $I^c = \{u \in X ; I(u) \leq c\}$  for  $c \in \mathbb{R}$  and  $\gamma_D$  stands as the Krasnoselskii's Genus relative to  $D$ , are critical values of  $I$  and  $c_k \rightarrow +\infty$ , as  $k \rightarrow +\infty$ . □

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EQUAÇÃO DE CHOQUARD: EXISTÊNCIA DE SOLUÇÕES DE ENERGIA MÍNIMA PARA UMA CLASSE DE PROBLEMAS NÃO LOCAIS ENVOLVENDO POTENCIAIS LIMITADOS OU ILIMITADOS

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### Abstract

Neste trabalho, apresentamos um estudo sobre a existência de solução de energia mínima para a seguinte equação de Choquard não linear

$$\begin{cases} -\Delta u + V(x)u = \left( \int_{\mathbb{R}^N} \frac{Q(y)F(u(y))}{|x-y|^\mu} dy \right) Q(x)f(u(x)) \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{cases} \quad (1)$$

onde  $N \geq 3$ ,  $0 < \mu < N$ ,  $V \in \mathcal{C}(\mathbb{R}^N, [0, +\infty))$ ,  $Q \in \mathcal{C}(\mathbb{R}^N, (0, +\infty))$ ,  $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$  e  $F(t) = \int_0^t f(s)ds$ . A não-linearidade  $f : \mathbb{R} \rightarrow \mathbb{R}$  é contínua e tem comportamento assintoticamente linear no infinito. Além disso, sobre certas condições da variedade de Nehari  $\mathcal{N}$  e algumas outras desigualdades, estabelecidas no trabalho, a equação (1) tem uma solução de energia mínima.

## 1 Introdução

Para a elaboração deste trabalho, seguimos os artigos [1], [2] e [3]. Em 2018, os autores Sitong Chen e Shuai Yuan estudaram a seguinte equação de Choquard não linear dado em (1). Consequentemente, expressaram o conjunto  $E$  de modo que

$$E := \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < +\infty \right\},$$

afim de obter solução fraca para (1), cujo objetivo é encontrar um ponto crítico não trivial para  $\Phi$ . Por meio de métodos variacionais, podemos definir o funcional energia natural associado ao problema (1),  $\Phi : E \rightarrow \mathbb{R}$  por

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{Q(y)F(u(y))}{|x-y|^\mu} dy Q(x)F(u(x)) dx.$$

Mostramos que  $\Phi$  é de classe  $\mathcal{C}^1(E, \mathbb{R})$ . Recentemente, muitos pesquisadores começaram a se concentrar na equação de Choquard com a não linearidade não homogênea satisfazendo as seguintes hipóteses:

(F0)  $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$  satisfaz

$$\lim_{t \rightarrow 0} \frac{F(t)}{|t|^{(2N-\mu)/N}} = 0 \quad \text{e} \quad \lim_{|t| \rightarrow +\infty} \frac{F(t)}{|t|^{(2N-\mu)/(N-2)}} = 0,$$

existe uma constante  $C_0 > 0$  tal que

$$|tf(t)| \leq C_0 \left( |t|^{(2N-\mu)/N} + |t|^{(2N-\mu)/(N-2)} \right), \quad \forall t \in \mathbb{R}.$$

(Q1)  $V(x), Q(x) > 0$ ;  $\forall x \in \mathbb{R}^N$ ,  $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$  e  $Q \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N, \mathbb{R})$ ;

(Q2) Se  $\{A_n\} \subset \mathbb{R}^N$  é uma sequência do conjunto de Borel tal que a medida de Lebesgue para  $A_n$  é menor do que  $\delta$ ,  $\forall n$  e algum  $\delta > 0$ , então

$$\lim_{r \rightarrow +\infty} \int_{A_n \cap B_r^c(0)} [Q(x)]^{\frac{2N}{2N-\mu}} dx = 0, \text{ uniformemente em } n \in \mathbb{N};$$

(Q3)  $\frac{Q}{V} \in L^\infty(\mathbb{R}^N)$ ;

(Q4) Existe  $p \in (2, 2^*)$  tal que

$$\frac{[Q(x)]^{\frac{2N}{2N-\mu}}}{[V(x)]^{\frac{2^*-p}{2^*-2}}} \longrightarrow 0, \quad |x| \rightarrow +\infty.$$

(F1)  $\lim_{|t| \rightarrow +\infty} \frac{F(t)}{|t|} = +\infty$ ;

(F2)  $\lim_{t \rightarrow 0} \frac{F(t)}{|t|^{\frac{2N-\mu}{N}}} = 0$ , se vale (Q3); ou  $\lim_{t \rightarrow 0} \frac{F(t)}{|t|^{\frac{p(2N-\mu)}{2N}}} = 0$ , se vale (Q4).

(F3)  $\lim_{|t| \rightarrow +\infty} \frac{F(t)}{|t|^{\frac{2N-\mu}{N-2}}} < +\infty$ , se vale (Q3);  $\lim_{|t| \rightarrow +\infty} \frac{F(t)}{|t|^{\frac{p(2N-\mu)}{2N}}} < +\infty$ , se vale (Q4).

(F4)  $f(t)$  é não-decrescente em  $\mathbb{R}$ .

Neste trabalho, explicitamos a existência de solução de energia mínima por meio do conjunto de Nehari,

$$\mathcal{N} := \{u \in E \setminus \{0\} : \langle \Phi'(u), u \rangle = 0\}.$$

Para isso, garantimos que o funcional  $\Phi$  associado ao problema (1) possui a Geometria do Passo da Montanha, utilizando as hipóteses de crescimento assumidas sobre a função  $f$  acima, e asseguramos a existência de uma sequência limitada de Cerami  $(u_n)_{n \in \mathbb{N}}$  para  $\Phi$ . Por fim, evidenciamos que o funcional  $\Phi$  possui ínfimo e é atingido para algum elemento  $\bar{u} \in H^1(\mathbb{R}^N)$ .

Note que  $(V, Q) \in \mathcal{K}$  significa o conjunto de todos os potenciais  $V$  e  $Q$  tais que (Q1)-(Q4) são satisfeitas.

## 2 Resultado Principal

O principal resultado deste trabalho pode ser descrito da seguinte forma:

**Teorema 2.1.** *Suponha que  $(V, Q) \in \mathcal{K}$  e  $f \in C^1(\mathbb{R}, \mathbb{R})$  satisfazendo (F1)-(F4). Então (1) tem uma solução de energia mínima  $\bar{u} \in E$  tal que  $\Phi(\bar{u}) = \inf_{\mathcal{N}} \Phi > 0$ .*

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FULLY NONLINEAR SINGULARLY PERTURBED MODELS WITH NON-HOMOGENEOUS  
DEGENERACY

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**Abstract**

This work is devoted to studying non-variational, nonlinear singularly perturbed elliptic model enjoying a double degeneracy character with prescribed boundary value domain. For each  $\epsilon > 0$  fixed, we seek a non-negative function  $u^\epsilon$  satisfying

$$\begin{cases} [|\nabla u^\epsilon|^p + \alpha(x)|\nabla u^\epsilon|^q]\Delta u^\epsilon = \zeta_\epsilon(x, u^\epsilon) & \text{em } \Omega \\ u^\epsilon(x) = g(x) & \text{em } \partial\Omega, \end{cases}$$

in the viscosity sense for suitable  $p, q \in (0, \infty)$ ,  $\alpha, g$ , where  $\zeta_\epsilon$  one behaves singularly of order  $O(\epsilon^{-1})$  near  $\epsilon$ -surfaces. In such context, we establish that solutions are locally (uniformly) Lipschitz continuous, and they grow in a linear fashion.

Keywords: Singular perturbation methods, doubly degenerate fully non-linear operators, geometric regularity theory.

## 1 Introduction

In this work we shall develop to study (locally) sharp and geometric estimates of one-phase solutions to a singularly perturbed problem having a non-homogeneous degeneracy, whose mathematical model is given by: Fixed a parameter  $\epsilon \in (0, 1)$ , we would like to find  $u^\epsilon \geq 0$  viscosity solution to

$$\begin{cases} [|\nabla u^\epsilon|^p + \alpha(x)|\nabla u^\epsilon|^q]\Delta u^\epsilon = \zeta_\epsilon(x, u^\epsilon) & \text{in } \Omega \\ u^\epsilon(x) = g(x) & \text{on } \partial\Omega, \end{cases} \quad (1)$$

for a bounded domain  $\Omega \subset \mathbb{R}^n$ , where  $0 \leq g \in C^0(\partial\Omega)$ .

In a few words, under appropriated hypothesis on data, we show that, for  $\epsilon \rightarrow 0^+$ , the family  $\{u^\epsilon\}_{\epsilon>0}$  to (1) are asymptotic approximations to a one-phase  $u_0$  of an inhomogeneous non-linear free boundary problem, which arises in the mathematical formulation of some issues in flame propagation and combustion theory.

We suppose that the exponents  $p, q$  and the modulating function  $\alpha(\cdot)$  fulfil

$$0 < p \leq q < \infty \quad \text{e} \quad \alpha \in C^0(\Omega, [0, \infty)). \quad (2)$$

The reaction term, i.e.,  $\zeta_\epsilon: \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , represents the singular perturbation of the model. In this point, we are interested in a singular behaviour or order  $O(\frac{1}{\epsilon})$  along  $\{u_\epsilon \sim \epsilon\}$ . Hence, we are led to consider reaction terms fulfilling

$$\mathcal{B}_0 \leq \zeta_\epsilon(x, t) \leq \frac{\mathcal{A}}{\epsilon} \chi_{(0, \epsilon)}(t) + \mathcal{B}, \quad \forall (x, t) \in \Omega \times \mathbb{R}_+, \quad (3)$$

for nonnegative constants  $\mathcal{A}, \mathcal{B}_0, \mathcal{B} \geq 0$ . Note that  $\zeta_\epsilon \equiv 0$  satisfies (3). Then, we shall also impose the following non-degeneracy assumption in order to ensure that such a reaction term enjoys an authentic singular character:

$$I = \inf_{\Omega \times [t_0, T_0]} \epsilon \zeta_\epsilon(x, \epsilon t) > 0, \quad (4)$$

for some constants  $0 \leq t_0 < T_0 < \infty$ , where  $\mathcal{J}$  does not depend on  $\epsilon$ .

## 2 Main Results

**Teorema 2.1 (Optimal Lipschitz estimate ).** *Let  $\{u^\epsilon\}_{\epsilon>0}$  be a solution(1). Dado  $\Omega' \Subset \Omega$ , there exists a constant  $C_0$  depending on dimension and on  $\Omega'$ , but independet of  $\epsilon > 0$ , such that*

$$\|\nabla u^\epsilon\|_{L^\infty(\Omega')} \leq C_0.$$

Additionaly, if  $\{u^\epsilon\}_{\epsilon>0}$  is a uniformly bounded family,<sup>1</sup>, then it is pre-compact in the Lipschitz topology

From now on, we will label the distance of a point in the non-coincidence set  $x_0 \in \Omega \cap \{u^\epsilon > 0\}$  to the approximation boundary,  $\Gamma_\epsilon$ , pby

$$d_\epsilon(x_0) = \text{dist}(x_0, \{u^\epsilon \leq \epsilon\}).$$

**Teorema 2.2 (Linear growth).** *Let  $\{u^\epsilon\}_{\epsilon>0}$  be a Perron's solution to(1). There exists a positive constant  $c(\text{parameters}) > 0$  such that, for  $x_0 \in \{u^\epsilon > \epsilon\}$  and  $0 < \epsilon \ll d_\epsilon(x_0) \ll 1$ , there holds*

$$u^\epsilon(x_0) \geq c \cdot d_\epsilon(x_0).$$

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<sup>1</sup>Such a bound will be universal, i.e, it will depend only on data of the problem. Moreover, this statement is obtained via the application of Aleandroff-Bakelmann-Pucci estimate adapted to our context.

## A GEOMETRIC APPROACH TO INFINITY LAPLACIAN WITH SINGULAR ABSORPTIONS

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This work is dedicated to the study of a nonvational class of singular elliptic equations ruled by the infinity Laplacian. We provide optimal regularity  $C^{1,\beta}$  regularity along the singular free boundary, where in particular, the optimality is designed by the magnitude of the singularity. By virtue of a suitable penalization scheme and a construction of radial super-solutions, existence and non-degeneracy properties are provided where fine geometric consequences for the singular free boundary are further obtained.

**1 Introduction**

The main purpose of this work is to study geometric and qualitative properties for nonnegative viscosity solutions of the following singular free boundary problem

$$\begin{cases} \Delta_\infty u = u^{-\gamma} & \text{in } \Omega \cap \{u > 0\}, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \quad (1)$$

with  $0 \leq \gamma < 1$ ,  $\Omega \subset \mathbb{R}^n$  is a bounded smooth domain and  $\varphi \geq 0$  is a given smooth boundary data. This type of singular free boundary problem brings extra difficulties. In this case, the *nonhomogeneity term blows up* along the a priori unknown set  $\partial\{u > 0\}$  – called free boundary. In order to circumvent this issue, we shall deal with the penalized problem

$$\begin{cases} \Delta_\infty u = \mathcal{B}_\varepsilon(u)u^{-\gamma} & \text{in } \{u > 0\}, \\ u = \varphi_\varepsilon & \text{on } \partial\Omega, \end{cases}$$

where the term  $\mathcal{B}_\varepsilon(s)$  is a suitable approximation for the function  $\chi_{\{s>0\}}$ , and for each parameter  $\varepsilon > 0$  the source term  $\mathcal{B}_\varepsilon(s)s^{-\gamma}$  is a Lipschitz function defined on  $\mathbb{R}$ . In addition, the boundary data  $\varphi_\varepsilon$  is assumed approximating  $\varphi$  given in (1).

**2 Main Results**

In view of this, we shall obtain regularity estimates for limiting solutions  $u$  of (1), given by

$$u = \lim_{\varepsilon \rightarrow 0} u_\varepsilon$$

Nonetheless, this approach still brings a delicate issue: since nontrivial nonnegative solutions  $u_\varepsilon$  may not present free boundaries  $\partial\{u_\varepsilon > 0\}$ , it would not be reasonable to get estimates at the free boundary  $\partial\{u > 0\}$  as limits of the ones at  $\partial\{u_\varepsilon > 0\}$ . This is well observed in the radial examples that satisfy

$$\inf_{x \in \Omega} u_\varepsilon(x) \gtrsim \varepsilon.$$

In order to overcome this problem, we prove the main ingredient of our analysis: a new oscillation estimate at floating level sets. The innovative feature concerns an interrelation between radii and appropriate nonzero level sets. More specifically, we prove the following theorem:

**Theorem 2.1 (Optimal oscillation estimates for floating level sets).** *There exist constants  $C$  and  $\kappa_0$  depending on universal parameters, with no dependence on  $\varepsilon > 0$ , such that if  $v$  is a nonnegative viscosity solution of*

$$\Delta_\infty v = \mathcal{B}_\varepsilon(v)v^{-\gamma} \quad \text{em } B_1$$

*then, for any*

$$v(x)^{\frac{1}{\alpha}} \leq \kappa \leq \kappa_0,$$

*there holds*

$$\sup_{B_\kappa(x)} v \leq C\kappa^\alpha,$$

*for the exponent*

$$\alpha = \frac{4}{3 + \gamma}.$$

Thanks to the Theorem (??), it was possible to prove that at points on the free boundary  $\partial\{u > 0\}$ , limiting solutions are precisely of the class  $C^{1,1-\alpha}$ . Surprisingly, such regularity is essentially superior than regularity results involving nonsingular infinity Laplacian equations, even considering the class of infinity harmonic functions. Consequently, this result allowed us to obtain estimates of non-degeneracy and geometric measurement property for the free boundary  $\partial\{u > 0\}$ .

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EXISTÊNCIA DE SOLUÇÕES POSITIVAS PARA O P-LAPLACIANO FRACIONÁRIO  
ENVOLVENDO NÃO LINEARIDADE CÔNCAVO CONVEXA

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**Abstract**

Neste trabalho, estabelecemos a existência de soluções positivas para o problema elíptico com não linearidades do tipo côncavo-convexa, dado por

$$\left\{ \begin{array}{l} (-\Delta)_p^s u + V(\varepsilon x) |u|^{p-2} u = \lambda f(\varepsilon x) |u|^{q-2} u + g(\varepsilon x) |u|^{r-2} u, \text{ em } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N). \end{array} \right. \quad (\tilde{P}_\varepsilon)$$

onde  $\varepsilon, \lambda > 0$  são parâmetros positivos,  $N > ps$  com  $s \in (0, 1)$  fixado,  $1 < q < p < r < p_s^*$ , e  $p_s^* = \frac{Np}{N-ps}$ . Consideramos hipóteses adequadas sobre as funções  $f$  e  $g$ , e para o potencial  $V$ , para concluir um resultado de existência de soluções positivas para o problema acima utilizando a conhecida variedade de Nehari. Mais especificamente demonstramos a existência de uma solução ground state positiva em  $\mathcal{N}_\varepsilon^+$  e outra solução positiva em  $\mathcal{N}_\varepsilon^-$ .

## 1 Introdução

Este trabalho é um recorte do trabalho final de dissertação o qual foi motivado pelo artigo dos autores Qingjun Lou e Hua Luo, [1]. Para demonstrarmos o resultado de existência de soluções para o problema (1), consideramos as seguintes hipóteses sobre as funções  $f$  e  $g$ :

$$(F) \quad f \geq 0, \not\equiv 0, f \in L^{\tilde{q}}(\mathbb{R}^N) \cap C(\mathbb{R}^N), \quad (\tilde{q} = \frac{r}{r-q}) \text{ onde } |f|_{\tilde{q}} > 0 \text{ e}$$

$$f_{\max} := \max_{x \in \mathbb{R}^N} f(x) = 1;$$

$$(G) \quad g \text{ é uma função contínua, positiva e definida em } \mathbb{R}^N. \text{ Além disso, } g(x) \leq 1 \text{ para todo } x \in \mathbb{R}^N.$$

Para o potencial  $V$  consideramos a seguinte hipótese:

$$(V) \quad V \in C(\mathbb{R}^N, \mathbb{R}) \text{ e satisfaz}$$

$$V_\infty := \liminf_{|x| \rightarrow +\infty} V(x) > V_0 := \inf_{x \in \mathbb{R}^N} V(x) > 0.$$

A hipótese (V) é muito comum em trabalhos dessa natureza e foi introduzida em [3] por Rabinowitz. Neste trabalho Rabinowitz demonstrou que se o potencial é coercivo, é possível garantir a existência de imersões compactas do espaço de trabalho para o espaço  $L^t(\mathbb{R}^N)$ , com  $t \in [2, 2^*]$ , onde  $2^* = \frac{2N}{N-2}$ , mesmo trabalhando sobre um domínio ilimitado. Ressaltamos que este resultado não se aplica em nosso caso, pois existem potenciais satisfazendo (V) que não são coercivos. Portanto a perda da compacidade foi um dos principais problemas abordados.

Ao problema (1) associamos o seguinte funcional energia

$$I_\varepsilon(u) = \frac{1}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dx dy + \frac{1}{p} \int_{\mathbb{R}^N} V(\varepsilon x) |u(x)|^p dx - \frac{\lambda}{q} \int_{\mathbb{R}^N} f(\varepsilon x) |u(x)|^q dx - \frac{1}{r} \int_{\mathbb{R}^N} g(\varepsilon x) |u(x)|^r dx.$$

Uma vez que estamos com potenciais gerais, afim de recuperar algumas propriedades importantes, definimos o seguinte espaço de trabalho:

$$X_\varepsilon = \left\{ u \in W^{s,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x) |u|^p dx < \infty \right\}.$$

Neste momento ressaltamos que as principais informações sobre o operador p-Laplaciano fracionário e o espaço de Sobolev fracionário  $W^{s,p}(\mathbb{R}^N)$ , foram obtidas de um modo geral por [2]. Para iniciarmos a construção do resultado, foi necessário definirmos as fibras ligadas ao funcional  $I_\varepsilon$  as quais podem ser escritas do seguinte modo,  $\gamma_u : \mathbb{R}_*^+ \rightarrow \mathbb{R}$  e para cada função  $u$  fixada, temos que  $\gamma_u(t) = I_\varepsilon(tu)$ . Na sequência introduzimos a famosa variedade de Nehari, dada por

$$\mathcal{N}_\varepsilon = \{u \in X_\varepsilon \setminus \{0\} : \gamma'_u(1) = 0\}.$$

Para alcançarmos nosso resultado, dividimos a variedade acima em três novos conjuntos,

$$\begin{aligned} \mathcal{N}_\varepsilon^+ &= \left\{ u \in \mathcal{N}_\varepsilon ; \gamma''_u(1) > 0 \right\}, \\ \mathcal{N}_\varepsilon^- &= \left\{ u \in \mathcal{N}_\varepsilon ; \gamma''_u(1) < 0 \right\}, \\ \mathcal{N}_\varepsilon^0 &= \left\{ u \in \mathcal{N}_\varepsilon ; \gamma''_u(1) = 0 \right\}, \end{aligned}$$

e exibimos condições suficientes para que  $\mathcal{N}_\varepsilon^0$  seja vazia. Deste modo, utilizando a coercividade do funcional  $I_\varepsilon$  sobre  $\mathcal{N}_\varepsilon$ , fomos capazes de demonstrar a existência de uma sequência de Palais-Smale em cada uma das variedades  $\mathcal{N}_\varepsilon^+$  e  $\mathcal{N}_\varepsilon^-$ , as quais nos forneceram um resultado de existência de solução para o problema (1). Na sequência, regularizamos estas soluções via iteração de Moser e por fim aplicarmos o princípio do máximo forte de onde estabelecemos a existência de soluções positivas para o problema (1).

## 2 Resultado Principal

O principal resultado deste trabalho pode ser descrito da seguinte forma:

**Teorema 2.1.** *Seja  $0 < \lambda < \frac{q}{p}\lambda_0$ , onde  $\lambda_0$  é um parâmetro suficientemente pequeno. Suponha ainda que  $f, g$  e  $V$  satisfaçam as condições (F), (G), e (V). Então o problema (1) possui pelo menos 2 soluções positivas, uma em  $\mathcal{N}_\varepsilon^+$  e outra em  $\mathcal{N}_\varepsilon^-$ .*

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## STABILIZED HYBRID FINITE ELEMENT METHODS FOR THE HELMHOLTZ PROBLEM

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### Abstract

<sup>1</sup> Stabilized hybrid finite element methods are proposed for the Helmholtz problem with Robin's condition using two different types of multipliers. These multipliers (continuous or discontinuous) are introduced to weakly impose continuity at the interfaces of the finite elements. We presented numerical results illustrating the great stability , precision and robustness of these formulations adopting polynomial spaces for the pressure and multipliers.

## 1 Introduction

Acoustic waves (sound) are small pressure fluctuations in an understandable fluid. These oscillations interact in such a way that the energy spreads through the medium. Assuming a linear constitutive law and considering the propagation of harmonic waves over time, we obtain the Helmholtz equation whose solutions depend on a parameter  $\kappa$ , called wave number [1], which characterizes the frequency of oscillations of harmonic solutions. As analyzed by Ihlenburg and Babuska [5], the finite element method with linear approximations presents adequate asymptotic behavior, with optimal convergence rates, only for extremely refined meshes, which obey the condition  $\kappa^2 h \leq 1$ , which makes this approach unviable for real problems with high numbers of waves  $\kappa$ . Loula and Fernandez [6] proposed a Petrov-Galerkin (**QOPG**) method whose weight functions are obtained by minimizing a local least squares functional truncation error. This method has good properties of stability, precision, generality and robustness.

To determine better approximations methods of discontinuous finite elements (**DG**) have been proposed. Despite the advantages offered by the **DG** methods, due to its formulation complexity, computational implementation and a high number of degrees of freedom have been proposed hybridizations for the **DG** methods in order to derive new finite element methods with better stability characteristics and reduced computational cost but preserving the robustness and flexibility of the **DG** Methods, [2].

We will study the Helmholtz equation

$$-\Delta p - \kappa^2 p = f, \quad \text{em } \Omega \quad (1)$$

with Robin's condition,

$$-\nabla p \cdot \mathbf{n} + i\kappa p = g, \quad \text{em } \partial\Omega \quad (2)$$

where  $\Omega \subset \mathbb{R}^2$  is a polygonal domain. We present three methods stabilized hybrid finite element methods, two with Lagrange multiplier associated with pressure, one with continuous multiplier denoted **LDGC-P** and another discontinuous denoted **LDGD-P**, and the third with multiplier associated with speed denoted **LDGF-P**.

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## 2 Main Results

The obtained approximations present reduced numerical pollution, optimal rates of convergence, flexibility and robustness. Numerical studies have shown that with the same number of global degrees of freedom, the **LDGC-P** method is more accurate than the **LDGD-P** method. In the case of the **LDGD-P** method projected, the choices  $Q_2 - p_1$  and  $\mathbb{P}_2 - p_1$  can recover the optimal convergence rates for the primal variable. It was also observed that this projection applied to the **LDGC-P** method, with a continuous multiplier, does not have the same stability and precision as the **LDGD-P** method, particularly with triangular elements. We also analyzed a stabilized primal hybrid formulation **LDGD-F**, where the multiplier is associated with the flow, as in the classic primal hybrid formulation of Raviart and Thomas [9] and in its stabilized version proposed by Ewing, Wang and Yang [3]. Compared with the formulation **LDGD-P** we can see that, choosing the degree  $s$  of the polynomial approximation of the multipliers equal or greater than the degree  $l$  of the polynomial approximation of the primal variable, that is  $s \geq l$ , the hybrid methods **LDGD-P** and **LDGD-F** provide the same approximations for the primal variable  $p_h$ . However, for  $s = l - 1$  these approaches differ. Results of convergence studies show that, for this choice  $s = l - 1$ , the **LDGD-P** method can present optimal convergence rates for the primal variable  $p_h$  when is used a projection of the terms of edges in the multiplier spaces. Already the **LDGD-F** method presented optimal convergence rates of the primal variable  $p_h$ , for the choice  $s = l - 1$ , without the need of this projection.

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## PROBLEMA QUASELINEAR DE AUTOVALOR COM NÃO-LINEARIDADE DESCONTÍNUA

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Neste trabalho apresentaremos um resultado de existência e não-existência de soluções positivas para uma classe de problemas quaselineares de autovalor. Mostramos a existência de uma aplicação contínua  $\Lambda : [0, a_*] \rightarrow \mathbb{R}$  tal que o gráfico dessa aplicação define a região de existência e não-existência de soluções positivas. A ferramenta principal usada são os métodos variacionais para funcionais localmente Lipschitz nos espaços de Orlicz-Sobolev.

**1 Introdução**

Neste presente trabalho estamos interessados em soluções positivas para a seguinte classe de problemas quaselineares

$$\begin{cases} -\Delta_\Phi u = \lambda f(x, u)\chi_{[u \geq a]} & \text{em } \Omega, \\ u = 0 & \text{sobre } \partial\Omega, \end{cases} \quad (1)$$

onde  $\Omega \subset \mathbb{R}^N$  é um domínio limitado,  $N \geq 2$ ,  $a$  e  $\lambda$  são parâmetros positivos,  $\chi$  é a função característica,  $f$  é uma função contínua satisfazendo condições apropriadas e  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  é uma N-função dada por  $\Phi(t) = \int_0^{|t|} s\phi(s)ds$ , onde  $\phi : (0, +\infty) \rightarrow (0, +\infty)$  é uma função de classe  $C^1$ . Com o intuito de utilizar métodos variacionais, inspirados por Fukagai e Narukawa [2], assumimos que  $\phi$  satisfaça as seguintes condições:

$(\phi_1)$   $(\phi(t)t)' > 0$ ,  $t > 0$ ;

$(\phi_2)$  existem  $l, m \in (1, N)$  com  $m \in [l, l^*)$  e  $l^* = \frac{lN}{N-l}$ , tais que

$$l \leq \frac{\Phi'(t)t}{\Phi(t)} \leq m \quad t > 0;$$

$(\phi_3)$  existem  $k_0, k_1 > 0$  tais que

$$k_0 \leq \frac{\Phi''(t)t}{\Phi'(t)} \leq k_1 \quad t > 0;$$

$(\phi_4)$   $\phi$  é uma função monótona não-decrescente em  $(0, \infty)$ ;

Da mesma forma, assumiremos que  $f : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$  é uma função contínua verificando:

$(f_1)$  para  $x \in \Omega$ ,  $f(x, t) \leq 0$  para  $t \leq 0$ , e  $f(x, t) > 0$  para  $t > 0$ ;

$(f_2)$  existe  $C_0 > 0$  tal que

$$f(x, t)t \leq C_0\Phi(t);$$

$(f_3)$   $\lim_{t \rightarrow +\infty} \frac{f(x, t)t}{\Phi(t)} = 0$ , uniformemente em  $\bar{\Omega}$ ;

$(f_4)$   $\lim_{t \rightarrow 0} \frac{f(x, t)t}{\Phi(t)} = 0$ , uniformemente em  $\bar{\Omega}$ ;

( $f_5$ ) para cada  $x \in \Omega$ , a função  $f(x, \cdot)$  é não-decrescente em  $\mathbb{R}_+$ .

Observamos que o problema (1), para o caso  $a = 0$ , é exatamente o estudo por Fukagai e Narukawa em [2]. Contudo, para  $a > 0$  a existência de soluções positivas não é tão simples, pois estamos com não-linearidade descontínua, logo temos que trabalhar com a teoria de funcionais Lipschitz e gradientes generalizados.

Enfatizamos que entendemos por **solução** do problema (1), uma função  $u_{\lambda,a} \in W_0^{1,\Phi}(\Omega)$  satisfazendo:

- i)  $||[u_{\lambda,a} \geq a]|| > 0$ ;
- ii) existe  $\zeta(\cdot, u_{\lambda,a}) \in L_{\tilde{\Phi}}(\Omega)$  tal que

$$\int_{\Omega} \phi(|\nabla u_{\lambda,a}|) \nabla u_{\lambda,a} \nabla v dx = \lambda \int_{\Omega} \zeta(x, u_{\lambda,a}) v dx, \quad v \in W_0^{1,\Phi}(\Omega).$$

Além disso,  $\zeta(x, u_{\lambda,a}) \in \partial F_a(x, u_{\lambda,a})$  q.t.p.  $x \in \Omega$ , onde  $F_a(x, t) = \int_0^t \chi_{[s \geq a]} f(x, s) ds$ .

Ademais, temos um segundo sentido de solução, o qual é apoiada por Gasiński e Papageorgiou em [1]. Dizemos que  $u_{\lambda,a}$  é uma **S-solução** para o problema (1) se,  $u_{\lambda,a}$  é solução de (1) e  $||[u_{\lambda,a} = a]|| = 0$ .

## 2 Resultado Principal

**Teorema 2.1.** Suponha que  $\phi$  e  $f$  são funções satisfazendo ( $\phi_1$ )-( $\phi_4$ ) e ( $f_1$ )-( $f_5$ ), respectivamente. Existem uma constante  $a_* > 0$  e uma aplicação contínua não-decrescente  $\Lambda : [0, a_*] \rightarrow \mathbb{R}_+$ , tais que, para cada  $a \in [0, a_*]$ :

- i) para todo  $\lambda \in (0, \Lambda(a))$ , o problema (1) não possui solução;
- ii) para  $\lambda = \Lambda(a)$ , o problema (1) possui pelo menos uma S-solução positiva;
- iii) para todo  $\lambda > \Lambda(a)$ , o problema (1) possui pelo menos duas soluções positivas  $u_{\lambda,a}$  e  $v_{\lambda,a}$  satisfazendo  $v_{\lambda,a} \leq u_{\lambda,a}$  e  $v_{\lambda,a} \neq u_{\lambda,a}$  em  $\Omega$ , onde  $u_{\lambda,a}$  é uma S-solução.

**Prova:** Inicialmente mostramos que para cada  $a_* > 0$  existe  $\lambda_* := \lambda(a_*) > 0$  (de modo que se  $a_* \rightarrow +\infty$ , então  $\lambda_* \rightarrow +\infty$ ) tal que para todo  $(a, \lambda) \in [0, a_*] \times [\lambda_*, +\infty)$  o problema (1) possui uma S-solução positiva em  $C_0^{1,\alpha}(\bar{\Omega})$ , para algum  $\alpha \in (0, 1)$ . Dessa forma, podemos definir a aplicação  $\Lambda : [0, a_*] \rightarrow \mathbb{R}_+$  dada por

$$\Lambda(a) := \inf \{ \lambda \in \mathbb{R}_+ : \text{existe uma S-solução positiva de (1)} \}.$$

Assim, o item (i) fica mostrado. Para os outros itens aplicamos os métodos de sub e supersolução e Teorema do Passo da Montanha para funcionais localmente Lipschitz ao funcional

$$I_{\lambda,a}(u) = \int_{\Omega} \Phi(|\nabla u|) dx - \lambda \int_{\Omega} F_a(x, u) dx, \quad u \in W_0^{1,\Phi}(\Omega).$$

■

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MULTIPLICITY OF SOLUTIONS TO A SCHRÖDINGER PROBLEM WITH SQUARE DIFFUSION TERM

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**Abstract**

In this paper we show multiplicity of solutions for a parameterized quasilinear Schrödinger equation in the presence of a square diffusion and indefinite superlinear term. Due to the presence of the quasilinear term, we can no longer work on the standard Sobolev spaces to show existence and non-existence of solutions. We overcome these difficulties by using perturbations arguments, Nehari sets and nonlinear Rayleigh quotients. As a by product of this approach we show that the associated energy functional has non-zero global minimizers only for small parameters.

## 1 Introduction

This work is concerned mainly with existence and multiplicity of solutions for the quasilinear Schrödinger equation

$$\begin{cases} -\Delta u - \frac{\kappa}{2} u \Delta u^2 = f(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where  $\Delta$  is the Laplacian operator,  $\kappa > 0$ ,  $p \in (2, 4)$ ,  $f \in L^\infty(\Omega)$  may change its sign, and  $\Omega \subset \mathbb{R}^N$  is a smooth bounded domain.

Due to the presence of the square diffusion term  $u \Delta u^2$ , it is well known that the natural functional space  $H_0^1(\Omega) := W_0^{1,2}(\Omega)$  is “too big” to look for variational solutions, while  $W_0^{1,4}(\Omega)$  may be “too small” and so a natural candidate would be the metric, but not vector space,

$$X = \{u \in H_0^1(\Omega) : \int u^2 |\nabla u|^2 < \infty\}$$

endowed with distance function given by

$$d_X(u, v) := \|u - v\|_{1,2} + \|\nabla u^2 - \nabla v^2\|_2.$$

Even though one makes sense to define a function  $u \in X$  as a weak solution of (1) whenever

$$\int (1 + \kappa u^2) \nabla u \nabla \varphi + \kappa \int u |\nabla u|^2 \varphi = \int f(x) |u|^{p-2} u \varphi$$

holds for all  $\varphi \in C_0^\infty(\Omega)$ , the lack of closedness of  $X$  with respect to its metric  $d_X$  leads  $X$  to be also “too big” to approach the problem (1) in a variational sense. So, after these points, we were led to infer that the framework  $Y := (Y, d_X)$ , defined by

$$Y = \overline{W_0^{1,4}(\Omega)}^{d_X}.$$

In spite of  $Y$  seems to be an appropriate space, we have no guarantee that it is a linear normed space, which prevents us to apply directly the usual minimax techniques to the energy functional

$$\Phi_\kappa(u) = \frac{1}{2} \int (1 + \kappa u^2) |\nabla u|^2 - \frac{1}{p} \int f |u|^p, \quad u \in Y,$$

to find its critical points (that we call weak solutions of (1)), that is, functions  $u \in Y$  such that  $\Phi'_\kappa(u)\varphi = 0$  for all  $\varphi \in C_0^\infty(\Omega)$ , where

$$\Phi'_\kappa(u)\varphi = \int (1 + \kappa u^2) \nabla u \nabla \varphi + \kappa \int u |\nabla u|^2 \varphi - \int f |u|^{p-2} u \varphi, \quad \varphi \in C_0^\infty(\Omega).$$

Inspired on ideas from [1, 3, 2], we approach our problem by using a perturbation of the original energy functional  $\Phi_\kappa$ , defined by

$$I_{\mu,\kappa}(u) := \frac{\mu}{4} \int |\nabla u|^4 dx + \Phi_\kappa(u), \quad u \in W_0^{1,4}(\Omega).$$

## 2 Main Results

**Theorem 2.1.** *Let  $p \in (2, 4)$ . Then:*

- (i) *the problem (1) admits two solutions  $w_\kappa, u_\kappa \in Y \cap L^\infty(\Omega)$ , for each  $\kappa \in (0, \kappa_0^*)$ , that satisfy  $\Phi_\kappa(w_\kappa) > 0$  and  $\inf_{u \in Y} \Phi_\kappa = \Phi_\kappa(u_\kappa) < 0$ . Moreover,  $\|u_\kappa\| \rightarrow \infty$  as  $\kappa \rightarrow 0$ ,*
- (ii) *the problem (1), for  $\kappa = \kappa_0^*$ , admits two solutions  $w_{\kappa_0^*}, u_{\kappa_0^*} \in Y \cap L^\infty(\Omega)$  that satisfy  $\Phi_{\kappa_0^*}(w_{\kappa_0^*}) > 0$  and  $\inf_{u \in Y} \Phi_{\kappa_0^*} = \Phi_{\kappa_0^*}(u_{\kappa_0^*}) = 0$ ,*
- (iii) *if  $\kappa > \kappa_0^*$ , then  $\inf_{u \in Y} \Phi_\kappa = 0$  and  $u = 0$  is the only minimizer. Moreover, the problem (1) does not admits any non-trivial solution for any  $\kappa > \kappa^*$ .*

Moreover,  $w_\kappa, u_\kappa$ , for  $\kappa \in (0, \kappa_0^*)$ , are bifurcations-solutions for the solutions  $w_{\mu,\kappa}, u_{\mu,\kappa}$  of Problem (??) at  $\mu = 0$ .

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# O PROBLEMA DE DIRICHLET PARA UMA CLASSE DE EQUAÇÕES DO TIPO P-LAPLACIANO

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## Abstract

Neste trabalho, estudamos o problema de Dirichlet para a seguinte equação diferencial parcial

$$\begin{cases} -\operatorname{div}\left(\frac{a(|\nabla u|)}{|\nabla u|} \nabla u\right) = F(x, u) & \text{em } \Omega \\ u = g & \text{em } \partial\Omega, \end{cases}$$

onde  $\Omega$  é um domínio limitado de classe  $C^{2,\alpha}$  contido em uma variedade Riemanniana completa  $M$ ,  $g \in C^{2,\alpha}(\bar{\Omega})$ ,  $F : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$  e  $a : [0, +\infty) \rightarrow \mathbb{R}$  são funções satisfazendo determinadas condições.

## 1 Introdução

Seja  $M$  uma variedade riemanniana completa e  $\Omega \subset M$  um domínio limitado de classe  $C^{2,\alpha}$ . Consideremos o problema de dirichlet

$$(P.D) = \begin{cases} -Q(u) = F(x, u) & \text{em } \Omega \\ u = g & \text{em } \partial\Omega \end{cases}$$

onde  $g \in C^{2,\alpha}(\bar{\Omega})$ ,  $Q(u) = \operatorname{div}\left(\frac{a(|\nabla u|)}{|\nabla u|} \nabla u\right)$ ,  $a : [0, +\infty) \rightarrow \mathbb{R}$  é tal que  $a \in C([0, +\infty)) \cap C^1((0, +\infty))$ ,  $a > 0$  e  $a' > 0$  em  $(0, +\infty)$  e  $a(0) = 0$ . Para garantir a elipticidade é exigido conforme [1] que

$$\min_{0 \leq s \leq s_0} \left\{ A(s), 1 + \frac{sA'(s)}{A(s)} \right\} > 0$$

para todo  $s_0 > 0$ , onde escrevemos  $a(s) = sA(s)$ .

Além disso supomos que  $F : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$  é não-crescente em  $t \in \mathbb{R}$ . Notemos que quando  $a(s) = s^{p-1}$ ,  $p > 1$ , temos que  $Q(u) = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  que é o operador do p-laplaciano.

O problema de dirichlet acima é uma generalização do caso em que  $F=0$ . Os autores em [1] estudam esse caso particular. Muitos resultados obtidos em [1] se estendem para o caso  $F \neq 0$ . Dentre eles, destacamos o resultado que segue abaixo.

## 2 Resultados Principais

**Teorema 2.1.** Fixemos  $p \in M$ . Seja  $\Omega$  domínio limitado de classe  $C^{2,\alpha}, \bar{\Omega}$  compacto e suponhamos que  $\bar{\Omega} \subset M - \{C_m(p) \cup \{p\}\}$ , onde  $C_m(p)$  é o lugar dos pontos mínimos. Suponhamos também que  $g \in C^{2,\alpha}(\bar{\Omega})$ . Suponhamos que  $F : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$  de classe  $C^1$  é tal que  $F_t(x, t) \leq 0$ ,  $\forall (x, t) \in \bar{\Omega} \times \mathbb{R}$ . Denotemos  $a(s) = sA(s)$  e assumimos que

(i)  $A \in C^{1,\alpha}([0, \infty)) \cap C^{2,\alpha}((0, \infty))$ ,

$$\min_{0 \leq s \leq s_0} \left\{ A(s), 1 + \frac{sA'(s)}{A(s)} \right\} > 0$$

para todo  $s_0 > 0$ .

(ii) Existe uma função não-decrescente  $\varphi : [s_0, +\infty) \rightarrow \mathbb{R}$  para algum  $s_0 > 0$  tal que

$$\int_{s_0}^{+\infty} \frac{\varphi(s)}{s^2} dx = +\infty$$

e

$$(1 + b^-)s^2 \geq \varphi(s)$$

onde  $b(s) = \frac{sa'(s)}{a(s)} - 1$  e  $b^-(s) = \min\{b, 0\}$ .

(iii) Existem  $\alpha_0 > 0$  e  $\alpha_1 > 0$  tal que

$$\alpha_1 \geq 1 + b(s) \geq \alpha_0, \quad \forall s \geq 0.$$

(iv) Existem  $s_0 > 0, \beta > 0$  e uma função  $\psi \in C([0, +\infty))$  com  $\lim_{s \rightarrow +\infty} \psi(s) = +\infty$  tal que

$$(b(s) + 1 - \beta b'^+(s)s - \beta(b(s) + 1)^2) s^2 \geq \psi(s), \quad \forall s \geq s_0$$

onde  $b'^+(s) = \max\{b'(s), 0\}$ .

Então o problema de Dirichlet, tem uma única solução  $u \in C^{2,\alpha}(\bar{\Omega})$ .

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EXISTÊNCIA E REGULARIDADE PARA A SOLUÇÃO DE UM SISTEMA MULTIFÁSICO DA  
ELETROHIDRODINÂMICA

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**Abstract**

Apresentaremos a análise matemática de um sistema de equações diferenciais parciais que modela um fluido multifásico sob o efeito de um campo elétrico. Provamos a existência de solução fraca global e mostramos resultados de regularidade, global no caso bidimensional e local no caso tridimensional.

## 1 Introdução

Estudamos o seguinte sistema de equações:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + 2\operatorname{div}(\mu(c)D(\mathbf{u})) + \rho\mathcal{T}(\rho) + \phi\nabla c \text{ em } Q_T, \quad (1)$$

$$\operatorname{div}(\mathbf{u}) = 0 \text{ em } Q_T, \quad (2)$$

$$\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla)\rho + \rho = 0 \text{ em } Q_T, \quad (3)$$

$$\frac{\partial c}{\partial t} + (\mathbf{u} \cdot \nabla)c = \operatorname{div}(M(c)\nabla\phi) \text{ em } Q_T, \quad (4)$$

$$\phi = \Psi'(c) - \Delta c \text{ em } Q_T, \quad (5)$$

$$\mathbf{u} = \frac{\partial c}{\partial \mathbf{n}} = \frac{\partial \rho}{\partial \mathbf{n}} = \frac{\partial \Delta c}{\partial \mathbf{n}} = 0 \text{ sobre } \partial\Omega \times (0, T), \quad (6)$$

$$\int_{\Omega} \rho = 0, \quad (7)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \rho(0) = \rho_0, c(0) = c_0 \text{ em } \Omega, \quad (8)$$

onde  $Q_T = \Omega \times (0, T)$ , com  $\Omega$  um domínio aberto e limitado de  $\mathbb{R}^n$ ,  $n = 2, 3$ . As incógnitas são: a função  $\mathbf{u}$  que representa o campo de velocidade do fluido, a pressão  $p$ , a densidade de carga livre  $\rho$ , o campo de fase  $c$  e o potencial químico  $\phi$ .  $M$  é a mobilidade do campo de fase,  $D(\mathbf{u}) := (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$  é a parte simétrica do gradiente e  $\mathcal{T} : L^2(\Omega) \rightarrow (H^1(\Omega))^n$  é um operador linear satisfazendo

$$\|\mathcal{T}(\rho)\|_{H^1} \leq C\|\rho\|_{L^2}. \quad (9)$$

O sistema (1)-(8) foi obtido por meio do modelo proposto por Yang, Li e Ding em [2]. Tomamos a densidade de massa volumétrica  $\rho_0$ , a constante dielétrica  $\epsilon_\gamma$  e a condutividade dielétrica  $\sigma$  constantes positivas. Assumimos que a mobilidade do campo de fase  $M$  depende de  $c$ . Admitimos a condição de contorno de Neumann homogênea para  $c$ ,  $V$  e  $\Delta c$ , já para  $\mathbf{u}$  colocamos a condição de Dirichlet homogênea. Além disso, supomos que a carga livre total é nula, isso é,  $\int_{\Omega} \rho = 0$ . Por fim, trocamos o operador  $\mathcal{T} : L^2(\Omega) \rightarrow (H^1(\Omega))^n$ ,  $n = 2, 3$ , dado por  $\mathcal{T}(\rho) = -\nabla V$ , onde  $V$  é a solução de  $\Delta V = -\rho$  em  $\Omega$ ,  $\frac{\partial V}{\partial \mathbf{n}} = 0$  sobre  $\partial\Omega$ , por um operador mais genérico, satisfazendo apenas a condição. Depois desse procedimento, as constantes que apareceram nas equações, por simplicidade, foram tomadas todas iguais a um.

As hipóteses assumidas sobre  $\Psi$  foram tais que o caso de um polinômio de quarta ordem com mínimos em 0 e 1, fosse contemplada. Este polinômio é conhecido como potencial de poç o duplo (*double well potential*) e é o potencial que aparece no modelo original de Yang e Ding.

## 2 Resultados Principais

Os principais resultados são os teoremas a seguir, cujas demonstrações podem ser encontradas em [1], no capítulo 4.

**Teorema 2.1.** *Suponha que  $\Omega \subset \mathbb{R}^n$ ,  $n = 2, 3$  e  $(\mathbf{u}_0, \rho_0, c_0) \in H \times L^2(\Omega) \times H^1(\Omega)$ . Então, existem  $\mathbf{u}, \rho, c$  e  $\phi$  tais que*

$$\begin{aligned}\mathbf{u} &\in L^\infty(0, T; H) \cap L^2(0, T; V), \quad \rho \in L^\infty(0, T; L^2(\Omega)) \cap H^1(0, T; W^{1,3}(\Omega)'), \\ c &\in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^3(\Omega)) \cap H^1(0, T; H^1(\Omega)'), \quad \phi \in L^2(0, T; H^1(\Omega)), \\ \frac{\partial \mathbf{u}}{\partial t} &\in L^2(0, T; V') \text{ se } n = 2 \quad \text{e} \quad \frac{\partial \mathbf{u}}{\partial t} \in L^{4/3}(0, T; V') \text{ se } n = 3.\end{aligned}$$

e satisfazem as seguintes equações:

$$\left\langle \frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \right\rangle + \int_{\Omega} \mu(c) D(\mathbf{u}) : D(\mathbf{v}) + b_u(\mathbf{u}; \mathbf{u}, \mathbf{v}) = \int_{\Omega} \rho \mathcal{T}(\rho) \cdot \mathbf{v} + \int_{\Omega} \phi \nabla c \cdot \mathbf{v}, \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{u} \rho) + \rho = 0 \text{ em } \mathcal{D}'(Q_T), \quad (2)$$

$$\left\langle \frac{\partial c}{\partial t}, z \right\rangle + b(\mathbf{u}; c, z) + \int_{\Omega} M(c) \nabla \phi \cdot \nabla z = 0, \quad (3)$$

$$\phi = \Psi'(c) - \Delta c \text{ q.t.p. em } Q_T, \quad (4)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad c(0) = c_0 \text{ q.t.p. em } \Omega \quad \text{e} \quad \rho(0) = \rho_0 \text{ em } H^1(\Omega)', \quad (5)$$

para todo  $\mathbf{v} \in V$  e  $z \in H^1(\Omega)$  e no sentido das distribuições em  $t$ . Na equação (1),  $D(\mathbf{u}) : D(\mathbf{v}) := \text{tr}(D(\mathbf{u})^T D(\mathbf{v}))$ . A solução  $(\mathbf{u}, \rho, c, \phi)$  é chamada solução fraca para o problema (1)-(8).

**Teorema 2.2** (Regularidade). *Seja  $\Omega \subset \mathbb{R}^n$ ,  $n = 2, 3$ . Suponha que  $\mathbf{u}_0 \in V$ ,  $\rho_0 \in L^p(\Omega)$ ,  $p \geq 3$ ,  $c_0 \in H^2(\Omega)$  e  $\frac{\partial c_0}{\partial n} = 0$  sobre  $\partial\Omega$ . Então,  $\mathbf{u}$ ,  $c$  e  $\rho$  dados pelo Teorema 2.1 satisfazem as seguintes regularidades:*

$$\begin{aligned}\mathbf{u} &\in L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)), \\ c &\in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^4(\Omega)) \cap H^1(0, T; L^2(\Omega)), \\ \rho &\in L^\infty(0, T; L^p(\Omega)) \cap H^1(0, T; H^1(\Omega)').\end{aligned}$$

para todo  $T > 0$  no caso em que  $\Omega \subset \mathbb{R}^2$  e para  $T = T_*$ , com  $T_*$  suficientemente pequeno, para o caso em que  $\Omega \subset \mathbb{R}^3$ .

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## TWO-DIMENSIONAL INCOMPRESSIBLE MICROPOLAR FLUIDS MODEL WITH SINGULAR INITIAL DATA

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### Abstract

This work deals with Cauchy problem for the two-dimensional incompressible micropolar fluids model through the velocity-vorticity formulation. It is assumed null angular viscosity and singular initial data, which includes the possibility of vortex sheets or measures as initial data in Morrey spaces. By means of integral techniques we establish the existence of weak solutions local and global in time. Also, the uniqueness and stability for these solutions are analyzed.

### 1 Introduction

The incompressible micropolar fluid motion in  $\mathbb{R}^2$ , in the velocity-vorticity formulation, is described by the following coupled system

$$\begin{aligned} \partial_t \omega - (\nu + \kappa) \Delta \omega + (\mathbf{u} \cdot \nabla) \omega &= -2\kappa \Delta b + \operatorname{curl} \mathbf{f}, \\ \partial_t b - \gamma \Delta b + 4\kappa b + (\mathbf{u} \cdot \nabla) b &= 2\kappa \omega + g, \\ \mathbf{u} &= \mathbf{K} * \omega, \\ \omega(\cdot, 0) &= \omega_0, \quad b(\cdot, 0) = b_0, \end{aligned} \tag{1}$$

where  $\mathbf{K}$  is the Biot-Savart kernel, that is,

$$\mathbf{K}(\mathbf{x}) = \frac{1}{2\pi} |\mathbf{x}|^{-2} (-x_2, x_1), \quad \mathbf{x} \in \mathbb{R}^2. \tag{2}$$

Here  $\mathbf{u}$  is the velocity field,  $b$  is the microrotation field interpreted as the angular velocity field of rotation of particles,  $\omega = \operatorname{curl} \mathbf{u}$  is the vorticity field,  $\mathbf{f}$  and  $g$  are given external fields, and the constants  $\nu$ ,  $\kappa$ ,  $\gamma$  are viscosities coefficients. We assume, without loss of generality,  $\mathbf{f} = 0$  and  $g = 0$ . The velocity fields given by (1) may include, in particular, the case of vortex sheet.

In the Navier-Stokes case ( $\kappa = 0$  and  $b = g = 0$  in the micropolar model), there are several works with singular initial data, for instance [2, 3, 5], and the references therein. In all these papers the parabolic character of the vorticity equation was of great importance. In the micropolar case, this was also a key argument in [4], in the particular case of null angular viscosity ( $\gamma = 0$ ), to prove the global existence and uniqueness of smooth solutions for the micropolar model with initial data in  $H^s(\mathbb{R}^2)$ ,  $s > 2$ . Motivated by these works, the goal of this paper is to analyze the Cauchy problem associated with the two dimensional micropolar fluids with partial viscosity, namely  $\gamma = 0$ , in terms of the evolution of the singular initial vorticity. We are also interested in the asymptotic behavior of the micropolar fluid motion with respect to time  $t > 0$  as well as in the case of vortex sheets structure of vorticity described above.

## 2 Main Results

We use the standard notation for the Lebesgue and Sobolev spaces and we denote the Morrey-type space of measures by  $\mathcal{M}^p(\mathbb{R}^n)$ .

**Theorem 2.1.** *Let  $\omega_0, b_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ . Then, the system (1) has a unique global mild solution and the inequality*

$$\|\mathbf{u}(\cdot, t)\|_\infty \leq C(\nu, \kappa) \{(\nu + \kappa)t\}^{-1/2} \quad (1)$$

holds, with  $t \in (0, T]$ , for all  $T > 0$ . Moreover, assume that  $(\omega, \mathbf{u}, b)$  and  $(\hat{\omega}, \hat{\mathbf{u}}, \hat{b})$  are mild solutions of system (1) with initial data  $(\omega_0, b_0)$  and  $(\hat{\omega}_0, \hat{b}_0)$ , respectively. Then, the following inequalities are verified

$$\|(\mathbf{u} - \hat{\mathbf{u}})(\cdot, t)\|_\infty \leq Ct^{-1/2}, \quad (2)$$

$$\|(\omega - \hat{\omega})(\cdot, t)\|_1 + \|(\omega - \hat{\omega})(\cdot, t)\|_\infty + \|(b - \hat{b})(\cdot, t)\|_1 + \|(b - \hat{b})(\cdot, t)\|_\infty \leq C\hat{\Pi}, \quad (3)$$

where  $\hat{\Pi} = \hat{\Pi}(\omega_0, b_0, \hat{\omega}_0, \hat{b}_0) = \max\{\|\omega_0 - \hat{\omega}_0\|_1, \|\omega_0 - \hat{\omega}_0\|_\infty, \|b_0 - \hat{b}_0\|_1, \|b_0 - \hat{b}_0\|_\infty\}$  and  $C > 0$  is a constant independent of  $\hat{\Pi}$ .

**Theorem 2.2.** *Let  $\omega_0 \in \mathcal{M}(\mathbb{R}^2)$  and  $b_0 \in \mathcal{M}(\mathbb{R}^2) \cap \mathcal{M}^p(\mathbb{R}^2)$ , with  $p > 2$ . The system (1) has a unique global weak solution such that*

$$\|\mathbf{u}(\cdot, t)\|_\infty \leq C(\nu, \kappa) \{(\nu + \kappa)t\}^{-1/2},$$

with  $t \in (0, T]$ , for all  $T > 0$ . Moreover, the solutions are also stable in an analogous sense to (2)-(3), where in this case  $\hat{\Pi}$  depends on the norms  $\mathcal{M}(\mathbb{R}^2)$  and  $\mathcal{M}^p(\mathbb{R}^2)$  of the initial data.

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## ESTABILIZAÇÃO NA FRONTEIRA NÃO LINEAR DE UM SISTEMA TERMOELÁSTICO

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### Resumo

Neste trabalho será apresentado a existência e estabilização da solução de um sistema termoelástico com dissipação não linear na fronteira. Provaremos inicialmente a existência através da Teoria de Semigrupos de Operadores Não Lineares. Posteriormente para análise da estabilização utilizamos um método que consiste em perturbar adequadamente a energia do sistema.

### 1 Introdução

Nosso objetivo é estudar o seguinte sistema termoelástico

$$u_{tt} - u_{xx} + \theta_x = 0 \tag{1}$$

$$\theta_t - \theta_{xx} + u_{xt} = 0 \tag{2}$$

Para  $0 < x < L$  e  $0 < t < +\infty$ , com as seguintes condições de fronteira.

$$\begin{aligned} u(0, t) &= 0, \\ u_x(L, t) &= -g(u_t(L, t)), \\ \theta(0, t) &= 0, \quad \theta(L, t) = 0, \end{aligned} \tag{3}$$

e condição inicial

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x). \tag{4}$$

Onde  $g : \mathbb{R} \rightarrow \mathbb{R}$  é uma função contínua decrescente que satisfaz,

$$\exists \lambda > 0, \quad c > 0; \quad |g(s)| \leq c|s|^\lambda; \quad |s| \leq 1, \tag{5}$$

$$\exists c > 0; \quad |g(s)| \leq c|s|; \quad |s| \geq 1, \tag{6}$$

$$\exists p > 1, \quad c > 0; \quad g(s)s \geq c|s|^{p+1}; \quad |s| \leq 1, \tag{7}$$

$$\exists c > 0; \quad g(s)s \geq c|s|^2; \quad |s| \geq 1. \tag{8}$$

A energia do sistema (1)-(4) é dada por

$$E(t) = \frac{1}{2} \int_0^L u_t^2 dx + \frac{1}{2} \int_0^L u_x^2 dx + \frac{1}{2} \int_0^L \theta^2 dx.$$

O Espaço de fase  $H$  é dado por

$$H = V \times L^2(0, L) \times L^2(0, L),$$

onde  $V = \{u \in H^1(0, L); \quad u(0) = 0\}$ .

Para obter a estabilização fizemos como em [2].

## 2 Resultados Principais

Para mostrar a existência de solução do sistema (1)-(4), utilizamos a teoria encontrada em [1] e [3].

Nosso principal resultado é que para o sistema (1)-(4), utilizando as hipóteses para  $g$  acima citadas e com o método da energia perturbada obtemos,

(i): Se  $\lambda = p = 1$ , existem constantes  $M > 1, \gamma > 0$  tais que

$$E(t) \leq ME(0)e^{-\gamma t} \quad \forall t \geq 0.$$

(ii): Se  $\lambda > 1$  e  $p > 1$ , existe uma constante  $M$  que depende de  $E(0)$  tal que

$$E(t) \leq 4 \left( Mt + (E(0))^{\frac{-(p-1)}{2}} \right)^{-2/(p-1)} \quad \forall t \geq 0.$$

(iii): Se  $\lambda < 1$  e  $p > 1$ , existe uma constante  $M$  que depende de  $E(0)$  tal que

$$E(t) \leq 4 \left( Mt + (E(0))^{\frac{-p+1-2\lambda}{2\lambda}} \right)^{-2\lambda/(p+1-2\lambda)} \quad \forall t \geq 0.$$

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## AN INVERSE PROBLEM FOR A SIR REACTION-DIFFUSION MODEL

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In this work we study an inverse problem that arises in the problem of determining coefficients for a reaction-diffusion system, originated in the theory of mathematical epidemiology. We consider a population lives in a three-dimensional space is subdivided into the subclasses of susceptible, infected and recovered. We assume that the dynamic process of disease transmission is governed by reaction-diffusion system. The inverse problem is the identification of reaction coefficients. We apply the optimal control theory approach: the inverse problem is reformulated as an optimization problem. Our results are the following: the existence and uniqueness of the solution of the direct problem, the existence of solution for the adjoint system, the existence of the solution of the optimization problem, a necessary optimality condition of first order, and the local uniqueness of the inverse problem.

**1 Introduction**

In recent decades there is a growing interest in inverse problems that arise in mathematical models from various applications and where the governing equations are given in terms of partial differential equations, see for example [1, 2, 3, 3, 5, 6]. Particularly, we have the following SIR-type reaction-diffusion system

$$S_t - \alpha \Delta S = \mu(N - S) - \beta SI, \quad I_t - \alpha \Delta I = -(\mu + \nu)I + \beta SI, \quad R_t - \alpha \Delta R = \nu I - \mu R \quad \text{in } Q_T, \quad (1)$$

$$\nabla S \cdot \eta = \nabla I \cdot \eta = \nabla R \cdot \eta = 0, \quad \text{on } \partial\Omega \times [0, T], \quad (2)$$

$$(S, I, R)(x, 0) = (S_0, I_0, R_0)(x), \quad \text{in } \bar{\Omega}, \quad (3)$$

where  $S, I$  and  $R$  are the susceptible, infected and recovered densities of a population;  $\alpha$  is the diffusion; and  $\beta, \mu$ , and  $\nu$  are space dependent coefficients. The inverse problem is the identification of the reaction coefficients from the final observation time of the state variables  $S, I$  and  $R$ : “Find the coefficients  $\beta, \mu, \nu$  such that at time  $t = T$  the solution of system (1)-(3) is very close to the observed data  $S^{obs}, I^{obs}$ , and  $R^{obs}$ ”. It can be reformulated as the following optimization problem

$$\inf J(S, I, R; \beta, \mu, \nu) : (\beta, \mu, \nu) \in U_{ad}(\Omega) \text{ y } (S, I, R) \text{ is solution of (1)-(3)}, \quad (4)$$

where

$$J = \|(S, I, R)(\cdot, T) - (S^{obs}, I^{obs}, R^{obs})\|_{L^2(\Omega)}^2 + \frac{\Gamma}{2} \|\nabla(\beta, \mu, \nu)\|_{L^2(\Omega)}^2, \quad U_{ad}(\Omega) = \mathcal{A}(\Omega) \cap \left[ H^{[[d/2]]+1}(\Omega) \right]^3$$

for  $d = 1, 2, 3$ ,  $\mathcal{A}(\Omega) = \left\{ (\beta, \mu, \nu) \in [C^\alpha(\bar{\Omega})]^3 : \text{Ran}(\beta, \mu, \nu) \subset \subset ]0, 1[^3 \quad \nabla(\beta, \mu, \nu) \in [L^2(\Omega)]^3 \right\}.$

**2 Main Results**

We consider the hypotheses: (S0) The bounded and convex open set  $\Omega$  is such that  $\partial\Omega$  is  $C^1$ ; (S1) The initial conditions  $S_0, I_0$  and  $R_0$  are of class  $C^{2,\alpha}(\bar{\Omega})$  and satisfy the inequalities

$$S_0(x) \geq 0, I_0(x) \geq 0, R_0(x) \geq 0; \int_{\Omega} I_0(x) dx > 0, \int_{\Omega} R_0(x) dx > 0, (S_0 + I_0 + R_0) \geq \phi_0 > 0,$$

on  $\Omega$ , for some positive constant  $\phi_0$ ; and (S2) the observation functions  $S^{obs}$ ;  $I^{obs}$  y  $R^{obs}$  are in  $L^2(\Omega)$ . Then, the main results of this work are the following.

**Theorem 2.1.** Suppose that hypotheses (S0)-(S2) are satisfied and further assume that  $(\beta, \mu, \nu) \in C^\alpha(\overline{\Omega}) \times C^\alpha(\overline{\Omega}) \times C^\alpha(\overline{\Omega})$ . Then the direct problem ((1)-(3)) admits a unique positive classical solution  $(S, I, R)$  such that  $S, I, R \in C^{2+\alpha, 1+\alpha/2}(\overline{Q}_T)$  and also  $S, I$  y  $R$  are bounded on  $\overline{Q}_T$  for any  $T \in \mathbb{R}^+$ .

**Theorem 2.2.** Suppose the hypotheses (S0)-(S2) hold. Then there is at least one solution to the optimization problem.

**Theorem 2.3.** Suppose the hypotheses (S0)-(S2) hold. Consider that  $(\bar{\beta}, \bar{\mu}, \bar{\nu})$  is the solution of the inverse problem and that  $(S, I, R)$  is the corresponding solution of the SIR model with  $(\bar{\beta}, \bar{\mu}, \bar{\nu})$  instead of  $(\beta, \mu, \nu)$ . Then  $(p_1, p_2, p_3)$  is bounded  $L^\infty(0, t; H^2(\Omega))$  for almost every time in  $]0, T]$ . In particular  $(p_1, p_2, p_3)$  is bounded in  $L^\infty(0, t; L^\infty(\Omega))$  for almost all times in  $]0, T]$ .

**Theorem 2.4.** Let  $(\bar{S}, \bar{I}, \bar{R})$  and  $(\bar{\beta}, \bar{\mu}, \bar{\nu})$  be as in theorem (2.3). Then

$$\begin{aligned} & \int_Q \left\{ (\hat{\mu} - \bar{\mu})[(N - \bar{S})p_1 - \bar{I}p_2 - \bar{R}p_3] + \bar{S}\bar{I}(\hat{\beta} - \bar{\beta})(p_2 - p_1) + \bar{I}(\hat{\nu} - \bar{\nu})(p_3 - p_2) \right\} dxdt \\ & + C \int_\Omega \left\{ |\nabla \bar{\beta} \nabla (\hat{\beta} - \bar{\beta})| + |\nabla \bar{\mu} \nabla (\hat{\mu} - \bar{\mu})| + |\nabla \bar{\nu} \nabla (\hat{\nu} - \bar{\nu})| \right\} dx \geq 0, \end{aligned}$$

for any  $\hat{\beta}, \hat{\mu}, \hat{\nu} \in U_{ad}$  holds.

**Theorem 2.5.** Given  $\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{R}_+^3$  (fixed) and defining  $\mathcal{U}_{\mathbf{c}}(\Omega) = \{(\beta, \mu, \nu) \in U_{ad}(\Omega) : \int_\Omega (\beta, \mu, \nu) d\mathbf{x} = \mathbf{c}\}$ . Then there exists  $\bar{\Gamma} \in \mathbb{R}^+$  such that the solution of the inverse problem is only defined, except for an additive constant, on  $\mathcal{U}_{\mathbf{c}}(\Omega)$  in the sense  $L^2(\Omega)$  for any regularization parameter  $\Gamma > \bar{\Gamma}$ .

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## SOLVABILITY OF THE FRACTIONAL HYPERBOLIC KELLER-SEGEL SYSTEM

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We study a new nonlocal approach to the mathematical modelling of the Chemotaxis problem, which describes the random motion of a certain population due a substance concentration. Considering the initial-boundary value problem for the fractional hyperbolic Keller-Segel model, we prove the solvability of the problem. The solvability result relies mostly on the kinetic method.

**1 Introduction**

We introduce and study in this paper the Fractional Hyperbolic Keller-Segel (FHKS for short) model for chemotaxis described by the following system

$$\begin{cases} \partial_t u + \operatorname{div}(g(u) \nabla \mathcal{K}c) = 0, & \text{in } (0, \infty) \times \Omega, \\ (-\Delta_N)^{1-s} c + c = u, & \text{in } \Omega, \\ u|_{\{t=0\}} = u_0, & \text{in } \Omega, \\ \nabla \mathcal{K}c \cdot \nu = 0, & \text{on } \Gamma, \end{cases} \quad (1)$$

where  $u(t, x)$  is the density of cells and  $c(t, x)$  is the chemoattractant concentration, which is responsible for the cell aggregation. The problem is posed in a bounded open subset  $\Omega \subset \mathbb{R}^n$ , ( $n = 1, 2$ , or  $3$ ), with  $C^2$ -boundary denoted by  $\Gamma$ , and as usual we denote by  $\nu(r)$  the outward normal to  $\Omega$  at  $r \in \Gamma$ . The given measurable bounded function  $u_0$  is the initial condition of the cells, and we assume

$$0 \leq u_0(x) \leq 1, \quad \text{for a.e. } x \in \Omega. \quad (2)$$

Moreover, since the normal flux in the equation on  $u$  vanishes on  $\Gamma$ , that is the boundary is characteristic, it is not necessary to prescribe boundary conditions for  $u$ , which prevents some specific difficulties related to the trace problem, see [5] for instance. Here for  $0 < s < 1$ ,  $(-\Delta_N)^s$  denotes the Neumann spectral fractional Laplacian (NSFL for short) operator, which characterizes long-range diffusion effects. We also consider the non-local operator  $\mathcal{K} \equiv (-\Delta_N)^{-s}$ .

The theory of chemotaxis modeling goes back to E. F. Keller and L. A. Segel [2, 3, 4], where a detailed description of the movement of cells oriented by chemical cues can be found. In fact, a nonlocal version of the Keller-Segel model has been proposed by Caffarelli, Vazquez in [1]. Although, the fractional model proposed here in (1) is a closer (fractional) generalization of the model considered in Dalibard, Perthame [6]. Indeed, in that paper they studied the following system

$$\begin{cases} \partial_t u + \operatorname{div}(g(u) \nabla S) = 0, & \text{in } (0, \infty) \times \Omega, \\ (-\Delta)S + S = u, & \text{in } \Omega, \\ u|_{\{t=0\}} = u_0, & \text{in } \Omega, \\ \nabla S \cdot \nu = 0, & \text{on } \Gamma, \end{cases} \quad (3)$$

which follows from the system (1), at least formally passing to the limit as  $s \rightarrow 0^+$ .

## 2 Main Results

We begin observing that, the first equation in (1) is a hyperbolic scalar conservation law, thus the density of cells function  $u$  may admit shocks. Therefore, in order to select the more correct physical solution, we need an admissible criteria, which is given by the entropy condition.

Now, we are able to state plainly the main result of this paper. Then, we have the following

**Theorem 2.1** (Main Theorem). *Let  $u_0 \in L^\infty(\Omega)$  be an initial data satisfying (2) and  $s \in (0, 1)$ . Then, there exists a pair of functions*

$$(u, c) \in L^\infty((0, \infty) \times \Omega) \times L^\infty((0, \infty); D((- \Delta_N)^{1-s})),$$

*which is a weak entropy solution to the **FHKS** system, and it satisfies*

$$0 \leq u(t, x) \leq 1, \quad 0 \leq c(t, x) \leq 1,$$

*for almost all  $t > 0$  and  $x \in \Omega$ .*

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## CONTROLLABILITY OF PHASE-FIELD SYSTEM WITH ONE CONTROL

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### Abstract

In this paper, we present some controllability results for linear and nonlinear phase-field systems of Caginalp type considered in a bounded interval of  $\mathbb{R}$  when the scalar control force acts on the temperature equation of the system by means of the Dirichlet condition on one of the endpoints of the interval. In order to prove the linear result we use the moment method providing an estimate of the cost of fast controls. Using this estimate we prove a local exact boundary controllability result to constant trajectories of the nonlinear phase-field system.

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## 1 Introduction

In this work, we present a controllability results for a nonlinear phase-field systems of Caginalp type considered in a bounded interval of  $\mathbb{R}$  when the scalar control force acts on the temperature equation of the system by means of the Dirichlet condition on one of the endpoints of the interval. We use the moment method providing an estimate of the cost to achieve a local exact boundary controllability result to constant trajectories of the nonlinear phase-field system.

The Phase-field system it is a model describing the transition between the solid and liquid phases in solidification/melting processes of a material occupying a domain. Given a time  $T > 0$  and the cylinder  $Q_T := (0, \pi) \times (0, T)$ , the system is described as follows by G. Caginalp in [1]:

$$\begin{cases} \tilde{\theta}_t - \xi \tilde{\theta}_{xx} + \frac{1}{2} \rho \xi \tilde{\phi}_{xx} + \frac{\rho}{\tau} \tilde{\theta} = \mathbf{f}(\tilde{\phi}) & \text{in } Q_T, \\ \tilde{\phi}_t - \xi \tilde{\phi}_{xx} - \frac{2}{\tau} \tilde{\theta} = -\frac{2}{\rho} \mathbf{f}(\tilde{\phi}) & \text{in } Q_T, \\ \tilde{\theta}(0, \cdot) = \mathbf{v}, \quad \tilde{\phi}(0, \cdot) = \mathbf{c}, \quad \tilde{\theta}(\pi, \cdot) = 0, \quad \tilde{\phi}(\pi, \cdot) = \mathbf{c} & \text{on } (0, T), \\ \tilde{\theta}(\cdot, 0) = \tilde{\theta}_0, \quad \tilde{\phi}(\cdot, 0) = \tilde{\phi}_0 & \text{in } (0, \pi), \end{cases}$$

In the Phase-field system above  $\tilde{\theta} = \tilde{\theta}(x, t)$  is the temperature of the material and  $\tilde{\phi} = \tilde{\phi}(x, t)$  identifies the phase transition of the material. When  $\tilde{\phi} = 1$  the material is in the solid state, and when  $\tilde{\phi} = -1$  it means that the material is in the liquid state.

Also,  $\tilde{\theta}_0, \tilde{\phi}_0$  represents the initial data;  $\mathbf{v} \in L^2(0, T)$  is the control;  $\mathbf{c}$  is a constant assuming the possible values in the set  $\{-1, 0, 1\}$ ; the constants  $\rho, \tau, \xi$  are, respectively, the latent heat, the relaxation time, and the thermal diffusivity; and  $\mathbf{f}(\tilde{\phi})$  is the nonlinear part of the system given by

$$\mathbf{f}(\tilde{\phi}) = -\frac{\rho}{4\tau} (\tilde{\phi} - \tilde{\phi}^3).$$

## 2 Main Results

The main result of the work is given in the following.

**Theorem 2.1.** Let us fix  $T > 0$  and assume that

$$\begin{cases} \xi^2\tau^2(\ell^2 - k^2)^2 - 2\xi\rho\tau(\ell^2 + k^2) - 2\rho - 1 \neq 0, & \forall k, \ell \geq 1, \quad \ell > k, \\ \xi \neq \frac{1}{j^2}\frac{\rho}{\tau}, & \forall j \geq 1. \end{cases}$$

Then, there exists  $\epsilon > 0$  such that, for any  $(\tilde{\theta}_0, \tilde{\phi}_0) \in H^{-1} \times (c + H_0^1)$ , with  $\|\tilde{\theta}_0\|_{H^{-1}} + \|\tilde{\phi}_0 - c\|_{H_0^1} \leq \epsilon$ , there exists  $v \in L^2(0, T)$  for which system (1) has a unique solution which satisfies  $\tilde{\theta}(\cdot, T), \tilde{\phi}(\cdot, T) = (0, \textcolor{red}{c})$  in  $(0, T)$ .

**Proof** The proof is developed using the following strategy.

First we prove the null controllability at time  $T > 0$  of the homogeneous linearized system (after a change of variables  $(\theta_0, \phi_0, \theta, \phi) = (\tilde{\theta}_0, \tilde{\phi}_0 - c, \tilde{\theta}, \tilde{\phi} - c)$ )

$$\begin{cases} \theta_t - \xi\theta_{xx} + \frac{1}{2}\rho\xi\phi_{xx} - \frac{\rho}{2\tau}\phi + \frac{\rho}{\tau}\theta = 0 & \text{in } Q_T, \\ \phi_t - \xi\phi_{xx} + \frac{1}{\tau}\phi - \frac{2}{\tau}\theta = 0 & \text{in } Q_T, \\ \theta(0, \cdot) = v, \quad \phi(0, \cdot) = \theta(\pi, \cdot) = \phi(\pi, \cdot) = 0 & \text{on } (0, T), \\ \theta(\cdot, 0) = \theta_0, \quad \phi(\cdot, 0) = \phi_0 & \text{in } (0, \pi), \end{cases}$$

The assumptions on  $\xi, \rho, \tau$  are used in the Moment Method for assuring that the eigenvalues of the space operator of the homogeneous linear system satisfy suitable properties to produce the following estimate of the control cost:

$$\|\textcolor{red}{v}\|_{L^2(0, T)} \leq C_0 e^{\frac{M}{T}} \|y_0\|_{H^{-1}}.$$

Next, we prove the null controllability at time  $T > 0$  of the non-homogeneous linearized system

$$\begin{cases} \theta_t - \xi\theta_{xx} + \frac{1}{2}\rho\xi\phi_{xx} - \frac{\rho}{2\tau}\phi + \frac{\rho}{\tau}\theta = f_1 & \text{in } Q_T, \\ \phi_t - \xi\phi_{xx} + \frac{1}{\tau}\phi - \frac{2}{\tau}\theta = f_2 & \text{in } Q_T, \\ \theta(0, \cdot) = v, \quad \phi(0, \cdot) = \theta(\pi, \cdot) = \phi(\pi, \cdot) = 0 & \text{on } (0, T), \\ \theta(\cdot, 0) = \theta_0, \quad \phi(\cdot, 0) = \phi_0 & \text{in } (0, \pi), \end{cases}$$

where  $f = (f_1, f_2)$  is a source with exponential decay when  $t \rightarrow T$ :

$$e^{\frac{C}{T-t}} f \in L^2(Q_T), \quad \text{for suitable } C > 0.$$

Finally, we apply a Fixed-Point argument to the non-homogeneous linear system with the operator

$$(f_1, f_2) \mapsto \left( \pm \frac{3\rho}{4\tau}\phi^2 + \frac{\rho}{4\tau}\phi^3, \mp \frac{3\rho}{2\tau}\phi^2 - \frac{1}{2\tau}\phi^3 \right)$$

in order to recover the local null controllability result at time  $T$  for the nonlinear Phase-Field system.

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CONTROLABILIDADE EXATA PARA A EQUAÇÃO KDV VIA ESTRATÉGIA  
STACKELBERG-NASH

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### Abstract

Neste trabalho abordamos um problema de controle hierárquico para a equação em Korteweg-de Vries (KdV) com controles distribuídos seguindo uma estratégia de Stackelberg-Nash. Tratamos de um problema de controle onde muitos objetivos devem ser alcançados de uma só vez, contamos com um controle principal chamado líder, e dois controles secundários chamados seguidores, onde cada um deles tem seu próprio objetivo. O objetivo do líder é conduzir as soluções da equação KdV para uma determinada trajetória, enquanto os seguidores devem ter um equilíbrio de Nash para seus objetivos.

## 1 Introdução

A equação de Korteweg-de Vries (KdV) é uma equação diferencial parcial de terceira ordem que modela a propagação das ondas em superfícies de águas rasas.

Neste trabalho consideramos um problema de controle multi-objetivo (isto é, muitos objetivos devem ser cumpridos de uma vez) o que pode tornar sua solução inviável. Para superar isto, aplicamos o conceito de Otimização de Stackelberg onde uma hierarquia para os controles é assumida. Consideramos um controle denominado líder e outros controles chamados de seguidores. Uma vez que a escolha do líder é fixada, os seguidores devem cumprir seus objetivos de forma otimizada.

Vamos ser mais específicos. Seja  $(0, L) \subset \mathbb{R}$  um intervalo aberto e  $T > 0$  um número real. Nós consideramos controles internos suportados em um subconjunto aberto não vazio  $\omega \subset (0, L)$  e condições homogêneas de fronteiras. Definimos  $Q = (0, L) \times (0, T)$  e para algum subconjunto aberto  $\omega \subset (0, L)$  definimos  $Q_\omega = \omega \times (0, T)$ .

Consideremos a equação KdV não linear

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = f\mathbf{1}_\mathcal{O} + v^1\chi_{\mathcal{O}_1} + v^2\chi_{\mathcal{O}_2} & \text{in } Q, \\ y(0, \cdot) = y(L, \cdot) = y_x(L, \cdot) = 0 & \text{in } (0, T), \\ y(x, \cdot) = y^0 & \text{in } (0, L), \end{cases} \quad (1)$$

onde  $y = y(x, t)$  é o estado e  $y^0$  é dado. Em (1), o conjunto  $\mathcal{O} \subset (0, L)$  é o domínio do controle líder  $f$  e  $\mathcal{O}_1, \mathcal{O}_2 \subset (0, L)$  são os domínios dos controles seguidores  $v_1$  e  $v_2$  (todos supostos bem pequenos e disjuntos). A função  $\mathbf{1}_A$  representa a função característica de um conjunto aberto  $A$ , onde  $\chi_A$  é uma função  $C_0^\infty(A)$ .

Sejam  $\mathcal{O}_{1,d}, \mathcal{O}_{2,d} \subset (0, L)$  conjuntos abertos e considere os funcionais

$$J_i(y^0, f, v^1, v^2) = \frac{\alpha_i}{2} \iint_Q \chi_{\mathcal{O}_{i,d}} |y - y_{i,d}|^2 dxdt + \frac{\mu_i}{2} \iint_Q \chi_{\mathcal{O}_i} |v^i|^2 dxdt, \quad i = 1, 2, \quad (2)$$

onde  $\alpha_i > 0$ ,  $\mu_i > 0$  são constantes e  $y_{i,d} = y_{i,d}(x, t)$  são funções dadas.

A controlabilidade exata de Stackelberg-Nash para equação KdV pode ser descrita em duas etapas. A primeira, para  $f$  fixado, os seguidores  $v^1$  e  $v^2$  buscam ser um equilíbrio de Nash para os funcionais custos  $J_i$  ( $i = 1, 2$ ). (Isto é, procuramos pelo par  $(v^1, v^2)$  com  $v^i \in L^2(\mathcal{O}_i \times (0, T))$  tal que satisfaça  $J_i(f; v^1, v^2) = \min_{\hat{v}^i} J_i(\hat{v}^i)$ ,  $i = 1, 2$ ).

Para a segunda etapa, fixamos uma trajetória  $\bar{y}$ , que é solução suficientemente regular de um sistema, sob mesmas condições de fronteira e dado inicial de (1).

Uma vez que o equilíbrio de Nash foi encontrado e para cada  $f$  fixado, procuramos por um controle  $\hat{f} \in L^2(\mathcal{O} \times (0, T))$  tal que  $y(\cdot, T) = \bar{y}(\cdot, T)$ , isto é, satisfaz a condição de controlabilidade exata em  $(0, L)$ .

Então definimos a nova variável  $z = y - \bar{y}$  e  $z_{i,d} = y_{i,d} - \bar{y}$ , e mostramos a controlabilidade nula para  $z$  que é  $z(x, T) = 0$  em  $(0, L)$  (Observe que mostrar isto é equivalente a mostrar a controlabilidade exata para  $y$ ). Onde  $z$  junto com  $\phi^i$  ( $i = 1, 2$ ) satisfazem um sistema de otimalidade do tipo:

$$\begin{cases} z_t + z_x + z_{xxx} + zz_x + (\bar{y}z)_x = f 1_{\mathcal{O}} - \sum_{i=1}^2 \frac{1}{\mu_i} \phi^i \chi_{\mathcal{O}_i} + f^0 & \text{em } Q, \\ -\phi_t^i - \phi_x^i - \phi_{xxx}^i - (z + \bar{y})\phi_x^i = \alpha_i(z - z_{i,d})\chi_{\mathcal{O}_{i,d}} + f^i, & \text{em } Q \\ z(0, \cdot) = z(L, \cdot) = z_x(L, \cdot) = 0 & \text{em } (0, T), \\ \phi^i(0, \cdot) = \phi^i(L, \cdot) = \phi_x^i(0, \cdot) = 0 & \text{em } (0, T), \\ z(\cdot, 0) = z^0, \quad \phi^i(\cdot, T) = 0 & \text{em } (0, L). \end{cases} \quad (3)$$

Deste modo, a estratégia que adotamos para encontrar o equilíbrio de Nash consiste em provar que o sistema (3) possui soluções. Uma vez o equilíbrio de Nash encontrado temos que provar que é possível resolver simultaneamente o objetivo do líder, ou seja, temos que provar a existência de  $f$  tal que a solução de (3) satisfaça a condição de controlabilidade nula de  $z$ , o que motiva o resultado principal deste trabalho.

## 2 Resultados Principais

**Teorema 2.1.** Para  $i = 1, 2$ , suponha que

$$\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset \quad (1)$$

e que  $\mu_i$  são suficientemente grandes. Além disso, suponha que uma das duas condições é satisfeita:

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d} \quad \text{ou} \quad \mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}. \quad (2)$$

Então, existe uma função positiva  $\hat{\rho} = \hat{\rho}(t)$  explodindo em  $t = T$  e  $\delta > 0$  tal que se

$$\|z^0\|_{H_0^1(0,L)}^2 + \sum_{i=1}^2 \iint_{\mathcal{O}_{i,d} \times (0,T)} \hat{\rho}^2 |z_{i,d}|^2 dx dt < \delta, \quad (3)$$

existem controles  $f \in L^2(\mathcal{O} \times (0, T))$  tal que a solução  $(y, \phi^1, \phi^2)$  de (3) satisfaz  $y(\cdot, T) = 0$ .

**Prova:** Existência - Para mostrar o resultado, primeiro mostramos um resultado de controlabilidade nula para um sistema linearizado do sistema (3). A prova do caso linear é feita através do Método de Unicidade de Hilbert (HUM), que consiste em uma equivalência a uma estimativa de observabilidade adequada para as soluções de um sistema adjunto, nesta etapa, novas estimativas de Carleman são demonstradas e para elas são construídas novas funções pesos pela necessidade de detalhes técnicos [1]. Por fim, para o caso não linear, utilizamos o Teorema da Função Inversa. ■

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## EXPONENTIAL ATTRACTOR FOR A CLASS OF NON LOCAL EVOLUTION EQUATIONS

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In this work we study the existence of exponential attractor for a non local evolution equation, which generalizes the model that describe the neuronal activity. Our results extend results obtained in [1] and in [7], where the studied models are configured as particular cases of the model presented here. Furthermore, from the existence of the exponential attractor, we conclude that the fractal dimension of the global attractor given in [2] has finite fractal dimension.

**1 Introduction**

We consider the non local evolution problem

$$\begin{cases} \partial_t u(x, t) = -u(x, t) + g(\beta K(f \circ u)(x, t) + \beta h), & x \in \Omega, t \in [0, \infty[; \\ u(x, t) = 0, & x \in \mathbb{R}^N \setminus \Omega, t \in [0, \infty[; \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (1)$$

where  $u(x, t)$  is a real function on  $\mathbb{R}^N \times [0, \infty[$ ,  $\Omega$  is a bounded smooth domain in  $\mathbb{R}^N$  ( $N \geq 1$ );  $h$  and  $\beta$  are nonnegative constants;  $K$  is an integral operator with symmetric kernel, given by  $Kv(x) := \int_{\mathbb{R}^N} J(x, y)v(y)dy$  where  $J$  is a non negative symmetric function of class  $C^1$ , with  $\int_{\mathbb{R}^N} J(x, y)dy = \int_{\mathbb{R}^N} J(x, y)dx = 1$ . The functions  $f$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  are locally Lipschitz continuous functions satisfying the growth conditions

$$|g(x)| \leq k_1|x| + k_2 \quad \text{and} \quad |f(x)| \leq c_1|x| + c_2, \quad \forall x \in \mathbb{R}$$

for some non-negative constants  $k_1, k_2, c_1, c_2$ .

In addition, we will also assume that  $g \in C^1(\mathbb{R})$  with

$$|g'(x)| \leq k_3|x| + k_4, \quad \forall x \in \mathbb{R},$$

for some non-negative constants  $k_3, k_4 > 0$ ; that  $g'$  is Lipschitz on bounded sets and that function  $f$  satisfy

$$|f(x) - f(y)| \leq c_0(1 + |x|^{p-1} + |y|^{p-1})|x - y|, \quad \forall (x, y) \in \mathbb{R} \times \mathbb{R},$$

where  $c_0$  is some non-negative constant.

**2 Main Results**

In [2] it has been proven that, under the hypotheses mentioned, the problem (1) is well posed in the phase space  $X = \{u \in L^p(\mathbb{R}^n) : u(x) = 0, \text{ if } x \in \mathbb{R}^n \setminus \Omega\}$ , with the induced norm of  $L^p(\Omega)$ , which is isometric to  $L^p(\Omega)$ . Furthermore, in [2] it has also been proven that the problem (1) generates a  $C^1$  flow,  $\{S(t)\}_{t \geq 0}$ , in  $L^p(\Omega)$ .

We prove that space  $W^{1,p}(\Omega)$  is positively invariant by the action of  $\{S(t)\}$  and admits a bounded subset  $\mathcal{B}_1$ , which is positively invariant and absorbing for  $\{S(t)\}$  (in the topology of  $W^{1,p}(\Omega)$ ). Using techniques of [4], [5] and [6] we obtain the main results of this work.

**Theorem 2.1.** *The flow  $\{S(t)\}_{t \geq 0}$  has a compact set  $\mathcal{M} \subset L^p(\Omega)$  with the following properties:*

- (i) *The set  $\mathcal{M}$  is positively invariant under semigroup  $\{S(t)\}$ , that is,  $S(t)\mathcal{M} \subset \mathcal{M}$  for any  $t \geq 0$ .*
- (ii) *The set  $\mathcal{M}$  has finite fractal dimension, that is,  $\dim_F(\mathcal{M}, L^p(\Omega)) < \infty$ .*
- (iii) *The set  $\mathcal{M}$  attracts exponentially  $\mathcal{B}_1$ , that is, there exist  $\alpha \geq 0$  and  $\omega > 0$  such that, for any  $t \geq 0$ ,*

$$\text{dist}_{L^p(\Omega)}(S(t)\mathcal{B}_1, \mathcal{M}) < \alpha e^{-\omega t},$$

where  $\text{dist}_{L^p(\Omega)}$  denotes the Hausdorff semidistance in  $L^p(\Omega)$ .

**Remark 2.1.** *The set  $\mathcal{M}$  given in the Theorem 2.1, attracts exponentially the action of  $\{S(t)\}$  on bounded subsets of  $\mathcal{B}_1$ . This is a sense of expoential attractor discussed in [3].*

If, in addition to the hypotheses initially considered, we assume that  $f$  and  $g$  are Lipschitzian, we obtain the following result, which gives an exponential attractor for  $\{S(t)\}$ , in the sense defined in [7] and in [6], which attracts exponentially any bounded subset of  $L^p(\Omega)$ .

**Theorem 2.2.** *The set  $\mathcal{M}$  given in the Theorem 2.1 is an exponential attractor for the semigroup  $\{S(t)\}$  generated by the solutions of the equation (1).*

**Proof** Due to the hypothese that  $f$  and  $g$  are Lipschitzian we have

$$\text{dist}_{L^p(\Omega)}(S(t)B, \mathcal{B}_1) \leq C_1(B)e^{[L_g\beta L_f \|J\|_1 - 1]t}.$$

Hence, using Theorem 2.1 and a result on transitivity of exponential attraction given in [5], the result follows. ■

**Corollary 2.1.** *The global attractor  $\mathcal{A}$  for the semigroup  $\{S(t)\}$  given in [2] has a finite fractal dimension.*

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## GLOBAL SOLUTIONS TO THE NON-LOCAL NAVIER-STOKES EQUATIONS

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### Abstract

We study the global well-posedness for a non-local-in-time Navier-Stokes equation. Our results recover in particular other existing well-posedness results for the Navier-Stokes equations and their time-fractional version. We show the appropriate manner to apply Kato's strategy and this context, with initial conditions in the divergence-free Lebesgue space  $L_d^\sigma(\mathbb{R}^d)$ .

## 1 Introduction

Consider the fractional-in-time Navier-Stokes equation

$$\begin{aligned} \partial_t^\alpha u - \Delta u + (u \cdot \nabla) u + \nabla p &= f, & t > 0, x \in \Omega \subset \mathbb{R}^d, \\ \nabla \cdot u &= 0, & t > 0, x \in \Omega \subset \mathbb{R}^d, \\ u(0, x) &= u_0(x), & x \in \Omega \subset \mathbb{R}^d, \end{aligned}$$

where  $\partial_t^\alpha u$  denotes the fractional derivative of  $u$  in the Caputo's sense with order  $\alpha \in (0, 1)$ . If the product  $(k * v)$  denotes the convolution on the positive halffline  $\mathbb{R}_+ := [0, \infty)$  with respect to time variable, then we have  $\partial_t^\alpha u = g_{1-\alpha} * u_t$ , for an absolutely continuous function  $u$ , where  $g_\beta$  is the standard notation for the function  $g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ ,  $t > 0$ ,  $\beta > 0$ . Toward the possibility of considering more general nonlocal-in-time effects, we will replace  $g_\alpha$  by  $k$ , and we assume as a general hypothesis that  $k$  is a kernel of type  $(\mathcal{PC})$ , by which we mean that the following condition is satisfied:

$(\mathcal{PC})$   $k \in L_{1,loc}(\mathbb{R}_+)$  is nonnegative and nonincreasing, and there exists a kernel  $\ell \in L_{1,loc}(\mathbb{R}_+)$  such that  $k * \ell = 1$  on  $(0, \infty)$ .

We also write  $(k, \ell) \in \mathcal{PC}$ . We point out that the kernels of type  $(\mathcal{PC})$  are called *Sonine kernels* and they have been successfully used to study integral equations of first kind in the spaces of Hölder continuous, Lebesgue and Sobolev functions, see [1].

Therefore, we consider the following problem for the following nonlocal-in-time Navier-Stoke-type equation

$$\partial_t(k * (u - u_0)) - \Delta u + (u \cdot \nabla) u + \nabla p = f, \quad t > 0, x \in \mathbb{R}^d, \quad (1)$$

$$\nabla \cdot u = 0, \quad t > 0, x \in \mathbb{R}^d, \quad (2)$$

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^d, \quad (3)$$

where  $u(t, x)$  represents the velocity field and  $p(t, x)$  is the associated pressure of the fluid. The function  $u_0(x) = u(0, x)$  is the initial velocity and  $f(t, x)$  represents an external force. The problem (1)-(3), can be written in an abstract form as

$$\partial_t(k * (u - u_0)) + \mathcal{A}_p u = F(u, u) + Pf, \quad t > 0, \quad (4)$$

where  $\mathcal{A}_p u := P(-\Delta)u$ ,  $P : L_p(\mathbb{R}^d) \rightarrow L_p^\sigma(\mathbb{R}^d)$  is well-known as Helmholtz-Leray's projection, and the nonlinear term  $F(u, v) := -P(u \cdot \nabla)v$ . Equation (4) can be written as a Volterra equation of the form

$$u + (\ell * \mathcal{A}_r u)(t) = u_0 + (\ell * [F(u, u) + Pf])(t), \quad t > 0, \quad (5)$$

by condition  $(k, \ell) \in \mathcal{PC}$ .

## 2 Main Results

We investigate the existence and uniqueness of global mild solutions for equation (5). Before we state the main result, we introduce space where the mild solution will dwell. Let  $d \in \mathbb{N}$ . For any  $2 \leq d < q < \infty$ , consider the space  $X_q$  of the functions  $v$  satisfying  $v \in C_b([0, \infty); L_d^\sigma(\mathbb{R}^d))$ ,  $(1 * \ell)^{\frac{1}{2} - \frac{d}{2q}}v \in C_b((0, \infty); L_q^\sigma(\mathbb{R}^d))$  and  $(1 * \ell)^{\frac{1}{2}}\nabla v \in C_b((0, \infty); L_d^\sigma(\mathbb{R}^d))$ , which is a Banach space with norm

$$\|v\|_{X_q} := \max\{\sup_{t>0} \|v(t)\|_{L_d^\sigma(\mathbb{R}^d)}, \sup_{t>0} [(1 * \ell)(t)]^{\frac{1}{2} - \frac{d}{2q}} \|v(t)\|_{L_q^\sigma(\mathbb{R}^d)}, \sup_{t>0} [(1 * \ell)(t)]^{\frac{1}{2}} \|\nabla v(t)\|_{L_d^\sigma(\mathbb{R}^d)}\}.$$

The existence of the mild solutions solution for (5) will be a consequence of the following fixed point lemma (see [2, Lemma 1.5]).

**Lemma 2.1.** *Let  $X$  be an abstract Banach Space and  $L : X \times X \rightarrow X$  a bilinear operator. Assume that there exists  $\eta > 0$  such that, given  $x_1, x_2 \in X$ , we have  $\|L(x_1, x_2)\| \leq \eta \|x_1\| \|x_2\|$ . Then for any  $y \in X$ , such that  $4\eta \|y\| < 1$ , the equation  $x = y + L(x, x)$  has a solution  $x$  in  $X$ . Moreover, this solution  $x$  is the only one such that*

$$\|x\| \leq \frac{1 - \sqrt{1 - 4\eta \|y\|}}{2\eta}. \quad (1)$$

**Theorem 2.1.** *Let  $d \in \mathbb{N}$ ,  $2 \leq d < q < \infty$ ,  $\eta$  an appropriate constant and  $f \in C_b([0, \infty); L_{\frac{qd}{q+d}}(\mathbb{R}^d))$  be such that  $\alpha := \{\sup_{t>0} [(1 * \ell)(t)]^{1 - \frac{d}{2q}} \|f(t)\|_{L_{\frac{qd}{q+d}}(\mathbb{R}^d)}\} < \infty$ . For  $u_0 \in L_d^\sigma(\mathbb{R}^d)$  and  $\alpha > 0$  sufficiently small, there exists  $0 < \lambda < \frac{1 - 4\alpha\vartheta C\eta}{4\eta}$ , where  $\vartheta$  and  $C$  are positive real constants, such that if  $\|u_0\|_{L_d(\mathbb{R}^d)} \leq \min\{1, C^{-1}\}\lambda$ , then the problem (5) has a global mild solution  $u \in X_q$  that is the unique one satisfying (1). In particular,*

$$\|u(t, \cdot)\|_{L_q(\mathbb{R}^d)} \leq \frac{1 - \sqrt{1 - 4\eta(\lambda + \alpha\vartheta C)}}{2\eta} [(1 * \ell)(t)]^{-\frac{1}{2} + \frac{d}{2q}} \text{ and } \|\nabla u(t, \cdot)\|_{L_d(\mathbb{R}^d)} \leq \frac{1 - \sqrt{1 - 4\eta(\lambda + \alpha\vartheta C)}}{2\eta} [(1 * \ell)(t)]^{-\frac{1}{2}}.$$

If, in addition,  $f \equiv 0$ , we have

$$[(1 * \ell)(t)]^{\frac{1}{2} - \frac{d}{2q}} \|u(t, \cdot)\|_{L_q(\mathbb{R}^d)} \rightarrow 0 \text{ and } [(1 * \ell)(t)]^{\frac{1}{2}} \|\nabla u(t, \cdot)\|_{L_d(\mathbb{R}^d)} \rightarrow 0,$$

as  $t \rightarrow 0^+$ . Furthermore, let  $u, v \in X_q$  be two solutions given by the existence part corresponding to the initial data  $u_0$  and  $v_0$ , respectively. Then,

$$\|u - v\|_{X_q} \leq \frac{C}{\sqrt{1 - 4\eta(\lambda + \alpha\vartheta C)}} \|u_0 - v_0\|_{L_d(\mathbb{R}^d)}.$$

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# CONTROLABILIDADE GLOBAL DO SISTEMA DE BOUSSINESQ COM CONDIÇÕES DE FRONTEIRA DO TIPO NAVIER

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## Abstract

Neste trabalho, apresentamos um resultado global de controlabilidade exata às trajetórias do sistema de Boussinesq. Consideraremos domínios limitados com fronteiras suaves. Completaremos o modelo considerando uma condição de fronteira do tipo *Navier slip-with-friction* para o campo velocidade e uma condição de fronteira do tipo *Robin* é imposta à temperatura. Assumiremos que se pode atuar livremente sobre a velocidade e a temperatura em uma parte arbitrária da fronteira. A prova se baseia em três argumentos principais. Primeiro, reformularemos o problema como um problema de controlabilidade distribuída usando um procedimento de extensão de domínio. Então, provaremos um resultado global de controlabilidade aproximado seguindo a estratégia de Coron et al [J. EUR. Math. Soc., 22 (2020), pp. 1625–1673], que trata das equações de Navier-Stokes (o argumento depende da controlabilidade do sistema inviscido de Boussinesq e das expansões assintóticas do *boundary layer*). Finalmente, concluiremos com um resultado de controlabilidade local que estabeleceremos por meio de um argumento de linearização e estimativas de Carleman apropriadas.

## 1 Introdução

Seja  $T > 0$ ,  $\Omega \subset \mathbb{R}^n$  ( $n = 2, 3$ ) um domínio limitado regular com  $\Gamma := \partial\Omega$  e  $\Gamma_c \subset \Gamma$  um subconjunto aberto não-vazio que intercepta todas as componentes conexas de  $\Gamma$ . Consideraremos um sistema de Boussinesq, tal que sobre a fronteira, o campo velocidade do fluido deve satisfazer uma condição *Navier slip-with-friction* e a temperatura uma condição do tipo *Robin*. Assumimos também que o controle pode atuar em  $\Gamma_c$ , obtendo:

$$\left\{ \begin{array}{ll} u_t - \Delta u + (u \cdot \nabla)u + \nabla p = \theta e_n & \text{em } (0, T) \times \Omega, \\ \theta_t - \Delta \theta + u \cdot \nabla \theta = 0 & \text{em } (0, T) \times \Omega, \\ \nabla \cdot u = 0 & \text{em } (0, T) \times \Omega, \\ u \cdot \nu = 0, \quad N(u) = 0 & \text{sobre } (0, T) \times (\Gamma \setminus \Gamma_c), \\ R(\theta) = 0 & \text{sobre } (0, T) \times (\Gamma \setminus \Gamma_c), \\ u(0, \cdot) = u_0, \quad \theta(0, \cdot) = \theta_0 & \text{em } \Omega. \end{array} \right. \quad (1)$$

As funções  $u = u(t, x)$ ,  $\theta = \theta(t, x)$  e  $p = p(t, x)$  são, respectivamente, vistas como o campo de velocidade, a temperatura e a pressão do fluido. Os termos das condições de fronteira de Navier e Robin são, respectivamente, dados pelas seguintes fórmulas:

$$N(u) := [D(u)\nu + Mu]_{tan} \quad \text{e} \quad R(\theta) := \frac{\partial \theta}{\partial \nu} + m\theta,$$

onde  $M = M(t, x)$  é uma matriz simétrica regular relacionada à rugosidade da fronteira, chamada *matriz de fricção* e  $m = m(t, x)$  é uma função regular, conhecida como *coeficiente de transferência de calor*. Com estas condições

temos a presença de *boundary layer*, devido o atrito na fronteira. Provamos que o sistema (1) é *controlável á trajetórias*, isto significa ser possível conduzir (por meio de controles) qualquer estado inicial á qualquer trajetória prescrita do sistema.

## 2 Resultados Principais

Vamos definir

$$L_c^2(\Omega)^n := \{u \in L^2(\Omega)^n : \nabla \cdot u = 0 \text{ in } \Omega, u \cdot \nu = 0 \text{ on } \Gamma \setminus \Gamma_c\},$$

$$W_T(\Omega) := [C_w^0([0, T]; L_c^2(\Omega)^n) \cap L^2(0, T; H^1(\Omega)^n)] \times [C_w^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))].$$

Temos o seguinte resultado principal:

**Teorema 2.1.** *Sejam  $T > 0$  um tempo positivo,  $(u_0, \theta_0) \in L_c^2(\Omega)^n \times L^2(\Omega)$  um dado inicial e  $(\bar{u}, \bar{\theta}) \in W_T(\Omega)$  uma trajetória fraca de (1). Então, existe uma solução fraca controlada para (1) em  $W_T(\Omega)$  que satisfaz*

$$(u, \theta)(T, \cdot) = (\bar{u}, \bar{\theta})(T, \cdot).$$

Este Teorema generaliza para o sistema de Boussinesq (onde os efeitos térmicos são considerados) o principal resultado do controle em [1], estabelecido para as equações de Navier-Stokes.

**Esquema da prova:** Principais ideias e resultados necessários para a prova do Teorema:

- Reduzimos o problema de controlabilidade distribuída aplicando uma técnica clássica de extensão de domínio. Em seguida, limitamos nossas considerações em dados iniciais regulares, usando o efeito de regularização do sistema Boussinesq não controlado.
- Partindo de dados iniciais suficientemente regulares, provamos um resultado *global de controlabilidade aproximada*. O *boundary layer* é tratado nesta etapa. Para isso, seguimos a estratégia realizada por Coron, Marbach e Sueur em [1] no caso Navier-Stokes.
- Provamos um resultado de *controlabilidade local* usando desigualdades de Carleman para o adjunto do sistema linearizado e uma estratégia de ponto fixo. Para isso, utilizamos ideias de [2] e [3].
- Combinamos todos esses argumentos e obtemos a prova.

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HIERARCHICAL EXACT CONTROLLABILITY OF SEMILINEAR PARABOLIC EQUATIONS  
WITH DISTRIBUTED AND BOUNDARY CONTROLS

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**Abstract**

We present some exact controllability results for parabolic equations in the context of hierarchic control using Stackelberg–Nash strategies. We analyze two cases: in the first one, the main control (the leader) acts in the interior of the domain and the secondary controls (the followers) act on small parts of the boundary; in the second one, we consider a leader acting on the boundary while the followers are of the distributed kind. In both cases, for each leader an associated Nash equilibrium pair is found; then, we obtain a leader that leads the system exactly to a prescribed (but arbitrary) trajectory. We consider linear and semilinear problems.

## 1 Introduction

Let  $\Omega \subset \mathbb{R}^N$  ( $N \geq 1$ ) be a bounded domain with boundary  $\Gamma$  of class  $C^2$ . Let  $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2 \subset \Omega$  be (small) nonempty open sets and let  $S, S_1$  and  $S_2$  be nonempty open subsets of  $\Gamma$ . Given  $T > 0$ , we will set  $Q := \Omega \times (0, T)$  and  $\Sigma := \Gamma \times (0, T)$ . In this paper,  $1_A$  denotes the characteristic function of the set  $A$ .

We will consider parabolic systems of the form

$$\begin{cases} y_t - \Delta y + a(x, t)y = F(y) + f1_{\mathcal{O}} & \text{in } Q, \\ y = v^1\rho_1 + v^2\rho_2 & \text{on } \Sigma, \\ y(\cdot, 0) = y^0 & \text{in } \Omega \end{cases} \quad (1)$$

and

$$\begin{cases} p_t - \Delta p + a(x, t)p = F(p) + u^11_{\mathcal{O}_1} + u^21_{\mathcal{O}_2} & \text{in } Q, \\ p = g\rho & \text{on } \Sigma, \\ p(\cdot, 0) = p^0 & \text{in } \Omega, \end{cases} \quad (2)$$

where  $y^0, p^0, f, g, v^i$  and  $u^i$  are given in appropriate spaces,  $F : \mathbb{R} \rightarrow \mathbb{R}$  is a locally Lipschitz-continuous function and  $\rho, \rho_i \in C^2(\Gamma)$ , with

$$0 < \rho \leq 1 \text{ on } S, \quad \rho = 0 \text{ on } \Gamma \setminus S, \quad 0 < \rho_i \leq 1 \text{ on } S_i, \quad \rho_i = 0 \text{ on } \Gamma \setminus S_i.$$

We will analyze the exact controllability to the trajectories of (1) and (2) following hierarchic control techniques, as introduced by J.-L. Lions [1]. More precisely, we will apply the Stackelberg–Nash method, which combines optimization techniques of the Stackelberg kind and non-cooperative Nash optimization techniques.

Let us define the secondary cost functionals for (1) and (2), respectively, as follows:

$$J_i(f; v^1, v^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d} \times (0,T)} |y - \xi_{i,d}|^2 dx dt + \frac{\mu_i}{2} \iint_{S_i \times (0,T)} |v^i|^2 d\sigma dt, \quad i = 1, 2, \quad (3)$$

and

$$K_i(g; u^1, u^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d} \times (0,T)} |p - \zeta_{i,d}|^2 dx dt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i \times (0,T)} |u^i|^2 dx dt, \quad i = 1, 2, \quad (4)$$

where  $\mathcal{O}_{1,d}, \mathcal{O}_{2,d} \subset \mathcal{O}$  be nonempty open,  $\xi_{i,d}, \zeta_{i,d}$  are given in  $L^2(\mathcal{O}_{i,d} \times (0, T))$ , and  $\alpha_i, \mu_i$  are positive constants.

Results based on Stackelberg–Nash strategies with one leader and several followers have been obtained in [3] (resp. in [1, 2]) in the context of approximate (resp. exact) controllability. In all these papers, distributed controls were considered.

The main goal in the present paper is to try to extend these results to systems of the kind (1) and (2), that is, parabolic semilinear systems partially controlled from the boundary.

## 2 Main Results

**Theorem 2.1.** Suppose  $\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset$ ,  $i = 1, 2$ . Assume that one of the following conditions holds: either

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d} \quad \text{and} \quad \xi_{1,d} = \xi_{2,d} \quad (5)$$

or

$$\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}. \quad (6)$$

If the  $\mu_i/\alpha_i$  ( $i = 1, 2$ ) are large enough and  $F \in W^{1,\infty}(\mathbb{R})$ , there exists a positive function  $\varsigma = \varsigma(t)$  blowing up at  $t = T$  with the following property: if  $\bar{y}$  is a trajectory of (1) associated to the initial state  $\bar{y}^0 \in L^2(\Omega)$  and

$$\iint_{\mathcal{O}_{i,d} \times (0,T)} \varsigma^2 |\bar{y} - \xi_{i,d}|^2 dx dt < +\infty, \quad i = 1, 2, \quad (7)$$

then, for any  $y^0 \in L^2(\Omega)$  there exist controls  $f \in L^2(\mathcal{O} \times (0, T))$  and associated Nash quasi-equilibria  $(v^1, v^2)$  such that the corresponding solutions to (1) satisfy  $y(T, \cdot) = \bar{y}(T, \cdot)$ .

**Theorem 2.2.** Suppose that

$$S \subset \overline{\mathcal{O}}_i \quad \text{and} \quad \overline{\mathcal{O}}_i \cap \overline{\mathcal{O}}_{j,d} = \emptyset, \quad i, j = 1, 2. \quad (8)$$

If the  $\mu_i/\alpha_i$  are large enough and  $F \in W^{1,\infty}(\mathbb{R})$ , there exists a positive function  $\hat{\varsigma} = \hat{\varsigma}(t)$  blowing up at  $t = T$  with the following property: if  $\bar{p}$  is a trajectory of (2) associated to the initial state  $\bar{p}^0 \in L^2(\Omega)$  and the  $\zeta_{i,d} \in L^2(\mathcal{O}_{i,d} \times (0, T))$  are such that

$$\iint_{\mathcal{O}_{i,d} \times (0,T)} \hat{\varsigma}^2 |\bar{p} - \zeta_{i,d}|^2 dx dt < +\infty, \quad i = 1, 2, \quad (9)$$

then, for any  $p^0 \in L^2(\Omega)$ , there exist a control  $g \in H^{1/2, 1/4}(S \times (0, T))$  and an associated Nash quasi-equilibria  $(u^1, u^2)$  such that the corresponding solution to (2) satisfies  $p(T, \cdot) = \bar{p}(T, \cdot)$ .

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INTERAÇÃO ENTRE DISSIPAÇÃO FRACIONÁRIA E NÃO-LINEARIDADE DE MEMÓRIA NA  
EXISTÊNCIA DE SOLUÇÕES PARA EQUAÇÕES DE TIPO PLACA

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**Abstract**

Consideramos uma equação de tipo de placas com inércia rotacional, sob os efeitos de um amortecimento fracionário e uma não-linearidade de tipo de memória, isto é, consideramos o seguinte problema de Cauchy

$$\begin{cases} u_{tt} - \Delta u_{tt} - \Delta u + \Delta^2 u + (-\Delta)^\theta u_t = \int_0^t (t-s)^{-\gamma} |u(s, \cdot)|^p \, ds, & (t, x) \in [0, \infty) \times \mathbb{R}^n, \\ (u, u_t)(0, x) = (u_0(x), u_1(x)), & x \in \mathbb{R}^n \times \mathbb{R}^n \end{cases} \quad (\text{P})$$

onde  $\theta \in [0, \frac{1}{2})$ ,  $\gamma \in (0, 1)$ ,  $p > 1$ . Nossa objetivo é determinar o expoente crítico  $\bar{p}$  que é limítrofe entre a existência e a não-existência de soluções globais para o problema dado, bem como entender como o amortecimento fracionário interage com a não-linearidade de memória e como essa interação pode interferir no valor de  $\bar{p}$ . Com esse fim, encontramos e utilizamos diversas estimativas de energia  $L^\eta - L^q$  com  $1 \leq \eta \leq 2 \leq q \leq \infty$ , bem como estimativas pontuais e de multiplicadores á Mikhlin-Hörmander,  $(L^1 \cap L^p) - L^p$  para  $p < 2$ , onde há uma perda na taxa de decaimento. Analisamos diversos cenários baseados na dimensão  $n \leq 4$  e nos intervalos admissíveis para os parâmetros  $\theta$  e  $\gamma$ , que caracterizam o *damping* fracionário e a não-linearidade de memória, respectivamente. A contraparte de resultados de não-existência é obtida utilizando uma técnica de “funções teste modificadas”, substituindo-se a condição de suporte compacto por funções em  $C_q^\infty(\mathbb{R}^n)$ , um espaço de funções infinitamente diferenciáveis com decaimento polinomial no infinito. Esta modificação se faz necessária devido à não-localidade do operador Laplaciano Fracionário  $(-\Delta)^\theta$ .

## 1 Introdução

Buscamos determinar o *expoente crítico* para (P). Nos resultados de existência, para  $n \leq 4$ , mostramos que  $\bar{p}$  é dado por uma competição entre três valores: um expoente de tipo de Fujita [1],

$$p_c(n, \gamma, \theta) := 1 + \frac{2(1 + (1 - \gamma)(1 - \theta))}{(n - 2 + 2\gamma(1 - \theta))_+}, \quad (1)$$

um expoente  $\gamma^{-1}$  independente da dimensão [2], devido à forte influência do termo de memória não-linear quando  $\gamma \nearrow 1$ , e um terceiro expoente,  $\tilde{p}_c = \frac{14 - 10\theta}{7 - 2\theta + 2\gamma(1 - \theta)}$ , fruto da perda de decaimento na região de alta frequência que é ocasionada pelo termo de inércia rotacional  $-\Delta u_{tt}$ .

## 2 Resultados Principais

**Teorema 2.1** (Existência de Soluções Globais). *Sejam  $n \in \mathbb{N}$ ,  $n \leq 4$ ,  $\theta \in [0, \frac{1}{2})$ ,  $\gamma \in (0, 1)$ ,  $p > \bar{p}$  e  $s = 2 + 2\gamma(1 - \theta)$ , com  $\bar{p} := \max \{p_c, \gamma^{-1}, \tilde{p}_c\}$ . Então, existe  $\varepsilon > 0$  tal que, para dados iniciais*

$$(u_0, u_1) \in \mathcal{A} := \begin{cases} \left(H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)\right) \times \left(H^{s-1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)\right), & \text{se } p \geq 2, \\ \left(H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap \dot{W}^{3,p}(\mathbb{R}^n)\right) \times \left(H^{s-1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap \dot{W}^{2,p}(\mathbb{R}^n)\right), & \text{se } p < 2, \end{cases} \quad (1)$$

com  $\|(u_0, u_1)\|_{\mathcal{A}} \leq \varepsilon$ , existe uma solução global para o problema (P),  $u \in \mathcal{C}([0, \infty), H^2) \cap \mathcal{C}^1([0, \infty), H^1)$ .

**Prova:** A boa colocação para o caso supercrítico é mostrada com base solução para o problema linear associado, escrita como  $u^{lin}(t, x) = K_0(t, x) * u_0(x) + K_1(t, x) * u_1(x)$ . Utiliza-se o Princípio de Duhamel para reescrever o problema como

$$u(t, \cdot) = u^{lin}(t, \cdot) + Gu(t, x) \quad \text{em } H^k(\mathbb{R}^n), \quad (2)$$

onde

$$Gu(t, x) := \int_0^t \int_0^\tau (\tau - s)^{-\gamma} (I - \Delta)^{-1} K_1(t - \tau, x) * |u(s, x)|^p \, ds \, d\tau. \quad (3)$$

Em seguida, define-se para  $T > 0$  o espaço de Banach de evolução  $X(T) := \mathcal{C}([0, T], H^2) \cap \mathcal{C}^1([0, T], H^1)$  e utiliza-se estimativas para  $u^{lin}$  e  $Gu$  em diversos espaços  $H^\kappa(\mathbb{R}^n)$ , de modo a provar que o operador  $G$  é bem-definido em  $X(T)$ , leva bolas de  $X(T)$  em bolas de  $X(T)$ , e é Lipschitz em  $X(T)$ . Com isso, pode-se aplicar o Princípio de Contração de Picard e garantir a existência e unicidade de solução local para o problema dado. Por fim, como as estimativas obtidas para  $u$  são uniformes com relação a  $T$ , toma-se o limite  $T \nearrow \infty$ , obtendo-se assim a globalidade das soluções com relação à variável temporal. ■

**Teorema 2.2** (Não-Existência de Soluções Globais). *Seja  $p_c$  como em (1), defina  $\bar{p} = \max\{p_c, \gamma^{-1}\}$  e fixe  $q = n + 2\theta$ . Assuma que  $u_0, u_1 \in L^1(\langle x \rangle^q dx)$  e ainda, que  $u_1$  satisfaça a seguinte condição de sinal,*

$$\int_{\mathbb{R}^n} u_1 \, dx > 0.$$

*Se existir uma solução fraca (não-trivial) global no tempo  $u \in L^p([0, \infty), L^p(\mathbb{R}^n, \langle x \rangle^{-q}))$  para o problema (P), então  $p \geq \bar{p}$ .*

**Prova:** Caso  $\gamma > \frac{n-2}{n}$ : Suponha que  $u \not\equiv 0$  seja uma solução global para (P). Para  $T > 0$  e  $R \gg 1$ , definimos  $\varphi_R(x) = \langle R^{-1}x \rangle^{-q}$  e  $\psi_T(t) = D_{t|T}^{1-\gamma}(\omega_T(t)^\beta)$ , onde  $D_{t|T}^\alpha$  denota o operador Derivada Fracionária de ordem  $\alpha$  de Riemann-Liouville à direita, e  $\omega_T(t) := (1 - t/T)\chi_{[0, T]}$ . Prova-se então que para  $\eta > 0$ , a solução  $u$  deve satisfazer

$$\int_0^{R^n} \int_{B_R} \omega_T(t)^\beta |u(t, x)|^p \varphi_R(x) \, dt \, dx < C_{\alpha, \varepsilon} R^{-g(\eta)p' + n + \eta}, \quad (4)$$

onde  $g(\eta)$  é uma função auxiliar que satisfaz  $g(\eta)p' - (n + \eta) > 0$  quando  $p < p_c$ . Tomando-se o limite  $T, R \nearrow \infty$ , conclui-se  $u \equiv 0$ , uma contradição. ■

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# EXPONENTES CRÍTICOS PARA UM SISTEMA PARABÓLICO ACOPLADO COM COEFICIENTES DEGENERADOS

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## Abstract

Neste trabalho estudamos o seguinte sistema parabólico acoplado  $u_t - \operatorname{div}(\omega(x)\nabla u) = t^r v^p$ ,  $v_t - \operatorname{div}(\omega(x)\nabla v) = t^s u^q$  em  $\mathbb{R}^N \times (0, T)$ , onde  $p, q > 0$ , com  $pq > 1$ ;  $r, s > -1$ , as condição inicial  $(u_0, v_0) \in (L^\infty(\mathbb{R}^N))^2$ , com  $u_0, v_0 \geq 0$ , e  $\omega$  é uma função de classe Muckenhoupt  $A_{1+\frac{2}{N}}$ . Estabelecemos resultados de existência local e global de soluções não negativas.

## 1 Introdução

Sejam  $T > 0$  e  $N \geq 0$ . Consideremos o seguinte sistema parabólico acoplado

$$\begin{cases} u_t - \operatorname{div}(\omega(x)\nabla u) = t^r v^p, & \mathbb{R}^N \times (0, T), \\ v_t - \operatorname{div}(\omega(x)\nabla v) = t^s u^q, & \mathbb{R}^N \times (0, T), \\ u(0) = u_0, & \mathbb{R}^N, \\ v(0) = v_0, & \mathbb{R}^N, \end{cases} \quad (1)$$

onde  $(u_0, v_0) \in (L^\infty(\mathbb{R}^N))^2$ , com  $u_0, v_0 \geq 0$ ;  $p, q > 0$ ,  $pq > 1$ ;  $r, s > -1$  e a função  $\omega$  ou é

(A)  $\omega(x) = |x|^a$  com  $a \in [0, 1)$  se  $N = 1, 2$  e  $a \in [0, \frac{2}{N})$  se  $N \geq 3$ , ou

(B)  $\omega(x) = |x|^b$  com  $b \in [0, 1)$ .

O problema (1) aparece em modelos térmicos com difusão degenerada em um meio não homogêneo e modelos populacionais, veja [3, 4].

Soluções para o problema (1) é entendida no seguinte sentido

**Definição 1.1.** Sejam  $(u_0, v_0) \in (L^\infty(\mathbb{R}^N))^2$ , com  $u_0, v_0 \geq 0$ ,  $r, s > -1$ ,  $pq > 1$  e  $T \in (0, \infty]$ . Então chamamos solução do problema (1), se  $(u, v) \in (L^\infty(0, T; L^\infty(\mathbb{R}^N)))^2$  e satisfaz

$$\begin{aligned} u(t) &= S(t)u_0 + \int_0^t S(t-\sigma)\sigma^r v^p(\sigma)d\sigma \\ v(t) &= S(t)v_0 + \int_0^t S(t-\sigma)\sigma^s u^q(\sigma)d\sigma, \end{aligned} \quad (2)$$

para  $t > 0$ . Quando  $T = \infty$ , então é dita solução global no tempo. Onde  $S(t)z(x) = \int_{\mathbb{R}^N} \Gamma(x, y, t)z(y)dy$  para  $t > 0$ , e  $\Gamma(x, y, t)$  é a solução fundamental do problema homogêneo  $u_t - \operatorname{div}(\omega(x)\nabla u) = 0$ .

## 2 Resultados Principais

Neste trabalho apresentamos resultados que garantem a existência local e global de soluções não negativas para o problema (1). O resultado que garante a existência local é o seguinte

**Teorema 2.1.** *Assuma (A) ou (B), e sejam  $(u_0, v_0) \in [L^\infty(\mathbb{R}^N)]^2$ , com  $u_0, v_0 \geq 0$ . Então, existe  $T > 0$  tal que o problema (1) possui uma única solução  $(u, v)$  definido sobre  $[0, T]$  e satisfazem*

$$\sup_{0 < t < T} (\|u(t)\|_\infty + \|v(t)\|_\infty) \leq C_0(\|u_0\|_\infty + \|v_0\|_\infty) \quad (1)$$

**Prova:** Para demonstrar este resultado, constuímos uma sequência e, em seguida, usamos as propriedades de  $\Gamma(x, y, t)$  e adaptamos as idéias que aparecem em [2].

**Teorema 2.2.** *Assuma que  $r, s > -1$  e  $p, q \geq 0$ , com  $pq > 1$ , e sejam  $\gamma_1 = \frac{(r+1)+(s+1)p}{pq-1}$ ,  $\gamma_2 = \frac{(s+1)+(r+1)q}{pq-1}$ ,  $\gamma = \max\{\gamma_1, \gamma_2\}$ ,  $r_{1*} = \frac{N}{(2-\alpha)\gamma_1}$ ,  $r_{2*} = \frac{N}{(2-\alpha)\gamma_2}$ , onde  $\alpha = a$  no caso (A) e  $\alpha = b$  no caso (B).*

(i) *Se  $\gamma \geq \frac{N}{2-\alpha}$ , então o problema (1) não tem soluções globais não triviais.*

(ii) *Se  $\gamma < \frac{N}{2-\alpha}$ , então existem soluções globais não triviais para o problema (1).*

**Prova:** Para demonstrar este resultado, utilizamos as propriedades de  $\Gamma(x, y, t)$  apresentadas em [2]. Logo, utilizamos as ideias que aparecem em [1], para o operador do problema (1).

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## SISTEMA DE BRESSE COM DISSIPAÇÃO NÃO-LINEAR NA FRONTEIRA

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Estudamos o sistema de Bresse com mecanismos de dissipação não-linear na fronteira. Utilizando conceitos e resultados da Teoria de Semigrupos não-lineares, provamos a existência e unicidade de solução para o sistema de Bresse com mecanismos de dissipação não-linear na fronteira e mostramos a estabilidade exponencial do sistema de Bresse sem qualquer condição sobre as velocidades de propagação das ondas.

**1 Introdução**

Considere o seguinte sistema de Bresse sem forças externas

$$\begin{aligned} \rho_1 \varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) &= 0 \text{ em } (0, L) \times (0, \infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) &= 0 \text{ em } (0, L) \times (0, \infty), \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) &= 0 \text{ em } (0, L) \times (0, \infty), \end{aligned} \quad (1)$$

Um problema delicado no estudo do sistema de Bresse consiste em mostrar a estabilidade exponencial com mecanismo de dissipação na fronteira.

Nosso objetivo é novo no estudo de sistema de Bresse pois, trabalharmos com mecanismo de dissipação na fronteira não-linear, o que é de grande relevância na literatura. Primeiramente, foi obtido a existência e unicidade do sistema de Bresse sem forças externas com as seguintes condições de fronteira

$$\begin{aligned} \varphi(0, t) = \psi(0, t) = w(0, t) &= 0, \forall t \geq 0, \\ k(\varphi_x + \psi + lw)(L, t) + g_1(\varphi_t(L, t)) &= 0, \forall t \geq 0 \\ b\psi_x(L, t) + g_2(\psi_t(L, t)) &= 0, \forall t \geq 0 \end{aligned} \quad (2)$$

onde  $g_i : \mathbb{R} \rightarrow \mathbb{R}$ , para  $i = 1, 2, 3$  são termos dissipativos não-lineares e condições iniciais

$$\begin{aligned} \varphi(\cdot, 0) &= \varphi_0(\cdot), \varphi_t(\cdot, 0) = \varphi_1(\cdot) \\ \psi(\cdot, 0) &= \psi_0(\cdot), \psi_t(\cdot, 0) = \psi_1(\cdot) \\ w(\cdot, 0) &= w_0(\cdot), w_t(\cdot, 0) = w_1(\cdot). \end{aligned} \quad (3)$$

O escopo do nosso trabalho está direcionado à estabilidade exponencial do sistema de Bresse sem forças externas com mecanismos de dissipação não-linear na fronteira agindo simultaneamente nas forças axial e de cisalhamento e no momento bending, sem a necessidade de velocidades iguais de propagação de ondas e sem condições adicionais. O trabalho que nos inspirou foi o de Lasiecka e Tatura [3], juntamente com a teoria de existência para semigrupos não-lineares abordada nos trabalhos de [2] e [3].

**2 Resultados Principais**

*Hipótese H-1:* As funções não lineares  $g_i : \mathbb{R} \rightarrow \mathbb{R}$ , para  $i = 1, 2, 3$  satisfazem as seguintes condições :

- (i)  $g_i$  são funções contínuas e crescentes sobre  $\mathbb{R}$ ;
- (ii)  $g_i(s)s > 0$  para  $s \neq 0$ ,
- (iii) Existem  $m$  e  $M$  constantes tais que  $0 < m < M$  e  $ms^2 \leq g_i(s)s \leq Ms^2$ ,  $|s| > 1$ .

**Teorema 2.1.** Assumindo a Hipótese H-1, temos que para cada  $Y_0 \in D(\mathcal{A})$  existe uma única solução forte para (1). Além disso, se  $Y_0 \in \mathcal{H}$  então (1), possui uma única solução generalizada.

Diversos trabalhos que envolvem sistema de Bresse tem como hipótese uma condição puramente matemática. Esta condição é a ou diferença entre as velocidades de propagações de ondas, a saber,

$$\frac{\rho_1}{\rho_2} = \frac{k}{b} \text{ e } k = k_0. \quad (4)$$

Mostramos que a energia associado à uma solução do sistema de Bresse com dissipações não lineares na fronteira decai exponencialmente sem a hipótese (4). Desssa forma, trabalhamos com sistemas fisicamente possíveis.

Com multiplicadores convenientes nos obtivemos a seguinte desigualdade:

$$\begin{aligned} \int_0^T E(t)dt \leq C \left[ \int_0^T [\rho_1(\varphi_t)^2(L) + \rho_2(\psi_t)^2(L) + \rho_1(w_t)^2(L) + (g_1(\varphi_t(L)))^2 + (g_2(\psi_t(L)))^2 \right. \\ \left. + (g_3(w_t(L)))^2] Ldt + \int_0^T \int_0^L (\varphi^2 + \psi^2 + w^2) dx dt + E(T) \right] \end{aligned} \quad (5)$$

**Lemma 2.1.** Para  $T$  suficientemente grande, existe uma constante  $C > 0$  tal que

$$\begin{aligned} \int_0^T \int_0^L (\varphi^2 + \psi^2 + w^2) dx dt \leq C \int_0^T [(g_1(\varphi_t(L)))^2 + (g_2(\psi_t(L)))^2 + (g_3(w_t(L)))^2 \\ + \rho_1(\varphi_t)^2(L) + \rho_2(\psi_t)^2(L) + \rho_1(w_t)^2(L)] Ldt, \end{aligned} \quad (6)$$

para toda solução forte  $U = (\varphi, \psi, w, \Phi, \Psi, W)$  do sistema de Bresse dado em (1)-(3).

Da desigualdade dada em (5) e do lema anterior, obtemos o resultado:

**Teorema 2.2.** Seja  $T > 0$  suficientemente grande. Então, a energia do sistema dado por (1)-(3) satisfaz

$$E(T) \leq C_T \int_0^T [(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1w_t^2 + (g_1(\varphi_t))^2 + (g_2(\psi_t))^2 + (g_3(w_t))^2)(L)] Ldt. \quad (7)$$

Utilizando as funções definidas por Lasieka e Tataru e procedendo de maneira análoga como em [3], tendo em vista o Teorema (2.2), a solução do sistema (1)-(3) satisfaz o Teorema 2 de [3].

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# SOBRE A CONTROLABILIDADE UNIFORME DOS SISTEMAS BURGERS- $\alpha$ NÃO-VISCOSO E VISCOSO

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## Abstract

Analisamos neste trabalho a controlabilidade global de certas famílias de EDP's chamadas de *sistemas Burgers- $\alpha$  não-viscoso e viscoso*. Nessas equações, o termo convectivo da famosa equação de Burgers é substituído por um termo regularizado, o qual é induzido por um filtro de Helmholtz de comprimento de onda característico  $\alpha$ . Provamos primeiramente um resultado de controlabilidade global exata (uniforme em relação a  $\alpha$ ) para o sistema Burgers- $\alpha$  não-viscoso usando, principalmente, o *método do retorno* e um *argumento de ponto-fixo*. Após isso, a controlabilidade global exata e uniforme à estados constantes é deduzida para o sistema viscoso. Para tal propósito, provamos primeiramente um resultado de controlabilidade local exata e, feito isso, estabelecemos um resultado de controlabilidade global aproximada para estados inicial e final regulares.

## 1 Introdução

Sejam  $L, T > 0$  dados. Neste trabalho, consideraremos as seguintes duas famílias de sistemas controlados:

$$\left\{ \begin{array}{ll} y_t + zy_x = p(t) & \text{em } (0, T) \times (0, L), \\ z - \alpha^2 z_{xx} = y & \text{em } (0, T) \times (0, L), \\ z(\cdot, 0) = v_l, \quad z(\cdot, L) = v_r & \text{em } (0, T), \\ y(\cdot, 0) = v_l & \text{em } I_l, \\ y(\cdot, L) = v_r & \text{em } I_r, \\ y(0, \cdot) = y_0 & \text{em } (0, L), \end{array} \right. \quad (1)$$

onde  $I_l = \{t \in (0, T) : v_l(t) > 0\}$  e  $I_r = \{t \in (0, T) : v_r(t) < 0\}$  e

$$\left\{ \begin{array}{ll} y_t - \mu y_{xx} + zy_x = p(t) & \text{em } (0, T) \times (0, L), \\ z - \alpha^2 z_{xx} = y & \text{em } (0, T) \times (0, L), \\ z(\cdot, 0) = y(\cdot, 0) = v_l & \text{em } (0, T), \\ z(\cdot, L) = y(\cdot, L) = v_r & \text{em } (0, T), \\ y(0, \cdot) = y_0 & \text{em } (0, L). \end{array} \right. \quad (2)$$

Os sistemas (1) e (2) são chamados, respectivamente, de *sistemas Burgers- $\alpha$  não-viscoso e viscoso*. Devemos destacar também que a terna  $(p, v_l, v_r)$  e o par  $(y, z)$  representam, respectivamente, os *controles* e os *estados associados*. O parâmetro  $\mu > 0$  é a viscosidade do fluido e  $\alpha > 0$  é o comprimento de onda característico do chamado *filtro de Helmholtz*.

## 2 Resultados Principais

**Teorema 2.1.** Seja  $\alpha > 0$  dado. O sistema Burgers- $\alpha$  não-viscoso (1) é globalmente exatamente controlável em  $C^1$ . Mais precisamente, dados  $y_0, y_1 \in C^1([0, L])$ , existe um controle fonte  $p^\alpha \in C^0([0, T])$ , um par de controles de fronteira  $(v_l^\alpha, v_r^\alpha) \in C^1([0, T]; \mathbb{R}^2)$  e um par de estados associados  $(y^\alpha, z^\alpha) \in C^1([0, T] \times [0, L]; \mathbb{R}^2)$  satisfazendo (1) e

$$y^\alpha(T, \cdot) = y_T \quad \text{in } (0, L).$$

Além disso, existe uma constante  $C > 0$  (dependendo de  $L, T, y_0$  e  $y_T$ , mas independente de  $\alpha$ ), tal que

$$\|(z^\alpha, y^\alpha)\|_{C^1([0, T] \times [0, L]; \mathbb{R}^2)} + \|p^\alpha\|_{C^0([0, T])} + \|(v_l^\alpha, v_r^\alpha)\|_{C^1([0, T]; \mathbb{R}^2)} \leq C.$$

**Prova:** Conforme vemos em [1], a prova baseia-se, essencialmente, em dois argumentos principais: *método do retorno* e *argumento de ponto fixo*. A aplicação do *método do retorno* consistiu em linearizar o sistema não-linear (1) ao redor de uma trajetória apropriada e provar que o sistema linearizado assim obtido é globalmente controlável a zero. Após isso, efetuamos uma leve perturbação nesse sistema linearizado e provamos, usando um *argumento de ponto fixo de Banach*, que o sistema perturbado é localmente controlável a zero. O resultado do teorema segue facilmente daí.

**Teorema 2.2.** Seja  $\alpha > 0$  dado. Então, o sistema Burgers- $\alpha$  viscoso é globalmente exatamente controlável, em  $L^\infty$ , à trajetórias constantes. Noutras palavras, para quaisquer  $y_0 \in L^\infty(0, L)$  e  $N \in \mathbb{R}$ , existe um controle fonte  $p^\alpha \in C^0([0, T])$ , um par de controles de fronteira  $(v_l^\alpha, v_r^\alpha) \in H^{3/4}(0, T; \mathbb{R}^2)$  e um par de estados associados  $(y^\alpha, z^\alpha) \in L^2(0, T; H^1(0, L; \mathbb{R}^2)) \cap L^\infty(0, T; L^\infty(0, L; \mathbb{R}^2))$ , satisfazendo (2),

$$y^\alpha(T, \cdot) = N \quad \text{in } (0, L),$$

e a seguinte estimativa

$$\|p^\alpha\|_{C^0([0, T])} + \|(v_l^\alpha, v_r^\alpha)\|_{H^{3/4}(0, T; \mathbb{R}^2)} \leq C,$$

onde  $C > 0$  é uma constante que depende de  $L, T, y_0$  e  $N$ , mas independe de  $\alpha$ . Além disso, se  $y_0 \in H_0^1(0, L)$  então a mesma conclusão ocorre com

$$(y^\alpha, z^\alpha) \in L^2(0, T; H^2(0, L; \mathbb{R}^2)) \cap H^1(0, T; L^2(0, L; \mathbb{R}^2)).$$

**Prova:** Conforme vemos em [1], a prova divide-se em três etapas: efeito regularizante, controlabilidade aproximada para dados regulares e controlabilidade local exata à trajetórias de classe  $C^1([0, T])$ .

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# SISTEMA DE BRESSE COM ACOPLAMENTO TERMOELÁSTICO NO MOMENTO FLETOR E LEI DE FOURIER

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## Abstract

Neste trabalho estuda-se um sistema de Bresse com acoplamento termoelástico no momento fletor considerando a lei de Fourier para o fluxo de calor. O principal objetivo é fazer uma apresentação mais detalhada da existência, unicidade e comportamento assintótico do problema descrito em [1]. A teoria de semigrupos de operadores lineares é utilizada para garantir a existência e unicidade de solução. Uma condição necessária e suficiente é dada para a obtenção da estabilidade exponencial do semigrupo e verifica-se que sob certas condições obtém-se decaimento polinomial da solução.

## 1 Introdução

O objetivo do presente trabalho é apresentar resultados descritos em [1] de uma forma didática e detalhada. Além disso, utilizando um resultado obtido em [2], melhoramos resultados estabelecidos em [1], no que concerne as taxas de decaimento polinomial, para o sistema termoelástico de Bresse.

O sistema estudado é dado por

$$\rho_1 \varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) = 0 \quad \text{em } (0, \infty) \times (0, L), \quad (1)$$

$$\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\theta_x = 0 \quad \text{em } (0, \infty) \times (0, L), \quad (2)$$

$$\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) = 0 \quad \text{em } (0, \infty) \times (0, L), \quad (3)$$

$$\rho_3 \theta_t - \alpha\theta_{xx} + \gamma\psi_{xt} = 0 \quad \text{em } (0, \infty) \times (0, L). \quad (4)$$

Considerando as seguintes condições iniciais:

$$\begin{aligned} \varphi(0, \cdot) &= \varphi_0, & \varphi_t(0, \cdot) &= \varphi_1, & \psi(0, \cdot) &= \psi_0, & \psi_t(0, \cdot) &= \psi_1, \\ w(0, \cdot) &= w_0, & w_t(0, \cdot) &= w_1, & \theta(0, \cdot) &= \theta_0. \end{aligned} \quad (5)$$

E condições de fronteira de Dirichlet

$$\begin{aligned} \varphi(t, 0) &= \varphi(t, L) = \psi(t, 0) = \psi(t, L) = w(t, 0) = w(t, L) = 0, \\ \theta(t, 0) &= \theta(t, L) = 0 \quad \text{para } t \in (0, \infty), \end{aligned} \quad (6)$$

ou condições de fronteira de Dirichlet-Neumann

$$\begin{aligned} \varphi(t, 0) &= \varphi(t, L) = \psi_x(t, 0) = \psi_x(t, L) = w_x(t, 0) = w_x(t, L) = 0, \\ \theta(t, 0) &= \theta(t, L) = 0 \quad \text{para } t \in (0, \infty). \end{aligned} \quad (7)$$

Onde os coeficientes  $\rho_1, k, \rho_2, \rho_3 = \frac{\gamma}{m}, \alpha = \frac{k_1\gamma}{m}, b, \gamma, l, k_0, k_1$  e  $m$  são constantes positivas e as funções  $\varphi, \psi, w$  e  $\theta$  descrevem, respectivamente, a oscilação vertical, o ângulo de rotação da seção transversal, a oscilação longitudinal e a variação de temperatura de uma viga fina, arqueada e com comprimento  $L$ .

O sistema (1)-(7) foi estudado por [1], onde os autores mostraram que a estabilidade da solução do sistema está diretamente ligada as seguintes constantes

$$\chi := \left| 1 - \frac{k}{k_0} \right| \text{ e } \chi_0 := \left| \frac{b}{k} \rho_1 - \rho_2 \right|. \quad (8)$$

Em [1], Fatori e Rivera mostraram que a solução do sistema é exponencialmente estável se, e somente se,  $\chi = \chi_0 = 0$ . Além disso, mostraram que se  $\chi = \chi_0 = 0$  não se satisfaz o semigrupo associado ao sistema (1)-(5) em geral possui uma taxa de decaimento  $t^{-1/6}$  e que para o caso em que  $\chi = 0$  e  $\chi_0 \neq 0$  a taxa é  $t^{-1/3}$ .

Obtemos neste trabalho uma melhora em relação ao artigo apresentado em [1], a saber, mostraremos que para o caso  $\chi_0 \neq 0$  e  $\chi = 0$  a taxa de decaimento da solução do semigrupo associado ao sistema (1)-(7) pode ser melhorada para  $t^{-1/2}$ , e que no caso de  $\chi \neq 0$  e o semigrupo associado ao sistema (1)-(5) com condições de fronteira (7) ser polinomialmente estável, a taxa de decaimento da solução não pode ser melhor que  $t^{-1/2}$ .

## 2 Resultados Principais

**Teorema 2.1.**

$$\frac{\rho_1}{\rho_2} \neq \frac{k}{b} \quad \text{ou} \quad k \neq k_0, \quad (1)$$

então o semigrupo associado ao sistema (1)-(5) com condições de fronteira (7) não é exponencialmente estável.

**Prova:** Para obter esta prova veja [3].

**Teorema 2.2.** Suponha que  $\rho_1, \rho_2, \rho_3, b, k, k_0, \alpha, \gamma > 0$ ,  $\chi_0 \neq 0$  e  $\chi = 0$ . Então, existe uma constante  $C > 0$ , independentes do dado inicial  $U_0 \in \mathcal{H}$ , tal que

$$\|U(t)\|_{\mathcal{H}} \leq \frac{C}{t^{1/2}} \|U_0\|_{D(\mathcal{A}_i)}; \quad t \rightarrow \infty. \quad (2)$$

**Prova:** Para obter esta prova veja [3].

**Teorema 2.3.** Se  $\chi \neq 0$  e o semigrupo associado ao sistema (1)-(5) com condições de fronteira (7) for polinomialmente estável, então a taxa de decaimento do semigrupo não pode ser melhor que

$$\|U(t)\|_{\mathcal{H}_2} \leq \frac{C}{t^{1/2}} \|U_0\|_{D(\mathcal{A}_2)}; \quad t \rightarrow \infty. \quad (3)$$

**Prova:** Para obter esta prova veja [3].

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## LINEABILITY OF MULTILINEAR SUMMING OPERATORS

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### Abstract

We present a result on the recent notion of directed/geometric lineability, introduced by Fávaro, Pellegrino and Tomáz (2020), related to the class of multilinear  $\Lambda$ -summing operators. Some applications are obtained, in particular, we prove that the set of  $m$ -linear operators on Banach spaces with values on  $\ell_p$  that are absolutely but not multiple summing is  $(1, \mathfrak{c})$ -spaceable. This is a joint work with N. G. Albuquerque and D. Tomáz.

### 1 Introduction

Let  $E_1, \dots, E_m$  be Banach spaces over  $\mathbb{K}$ , the complex or real scalar field. The Bohnenblust-Hille multilinear inequality [1], provides that every  $m$ -linear form  $E_1 \times \dots \times E_m \rightarrow \mathbb{K}$  is multiple  $(\frac{2m}{m+1}; 1)$ -summing and, moreover,  $\frac{2m}{m+1}$  is optimal. The Defant-Voigt theorem (see [2]) also tells us that every multilinear form  $E_1 \times \dots \times E_m \rightarrow \mathbb{K}$  is  $(r; 1)$ -summing, for  $r \geq 1$ . Combining these results we concluded that

$$\Pi_{(r;1)}^{\text{abs}}(E_1, \dots, E_m, \mathbb{K}) \setminus \Pi_{(r;1)}^{\text{mult}}(E_1, \dots, E_m, \mathbb{K}) \neq \emptyset$$

whenever  $1 \leq r < \frac{2m}{m+1}$ . Here  $\Pi_{(p;q)}^{\text{abs}}$  denotes the class of absolutely  $(p, q)$ -summing operators and  $\Pi_{(p;q)}^{\text{mult}}$  the class of multiple  $(p, q)$ -summing operators. Therefore is natural to investigate the lineability (and also spaceability) of the set of absolutely but not multiple summing multilinear operators. We deal with these problems in the general concept of multilinear  $\Lambda$ -summing operators (see [1] and [3]), providing results in the more restrictive variant of lineability/spaceability notion recently presented in [4]. Next we present a precise definition of these concepts. As usual, the topological dual and the closed unit ball of a Banach space  $E$  will be denoted by  $E'$  and  $B_E$ , respectively,  $\mathfrak{c}$  stands for the cardinality of  $\mathbb{R}$  and  $m$  will always be a positive integer.

**Definition 1.1.** Let  $E_1, \dots, E_m, F$  Banach spaces,  $\mathbf{r} := (r_1, \dots, r_m), \mathbf{s} = (s_1, \dots, s_m) \in [1, +\infty)^m$  and  $\Lambda \subset \mathbb{N}^m$  a set of indexes, a  $m$ -linear operator  $T : E_1 \times \dots \times E_m \rightarrow F$  is  $\Lambda$ -( $\mathbf{r}, \mathbf{s}$ )-summing, if there is a constant  $C > 0$  such that

$$\left( \sum_{i_1=1}^N \left( \dots \left( \sum_{i_m=1}^N \left\| T(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)}) \mathbf{1}_\Lambda(i_1, \dots, i_m) \right\|_F^{r_m} \right)^{\frac{r_{m-1}}{r_m}} \dots \right)^{\frac{r_1}{r_2}} \right)^{\frac{1}{r_1}} \leq C \prod_{k=1}^m \sup_{\phi_k \in B_{E'_k}} \left( \sum_{i=1}^N |\phi(x_i^{(k)})|^{p_k} \right)^{\frac{1}{p_k}},$$

for all  $N \in \mathbb{N}$  and  $x_i^{(k)} \in E_k$ ,  $k = 1, \dots, m$ ,  $i = 1, \dots, N$ , where  $\mathbf{1}_\Lambda$  is the characteristic function of  $\Lambda$ .

The set of operators that fulfill the previous inequality is denoted by  $\Pi_{(\mathbf{r}, \mathbf{s})}^\Lambda(E_1, \dots, E_m; F)$ , which is a Banach space endowed with the usual norm taken as the infimum of the constants  $C > 0$ . Notice that when  $\Lambda = \mathbb{N}^m$  and  $\Lambda = \{(i, \dots, i) : i \in \mathbb{N}\}$ , the class of multiple, absolutely summing operators is recovered, respectively. It is worth pointing out that the concepts of summing operators can be investigated on quasi-Banach spaces which topological dual is nontrivial (see [5]).

**Definition 1.2.** Let  $\alpha, \beta, \lambda$  be cardinal numbers and  $V$  be a vector space, with  $\dim V = \lambda$  and  $\alpha < \beta \leq \lambda$ . A set  $A \subset V$  is  $(\alpha, \beta)$ -lineable (respec.  $(\alpha, \beta)$ -spaceable), if it is  $\alpha$ -lineable and for every subspace  $W_\alpha \subset V$ , with  $W_\alpha \subset A \cup \{0\}$  and  $\dim W_\alpha = \alpha$ , there is a subspace (respec. closed subspace)  $W_\beta \subset V$ , with  $\dim W_\beta = \beta$  and  $W_\alpha \subset W_\beta \subset A \cup \{0\}$ .

Observe that this definition encompass and refine the original lineability notion when  $\alpha = 0$ .

## 2 Main Results

Let  $\Lambda \subset \Lambda^*$  subsets of  $\mathbb{N}^m$ ,  $\mathbf{E} =: E_1 \times \dots \times E_m$ , with  $E_1, \dots, E_m$  Banach spaces and  $p \in (0, \infty)$ . Our main result provides that the set  $\Pi_{(\mathbf{r}, \mathbf{s})}^\Lambda(\mathbf{E}; \ell_p) \setminus \Pi_{(\mathbf{r}, \mathbf{s})}^{\Lambda^*}(\mathbf{E}; \ell_p)$  is either empty or  $(1, \mathfrak{c})$ -spaceable. Among others applications, we provide the spaceability of the class of absolutely but not multiple summing operators and also the class of linear operators that fails to be absolutely summing.

**Theorem 2.1.** Let  $E_1, \dots, E_m$  be Banach spaces,  $\mathbf{E} =: E_1 \times \dots \times E_m$ ,  $p \in (0, \infty)$ ,  $\mathbf{r} := (r_1, \dots, r_m), \mathbf{s} := (s_1, \dots, s_m) \in [1, +\infty)^m$  and  $\Lambda \subset \Lambda^* \subset \mathbb{N}^m$  sets of indexes. Let us consider the spaces of  $m$ -linear summing operators  $\Pi_{(\mathbf{r}, \mathbf{s})}^\Lambda(\mathbf{E}; \ell_p)$  and  $\Pi_{(\mathbf{r}, \mathbf{s})}^{\Lambda^*}(\mathbf{E}; \ell_p)$ . Then

$$\Pi_{(\mathbf{r}, \mathbf{s})}^\Lambda(\mathbf{E}; \ell_p) \setminus \Pi_{(\mathbf{r}, \mathbf{s})}^{\Lambda^*}(\mathbf{E}; \ell_p)$$

is either nonempty or  $(1, \mathfrak{c})$ -spaceable.

**Corollary 2.1.** Let  $\mathbf{r} := (r_1, \dots, r_m), \mathbf{s} := (s_1, \dots, s_m) \in [1, +\infty)^m$  and  $p \in (0, +\infty)$ . Then

$$\Pi_{(\mathbf{r}, \mathbf{s})}^{abs}(E_1, \dots, E_m; \ell_p) \setminus \Pi_{(\mathbf{r}, \mathbf{s})}^{mult}(E_1, \dots, E_m; \ell_p)$$

is either empty or  $(1, \mathfrak{c})$ -spaceable.

It is well known that, for  $0 < p < 1$ , the identity  $I : \ell_p \rightarrow \ell_p$  is a non- $(r, s)$ -absolutely summing operator for any  $1 \leq s \leq r < \infty$ . Hence, the set  $\mathcal{L}(\ell_p, \ell_p) \setminus \Pi_{(r,s)}(\ell_p, \ell_p)$  is not empty. Using this fact and a direct application of the technique used in Theorem 2.1, we obtain the following result.

**Proposition 2.1.** Let  $0 < p < 1$  and let  $1 \leq s \leq r < \infty$ . Then  $\mathcal{L}(\ell_p; \ell_p) \setminus \bigcup_{1 \leq s \leq r < \infty} \Pi_{(r,s)}(\ell_p; \ell_p)$  is  $(1, \mathfrak{c})$ -spaceable.

In the next result we deal with multilinear operators with values in  $\ell_p(\Gamma)$ .

**Proposition 2.2.** Under the same assumptions of the Theorem 2.1, if the set  $\Pi_{(\mathbf{r}, \mathbf{s})}^\Lambda(\mathbf{E}; \ell_p) \setminus \Pi_{(\mathbf{r}, \mathbf{s})}^{\Lambda^*}(\mathbf{E}; \ell_p)$  is nonempty, it is  $(\alpha, \text{card}(\Gamma))$ -lineable for all  $\alpha < \text{card}(\Gamma)$ .

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# POLINÔMIOS HOMOGÊNEOS NÃO ANALÍTICOS E UMA APLICAÇÃO ÀS SÉRIES DE DIRICHLET

MIKAELA A. OLIVEIRA<sup>1</sup><sup>1</sup>ICE, UFAM, AM, Brasil, mado11318@gmail.com**Abstract**

Na dissertação estuda-se polinômios homogêneos contínuos que não são analíticos. Os principais resultados referem-se à existência de estruturas lineares constituídas por polinômios não analíticos e, também, uma aplicação desses polinômios às séries de Dirichlet. Com esse fim, começamos com o estudo dos polinômios homogêneos entre espaços de Banach e suas principais propriedades. Em seguida, são exibidas as construções do polinômio 2-homogêneo dada por Toeplitz e do polinômio  $m$ -homogêneo,  $m \geq 2$ , devida à Bohnenblust e Hille. Com o auxílio desses polinômios é gerado um subespaço vetorial isomorfo ao espaço  $\ell_1$ , gozando da propriedade de que os seus elementos (não nulos) são polinômios homogêneos que não são analíticos num determinado vetor. Em particular, o conjunto dos polinômios homogêneos não analíticos em  $c_0$  é espaçável. Por fim, como uma aplicação exibimos a solução do Problema de Convergência Absoluta de Bohr, que consiste na determinação da distância máxima entre as abscissas de convergência absoluta e uniforme de uma série de Dirichlet, tendo como ferramenta útil em sua solução o polinômio de Bohnenblust e Hille.

## 1 Introdução e resultados principais

é sabido dos cursos introdutórios de análise complexa que uma função de uma variável complexa é holomorfa se e somente se é analítica, i.e., pode ser representada localmente como uma série infinita de monômios de uma variável. Para funções de finitas variáveis complexas é possível mostrar que esse resultado continua válido. Por muito tempo se acreditou que para funções em infinitas variáveis isso também valeria.

Em 1913 o matemático Toeplitz exibiu um exemplo de uma função holomorfa em que sua representação por série de potências não convergia em todo ponto. Mais precisamente ele construiu um polinômio 2-homogêneo em  $c_0$  que não era analítico em todos os pontos do seu domínio. Denotando  $\mathcal{P}(^2c_0)$  o espaço dos polinômios 2-homogêneos contínuos em  $c_0$ , Toeplitz mostrou o seguinte resultado

**Teorema 1.1.** *Existe  $P \in \mathcal{P}(^2c_0)$  de modo que para cada  $\varepsilon > 0$ , existe  $z \in \ell_{4+\varepsilon}$   $P$  tal que não é analítico.*

Posteriormente, Bohnenblust e Hille em [1] para resolverem um problema na série de Dirichlet estenderam a construção de Toeplitz a polinômios  $m$ -homogêneos ( $m \geq 2$ ) e mostraram o seguinte

**Proposição 1.1.** *Para cada  $m \geq 2$  fixo, existe  $P \in \mathcal{P}(^m c_0)$  tal que para cada  $\varepsilon > 0$ , existe  $z \in \ell_{\frac{2m}{m-1}+\varepsilon}$  de modo que  $P$  não é analítico.*

O trabalho de Bohnenblust e Hille [1] tem sido bastante explorado nos últimos anos por ter implicações no estudo da analiticidade de polinômios  $m$ -homogêneos contínuos. Em 2016 J. Alberto Conejero, Juan B. Seoane-Sepúlveda e Pablo Sevilla Peris com base nos polinômios de Bohnenblust e Hille mostraram em [2] que o conjunto dos polinômios  $m$ -homogêneos contínuos não analíticos em  $c_0$ , (que denotaremos por  $N_m$ ) é espaçável em  $\mathcal{P}(^m c_0)$ , ou seja,  $N_m \cup \{0\}$  contém um espaço vetorial fechado de dimensão infinita.

**Teorema 1.2.** *Para cada  $m \geq 2$ , o conjunto  $N_m \cup \{0\}$  contém uma cópia isomorfa de  $\ell_1$ . Em particular  $N_m$  é espaçável em  $\mathcal{P}(^m c_0)$ .*

Como aplicação estudamos o *Problema de convergência absoluta de Bohr* que consiste na determinação da largura máxima da faixa em que uma série de Dirichlet converge uniformemente mas não absolutamente. Este problema foi considerado por Harold Bohr em 1913 enquanto investigava a distância máxima entre as abscissas de convergência de uma série de Dirichlet. Bohr considerou o número

$$S = \sup\{\sigma_a(D) - \sigma_u(D) : D \text{ é uma série de Dirichlet}\}$$

onde  $\sigma_a(D)$  e  $\sigma_u(D)$  denotam as abscissas de convergência absoluta e uniforme de uma série de Dirichlet  $D$ , respectivamente. Bohr mostrou que  $S \leq \frac{1}{2}$ , entretanto ele não conseguiu nenhum exemplo de modo que

$$\sigma_a(D) - \sigma_u(D) = \frac{1}{2}.$$

Apesar de não ter resolvido este problema, Bohr forneceu ferramentas que levaram a sua solução. Ele percebeu que as séries de Dirichlet e as séries de potências formais estavam relacionadas por meio dos números primos. Dada uma série de Dirichlet  $D(s) = \sum a_n n^{-s}$  considere para cada  $n \in \mathbb{N}$  sua decomposição em números primos  $n = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ . Pela unicidade dessa decomposição cada  $n$  corresponde um único  $\alpha = (\alpha_1, \dots, \alpha_N)$ . Então, definindo  $c_\alpha = a_{p^\alpha}$  cada série de Dirichlet corresponde uma única série de potências  $\sum c_\alpha z^\alpha$ , onde  $z^\alpha = z^{\alpha_1} \cdots z^{\alpha_N}$ . Essa correspondência é chamada de *Transformada de Bohr* e permite traduzir problemas sobre séries de Dirichlet em termos de séries de potências.

O problema de convergência absoluta de Bohr foi resolvido apenas em 1931 por Hille e Bohnenblust, e uma das ferramentas usadas foi o polinômio  $m$ -homogêneo que tinham construído. Com isso eles mostraram em [2] que para cada  $m \in \mathbb{N}$ , existem séries de Dirichlet tais que

$$\sigma_a - \sigma_u = \frac{m-1}{2m}.$$

Isto implicava no seguinte resultado

**Proposição 1.2.** *Temos*

$$S = \frac{1}{2},$$

e o supremo é atingido, ou seja, existe uma série de Dirichlet tal que  $\sigma_u(D) = 0$  e  $\sigma_a(D) = \frac{1}{2}$ .

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## IDEAIS INJETIVOS DE POLINÔMIOS HOMOGÊNEOS ENTRE ESPAÇOS DE BANACH

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Neste trabalho estudamos os ideais injetivos de polinômios homogêneos, com ênfase na envoltória injetiva. Posteriormente, apresentamos a descrição da envoltória injetiva de um ideal de composição e aplicações desta descrição são fornecidas.

## 1 Introdução

As noções de ideal injetivo e envoltória injetiva aparecem inicialmente para ideais de operadores lineares (veja [3]), e posteriormente são generalizados de forma natural para ideais de polinômios homogêneos. Os ideais injetivos são importantes por possuir estreita relação com as injeções métricas (ou isometrias lineares) e com a restrição de contradomínio de um operador. São importantes também porque muitos ideais interessantes de operadores e de polinômios são injetivos. Este trabalho é baseado nos resultados principais de [2], artigo no qual os ideais injetivos de polinômios foram primeiramente estudados.

## 2 Resultados Principais

Ao longo deste trabalho,  $m$  denota um número natural qualquer e as letras  $E, F, G$  e  $H$  denotam espaços de Banach quaisquer, reais ou complexos. Por  $E'$  denotamos o dual topológico do espaço  $E$ , por  $\mathcal{L}(E; F)$  denotamos o espaço dos operadores lineares de  $E$  em  $F$  e por  $\mathcal{P}^{(m)}E; F)$  o espaço dos polinômios  $m$ -homogêneos contínuos de  $E$  em  $F$ . Para comodidade do leitor, apresentamos a definição de ideal de polinômios.

**Definição 2.1.** Um *ideal de polinômios (homogêneos)* é uma subclasse  $\mathcal{Q}$  da classe de todos os polinômios homogêneos contínuos entre espaços de Banach tal que suas componentes  $\mathcal{Q}^{(m)}E; F) = \mathcal{P}^{(m)}E; F) \cap \mathcal{Q}$ , onde  $m \in \mathbb{N}$  e  $E$  e  $F$  são espaços de Banach arbitrários, satisfazem as seguintes condições:

- (1)  $\mathcal{Q}^{(m)}E; F)$  é um subespaço vetorial de  $\mathcal{P}^{(m)}E; F)$  que contém os polinômios  $m$ -homogêneos de tipo finito.
- (2) Se  $u \in \mathcal{L}(G; E)$ ,  $P \in \mathcal{Q}^{(m)}E; F)$  e  $t \in \mathcal{L}(F; H)$ , então  $t \circ P \circ u \in \mathcal{Q}^{(m)}G; H)$ .

Se  $\|\cdot\|_{\mathcal{Q}}: \mathcal{Q} \rightarrow \mathbb{R}$  é uma função tal que  $(\mathcal{Q}^{(m)}E; F), \|\cdot\|_{\mathcal{Q}}$  é um espaço normado (de Banach) para quaisquer espaços de Banach  $E$  e  $F$  e numero natural  $m$ , e satisfaz as seguintes condições:

- (I)  $\|\text{id}_m: \mathbb{K} \rightarrow \mathbb{K}, \text{id}_m(\lambda) = \lambda^m\|_{\mathcal{Q}} = 1$  para todo  $m \in \mathbb{N}$ , e
  - (II) Se  $u \in \mathcal{L}(G; E)$ ,  $P \in \mathcal{Q}^{(m)}E; F)$  e  $t \in \mathcal{L}(F; H)$ , então  $\|t \circ P \circ u\|_{\mathcal{Q}} \leq \|t\| \cdot \|P\|_{\mathcal{Q}} \cdot \|u\|^m$ ,
- então  $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$  é chamado de *ideal normado (de Banach)* de polinômios.

Dado um ideal de polinômios  $\mathcal{Q}$ , por  $\mathcal{Q}_m$  denotamos a sua *componente m-linear*, isto é,  $\mathcal{Q}_m(E; F) := \mathcal{Q}^{(m)}E; F)$  para todos  $E$  e  $F$  espaços de Banach.  $\mathcal{Q}_m$  é chamado também de *ideal de polinômios m-homogêneos*. é claro que  $\mathcal{Q}_1$  é um ideal de operadores lineares. Para a teoria básica de ideais de operadores, veja [3].

Uma *injeção métrica* é um operador linear  $j: E \rightarrow F$  tal que  $\|j(x)\| = \|x\|$  para todo  $x \in E$ . Por  $I_E: E \rightarrow \ell_\infty(B_{E'})$  denotamos a *injeção métrica canônica* (veja [3, C.3.3]).

**Definição 2.2.** (i) Dizemos que um ideal de polinômios  $\mathcal{Q}$  é *injetivo* se dados  $P \in \mathcal{P}(^m E; F)$  e uma injeção métrica  $j: F \rightarrow G$  tais que  $j \circ P \in \mathcal{Q}(^m E; G)$ , tem-se que  $P \in \mathcal{Q}(^m E; F)$ .

(ii) Um ideal normado de polinômios  $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$  é *injetivo* se  $\mathcal{Q}$  é um ideal injetivo de polinômios e, na situação acima,  $\|P\|_{\mathcal{Q}} = \|j \circ P\|_{\mathcal{Q}}$ .

A seguir veremos as propriedades principais da envoltória injetiva de um ideal de polinômios.

**Proposição 2.1.** *Seja  $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$  um ideal normado de polinômios. Então existe um (único) menor ideal normado injetivo de polinômios  $(\mathcal{Q}^{\text{inj}}, \|\cdot\|_{\mathcal{Q}^{\text{inj}}})$  que contém  $\mathcal{Q}$  e tal que  $\|\cdot\|_{\mathcal{Q}^{\text{inj}}} \leq \|\cdot\|_{\mathcal{Q}}$ . Para  $P \in \mathcal{P}(^m E; F)$ ,*

$$P \in \mathcal{Q}^{\text{inj}}(^m E; F) \iff I_F \circ P \in \mathcal{Q}(^m E; \ell_{\infty}(B_{F'})) \text{ e } \|P\|_{\mathcal{Q}^{\text{inj}}} := \|I_F \circ P\|_{\mathcal{Q}}.$$

Mais ainda, o ideal  $(\mathcal{Q}^{\text{inj}}, \|\cdot\|_{\mathcal{Q}^{\text{inj}}})$  é de Banach se o ideal  $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$  for de Banach. O ideal  $\mathcal{Q}^{\text{inj}}$  (ideal normado  $(\mathcal{Q}^{\text{inj}}, \|\cdot\|_{\mathcal{Q}^{\text{inj}}})$ ) é chamado de envoltória injetiva do ideal  $\mathcal{Q}$  (ideal normado  $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ ).

**Corolário 2.1.** (a) Um ideal de polinômios  $\mathcal{Q}$  é injetivo se, e somente se,  $\mathcal{Q} = \mathcal{Q}^{\text{inj}}$ .

(b) Um ideal normado de polinômios  $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$  é injetivo se, e somente se,  $\mathcal{Q} = \mathcal{Q}^{\text{inj}}$  e  $\|\cdot\|_{\mathcal{Q}} = \|\cdot\|_{\mathcal{Q}^{\text{inj}}}$ .

Os conceitos e propriedades de ideais injetivos de operadores (veja [3, 4.6]) são naturalmente recuperados do caso polinomial ao se considerar o caso linear  $m = 1$  no que foi apresentado acima. Neste caso, denotamos também por  $\mathcal{I}^{\text{inj}}$  a envoltória injetiva de um ideal de operadores  $\mathcal{I}$ . Analogamente, obtemos a definição e as propriedades de ideal injetivo de polinômios  $m$ -homogêneos ao considerarmos  $m$  fixo no que foi apresentado acima.

Seja  $\mathcal{I}$  um ideal de operadores. Um polinômio  $P \in \mathcal{P}(^m E; F)$  pertence a  $\mathcal{I} \circ \mathcal{P}(^m E; F)$  se existem um espaço de Banach  $G$ , um polinômio  $Q \in \mathcal{P}(^m E; G)$  e um operador linear  $u \in \mathcal{I}(G; F)$  tais que  $P = u \circ Q$ . O ideal de polinômios  $\mathcal{I} \circ \mathcal{P}$  é chamado de *ideal de composição*. Para maiores informações sobre esse ideal, veja [1]. O resultado a seguir descreve a envoltória injetiva de um ideal de composição.

**Teorema 2.1.** *Seja  $\mathcal{I}$  um ideal de operadores. Então  $(\mathcal{I} \circ \mathcal{P})^{\text{inj}} = \mathcal{I}^{\text{inj}} \circ \mathcal{P}$ .*

Muitas consequências decorrem da fórmula acima. Vejamos algumas.

**Corolário 2.2.** *As seguintes afirmações são equivalentes para um ideal de operadores  $\mathcal{I}$ :*

(a)  $\mathcal{I}$  é injetivo.

(b)  $\mathcal{I} \circ \mathcal{P}$  é um ideal injetivo de polinômios.

(c)  $(\mathcal{I} \circ \mathcal{P})_m$  é um ideal injetivo de polinômios  $m$ -homogêneos para algum  $m \in \mathbb{N}$ .

Por  $\mathcal{A}$  denotamos o ideal dos operadores lineares que podem ser aproximados, na norma usual, por operadores lineares de posto finito, por  $\mathcal{K}$  denotamos o ideal dos operadores lineares compactos, por  $\mathcal{P}_{\mathcal{A}}$  denotamos o ideal dos polinômios que podem ser aproximados, na norma usual, por polinômios de posto finito e por  $\mathcal{P}_{\mathcal{K}}$  o ideal dos polinômios compactos. De [3, 4.6.13] sabemos que  $\mathcal{A}^{\text{inj}} = \mathcal{K}$ . Usando o Teorema 2.1 consegue-se a versão polinomial desse resultado:

**Corolário 2.3.**  $(\mathcal{P}_{\mathcal{A}})^{\text{inj}} = \mathcal{P}_{\mathcal{K}}$ .

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EXISTENCE OF POSITIVE SOLUTIONS FOR BOUNDARY VALUE PROBLEMS WITH  
*P*-LAPLACIAN OPERATOR

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**Abstract**

This paper is concerned with the existence of positive solutions for three point boundary value problems of Riemann-Liouville fractional differential equations with *p*-Laplacian operator. By means of the properties of the Green's function and Avery-Peterson fixed point theorem, we establish a condition ensuring the existence of at least three positive solutions for the problem.

## 1 Introduction

This paper investigates the existence of at least three positive solutions for the following nonlinear fractional boundary value problem, (FBVP in short),

$$D_{0+}^\beta (\varphi_p(D_{0+}^\alpha u(t))) + a(t)f(t, u, u') = 0, \quad \text{for each } t \in [0, 1],$$

$$D_{0+}^\alpha u(0) = u(0) = u'(0) = 0, \quad D_{0+}^{\alpha-2}u(0) = D_{0+}^{\alpha-2}u(1) = \gamma u(\eta),$$

where  $\eta \in (0, 1)$ ,  $\gamma \in \left(0, \frac{\Gamma(\alpha-1)}{\eta^{\alpha-2}}\right)$ ,  $\varphi_p(s) = |s|^{p-2}s$ ,  $p > 1$ ,  $D_{0+}^\alpha$ ,  $D_{0+}^\beta$  are the Riemann-Liouville fractional derivatives with  $\alpha \in (3, 4]$  and  $\beta \in (0, 1]$ .

To establish the existence of multiple positive solutions of FBVP, throughout this paper, we assume that  $f$  and  $a$  satisfy the following conditions:

(H<sub>1</sub>)  $f \in C([0, 1] \times [0, \infty) \times [0, \infty), [0, \infty))$  is a given nonlinear function.

(H<sub>2</sub>)  $a \in L^\infty[0, 1]$  and there exists  $m > 0$  such that  $a(t) \geq m$  a.e.  $t \in [0, 1]$ .

## 2 Main Results

In this section we deduce the existence of at least three positive solutions of the FBVP by using the well known Avery-Peterson fixed point theorem; see [1].

Let  $\gamma$  and  $\theta$  be nonnegative continuous convex functionals on  $P$ ,  $\omega$  be nonnegative continuous concave functional on  $P$  and  $\psi$  be a nonnegative continuous functional en  $P$ . Then for positive numbers  $a$ ,  $b$ ,  $c$  and  $d$ , we define the following convex sets:

$$P(\gamma, d) = \{x \in P : \gamma(x) < d\},$$

$$P(\gamma, \omega, b, d) = \{x \in P : b \leq \omega(x), \gamma(x) \leq d\},$$

$$P(\gamma, \theta, \omega, b, c, d) = \{x \in P : b \leq \omega(x), \theta(x) \leq c, \gamma(x) \leq d\},$$

and a closed set

$$R(\gamma, \psi, a, d) = \{x \in P : a \leq \psi(x), \gamma(x) \leq d\}.$$

**Theorem 2.1.** [1] Let  $P$  be a cone in Banach space  $E$ . Let  $\gamma, \theta$  be nonnegative continuous convex functionals on  $P$ ,  $\omega$  be a nonnegative continuous concave functional on  $P$ , and  $\psi$  be a nonnegative continuous functional on  $P$  satisfying  $\psi(\lambda x) \leq \lambda\psi(x)$  for  $0 \leq \lambda \leq 1$ , such that for some positive numbers  $M$  and  $d$ ,  $\omega(x) \leq \psi(x)$  and  $\|x\| \leq M\gamma(x)$  for  $x \in \overline{P(\gamma, d)}$ .

Suppose that  $T : \overline{P(\gamma, d)} \rightarrow \overline{P(\gamma, d)}$  is completely continuous and there exist positive numbers  $a, b, c$  with  $a < b$  such that

(S<sub>1</sub>)  $\{x \in P(\gamma, \theta, \omega, b, c, d) : \omega(x) > b\} \neq \emptyset$  and  $\omega(Tx) > b$  for  
 $x \in P(\gamma, \theta, \omega, b, c, d)$ ;

(S<sub>2</sub>)  $\omega(Tx) > b$  for  $x \in P(\gamma, \omega, b, d)$  with  $\theta(Tx) > c$ ;

(S<sub>3</sub>)  $0 \notin R(\gamma, \psi, a, d)$  and  $\psi(Tx) < a$  for  $x \in R(\gamma, \psi, a, d)$  with  $\psi(x) = a$ .

Then  $T$  has at least three fixed points  $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$  such that  $\gamma(x_i) \leq d$ ,  $i = 1, 2, 3$ ;  $\omega(x_1) > b$ ;  $\psi(x_2) > a$ ,  $\omega(x_2) < b$ ;  $\psi(x_3) < a$ .

Now, for convenience, we denote

$$\begin{aligned} r_1 &= \left( \frac{\|a\|_\infty}{\Gamma(\beta+1)} \right)^{1-q} \frac{\mathcal{N}}{\mathfrak{B}(2, \beta(q-1)+1)}, \\ r_2 &= \left( \frac{m}{\Gamma(\beta+1)} \right)^{1-q} \tau^{1-\alpha} \left[ \int_\tau^1 G(1, s) s^{\beta(q-1)} ds + \frac{\gamma}{\mathcal{N}} \int_0^1 G(\eta, s) s^{\beta(q-1)} ds \right]^{-1}, \\ r_3 &= \left( \frac{\|a\|_\infty}{\Gamma(\beta+1)} \right)^{1-q} \frac{\Gamma(\alpha) - (\alpha-1)\gamma\eta^{\alpha-2}}{\mathfrak{B}(2, \beta(q-1)+1)}, \end{aligned}$$

where  $\mathcal{N} = \Gamma(\alpha-1) - \gamma\eta^{\alpha-2}$ .

**Theorem 2.2.** Suppose that (H<sub>1</sub> – H<sub>2</sub>) hold and there exist constants  $0 < a < b < b\tau^{1-\alpha} < d$ , such that

(A<sub>1</sub>)  $f(t, u, u') \leq \varphi_p(r_1 d)$ ,  $(t, u, u') \in [0, 1] \times [0, d] \times [0, d]$ ,

(A<sub>2</sub>)  $f(t, u, u') > \varphi_p(r_2 b)$ ,  $(t, u, u') \in [\tau, 1] \times [b, b\tau^{1-\alpha}] \times [0, d]$ ,

(A<sub>3</sub>)  $f(t, u, u') < \varphi_p(r_3 a)$ ,  $(t, u, u') \in [0, 1] \times [\tau^{\alpha-1} a, a] \times [0, d]$ .

Then the FBVP has at least three positive solutions  $u_1, u_2$  and  $u_3$  satisfying

$$\begin{aligned} \max_{0 \leq t \leq 1} u'_i(t) &\leq d, i = 1, 2, 3; \quad \min_{\tau \leq t \leq 1} u_1(t) > b; \\ \max_{0 \leq t \leq 1} u_2(t) &\geq a \quad \text{with} \quad \min_{\tau \leq t \leq 1} u_2(t) < b; \quad \text{and} \quad \max_{0 \leq t \leq 1} u_3(t) < a. \end{aligned}$$

## References

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## ESQUEMAS DE DIFERENÇAS FINITAS PARA SEÇÃO CIRCULAR

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## Abstract

Neste trabalho apresenta-se um esquema de diferenças finitas em coordenadas polares para o laplaciano de uma função com condição de contorno de Dirichlet e, a partir dele, é desenvolvido um esquema semelhante para a norma do gradiente. A principal dificuldade é tratar a singularidade na origem que surge devido à mudança para o sistema de coordenadas polares. Além disso, são montadas matrizes auxiliares para facilitar a implementação numérica dessa aproximação. Uma aplicação dos esquemas desenvolvidos é feita utilizando um modelo de torção elastoplástica para avaliar a qualidade dos resultados e discutir sobre sua importância.

## 1 Introdução

O tema deste trabalho foi motivado por estudos acerca do problema da torção elastoplástica (PTE), que consiste em definir regiões de plasticidade formadas na seção transversal  $\Omega \subset \mathbb{R}^2$  de uma barra submetida a torção. A solução desse problema – descrito detalhadamente em [3] – satisfaz a seguinte desigualdade variacional:

$$u \in \mathcal{K}_\nabla : \int_{\Omega} \nabla u \cdot \nabla(v - u) \, dx dy \geq -\tau \int_{\Omega} \nabla(v - u) \, dx dy, \quad \forall v \in \mathcal{K}_\nabla, \quad (1)$$

com condição de contorno de Dirichlet  $u = 0$  em  $\partial\Omega$ . O conjunto  $\mathcal{K}_\nabla = \{v \in \mathcal{H}_0^1(\Omega) : \|\nabla v\| \leq 1\}$  define os deslocamentos admissíveis e  $\tau$  é uma constante física. A resolução numérica do problema envolve a discretização via método das diferenças finitas do laplaciano e da norma do gradiente de  $u$ . Assim, surgiram dificuldades para alguns formatos de barra, como seções em L (abordada em [1]) e circulares. Embora a utilização de diferenças finitas com coordenadas polares seja antiga, há detalhes que precisam ser tratados com atenção.

Seja o disco de raio  $R$  definido por  $\Omega = \{(x, y) : x^2 + y^2 < R\}$ , aplica-se a mudança de coordenadas  $x = r \cos \theta$ ,  $y = r \sin \theta$ , na qual  $r = \sqrt{x^2 + y^2}$  e  $\theta = \arctan(y/x)$ . Para  $\Omega_p = \{(r, \theta) : 0 < r < R, 0 \leq \theta < 2\pi\}$ , tem-se::

$$\Delta u(r, \theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial u^2}{\partial \theta^2}, \quad \|\nabla u(r, \theta)\|^2 = \left( \frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial u}{\partial \theta} \right)^2. \quad (2)$$

Note que surge uma singularidade na origem nas equações (2). Diferentes estratégias podem ser adotadas para trazer a regularidade desejada, como o método de deslocamento da grade descrito em [2]. Neste caso, a malha de diferenças finitas é definida de forma que o eixo radial seja composto por semi-inteiros, ou seja,  $r_i = (i - 1/2) h_r$  e  $\theta_j = (j - 1) h_\theta$ , sendo  $h_r = R/(N_r + 1/2)$  e  $h_\theta = 2\pi/N_\theta$ , para  $i = 1, 2, \dots, N_r + 1$ ,  $j = 1, 2, \dots, N_\theta + 1$ .

Utilizando diferenças centradas para aproximar o laplaciano, para  $i = 2, 3, \dots, N_r$  e  $j = 1, 2, \dots, N_\theta$ , tem-se:

$$\Delta u(r_i, \theta_j) \approx \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h_r^2} + \frac{1}{r_i} \frac{U_{i+1,j} - U_{i-1,j}}{2h_r} + \frac{1}{r_i^2} \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h_\theta^2}, \quad (3)$$

onde  $U_{i,j}$  denota a solução aproximada de  $u(r_i, \theta_j)$ . Quanto aos pontos de fronteira, segue da condição de contorno e da periodicidade do disco que  $U_{N_r+1,j} = 0$  e  $U_{i,N_\theta+1} = U_{i,1}$ . Para  $i = 1$  o termo  $U_{0,j}$  se anula na equação (3), pois  $r_1 = h_r/2$ . Logo esse método permite que o laplaciano seja resolvido sem nenhuma condição de polo.

O objetivo principal deste trabalho é desenvolver um esquema de diferenças finitas semelhante para a norma do gradiente, já que os resultados encontrados dizem respeito apenas ao laplaciano. Aplicando diferenças centradas:

$$\|u(r_i, \theta_j)\|^2 \approx \frac{U_{i+1,j}^2 - 2U_{i+1,j}U_{i-1,j} + U_{i-1,j}^2}{4h_r^2} + \frac{1}{r_i^2} \frac{U_{i,j+1}^2 - 2U_{i,j+1}U_{i,j-1} + U_{i,j-1}^2}{4h_\theta^2}, \quad (4)$$

para  $i = 2, 3, \dots, N_r$  e  $j = 1, 2, \dots, N_\theta$ . A aproximação para os pontos de fronteira é análoga ao caso anterior, mas para  $i = 1$  o termo  $U_{0,j}$  não se anula. Para contornar essa questão, optou-se por diferenças progressivas para aproximar  $U_{1,j}$ . Para implementar numericamente, duas matrizes  $A$  e  $B$  foram montadas tais que  $\|u\|^2 \approx (AU)^2 + (BU)^2$ , sendo  $U = [U^1 \ U^2 \ \dots \ U^{N_r+1}]^t$  e  $U^i = [U_{i,1} \ U_{i,2} \ \dots \ U_{i,N_\theta+1}]^t$ . São elas:

$$A = \begin{bmatrix} -2L & 2L & & & \\ -L & 0 & L & & \\ \ddots & \ddots & \ddots & & \\ & -L & 0 & L & \\ & -L & 0 & 0 & \\ & & -2L & 0 & \end{bmatrix}, \quad B = \begin{bmatrix} M & & & & \\ & \ddots & & & \\ & & M & & \\ & & & & 0 \end{bmatrix}, \quad (5)$$

nas quais:

$$L = \begin{bmatrix} 1/(2h_r) & & & & \\ & \ddots & & & \\ & & 1/(2h_r) & & \\ & & & 0 & \\ 1/(2h_r) & & & & \end{bmatrix}, \quad M = \begin{bmatrix} -1/h_\theta & 1/h_\theta & & & \\ -1/(2h_\theta) & 0 & 1/(2h_\theta) & & \\ & \ddots & \ddots & \ddots & \\ & & -1/(2h_\theta) & 0 & 1/(2h_\theta) \\ & & 1/(2h_\theta) & -1/(2h_\theta) & 0 & 0 \\ & & 1/h_\theta & & 1/h_\theta & 0 \end{bmatrix}. \quad (6)$$

## 2 Resultados Principais

O PTE foi resolvido numericamente através de dois modelos, via I. complementaridade (utilizando  $\Delta u$ ) e II. minimização (utilizando  $\|\nabla u\|$ ). O caso circular com  $\tau$  constante possui solução analítica, então foi possível calcular o erro. Para (I) o erro relativo médio foi de 0,0554% e para (II) foi de 0,8010%. O esquema de complementaridade teve melhor desempenho, mas no geral pode-se dizer que ambos apresentaram resultados satisfatórios.

A equivalência entre as regiões plásticas definidas por  $\|\nabla u\| = 1$  ou  $|u| = d$  foi um marco na área, sendo  $d : \Omega \rightarrow \mathbb{R}$  a função que mede a menor distância de cada ponto até a fronteira. Esse resultado configura o PTE como um problema tipo obstáculo, com  $u$  sendo representado por uma membrana e  $d$  o obstáculo que restringe seu deslocamento. Assim, é possível encontrar as regiões sem envolver o gradiente. Essa equivalência, entretanto, é válida para o PTE clássico, mas existem outras variações de  $\tau$  compatíveis com a formulação matemática que fogem ao significado físico do problema. Nesses casos, há o surgimento de novas regiões (ou o alargamento de regiões existentes) em que a norma do gradiente iguala ou supera a unidade sem que haja contato com o obstáculo. Por isso um esquema em diferenças finitas para  $\|\nabla u\|$  é importante, ampliando o campo de aplicações matemáticas.

## References

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