Phase Space Analysis for Evolution PDE's and Applications [1]

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Main goal and program

The main goal of this course is to apply tools from harmonic analysis to study some important partial differential equations from mathematical physics.

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Topics of the lectures:

1. Phase Space Analysis for Evolutions PDE's;
2. The Method of Stationary Phase and Applications;
3. Global existence (in time) for semilinear models;

Main Reference:

M. R. Ebert and M. Reissig, Methods for Partial Differential Equations. 1. ed. Cham, Switzerland: Springer International Publishing, 2018, 479p.

Summary

Evolution PDE's

Let us introduce the notion *p-evolution operator*. For this reason we consider for a fixed integer $p \geq 1$, the linear partial differential equation

$$D_t^m u + \sum_{j=1}^{m} A_j(t, x, D_x) D_t^{m-j} u = f(t, x),$$

where where $D_t = -i\partial_t$, $D_{x_k} = -i\partial_{x_k}$, $k = 1, \cdots, n$, $i^2 = -1$, $A_j = A_j(t, x, D_x) = \sum_{k=0}^{jp} A_{j,k}(t, x, D_x)$ are linear partial differential operators of order jp and, $A_{j,k} = A_{j,k}(t, x, D_x)$ are linear partial differential operators of order k. The *principal part* of this linear partial differential operator *in the sense of Petrovsky* is defined by

$$D_t^m + \sum_{j=1}^{m} A_{j,jp}(t, x, D_x) D_t^{m-j}.$$

Definition1

The given linear partial differential operator

$$D_t^m + \sum_{j=1}^{m} A_j(t, x, D_x) D_t^{m-j}$$

is called a *p-evolution operator* if the principal symbol in the sense of Petrovsky

$$\tau^m + \sum_{j=1}^{m} A_{j,jp}(t, x, \xi)\tau^{m-j}$$

has only real and distinct roots
$\tau_1 = \tau_1(t, x, \xi), \cdots, \tau_m = \tau_m(t, x, \xi)$ for all points (t, x) from the domain of definition of coefficients and for all $\xi \neq 0$.

Remark

The set of 1-evolution operators coincides with the set of strictly hyperbolic operators. The p-evolution operators with $p \geq 2$ represent generalizations of Schrödinger operators

$$i\partial_t u + \triangle u = 0,$$

and plate equation

$$\partial_t^2 u + \triangle^2 u = 0.$$

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Remark

One of the later goals is to study Cauchy problems for p-evolution equations. Taking account of the Lax-Mizohata theorem, the assumption that the characteristic roots are real in Definition 1, is necessary for proving well-posedness for the Cauchy problem.

Function Spaces

Definition

Let $0 < p \leq \infty$. Then the Lebesgue space $L^p(\mathbb{R}^n)$ is the set of all Lebesgue measurable complex-valued functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p} = \Big(\int_{\mathbb{R}^n} |f(x)|^p \, dx \Big)^{\frac{1}{p}} < \infty \quad \text{for} \quad p \in [1,\infty),$$

$$\|f\|_{L^\infty} = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |f(x)| < \infty.$$

Now we define Sobolev spaces of integer and fractional order:

Definition

Let $1 \leq p \leq \infty$ and $m \in \mathbb{N}$. Then the Sobolev spaces $W_p^m(\mathbb{R}^n)$ are defined as

$$W_p^m(\mathbb{R}^n) := \Big\{ f \in L^p(\mathbb{R}^n) : \|f\|_{W_p^m} := \sum_{|\alpha| \leq m} \|\partial_x^\alpha f\|_{L^p} < \infty \Big\}.$$

If $p = 2$, then we also use the notation $W^s(\mathbb{R}^n)$.

We recall the classical definition of Fourier transformation in L^p spaces for $p \in [1,\infty)$:

$$F(f)(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx.$$

Definition

Let $1 < p < \infty$ and $s \in \mathbb{R}^1$. Then the Sobolev spaces of fractional order $H^s_p(\mathbb{R}^n)$ are defined as

$$H^s_p(\mathbb{R}^n) := \{f \in S'(\mathbb{R}^n) : \|f\|_{H^s_p} := \|F^{-1}(\langle\xi\rangle^s F(f))\|_{L^p} < \infty\}.$$

Here $\langle\xi\rangle$ denotes the Japanese brackets with $\langle\xi\rangle^2 := 1 + |\xi|^2$. If $p = 2$, then we also use the notation $H^s(\mathbb{R}^n)$.

Phase space analysis for the heat equation

Theorem 1

Let us consider the Cauchy problem

$$u_t - \triangle u = 0, \;\; u(0,x) = \varphi(x).$$

Then we have the following estimates for the derivatives $\partial_t^k \partial_x^\alpha u(t,\cdot)$ of the solution u ($k + |\alpha| \geq 0$):

$$\|\partial_t^k \partial_x^\alpha u(t,\cdot)\|_{L^2} \leq C_{k,\alpha} t^{-k-\frac{|\alpha|}{2}} \|\varphi\|_{L^2},$$

$$\|\partial_t^k \partial_x^\alpha u(t,\cdot)\|_{L^2} \leq C_{k,\alpha} (1+t)^{-k-\frac{|\alpha|}{2}} \|\varphi\|_{H^{2k+|\alpha|}}.$$

Remark

The first estimate requires only L^2 regularity for φ. We get for large t the decay $t^{-k-\frac{|\alpha|}{2}}$. But this term becomes unbounded for $t \to +0$. To avoid this singular behavior we assume additional regularity $H^{2k+|\alpha|}$.

Proof.

Using the properties of the Fourier transformation it holds

$$\partial_t^k \partial_x^\alpha u(t,x) = F^{-1}_{\xi\to x}\big((-1)^k i^{|\alpha|} |\xi|^{2k} \xi^\alpha e^{-|\xi|^2 t} F(\varphi)(\xi)\big).$$

By Parseval-Plancharel formula we get for $t > 0$

$$\|\partial_t^k \partial_x^\alpha u(t,\cdot)\|^2_{L^2} = \Big\| |\xi|^{2k}\xi^\alpha e^{-|\xi|^2 t} F(\varphi)\Big\|^2_{L^2} \le \Big\| |\xi|^{2k+|\alpha|} e^{-|\xi|^2 t} F(\varphi)\Big\|^2_{L^2}$$
$$= \Big\| \frac{|\xi|^{2k+|\alpha|} t^{k+\frac{|\alpha|}{2}}}{t^{k+\frac{|\alpha|}{2}}} e^{-|\xi|^2 t} F(\varphi)\Big\|^2_{L^2}.$$

The conclusion follows thanks to

$$|\xi|^{2k+|\alpha|} t^{k+\frac{|\alpha|}{2}} e^{-|\xi|^2 t} = \big(|\xi|^2 t\big)^{k+\frac{|\alpha|}{2}} e^{-|\xi|^2 t}$$

be uniformly bounded. □

Lemma 1 (Young's inequality)

Let $f \in L^r(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ be two given functions. Then the following estimates hold for the convolution $u := f * g$:

$$\|u\|_{L^q} \leq \|f\|_{L^r}\|g\|_{L^p} \text{ for all } 1 \leq p \leq q \leq \infty \text{ and } 1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}.$$

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$\|u\|_{L^q} \leq \|f\|_{L^r} \|g\|_{L^p}$ for all $1 \leq p \leq q \leq \infty$ and $1 + \dfrac{1}{q} = \dfrac{1}{r} + \dfrac{1}{p}$.

Lemma 2

The following estimates hold in $\mathbb{R}^n$ for $n \geq 1$:

$$\|F^{-1}(|\xi|^a e^{-|\xi|^{2\kappa} t})\|_{L^r(\mathbb{R}^n)} \leq C t^{-\frac{n}{2\kappa}(1-\frac{1}{r})-\frac{a}{2\kappa}}$$

for $\kappa > 0$, $r \in [1, \infty]$ and $t > 0$ provided that

$$a + n\Big(1 - \frac{1}{r}\Big) > 0.$$

In particular, if $a > 0$, then the statement is true for all $r \in [1, \infty]$ and for $r = 1$ if $a = 0$.

By using the representation of solution

$$u(t,x) = F^{-1}_{\xi\to x}\big(e^{-|\xi|^2 t}F(\varphi)(\xi)\big) = F^{-1}_{\xi\to x}\big(e^{-|\xi|^2 t}\big) * \varphi$$

and Lemmas 1 and 2, one can also derive:

Theorem 2

We study the Cauchy problem

$$u_t - \triangle u = 0, \;\; u(0,x) = \varphi(x).$$

Then we have the following estimates for the derivatives $\partial_t^k \partial_x^\alpha u(t,\cdot)$ of the solution u ($k + |\alpha| \geq 0$):

$$\|\partial_t^k \partial_x^\alpha u(t,\cdot)\|_{L^q} \leq C_{k,\alpha} t^{-k-\frac{|\alpha|}{2}-\frac{n}{2}(1-\frac{1}{r})} \|\varphi\|_{L^p}$$

for all $1 \leq p \leq q \leq \infty$ and $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$.

Corollary

Under the assumptions of Theorem 2 we conclude the estimates

$$\|\partial_t^k \partial_x^\alpha u(t,\cdot)\|_{L^q} \leq C_{k,\alpha} t^{-k-\frac{|\alpha|}{2}} \|\varphi\|_{L^q}$$

for all $1 \leq q \leq \infty$.

This corollary is used to derive, for example, $L^p - L^q$ decay estimates with decay function $1+t$ instead of t.
Indeed, using the embedding

$$W_p^{n(\frac{1}{p}-\frac{1}{q})}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) \text{ for } 1 \leq p \leq q < \infty.$$

and the $L^q - L^q$ decay estimate from Corollary gives

$$\|u(t,\cdot)\|_{L^q} \leq C\|\varphi\|_{W_p^{n(\frac{1}{p}-\frac{1}{q})}}, \quad \|\partial_t^k \partial_x^\alpha u(t,\cdot)\|_{L^q} \leq C\|\varphi\|_{W_p^{n(\frac{1}{p}-\frac{1}{q})+2k+|\alpha|}},$$

respectively, for $1 \leq p \leq q < \infty$.

Summarizing all these estimates implies the following statement.

Theorem 3

We study the Cauchy problem

$$u_t - \triangle u = 0, \quad u(0,x) = \varphi(x).$$

Then we have the following estimates for the derivatives $\partial_t^k \partial_x^\alpha u(t,\cdot)$ of the solution u ($k + |\alpha| \geq 0$):

$$\|\partial_t^k \partial_x^\alpha u(t,\cdot)\|_{L^q} \leq C_{k,\alpha}(1+t)^{-k-\frac{|\alpha|}{2}-\frac{n}{2}(1-\frac{1}{r})}\|\varphi\|_{W_p^{n(1-\frac{1}{r})+2k+|\alpha|}}$$

for all $1 \leq p \leq q < \infty$ and $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$.

Phase space analysis for the Schrödinger equation

Let us consider the Cauchy problem for the Schrödinger equation

$$i\partial_t u + \triangle u = 0, \;\; u(0,x) = \varphi(x).$$

After application of inverse Fourier transformation (we assume the validity of Fourier's inversion formula) we obtain

$$u(t,x) = F^{-1}_{\xi \to x}\big(e^{-i|\xi|^2 t} F(\varphi)(\xi)\big).$$

Theorem 4

We study the Cauchy problem

$$i\partial_t u + \triangle u = 0, \;\; u(0,x) = \varphi(x).$$

Then we have the following estimates for the derivatives $\partial_t^k \partial_x^\alpha u(t,\cdot)$ of the solution u ($k + |\alpha| \geq 0$):

$$\|\partial_t^k \partial_x^\alpha u(t,\cdot)\|_{L^2} = \||D|^{2k} \partial_x^\alpha \varphi\|_{L^2} \leq \|\varphi\|_{H^{2k+|\alpha|}}.$$

Using the representation of solution

$$u(t,x) = \frac{1}{(2\sqrt{\pi i t})^n} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} \varphi(y)\, dy$$

we immediately obtain $\|u(t,\cdot)\|_{L^\infty} \leq Ct^{-\frac{n}{2}} \|\varphi\|_{L^1}$. Moreover, using that $\partial_t^k \partial_x^\alpha u$ solves the Cauchy problem

$$i\partial_t(\partial_t^k \partial_x^\alpha u) + \triangle(\partial_t^k \partial_x^\alpha u) = 0, \;\; (\partial_t^k \partial_x^\alpha u)(0,x) = i^k \triangle^k \partial_x^\alpha \varphi(x)$$

we conclude:

Theorem 5

We study the Cauchy problem

$$i\partial_t u + \triangle u = 0, \;\; u(0,x) = \varphi(x).$$

Then we have the following $L^1 - L^\infty$ estimates for the derivatives $\partial_t^k \partial_x^\alpha u(t,\cdot)$ of the solution u ($k + |\alpha| \geq 0$):

$$\|\partial_t^k \partial_x^\alpha u(t,\cdot)\|_{L^\infty} \leq Ct^{-\frac{n}{2}} \||D|^{2k} \partial_x^\alpha \varphi\|_{L^1}.$$

Interpolation argument implies the following:

Theorem 6

We study the Cauchy problem

$$i\partial_t u + \triangle u = 0, \;\; u(0,x) = \varphi(x).$$

Then we have the following $L^p - L^q$ estimates on the conjugate line for the derivatives $\partial_t^k \partial_x^\alpha u(t,\cdot)$ of the solution u $(k + |\alpha| \geq 0)$ for $q \in [2,\infty]$

$$\|\partial_t^k \partial_x^\alpha u(t,\cdot)\|_{L^q} \leq C t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \||D|^{2k} \partial_x^\alpha \varphi\|_{L^p},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

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where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark

Contrary to the heat equation, one may not expect $L^p - L^p$ estimates for $p \neq 2$ for solutions to the Schrödinger equation.

Phase space analysis for Wave models

We are interested in the Cauchy problem

$$u_{tt} - \triangle u = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x), \quad x \in \mathbb{R}^n, \quad n \geq 1.$$

We arrive at the following representation for u:

$$u(t,x) = F^{-1}_{\xi \to x}\big(\cos(|\xi|t)F(\varphi)(\xi)\big) + F^{-1}_{\xi \to x}\Big(\frac{\sin(|\xi|t)}{|\xi|}\, F(\psi)(\xi)\Big).$$

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We denote the total energy by

$$E_W(u)(t) := \frac{1}{2}\int_{\mathbb{R}^n} \Big(|u_t(t,x)|^2 + |\nabla u(t,x)|^2\Big)dx$$
$$= \frac{1}{2}\|u_t(t,\cdot)\|^2_{L^2} + \frac{1}{2}\|\nabla u(t,\cdot)\|^2_{L^2}.$$

Theorem 7 (Conservation of energy)

Let

$$u \in C([0,T], H^1(\mathbb{R}^n)) \cap C^1([0,T], L^2(\mathbb{R}^n))$$

be a Sobolev solution of

$$u_{tt} - \triangle u = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x)$$

with data $\varphi \in H^1(\mathbb{R}^n)$ and $\psi \in L^2(\mathbb{R}^n)$. Then it holds

$$E_W(u)(t) = E_W(u)(0) = \frac{1}{2}(\|\psi\|_{L^2}^2 + \|\nabla\varphi\|_{L^2}^2) \quad \text{for all} \quad t \geq 0.$$

Moreover,

$$\|u(t,\cdot)\|_{L^2} \leq C(t\|\psi\|_{L^2} + \|\varphi\|_{L^2}).$$

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with data $\varphi \in H^1(\mathbb{R}^n)$ and $\psi \in L^2(\mathbb{R}^n)$. Then it holds

$$E_W(u)(t) = E_W(u)(0) = \frac{1}{2}(\|\psi\|^2_{L^2} + \|\nabla\varphi\|^2_{L^2}) \quad \text{for all} \quad t \geq 0.$$

Moreover,

$$\|u(t,\cdot)\|_{L^2} \leq C(t\|\psi\|_{L^2} + \|\varphi\|_{L^2}).$$

The goal to derive $L^p - L^q$ decay estimates for solutions to the Cauchy problem for the wave equation requires a deeper understanding of oscillating integrals with localized amplitudes in different parts of the extended phase space. One basic tool to get such estimates is the method of stationary phase.

Phase space analysis for damped wave model

Let us turn to the Cauchy problem

$$u_{tt} - \triangle u + u_t = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x).$$

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Let us turn to the Cauchy problem

$$u_{tt} - \triangle u + u_t = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x).$$

We know that the wave energy $E_W(u)(t)$ of Sobolev solutions is a decreasing function if $E_W(u)(0)$ is finite.
Applying phase space analysis allows to verify that the energy $E_W(u)(t)$ is even decaying for $t \to \infty$. We are able to derive for $E_W(u)(t)$ an *optimal decay behavior* with an optimal *decay rate*.

Step 1 Transformation of the dissipation into a mass term

We introduce a new function $w = w(t,x)$ by $w(t,x) := e^{\frac{1}{2}t}u(t,x)$. Then w satisfies the Cauchy problem

$$w_{tt} - \triangle w - \frac{1}{4}w = 0, \quad w(0,x) = \varphi(x), \quad w_t(0,x) = \frac{1}{2}\varphi(x) + \psi(x).$$

Applying Fourier transform $v = v(t,\xi) = F_{x\to\xi}(w(t,x))(t,\xi)$:

$$v_{tt} + \Big(|\xi|^2 - \frac{1}{4}\Big)v = 0, \quad v(0,\xi) = v_0(\xi) := F(\varphi)(\xi),$$
$$v_t(0,\xi) = v_1(\xi) := \frac{1}{2}\,F(\varphi)(\xi) + F(\psi)(\xi).$$

We make a distinction of cases for $\{\xi \in \mathbb{R}^n : |\xi| > \frac{1}{2}\}$ and for $\{\xi \in \mathbb{R}^n : |\xi| < \frac{1}{2}\}$:

Step 2 *Representation of solutions in the phase space*

Case 1 $\{\xi : |\xi| > \frac{1}{2}\}$

Using $|\xi|^2 > \frac{1}{4}$ we obtain immediately the following representation of solution $v(t,\xi)$:

$$v(t,\xi) = \cos\Big(\sqrt{|\xi|^2 - \frac{1}{4}}\; t\Big)\, v_0(\xi) + \frac{\sin\Big(\sqrt{|\xi|^2 - \frac{1}{4}}\; t\Big)}{\sqrt{|\xi|^2 - \frac{1}{4}}}\, v_1(\xi).$$

Case 2 $\{\xi : |\xi| < \frac{1}{2}\}$

The solution to the transformed differential equation is

$$\begin{aligned} v(t,\xi) &= \Big(\frac{v_0(\xi)}{2} - \frac{v_1(\xi)}{\sqrt{1-4|\xi|^2}}\Big) e^{-\frac{1}{2}\sqrt{1-4|\xi|^2}\,t} \\ &\quad + \Big(\frac{v_0(\xi)}{2} + \frac{v_1(\xi)}{\sqrt{1-4|\xi|^2}}\Big) e^{\frac{1}{2}\sqrt{1-4|\xi|^2}\,t} \\ &= v_0(\xi)\cosh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}\,t\Big) + \frac{2v_1(\xi)}{\sqrt{1-4|\xi|^2}}\sinh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}\,t\Big). \end{aligned}$$

Theorem 8

The solution to the Cauchy problem

$$u_{tt} - \triangle u + u_t = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x)$$

with data $\varphi \in H^1(\mathbb{R}^n)$ and $\psi \in L^2(\mathbb{R}^n)$ satisfies the following estimates for $t \geq 0$:

$$\begin{aligned} \|u(t,\cdot)\|_{L^2} &\leq C(\|\varphi\|_{L^2} + \|\psi\|_{H^{-1}}), \\ \|\nabla u(t,\cdot)\|_{L^2} &\leq C(1+t)^{-\frac{1}{2}}(\|\varphi\|_{H^1} + \|\psi\|_{L^2}), \\ \|u_t(t,\cdot)\|_{L^2} &\leq C(1+t)^{-1}(\|\varphi\|_{H^1} + \|\psi\|_{L^2}). \end{aligned}$$

Consequently, the wave energy satisfies the estimate

$$E_W(u)(t) \leq C(1+t)^{-1}(\|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2).$$

Proof.

Let us discuss the estimate only for $\|\nabla u(t,\cdot)\|_{L^2}$.
It is clear that there exists $\delta \in (0,1)$ such that

$$\|\nabla u(t,\cdot)\|^2_{L^2(|\xi|\geq\frac{1}{4})} \leq Ce^{-\delta t}\int_{\mathbb{R}^n}\left(|\xi|^2|v_0(\xi)|^2 + |v_1(\xi)|^2\right) d\xi.$$

By using the property

$$\sqrt{x+y} \leq \sqrt{x} + \frac{y}{2\sqrt{x}} \text{ for any } x>0 \text{ and } y \geq -x$$

it follows the inequality

$$-4|\xi|^2 \leq -1 + \sqrt{1-4|\xi|^2} \leq -2|\xi|^2 \text{ for } |\xi| < \frac{1}{2}.$$

With this inequality we proceed as follows:

$$\int_{|\xi|<\frac{1}{4}} |\xi|^2|\hat{u}(t,\xi)|^2\, d\xi \leq Ce^{-t}\int_{|\xi|<\frac{1}{4}}\left(|v_0(\xi)|^2|\xi|^2 + |v_1(\xi)|^2|\xi|^2\right) d\xi$$

$$+\, C\int_{|\xi|<\frac{1}{4}}\left(|v_0(\xi)|^2 + |v_1(\xi)|^2\right)|\xi|^2 e^{-2|\xi|^2 t}\, d\xi.$$

Decay behavior under additional regularity of data

Theorem 9

Under the regularity assumption

$$(\varphi,\psi)\in H^1(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\cap L^m(\mathbb{R}^n),\quad m\in[1,2)$$

the solution to the Cauchy problem

$$u_{tt}-\triangle u+u_t=0,\quad u(0,x)=\varphi(x),\quad u_t(0,x)=\psi(x),$$

satisfies the following estimates for $t\geq 0$:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2} &\leq C_m(1+t)^{-\frac{n(2-m)}{4m}}\big(\|\varphi\|_{H^1\cap L^m}+\|\psi\|_{L^2\cap L^m}\big),\\
\|\nabla u(t,\cdot)\|_{L^2} &\leq C_m(1+t)^{-\frac{1}{2}-\frac{n(2-m)}{4m}}\big(\|\varphi\|_{H^1\cap L^m}+\|\psi\|_{L^2\cap L^m}\big),\\
\|u_t(t,\cdot)\|_{L^2} &\leq C_m(1+t)^{-1-\frac{n(2-m)}{4m}}\big(\|\varphi\|_{H^1\cap L^m}+\|\psi\|_{L^2\cap L^m}\big).
\end{aligned}$$

Proof.

Let us discuss the estimate only for $\|\nabla u(t,\cdot)\|_{L^2}$.
Setting

$$\frac{1}{2} = \frac{1}{r} + \frac{1}{m'}$$

and after using Hölder's inequality we get

$$\begin{aligned}
\||\xi|\hat{u}(t,\xi)\|^2_{L^2\{|\xi|<\frac{1}{4}\}} &\le C\int_{|\xi|<\frac{1}{4}} |\xi|^2 e^{-|\xi|^2 t}(|v_0(\xi)|^2 + |v_1(\xi)|^2)\, d\xi \\
&\le C(\|v_0\|^2_{L^{m'}} + \|v_1\|^2_{L^{m'}})\Big(\int_{|\xi|<\frac{1}{4}} (|\xi|^2 e^{-|\xi|^2 t})^{\frac{r}{2}}\, d\xi\Big)^{\frac{2}{r}} \\
&\le C(\|\varphi\|^2_{L^m} + \|\psi\|^2_{L^m})\Big(\int_{|\xi|<\frac{1}{4}} (|\xi|^2 e^{-|\xi|^2 t})^{\frac{m}{2-m}}\, d\xi\Big)^{\frac{2-m}{m}} \\
&\le C_m(1+t)^{-\frac{n(2-m)}{2m}-1}(\|\varphi\|^2_{L^m} + \|\psi\|^2_{L^m}).
\end{aligned}$$

□

The diffusion phenomenon for damped wave models

Let us turn to the Cauchy problems

$$\begin{aligned} & u_{tt} - \triangle u + u_t = 0 \\ & u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x) \end{aligned} \quad \text{and} \quad \begin{aligned} & w_t - \triangle w = 0 \\ & w(0,x) = \varphi(x) + \psi(x). \end{aligned}$$

Then we have the following remarkable result:

Theorem 10

The difference of solutions to the above Cauchy problems satisfies the following estimate:

$$\Big\| F^{-1}_{\xi\to x}\Big(\chi(\xi) F_{x\to\xi}(u(t,x) - w(t,x))\Big)\Big\|_{L^2} \leq C(1+t)^{-1}\|(\varphi,\psi)\|_{L^2}.$$

Here $\chi \in C_0^\infty(\mathbb{R}^n)$ is a cut-off function, with $\chi(s) = 1$ for $|s| \leq \frac{\varepsilon}{2} \ll 1$ and $\chi(s) = 0$ for $|s| \geq \varepsilon$ which localizes to small frequencies.

Proof.

We use for small frequencies $|\xi| < \frac{1}{2}$ the following representation for the solutions $u = u(t,x)$

$$F_{x\to\xi}(u)(t,\xi)$$
$$= e^{-\frac{1}{2}t}\Big(\Big(\frac{1}{2}F(\varphi)(\xi) - \frac{\frac{1}{2}F(\varphi)(\xi) + F(\psi)(\xi)}{\sqrt{1-4|\xi|^2}}\Big)e^{-\frac{1}{2}\sqrt{1-4|\xi|^2}\,t}$$
$$+\Big(\frac{1}{2}F(\varphi)(\xi) + \frac{\frac{1}{2}F(\varphi)(\xi) + F(\psi)(\xi)}{\sqrt{1-4|\xi|^2}}\Big)e^{\frac{1}{2}\sqrt{1-4|\xi|^2}\,t}\Big).$$

and
$$F_{x\to\xi}(w)(t,\xi) = e^{-|\xi|^2 t}\big(F(\varphi)(\xi) + F(\psi)(\xi)\big).$$

Then we take into consideration

$$\sqrt{1+s} = 1 + \frac{s}{2} - \frac{s^2}{8} + O(s^3)$$

$$\text{and}\quad \frac{1}{\sqrt{1+s}} = 1 - \frac{s}{2} + O(s^2) \ \text{ for } \ s \to +0.$$

Thanks for your attention!