Phase Space Analysis for Evolution PDE's and Applications [1]

The Method of Stationary Phase and Applications

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XIII ENAMA
UFSC, Florianópolis, November, 2019

[1] This work was partially supported by FAPESP

Summary

1. The Method of Stationary Phase
2. $L^p - L^q$ estimates for the Wave equation
3. $L^p - L^q$ estimates for the Plate equation

The Method of Stationary Phase

In order to derive $L^p - L^q$ estimates on the conjugate line $\frac{1}{p} + \frac{1}{q} = 1$ for the wave equation

$$u_{tt} - \triangle u = 0$$

Schrödinger equation

$$i\partial_t u + \triangle u = 0,$$

or plate equation

$$\partial_t^2 u + \triangle^2 u = 0,$$

one has to deal with the convolution

$$F^{-1}_{\xi \to x}(e^{-i|\xi|^\sigma t}) * \varphi, \qquad \sigma = 1, 2.$$

Even if we choose φ very smooth, there is no hope to just apply Young's inequality.

The main ideas are the followings:

1. We add an amplitude function in our model Fourier multiplier. Instead, our model Fourier multiplier takes the form

$$F^{-1}_{\xi\to x}\Big(e^{-i|\xi|^\sigma t}\frac{1}{|\xi|^{2r}}F(\varphi)(\xi)\Big),$$

   where the parameter r is determined later.

2. We decompose the extended phase space $(0,\infty)\times\mathbb{R}^n_\xi$ into two zones. For this reason we introduce a function $\chi\in C^\infty(\mathbb{R}^n_\xi)$ satisfying $\chi(\xi)\equiv 0$ for $|\xi|\le\frac{1}{2}$, $\chi(\xi)\equiv 1$ for $|\xi|\ge\frac{3}{4}$, and $\chi(\xi)\in[0,1]$. Then we define the *pseudo-differential zone*

$$Z_{pd}=\{(t,\xi)\in(0,\infty)\times\mathbb{R}^n_\xi : t|\xi|^\sigma\le 1\},$$

   and the *evolution zone*

$$Z_{hyp}=\{(t,\xi)\in(0,\infty)\times\mathbb{R}^n_\xi : t|\xi|^\sigma\ge 1\}.$$

To derive $L^p - L^q$ estimates for the model Fourier multiplier

$$F^{-1}_{\xi\to x}\Big(e^{-i|\xi|^\sigma t}\frac{1-\chi(t|\xi|^\sigma)}{|\xi|^{2r}}F(\varphi)(\xi)\Big),$$

we use the following result about translation invariant operators in $L^p = L^p(\mathbb{R}^n)$ spaces:[2]

Lemma 1

Let f be a measurable function. Moreover, we suppose the following relation with suitable positive constants C and $b \in (1,\infty)$:

$$\text{meas}\,\{\xi \in \mathbb{R}^n : |f(\xi)| \geq \ell\} \leq C\ell^{-b}.$$

Then $f \in M_p^q$ if $1 < p \leq 2 \leq q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{b}$.

[2] L. Hörmander, Estimates for translation invariant operators in L^p spaces. Acta Mathematica 104 (1960), 93–140.

Theorem 1

Let us consider for $\varphi \in C_0^\infty(\mathbb{R}^n)$ the Fourier multiplier

$$F^{-1}_{\xi \to x}\Big(e^{-i|\xi|^\sigma t}\frac{1-\chi(t|\xi|^\sigma)}{|\xi|^{2r}}F(\varphi)(\xi)\Big).$$

We assume $1 < p \le 2 \le q < \infty$ and $0 \le 2r \le n(\frac{1}{p} - \frac{1}{q})$. Then we have the $L^p - L^q$ estimates

$$\Big\|F^{-1}_{\xi \to x}\Big(e^{-i|\xi|^\sigma t}\frac{1-\chi(t|\xi|^\sigma)}{|\xi|^{2r}}F(\varphi)(\xi)\Big)\Big\|_{L^q(\mathbb{R}^n)} \le Ct^{\frac{2r}{\sigma}-\frac{n}{\sigma}(\frac{1}{p}-\frac{1}{q})}\|\varphi\|_{L^p(\mathbb{R}^n)}$$

for all admissible p, q. The constant C depends on p and q.

Proof.

After the change of variables $\eta := t^{\frac{1}{\sigma}}\xi$ and $t^{\frac{1}{\sigma}}y := x$, applying Lemma 1 with $b = \frac{n}{2r}$ we conclude

$$m(\eta) = e^{-i|\eta|}\frac{1-\chi(|\eta|)}{|\eta|^{2r}} \in M_p^q.$$

Lemma 2 (Littman' type)

Let us consider for $\tau \geq \tau_0$, τ_0 is a large positive number, the oscillating integral

$$F^{-1}_{\eta \to x}\big(e^{-i\tau\rho(\eta)}v(\eta)\big).$$

The amplitude function $v = v(\eta)$ is supposed to belong to $C_0^\infty(\mathbb{R}^n)$ with support in $\{\eta \in \mathbb{R}^n : |\eta| \in [\frac{1}{2}, 2]\}$. The function $\rho = \rho(\eta)$ is C^∞ in a neighborhood of the support of v. Moreover, the rank of the Hessian $H_\rho(\eta)$ is supposed to satisfy the assumption rank $H_\rho(\eta) \geq \kappa$ on the support of v. Then the following $L^\infty - L^\infty$ estimate holds[a]:

$$\|F^{-1}_{\eta \to x}\big(e^{-i\tau\rho(\eta)}v(\eta)\big)\|_{L^\infty(\mathbb{R}^n_x)} \leq C(1+\tau)^{-\frac{\kappa}{2}} \sum_{|\alpha| \leq L} \|D^\alpha_\eta v(\eta)\|_{L^\infty(\mathbb{R}^n_\eta)},$$

where L is a suitable entire number.

[a] H. Pecher, L^p-Abschätzungen und klassische Lösungen für nichtlineare Wellengleichungen. I. Math. Z. 150 (1976), 159–183.

If $\rho(\eta) = |\eta|^\sigma$ in Lemma 2, then

$$\kappa \doteq \operatorname{rank} H_\rho(\eta) = \begin{cases} n, & \sigma > 1 \\ n-1, & \sigma = 1 \end{cases} .$$

By using Littman's type lemma we derive:

Theorem 2

Let us consider for $\varphi \in C_0^\infty(\mathbb{R}^n)$ the Fourier multiplier

$$F^{-1}_{\xi \to x}\Big(e^{-i|\xi|^\sigma t}\frac{\chi(t|\xi|^\sigma)}{|\xi|^{2r}}F(\varphi)(\xi)\Big)$$

We assume $(n - \frac{\kappa\sigma}{2})(\frac{1}{p} - \frac{1}{q}) \leq 2r$. Then we have the following $L^p - L^q$ estimates on the conjugate line:

$$\Big\|F^{-1}_{\xi \to x}\Big(e^{-i|\xi|^\sigma t}\frac{\chi(t|\xi|^\sigma)}{|\xi|^{2r}}F(\varphi)(\xi)\Big)\Big\|_{L^q(\mathbb{R}^n)} \leq C t^{2r - n(\frac{1}{p} - \frac{1}{q})}\|\varphi\|_{L^p(\mathbb{R}^n)}$$

for all admissible $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof of Theorem 2: 1/3

$L^1 - L^\infty$ estimates: Apply Young's inequality

$$\Big\| F^{-1}_{\xi\to x}\Big(e^{-i|\xi|^\sigma t}\frac{\chi(t|\xi|^\sigma)}{|\xi|^{2r}}F(\varphi)(\xi)\Big)\Big\|_{L^\infty(\mathbb{R}^n)}$$
$$\le C\Big\| F^{-1}_{\xi\to x}\Big(e^{-i|\xi|^\sigma t}\frac{\chi(t|\xi|^\sigma)}{|\xi|^{2r}}\Big)\Big\|_{L^\infty(\mathbb{R}^n)}\|\varphi\|_{L^1(\mathbb{R}^n)}.$$

The last oscillating integral is estimated by the Littman type lemma. We choose $0 \le \phi \in C_c^\infty(\mathbb{R}^n)$ supported in $\{\xi \in \mathbb{R}^n : |\xi| \in [\frac{1}{2}, 2]\}$. We set $\phi_k(\xi) := \phi(2^{-k}|\xi|)$ for $k \ge 1$ and $\phi_0(\xi) := 1 - \sum_{k=1}^\infty \phi_k(\xi)$. We put $t^{\frac{1}{\sigma}}\xi =: 2^{\frac{k}{\sigma}}\eta$ to obtain

$$\Big\| F^{-1}_{\xi\to x}\Big(e^{-i|\xi|^\sigma t}\frac{\chi(t|\xi|^\sigma)\phi_k(t|\xi|^\sigma)}{|\xi|^{2r}}\Big)\Big\|_{L^\infty}$$
$$= 2^{\frac{k}{\sigma}(n-2r)}t^{\frac{1}{\sigma}(2r-n)}\Big\| F^{-1}_{\eta\to x}\Big(e^{-i2^k|\eta|^\sigma}\frac{\phi_k(2^k|\eta|^\sigma)}{|\eta|^{2r}}\Big)\Big\|_{L^\infty}$$
$$= 2^{\frac{k}{\sigma}(n-2r)}t^{\frac{1}{\sigma}(2r-n)}\Big\| F^{-1}_{\eta\to x}\Big(e^{-i2^k|\eta|^\sigma}\frac{\phi(|\eta|^\sigma)}{|\eta|^{2r}}\Big)\Big\|_{L^\infty}.$$

Proof of Theorem 2: 2/3

Now we aplly Lemma 2 with $\tau := 2^k$. Taking into account of the properties of ϕ (smoothness and compact support) we may conclude

$$\Big\| F^{-1}_{\eta \to x}\Big(e^{-i2^k|\eta|^\sigma} \frac{\phi(|\eta|^\sigma)}{|\eta|^{2r}}\Big)\Big\|_{L^\infty} \leq C(1+2^k)^{-\frac{\kappa}{2}}, \quad \kappa = \begin{cases} n, & \sigma > 1 \\ n-1, & \sigma = 1 \end{cases} .$$

Summarizing all estimates gives

$$\Big\| F^{-1}\Big(e^{-i|\xi|^\sigma t} \frac{\chi(t|\xi|^\sigma)\phi_k(t|\xi|^\sigma)}{|\xi|^{2r}} F(\varphi)(\xi)\Big)\Big\|_{L^\infty} \leq C 2^{\frac{k}{\sigma}(n-2r-\frac{\kappa\sigma}{2})} t^{\frac{1}{\sigma}(2r-n)} \|\varphi\|_{L^1}$$

$L^2 - L^2$ estimates:

After application of the formula of Parseval-Plancherel we immediately get the desired $L^2 - L^2$ estimate

$$\Big\| F^{-1}_{\xi \to x}\Big(e^{-i|\xi|^\sigma t} \frac{\chi(t|\xi|^\sigma)\phi_k(t|\xi|^\sigma)}{|\xi|^{2r}} F(\varphi)(\xi)\Big)\Big\|_{L^2} \leq C 2^{-\frac{2kr}{\sigma}} t^{\frac{2r}{\sigma}} \|\varphi\|_{L^2}.$$

Proof of Theorem 2: 3/3

$L^p - L^q$ estimates on the conjugate line:

Using the $L^1 - L^\infty$ and $L^2 - L^2$ estimates the application of the Riesz-Thorin interpolation theorem yields $L^p - L^q$ estimates on the conjugate line

$$\Big\| F^{-1}_{\xi \to x}\Big(e^{-i|\xi|^\sigma t} \frac{\chi(t|\xi|^\sigma)\phi_k(t|\xi|^\sigma)}{|\xi|^{2r}} F(\varphi)(\xi)\Big)\Big\|_{L^q}$$
$$\leq C 2^{\frac{k}{\sigma}\left((n-\frac{\kappa\sigma}{2})(\frac{1}{p}-\frac{1}{q})-2r\right)} t^{\frac{2r}{\sigma}-\frac{n}{\sigma}(\frac{1}{p}-\frac{1}{q})} \|\varphi\|_{L^p},$$

where $p \in (1,2]$.

After choosing $(n - \frac{\kappa\sigma}{2})(\frac{1}{p} - \frac{1}{q}) \leq 2r$ and using the dyadic decomposition, the desired $L^p - L^q$ estimates follow. This completes the proof.

$L^p - L^q$ estimates for the Wave equation

Now, let us come back to the representation of solutions to the Cauchy problem for the free wave equation

$$u(t,x) = F^{-1}_{\xi\to x}\Big(\Big(e^{i|\xi|t} + e^{-i|\xi|t}\Big)\frac{1}{2}F(\varphi)(\xi)\Big)$$
$$+F^{-1}_{\xi\to x}\Big(\Big(e^{i|\xi|t} - e^{-i|\xi|t}\Big)\frac{1}{2i|\xi|}F(\psi)(\xi)\Big).$$

Remark

If $\varphi \neq 0$, we have $L^2 - L^2$ estimates for $u(t,x)$, but not $L^p - L^q$ with $p \neq 2$ and $q \neq 2$(see Theorem 2 for the conjugate line). If $\varphi \equiv 0$, for small frequencies it is helpful to use recall

$$u(t,x) = F^{-1}_{\xi\to x}\Big(\frac{\sin(|\xi|t)}{|\xi|}\,F(\psi)(\xi)\Big).$$

By putting $\sigma = 1$, $r = 0$ in Theorem 1 and $r = \frac{1}{2}$ in Theorem 2 we conclude:

Theorem 3

Let ψ belong to $L^p(\mathbb{R}^n)$. Then we have the following $L^p - L^q$ decay estimates on the conjugate line:

$$\left\| F^{-1}_{\xi\to x}\left(\frac{e^{i|\xi|t} - e^{-i|\xi|t}}{2i|\xi|} F(\psi)(\xi)\right)\right\|_{L^q(\mathbb{R}^n)} \leq Ct^{1-n(\frac{1}{p}-\frac{1}{q})}\|\psi\|_{L^p(\mathbb{R}^n)}, \quad (1)$$

where $\frac{n+1}{2}(\frac{1}{p} - \frac{1}{q}) \leq 1$, $1 < p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Indeed, (1) holds [3] uniformly for any $t > 0$, if, and only if, the point $(\frac{1}{p}, \frac{1}{q})$ belongs to the closed triangle with vertices

$$P_1 = (\tfrac{1}{2} + \tfrac{1}{n+1}, \tfrac{1}{2} - \tfrac{1}{n+1}),\ P_2 = (\tfrac{1}{2} - \tfrac{1}{n-1}, \tfrac{1}{2} - \tfrac{1}{n-1}),$$
$$\text{and } P_3 = (\tfrac{1}{2} + \tfrac{1}{n-1}, \tfrac{1}{2} + \tfrac{1}{n-1}).$$

[3] J. Peral, L^p estimates for the Wave Equation, J. Funct. Anal. 36, 114–145 (1980).

$L^p - L^q$ estimates for the Plate equation

Let us consider the Cauchy problem for the plate equation,

$$\begin{cases} u_{tt} + (-\triangle)^\sigma u = 0\,, & t \geq 0,\ x \in \mathbb{R}^n \\ u(0,x) = \varphi(x), \qquad u_t(0,x) = \psi(x). \end{cases} \tag{2}$$

Theorem 4

Let $\sigma > 1$. If $\varphi \equiv 0$ and $\psi \in L^p$, then the solution u to the Cauchy problem (2) satisfies the following estimate[a]

$$\|u(t,\cdot)\|_{L^q} \lesssim t^{1-\frac{n}{\sigma}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\psi\|_{L^p(\mathbb{R}^n)}, \qquad \forall t > 0, \tag{3}$$

for all $1 \leq p \leq q < \infty$, with $\frac{1}{p} + \frac{1}{q} \leq 1$ and $\frac{1-\sigma}{p} - \frac{1}{q} \leq \sigma\left(\frac{1}{n} - \frac{1}{2}\right)$ or $\frac{1}{p} + \frac{1}{q} \geq 1$ and $\frac{1}{p} + \frac{\sigma-1}{q} \leq \sigma\left(\frac{1}{n} + \frac{1}{2}\right)$.

[a] M. R. Ebert, L. M. Lourenço, The Critical Exponent for Evolution Models with Power Non-linearity.New Tools for Nonlinear PDEs and Application. Trends in Mathematics. Birkhäuser, Cham 2019, p. 153-177.

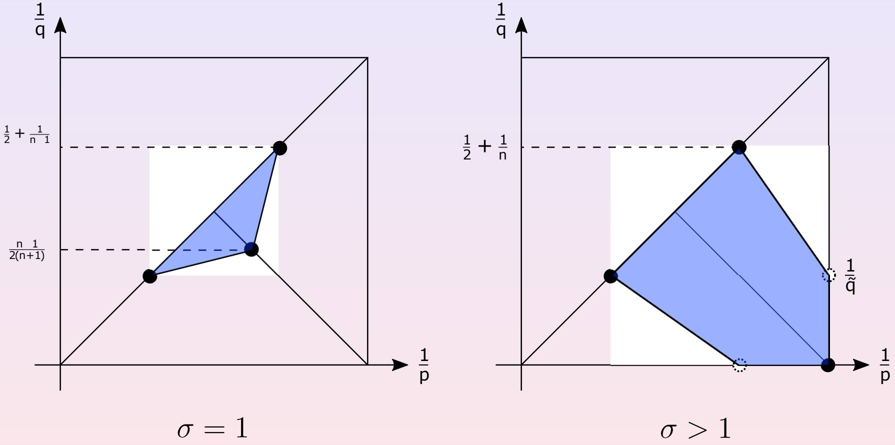

Figure : $L^p - L^q$ estimates

Notation

By $L_p^q = L_p^q(\mathbb{R}^n)$ we denote the space of tempered distributions T satisfying the estimate

$$\|T * f\|_{L^q} \leq C\|f\|_{L^p}$$

for all f in the Schwartz space $S(\mathbb{R}^n)$ with a constant C which is independent of f.
The set of Fourier transforms $\hat{T}$ of distributions $T \in L_p^q$ is denoted by $M_p^q = M_p^q(\mathbb{R}^n)$. The elements in M_p^q are called multipliers of type (p, q). We define in $M_p^q(\mathbb{R}^n)$ the following norm

$$\|m\|_{M_p^q} := \sup\{\|\mathfrak{F}^{-1}(m\mathfrak{F}(f))\|_q : f \in \mathcal{S}, \|f\|_p = 1\}.$$

In the case $p = q$, we denote M_p^p by M_p.

Proof of Theorem 4: 1/4

After applying the Fourier transform, the solution to (2) can be written as

$$u(t,x) = \mathfrak{F}^{-1}_{\xi\to x}\Big(\Big(e^{i|\xi|^\sigma t} - e^{-i|\xi|^\sigma t}\Big)\frac{1}{2i|\xi|^\sigma}\mathfrak{F}(\psi)(\xi)\Big),$$

or

$$u(t,x) = \mathfrak{F}^{-1}_{\xi\to x}\Big(\sin(t|\xi|^\sigma)\frac{1}{|\xi|^\sigma}\mathfrak{F}(\psi)(\xi)\Big).$$

By homogeneity, it is sufficient to prove (3) for $t = 1$. By using that $\frac{\sin(|\xi|^\sigma)}{|\xi|^\sigma}$ is a bounded function, similar to Theorem 1, one may conclude the following:

Proposition 1

For $1 < p \leq 2 \leq q < \infty$, we have

$$\frac{(1-\chi(\xi))\sin(|\xi|^\sigma)}{|\xi|^\sigma} \in M_p^q. \tag{4}$$

Proof of Theorem 4: 2/4

Theorem (Mikhlin-Hörmander)

Let $1 < p < \infty$ and $k = \max\{[n(1/p - 1/2)] + 1, [n/2] + 1\}$. Suppose that $m \in \mathcal{C}^k(\mathbb{R}^n \setminus \{0\})$ and

$$\left|\partial_\xi^\beta m(\xi)\right| \leq C\,|\xi|^{-|\beta|}, \quad |\beta| \leq k.$$

Then $m \in M_p \doteq M_p^p$.

As a consequence of the Mikhlin-Hörmander multiplier theorem we have:

Proposition 2

For all $1 < q < \infty$ we have

$$m(\xi) := \frac{(1 - \chi(\xi))\sin(|\xi|^\sigma)}{|\xi|^\sigma} \in M_q^q. \tag{5}$$

Proof of Theorem 4: 3/4

Proposition 3

Let us consider Fourier multiplier

$$m(\xi) = \frac{\chi(\xi) e^{i|\xi|^\sigma}}{|\xi|^\sigma}, \qquad \xi \in \mathbb{R}^n, \qquad \sigma > 0, \sigma \neq 1, \qquad b \in \mathbb{R}.$$

If $p \leq q$, $\frac{1}{p} + \frac{1}{q} \geq 1$ and $\frac{1}{p} + \frac{\sigma-1}{q} < \sigma\left(\frac{1}{n} + \frac{1}{2}\right)$, then $m \in M_p^q$.
The same conclusion is true if $p \leq q$, $\frac{1}{p} + \frac{1}{q} \leq 1$ and $\frac{1-\sigma}{p} - \frac{1}{q} < \sigma\left(\frac{1}{n} - \frac{1}{2}\right)$.

Proof of Theorem 4: 3/4

Proposition 3

Let us consider Fourier multiplier

$$m(\xi) = \frac{\chi(\xi)e^{i|\xi|^\sigma}}{|\xi|^\sigma}, \qquad \xi \in \mathbb{R}^n, \qquad \sigma > 0, \sigma \neq 1, \qquad b \in \mathbb{R}.$$

If $p \le q$, $\frac{1}{p} + \frac{1}{q} \ge 1$ and $\frac{1}{p} + \frac{\sigma-1}{q} < \sigma\left(\frac{1}{n} + \frac{1}{2}\right)$, then $m \in M_p^q$.
The same conclusion is true if $p \le q$, $\frac{1}{p} + \frac{1}{q} \le 1$ and $\frac{1-\sigma}{p} - \frac{1}{q} < \sigma\left(\frac{1}{n} - \frac{1}{2}\right)$.

Remark

By using duality argument, it is sufficient to prove Proposition 3 for $\frac{1}{p} + \frac{1}{q} \ge 1$.

Proof of Theorem 4: 4/4

Let us sketch the proof of Proposition 3. By Littman' lemma

$$\|\phi_k \cdot m\|_{M_p^q} \leq C2^{k\left(-\sigma+(\frac{1}{p}-\frac{1}{2})(n(2-\sigma))\right)}, \quad \frac{1}{p}+\frac{1}{q}=1.$$

Applying the Berstein's inequality [4] for $M > \frac{n}{2}$ we get

$$\|\phi_k \cdot m\|_{M_1} \leq \|\phi_k \cdot m\|_{L^2}^{(1-\frac{n}{2M})} \|D^M(\phi_k \cdot m)\|_{L^2}^{\frac{n}{2M}} \leq C2^{k\sigma(\frac{n}{2}-1)}.$$

Hence, by Riesz-Thorin interpolation theorem we conclude that

$$\|\phi_k \cdot m\|_{M_p^q} \leq C2^{kn\left(\frac{1}{p}+\frac{\sigma-1}{q}-\sigma\left(\frac{1}{n}+\frac{1}{2}\right)\right)},$$

and the series in k is convergent if $\frac{1}{p}+\frac{\sigma-1}{q}-\sigma\left(\frac{1}{n}+\frac{1}{2}\right)<0$ and the proof is concluded.

[4] Bernstein' result: Let $n \geq 1$ and $N > n\left(\frac{1}{p}-\frac{1}{2}\right)$. Assume that $\hat{a} \in H^N$, then $F^{-1}(\hat{a}) \in L^p$ and there exists a constant $C > 0$ such that

$$\|F^{-1}(\hat{a})\|_{L^p} \leq C\|\hat{a}\|_{L^2}^{1-\frac{n}{N}(\frac{1}{p}-\frac{1}{2})} \|D^N\hat{a}\|_{L^2}^{\frac{n}{N}(\frac{1}{p}-\frac{1}{2})}$$

Thanks for your attention!