Phase Space Analysis for Evolution PDE's and Applications [1]

Global existence (in time) for semilinear models

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Summary

A general problem

Let us consider the semilinear Cauchy problem

$$L(\partial_t, \partial_x, t, x)u = f(u), \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x),$$

where $f(u) = |u|^p$ and L is a linear partial differential operator.

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1. The existence of global (in time) small data solutions. Since $f(u_0)$ becomes small for large p, one is often able to prove global existence result for some $p > p_c$;

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Depending on the regularity of initial data, $p = p_c$ belongs to the supercritical or subcritical case.

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Semilinear heat model

Let us consider the semilinear heat model with power nonlinearity

$$u_t - \triangle u = |u|^{p-1}u, \quad u(0,x) = u_0(x).$$

Fujita [3] shown that $p_{Fuj}(n) = 1 + \frac{2}{n}$ is the threshold between global existence of small data solutions for exponents larger and blow up behavior of solutions for exponents smaller or equal to the Fujita exponent.

[3] H. Fujita, *On the blowing up of solutions of the Cauchy problem for* $u_t = \triangle u + u^{1+\alpha}$. J. Fac. Sci. Univ. Tokyo Sect. I 13 (1966), 109–124.

Semilinear wave equation

Much has been devoted to the Cauchy problem

$$\begin{cases} u_{tt} + (-\triangle)^{\sigma} u = |u|^p \,, & t \geq 0,\; x \in \mathbb{R}^n \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \end{cases} \tag{1}$$

- If $\sigma = 1$, it was proved [4] that the critical index $p_0(n)$ to (1) is the Strauss exponent, i.e., the positive root of

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

- If $\sigma > 1$, then the critical exponent to (1) is given by E. & Lourenço (2019)

$$p_c \doteq 1 + \frac{2\sigma}{[n-\sigma]_+}.$$

[4] V. Georgiev, H. Lindblad, and C. Sogge, Weighted Strichartz estimates and global existence for semilinear wave equations. Amer. J. Math. 119 (1997), no. 6, 1291–1319.

Semilinear exterior damped wave equations: 1/3

A priori estimates for solutions to the associate linear problem [5] may be applied to look for global-in-time small data solutions to

$$u_{tt} - \triangle u + u_t = f(u), \qquad (u, u_t)(0, x) = (u_0, u_1)(x),$$

where $|f(u)| \approx |u|^p$ for some $p > 1$.

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where $|f(u)| \approx |u|^p$ for some $p > 1$.
By using comp. supp. data in $H^1 \times L^2$, Todorova & Yordanov (2001) showed that critical exponent is given by Fujita index $p_c = 1 + 2/n$ for any $n \geq 1$:

- Global existence for small data if $p > 1 + 2/n$ (and also $p \leq 1 + 2/(n-2)$ if $n \geq 3$),
- Nonexistence if $p < 1 + 2/n$ ($p = 1 + 2/n$, in Zhang (2001))

[5] A. Matsumura, On the asymptotic behavior of solutions of semi-linear wave equations, Publ. Res. Inst. Math. Sci. **12**, 169–189 (1976).

Removing compact support: 2/3

- In [6] the authors showed that comp. supp. is not necessary. If data are small in $H^1 \times L^2$ and in L^1, critical exponent remains $1 + 2/n$ in space dimension $n = 1, 2$. For $n = 3$, they could proved a Global existence only for $2 < p \leq 3$.

[6] R. Ikehata, Y. Mayaoka, T. Nakatake, Decay estimates of solutions for dissipative wave equations in $\mathbb{R}^N$ with lower power nonlinearities, J. Math. Soc. Japan **56**, 365–373 (2004).

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- Then Narazaki (2004) introduced estimates on $L^p - L^q$ basis, extending to space dimension $n \leq 5$ if data are small in: energy space, L^1 and $W^{1,p} \times L^p$ when $p < 2$.

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- Ikehata & Ohta (2002) for $n = 1, 2$, proved that if one removes the assumption that the initial data are in $L^1(\mathbb{R}^n)$ and only assume that they are in the energy space, then the critical exponent is modified to $1 + \frac{2m}{n}$ under additional regularity $L^m(\mathbb{R}^n)$, with $m \in [1, 2]$.

[6] R. Ikehata, Y. Mayaoka, T. Nakatake, Decay estimates of solutions for dissipative wave equations in $\mathbb{R}^N$ with lower power nonlinearities, J. Math. Soc. Japan **56**, 365–373 (2004).

Recently: 3/3

- In [7], under additional regularity $L^m(\mathbb{R}^n)$, with $m \in (m_0, 2]$, $m_0 > 1$, the authors derived $L^p - L^q$ estimates and proved global existence results for $p \geq 1 + \frac{2m}{n}$ and $n < n_0$.

[7] M. Ikeda, T. Inui, M. Okamoto, Y. Wakasugi, $L^p - L^q$ estimates for the damped wave equation and the critical exponent for the nonlinear problem with slowly decaying data, Communications on Pure and Applied Analysis, 18 (2019), 1967–2008.

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- E.& Girardi & Reissig (Math. Ann. 2019) considered

$$u_{tt} - \triangle u + u_t = |u|^{1+\frac{2}{n}}\mu(|u|), \qquad (u, u_t)(0, x) = (u_0, u_1)(x),$$

where μ is a modulus of continuity. Under additional regularity $L^1(\mathbb{R}^n)$ for initial data, the threshold between global (in time) existence of small data solutions and blow-up behavior even of small data solutions is given by

$$\int_0^\epsilon \frac{\mu(s)}{s}\, ds < \infty \ \text{ or } \ \int_0^\epsilon \frac{\mu(s)}{s}\, ds = \infty, \quad \epsilon > 0 \textit{ sufficiently small}.$$

[7] M. Ikeda, T. Inui, M. Okamoto, Y. Wakasugi, $L^p - L^q$ estimates for the damped wave equation and the critical exponent for the nonlinear problem with slowly decaying data, Communications on Pure and Applied Analysis, 18 (2019), 1967–2008.

Non-existence result via Test function method

Theorem 1

Let us consider the Cauchy problem for the classical damped wave equation with power nonlinearity

$$u_{tt} - \triangle u + u_t = |u|^p, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x)$$

in $[0,\infty) \times \mathbb{R}^n$ *with* $n \geq 1$ *and* $p \in (1, 1 + \frac{2}{n}]$*. Let* $(\varphi,\psi) \in \mathcal{A}_{1,1}$ *satisfy the assumption*

$$\int_{\mathbb{R}^n} (\varphi(x) + \psi(x))\, dx > 0.$$

Then there exists a locally (in time) defined energy solution

$$u \in C([0,T), H^1(\mathbb{R}^n)) \cap C^1([0,T), L^2(\mathbb{R}^n)).$$

This solution can not be continued to the interval $[0,\infty)$ *in time.*

Proof of Theorem 1: 1/4

We first introduce test functions $\eta = \eta(t)$ and $\phi = \phi(x)$ having the following properties:

1. $\eta \in C_0^\infty[0,\infty)$, $0 \le \eta(t) \le 1$,

$$\eta(t) = \begin{cases} 1 & \text{for } 0 \le t \le \frac{1}{2}, \\ 0 & \text{for } t \ge 1, \end{cases}$$

2. $\phi \in C_0^\infty(\mathbb{R}^n)$, $0 \le \phi(x) \le 1$,

$$\phi(x) = \begin{cases} 1 & \text{for } |x| \le \frac{1}{2}, \\ 0 & \text{for } |x| \ge 1, \end{cases}$$

3. $\frac{\eta'(t)^2}{\eta(t)} \le C$ for $\frac{1}{2} < t < 1$, and $\frac{|\nabla\phi(x)|^2}{\phi(x)} \le C$ for $\frac{1}{2} < |x| < 1$.

Let $R \in [0,\infty)$ be a large parameter. We define the test function

$$\chi_R(t,x) := \eta_R(t)\phi_R(x) := \eta\Big(\frac{t}{R^2}\Big)\phi\Big(\frac{x}{R}\Big).$$

Proof of Theorem 1: 2/4

We put

$$Q_R := [0, R^2] \times B_R, \quad B_R := \{x \in \mathbb{R}^n : |x| \leq R\}.$$

We suppose that the energy solution $u = u(t, x)$ exists globally in time. We define the functional

$$I_R := \int_{Q_R} |u(t,x)|^p \chi_R(t,x)^q \, d(x,t) = \int_{Q_R} (u_{tt} - \triangle u + u_t) \chi_R(t,x)^q \, d(x,t).$$

Here q is the Sobolev conjugate of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$. After integration by parts we obtain

$$\begin{aligned} I_R = &- \int_{B_R} (\varphi + \psi) \phi_R^q \, dx + \int_{Q_R} u \partial_t^2 (\chi_R^q) \, d(x,t) \\ &- \int_{Q_R} u \partial_t (\chi_R^q) \, d(x,t) - \int_{Q_R} u \triangle (\chi_R^q) \, d(x,t) \\ := &- \int_{B_R} (\varphi + \psi) \phi_R^q \, dx + J_1 + J_2 + J_3. \end{aligned}$$

Proof of Theorem 1: 3/4

We shall estimate separately J_1, J_2 and J_3. Here we use the notations

$$\hat{Q}_{R,t} := \left[\frac{R^2}{2}, R^2\right] \times B_R, \quad \hat{Q}_{R,x} := [0, R^2] \times \left(B_R \setminus B_{\frac{R}{2}}\right).$$

Application of Hölder's inequality implies

$$\begin{aligned} |J_3| &\le CR^{-2}\Big(\int_{\hat{Q}_{R,x}} |u|^p \chi_R^q(t,x)\, d(x,t)\Big)^{1/p}\Big(\int_{\hat{Q}_{R,x}} 1\, d(x,t)\Big)^{1/q} \\ &\le CR^{-2} I_{R,x}^{\frac{1}{p}}\Big(\int_{\hat{Q}_{R,x}} 1\, d(x,t)\Big)^{1/q} \le C I_{R,x}^{\frac{1}{p}} R^{\frac{n+2}{q}-2}, \end{aligned}$$

where

$$I_{R,x} := \int_{\hat{Q}_{R,x}} |u|^p \chi_R^q(t,x)\, d(x,t).$$

Since $1 < p \le 1 + 2/n$ the last inequality gives $|J_3| \le C I_{R,x}^{\frac{1}{p}}$.

Proof of Theorem 1: 4/4

By

$$\partial_t(\chi_R^q) = \frac{1}{R^2} q\phi_R^q(x)\eta_R^{q-1}(t)\eta'\Big(\frac{t}{R^2}\Big),$$

we have

$$\begin{aligned}|J_2| &\le C\frac{1}{R^2}\int_{\hat{Q}_{R,t}} |u|\chi_R^{q-1}\, d(x,t)\\ &\le C\frac{1}{R^2}\Big(\int_{\hat{Q}_{R,t}} |u|^p\chi_R^q\, d(x,t)\Big)^{1/p}\Big(\int_{\hat{Q}_{R,t}} 1\, d(x,t)\Big)^{1/q}\\ &\le CI_{R,t}^{\frac{1}{p}}\frac{1}{R^2}R^{\frac{n}{q}}\Big(\int_{\frac{R^2}{2}}^{R^2} 1\, dt\Big)^{1/q} \le CI_{R,t}^{\frac{1}{p}}R^{\frac{n+2}{q}-2} \le CI_{R,t}^{\frac{1}{p}}.\end{aligned}$$

Putting the estimates of J_1, J_2 and J_3 together we obtain

$$I_R \le CI_{R,t}^{1/p} + I_{R,x}^{1/p} \to 0$$

as $R \to \infty$. Hence $u \equiv 0$, a contradiction with the assumptions on initial data.

Global existence small data solutions

Theorem 2

Consider the Cauchy problem

$$u_{tt} - \triangle u + u_t = |u|^p, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x).$$

Let $n \leq 4$, $p_{GN}(n) = \frac{n}{n-2}$ *for* $n \geq 3$ *and let*

$$\begin{cases} p > p_{Fuj}(n) & \text{if } n = 1, 2, \\ 2 \leq p \leq 3 = p_{GN}(3) & \text{if } n = 3, \\ p = 2 = p_{GN}(4) & \text{if } n = 4. \end{cases}$$

Let $\mathcal{A} := (H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \times (L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n))$. *Then the following statement holds with a suitable constant* $\varepsilon_0 > 0$*: if* $\|(\varphi, \psi)\|_{\mathcal{A}} \leq \varepsilon_0$, *then there exists a unique globally (in time) energy solution* $u \in C([0,\infty), H^1(\mathbb{R}^n)) \cap C^1([0,\infty), L^2(\mathbb{R}^n))$.

Remark

There exists a constant $C > 0$ such that the solution and its energy terms satisfy the same decay estimates as the solutions to the associate linear problem

$$
\begin{aligned}
\|u(t,\cdot)\|_{L^2} &\leq C(1+t)^{-\frac{n}{4}}\|(\varphi,\psi)\|_{\mathcal{A}},\\
\|\nabla u(t,\cdot)\|_{L^2} &\leq C(1+t)^{-\frac{n}{4}-\frac{1}{2}}\|(\varphi,\psi)\|_{\mathcal{A}},\\
\|u_t(t,\cdot)\|_{L^2} &\leq C(1+t)^{-\frac{n}{4}-1}\|(\varphi,\psi)\|_{\mathcal{A}}.
\end{aligned}
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Remark

Let us consider the corresponding linear Cauchy problem

$$w_{tt} - \triangle w + w_t = f(t,x), \;\; w(0,x) = \varphi(x), \;\; w_t(0,x) = \psi(x).$$

Using Duhamel's principle $w = w(t,x)$ can be written as

$$\begin{aligned} w(t,x) &= K_0(t,x) *_{(x)} \varphi(x) + K_1(t,x) *_{(x)} \psi(x) \\ &\quad + \int_0^t K_1(t-s,x) *_{(x)} f(s,x)\,ds. \end{aligned}$$

Proof of Theorem 2: 1/4

The space of energy solutions is

$$X(t) = C([0,t], H^1(\mathbb{R}^n)) \cap C^1([0,t], L^2(\mathbb{R}^n))$$

equipped with the norm

$$\|u\|_{X(t)} := \sup_{0\le\tau\le t} \Big((1+\tau)^{\frac{n}{4}} \|u(\tau,\cdot)\|_{L^2} + (1+\tau)^{\frac{n+2}{4}} \|\nabla u(\tau,\cdot)\|_{L^2} + (1+\tau)^{\frac{n+4}{4}} \|u_\tau(\tau,\cdot)\|_{L^2} \Big).$$

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We define the operator

$$N : u \in X(t) \to Nu := K_0(t,x) *_{(x)} \varphi(x) + K_1(t,x) *_{(x)} \psi(x) + \int_0^t K_1(t-s,x) *_{(x)} |u(s,x)|^p \, ds.$$

Then we show that the following estimates are satisfied:

$$\|Nu\|_{X(t)} \le C_0 \, \|(\varphi,\psi)\|_{\mathcal{A}} + C_1(t) \|u\|^p_{X(t)},$$

$$\|Nu - Nv\|_{X(t)} \le C_2(t) \|u-v\|_{X(t)} \big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)} \big).$$

Proof of Theorem 2: 2/4

Proposition

Let u and v be elements of $X(t)$. Then under the assumptions of Theorem 2 the following estimates hold for $j + l = 0, 1$:

$$(1+t)^l (1+t)^{\frac{n}{4}+\frac{j}{2}} \|\nabla^j \partial_t^l Nu(t,\cdot)\|_{L^2} \leq C\,\|(\varphi,\psi)\|_{\mathcal{A}} + C\|u\|^p_{X(t)},$$

$$(1+t)^l (1+t)^{\frac{n}{4}+\frac{j}{2}} \|\nabla^j \partial_t^l (Nu(t,\cdot) - Nv(t,\cdot))\|_{L^2}$$
$$\leq C\|u-v\|_{X_0(t)} \big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big).$$

Here the nonnegative constant C is independent of $t \in [0,\infty)$.

The derived estimates for solutions to the linear problem implies

$$\|\nabla^j \partial_t^l K_0(t,0,x) *_{(x)} \varphi(x) + \nabla^j \partial_t^l K_1(t,0,x) *_{(x)} \psi(x)\|_{L^2}$$
$$\leq C\,(1+t)^{-l}(1+t)^{-\frac{n}{4}-\frac{j}{2}} \|(\varphi,\psi)\|_{\mathcal{A}}.$$

Proof of Theorem 2: 3/4

Using $K_1(0,0,x) = 0$ it follows

$$\nabla^j \partial_t^l \int_0^t K_1(t-s,0,x) *_{(x)} |u(s,x)|^p \, ds$$
$$= \int_0^t \nabla^j \partial_t^l K_1(t-s,0,x) *_{(x)} |u(s,x)|^p \, ds,$$

$$\Big\| \int_0^t \nabla^j \partial_t^l K_1(t-s,0,x) *_{(x)} |u(s,x)|^p \, ds \Big\|_{L^2}$$
$$\leq C \int_0^{\frac{t}{2}} (1+t-s)^{-(\frac{n}{4}+\frac{j}{2}+l)} \big\| |u(s,x)|^p \big\|_{L^2 \cap L^1} \, ds$$
$$+ C \int_{\frac{t}{2}}^t (1+t-s)^{-\frac{j}{2}-l} \big\| |u(s,x)|^p \big\|_{L^2} \, ds.$$

We use

$$\big\| |u(s,x)|^p \big\|_{L^1 \cap L^2} \leq C \|u(s,\cdot)\|_{L^p}^p + \|u(s,\cdot)\|_{L^{2p}}^p,$$
$$\big\| |u(s,x)|^p \big\|_{L^2} \leq C \|u(s,\cdot)\|_{L^{2p}}^p.$$

Proof of Theorem 2: 4/4

Applying Gagliardo-Nirenberg inequality with $\theta(q) = \frac{n(q-2)}{2q}$:

$$\|u(s,\cdot)\|_{L^q}^p \leq C\|u(s,\cdot)\|_{L^2}^{p(1-\theta(q))}\,\|\nabla u(s,\cdot)\|_{L^2}^{p\,\theta(q)} \leq C\|u\|_{X(s)}^p(1+s)^{-\frac{(p-1)n}{2}}.$$

We remark that the restriction $\theta(p) \geq 0$ implies that $p \geq 2$, whereas the restriction $\theta(2p) \leq 1$ implies that $p \leq p_{GN}(n)$ if $n \geq 3$. Hence, thanks to $p > p_{Fuj}(n)$ we conclude

$$\Big\| \int_0^t \nabla^j \partial_t^l K_1(t-s,0,x) *_{(x)} |u(s,x)|^p\,ds \Big\|_{L^2}$$

$$\leq C\|u\|_{X(t)}^p \int_0^{\frac{t}{2}} (1+t-s)^{-(\frac{n}{4}+\frac{j}{2}+l)}(1+s)^{-\frac{(p-1)n}{2}}\,ds$$

$$+C\|u\|_{X(t)}^p \int_{\frac{t}{2}}^t (1+t-s)^{-\frac{j}{2}-l}(1+s)^{-\frac{(2p-1)n}{4}}\,ds$$

$$\leq C(1+t)^{-(\frac{n}{4}+\frac{j}{2}+l)}\|u\|_{X(t)}^p.$$

Open problem 1: Klein-Gordon

For the Cauchy problem to semi-linear Klein-Gordon equation

$$u_{tt} - \triangle u + m^2 u = |u|^p, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x),$$

Lindblad-Sogge [8] proved global existence in time for small data energy solution for $n \leq 3$ and $p > 1 + \frac{2}{n}$(Fujita). For $n \leq 3$ and $p \leq 1 + \frac{2}{n}$ blow-up results are established.
In [9] Keel-Tao conjectured that for sufficiently large dimensions the solution for the semi-linear Cauchy problem has a blow-up for $p \leq 1 + \frac{2}{n} + \epsilon$, $\epsilon > 0$.

[8] H. Lindblad and D. Sogge. Restriction theorems and semilinear Klein-Gordon equations in $(1+3)-$dimensions. Duke Math. J. 85 (1996) 227 – 252.

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In [9] Keel-Tao conjectured that for sufficiently large dimensions the solution for the semi-linear Cauchy problem has a blow-up for $p \leq 1 + \frac{2}{n} + \epsilon$, $\epsilon > 0$.
Question: What about the critical exponent for large dimensions?

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[9] M. Keel and T. Tao. Small Data Blow-up for Semi-linear Klein-Gordon Equations. Amer.J.Math. 121 (1997) 629 – 669.

Open problem 2: Nakao problem

Let us consider the Cauchy problem for the weakly coupled system of wave equations

$$u_{tt} - \triangle u + bu_t = |v|^p, \quad v_{tt} - \triangle v = |u|^q. \tag{2}$$

If $b = 0$, the critical behavior (a curve in the p-q plane) of this system is described by the relation

$$\max\left\{\frac{q+2+p^{-1}}{pq-1}; \frac{p+2+q^{-1}}{pq-1}\right\} = \frac{n-1}{2}. \tag{3}$$

If in (3) the left-hand side is smaller than the right-hand side, then we have GESDS. If in (3) the left-hand side is larger or equal than the right-hand side, then we have, in general, blow-up under some restrictions to the data [10].

[10] D. del Santo, V. Georgiev, E. Mitidieri, Global existence of the solutions and formation of singularities for a class of hyperbolic systems, Geometrical Optics and Related Topics, Eds. F. Colombini and N. Lerner, Progress in Nonlinear Differential Equations and Their Applications, 32 (1997).

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What about the critical exponent for $b \neq 0$ in (2)?

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Open problem 3: Damped wave Scale invariant model

There exists a class of damped wave models for which the critical exponent depends somehow on the Fujita exponent and the Strauss exponent as well. This class is described by the following scale-invariant linear damped wave operators:

$$u_{tt} - \triangle u + \frac{\mu}{1+t} u_t = |u|^p, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x),$$

where $\mu > 0$ is a real parameter. It was recently shown by D'Abbicco'14 that $p_{Fuj}(n)$ is still the critical exponent when $\mu \geq \mu_0(n)$, with $\mu_0(n)$ sufficiently large.

[11] M. D'Abbicco, S. Lucente, M. Reissig, A shift in the Strauss exponent for semilinear wave equations with a not effective damping. J. Differ. Equ. 259, 5040–5073 (2015)

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If $\mu = 2$, it was proved [11] that $p_{crit}(n) = p_0(n+2)$, so we have a shift of the Strauss exponent. We still feel the influence of the dissipation term because of $p_{Fuj}(2) = p_0(4) = 2 < p_0(2)$ for $n = 2$ and $p_{Fuj}(3) < p_0(5) < p_0(3)$ for $n = 3$.

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It seems to be a challenge to determined the critical exponent in the case $\mu \in (0, \mu_0(n))$.

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Open problem 4: Evolution equations with structural damping

Let us consider the CP

$$u_{tt} + (-\triangle)^\sigma u + (-\triangle)^\theta u_t = |u|^p, \qquad (u, u_t)(0, x) = (u_0, u_1)(x),$$

with $p > 1$ and $2\theta \in (0, \sigma]$. Several papers are devoted to determined the critical exponent to this problem: D'Abbicco & Reissig (2014), D'Abbicco & E. (2014), Duong & Kainane & Reissig (2015), D'Abbicco & E. (2017).

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Under additional regularity $L^1(\mathbb{R}^n)$ for initial data, it was proved that the critical exponents p_c is

$$p_c \doteq 1 + \frac{2\sigma}{n - 2\theta}, \qquad 2\theta < n.$$

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What about the critical exponent for $\sigma < 2\theta \leq 2\sigma$? ($\sigma = 1 = \theta$ is well known as viscoelastic damped wave eq.)

Thanks for your attention!