# INTRODUCTION TO THE CONTROLLABILITY OF COUPLED PARABOLIC EQUATIONS

XIII ENAMA-FLORIANÓPOLIS

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- background of controllability in the ode case

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- background of controllability in the ode case
- present some of the problems and techniques used in the controllability of pde's
- mainly examples with the one dimensional heat equation
- present some results related to coupled parabolic equations

# (Linear) Ordinary Differential Equations framework

# Controllability of systems: The finite dimensional case

$$\begin{cases} \partial_t y = Ly + Bv \\ y(0) = y^0 \end{cases} \tag{1}$$

$L \in \mathcal{M}_n(\mathbb{R})$, $B \in \mathcal{M}_{n,m}(\mathbb{R})$, $m \leq n$.

**Definition**

*System (1) is controllable at time $T > 0$ if*

$$\forall y^0, y^1 \in \mathbb{R}^n, \exists\, v \in L^2(0,T)^m \text{ such that } y(T; y^0, v) = y^1$$

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# Example

$$\begin{cases} \frac{dy_1}{dt} = \frac{-1}{L} y_1 + v(t) \\ \frac{dy_2}{dt} = \frac{-1}{L} y_2 \\ (y_1(0), y_2(0)) = (y_1^0, y_2^0) \end{cases}$$

$$y(t) = \begin{bmatrix} e^{-t/L} y_1^0 \\ e^{-t/L} y_2^0 \end{bmatrix} + \begin{bmatrix} e^{-t/L} & 0 \\ 0 & e^{-t/L} \end{bmatrix} \begin{bmatrix} \int_0^t e^{\tau/L} v(\tau) d\tau \\ 0 \end{bmatrix}$$

this implies that the solution is:

$$y_1(t) = e^{-t/L} y_1^0 + e^{-t/L} \int_0^t e^{\tau/L} v(\tau) d\tau,$$

$$y_2(t) = e^{-t/L} y_2^0.$$

System is not exactly controllable!

We cannot act on $y_2$.

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# Controllability of systems: The finite dimensional case

$$\begin{cases} \partial_t y = Ly + Bv \\ y(0) = y^0 \end{cases} \tag{2}$$

$L \in \mathcal{M}_n(\mathbb{R})$, $B \in \mathcal{M}_{n,m}(\mathbb{R})$.

# Controllability of systems: The finite dimensional case

$$\begin{cases} \partial_t y = Ly + Bv \\ y(0) = y^0 \end{cases} \tag{2}$$

$L \in \mathcal{M}_n(\mathbb{R})$, $B \in \mathcal{M}_{n,m}(\mathbb{R})$.

**Proposition (Kalman rank condition )**

*System (2) (or $(L, B)$) is controllable if and only if*

$$rank\ [B|\, L] = n,$$

*where*

$$[B\,|\,L] = \Big[B,\ LB,\ \cdots\ ,\ L^{n-1}B\Big] \in \mathcal{M}_{n\times nm}(\mathbb{R})$$

$$\begin{cases} \partial_t y = Ly + Bv \\ y(0) = y^0 \end{cases} \tag{2}$$

$L \in \mathcal{M}_n(\mathbb{R})$, $B \in \mathcal{M}_{n,m}(\mathbb{R})$.

**Proposition**

*System (2) is controllable at time $T > 0$ if and only if it is controllable at any time.*

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# Finite dimensional systems

$$\begin{cases} \partial_t y = -k^2 (D + A)\, y + Bv \\ y(0) = y^0 \end{cases}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$rank\ [B|\, L] = 2 \Leftrightarrow b_2[-k^2(d-1)b_1 - b_2] \neq 0$$

# Null controllability

*Linearity* of the system allows to consider instead of *ANY* final state $y^1 = 0$.

In fact, let us assume that

$$\begin{cases} \frac{dy}{dt} = Ay + Bv(t) \\ y(0) = y^0 \end{cases}$$

is exactly controllable at time $T > 0$. That means that for every $y^0 \in \mathbb{R}^n$ and $y^1 \in \mathbb{R}^n$ it exists $v \in \mathcal{U}_{ad}$ such that $y(T) = y^1$. We can choose in particular $y^1 = 0$.

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# Null controllability

Reciprocally, let us assume that for every $y^0$ it exists $v$ such that

$$y(T) = 0.$$

We consider the equation

$$\begin{cases} \frac{dz}{dt} = Az \\ z(T) = y^1 \text{ the target state} \end{cases}$$

# Null controllability

If I choose $v$ such that the solution to

$$\begin{cases} \frac{dx}{dt} = Ax + Bv \\ x(0) = y^0 - z(0) \end{cases}$$

satisfies

$$x(T) = 0,$$

we get that

$$y(t) = x(t) + z(t)$$

verifies

$$\begin{cases} \frac{dy}{dt} = Ay + Bv \\ y(0) = y^0 \\ y(T) = y^1 \end{cases}$$

# Using the adjoint

Let $A^*$ be the adjoint matrix to $A$ that is, the matrix satisfying

$$(Ax, y) = (x, A^* y)$$

for every $x, y \in \mathbb{R}^n$ and $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^n$. We consider the *adjoint system*:

$$(Adj) \qquad \begin{cases} -\dot{\varphi} = A^* \varphi \\ \varphi(T) = \varphi^T \end{cases}$$

# Controllability condition

**Lemma**

*An initial datum* $y^0 \in \mathbb{R}^n$ *can be driven to zero at time* $T > 0$ *with* $v \in L^2(0, T)$ *if and only if*

$$\int_0^T (v, B^*\varphi)dt + (y^0, \varphi(0)) = 0$$

*for every* $\varphi^T \in \mathbb{R}^n$ *and* $\varphi$ *the corresponding solution to (Adj).*

Objective: minimize a quadratic functional $J : \mathbb{R}^n \to \mathbb{R}^n$,

$$J(\varphi^T) = \frac{1}{2}\int_0^T |B^*\varphi|^2 dt + (y^0, \varphi(0))$$

where $\varphi$ is the solution to (Adj) corresponding to the datum $\varphi^T$. Recall that we are looking for

$$\int_0^T (v, B^*\varphi) dt + (y^0, \varphi(0)) = 0$$

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We say that

$$(Adj) \qquad \begin{cases} -\dot{\varphi} = A^* \varphi \\ \varphi(T) = \varphi^T \end{cases}$$

is $B^*$-observable if it exists $C > 0$ such that for every $\varphi^T \in \mathbb{R}^n$ we get

$$(DO) \qquad \int_0^T |B^*\varphi|^2 dt \geq C|\varphi(0)|^2.$$

# Equivalences

We have that

$$\int_0^T |B^*\varphi|^2 dt \geq C|\varphi(0)|^2.$$

if and only if

$$(DOT) \qquad \int_0^T |B^*\varphi|^2 dt \geq C|\varphi^T|^2.$$

for every $\varphi^T$ and $\varphi$ the corresponding solution (Adj) .

**Proposition**

*The observability inquequality (DO) is equivalent to the following unique continuation property:*

$$(CU) \qquad B^*\varphi(t) = 0, \quad \forall t \in [0, T] \Rightarrow \varphi^T = 0.$$

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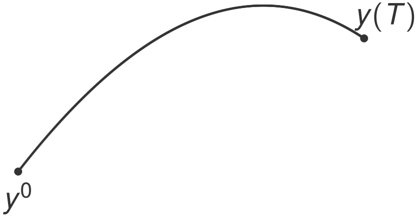
# Single one-dimensional heat equation

# Heat Equation

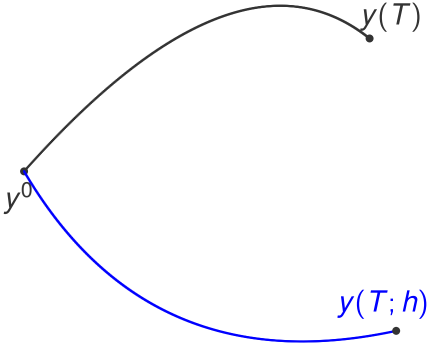
We consider for $y^0 \in L^2(0,\pi)$,

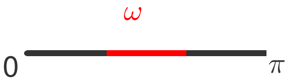
$$(H) \qquad \begin{cases} y_t - y_{xx} = 0 \quad (t,x) \in (0,T) \times (0,\pi) = \Omega_T, \\ y(t,0) = y(t,\pi) = 0, \quad t \in (0,T), \\ y(0,x) = y^0, \quad x \in (0,\pi) = \Omega. \end{cases}$$



- $\chi_\omega$ is the characteristic function of $\omega \subset (0, \pi)$
- $h \in L^2((0, T) \times (0, \pi))$ is a control to be determined.

$$(Hc) \qquad \begin{cases} y_t - y_{xx} = h\chi_\omega & (t, x) \in \Omega_T, \\ y(t, 0) = y(t, \pi) = 0, & t \in (0, T), \\ y(0, x) = y^0, & x \in \Omega. \end{cases}$$



# Heat
# No control

# Approximate control

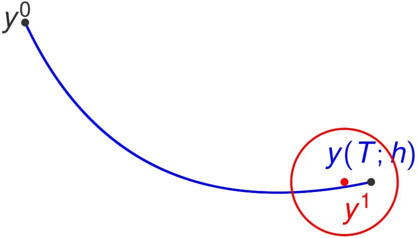
We say that (Hc) is *approximately controllable* at time $T > 0$ in $L^2(0,\pi)$ if for every $y^0, y^1 \in L^2(0,\pi)$ and $\varepsilon > 0$ there exists $h = h(y^0, y^1, \varepsilon)$ such that

$$\|y(T;h) - y^1\|_{L^2} \leq \varepsilon.$$

In other words, if for every $y^0 \in L^2(0,\pi)$ the set of *reachable states*

$$\mathcal{R}(y^0;T) = \{y(T;h),\ y \text{ solution to (Hc) with } h \in L^2((0,T)\times\omega)\}$$

is dense in $L^2(0,\pi)$.

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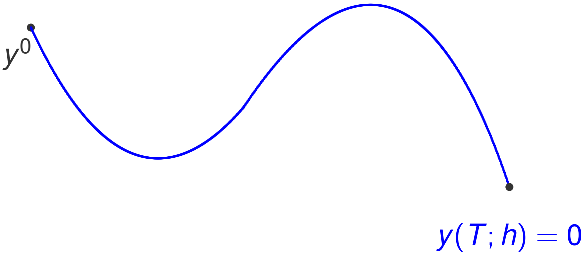
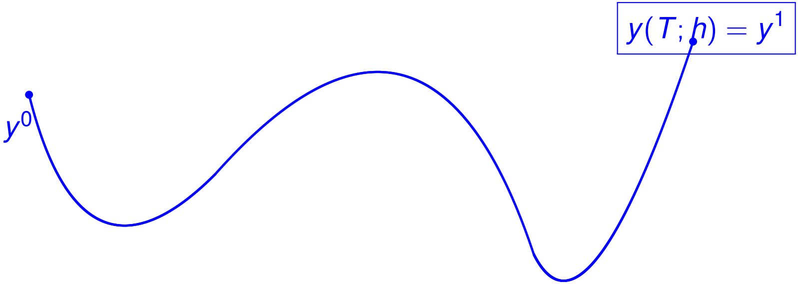
We say that (Hc) is *null controllable* if it exists $h = h(y^0)$ such that

$$y(T; h) = 0.$$



We say that (Hc) is *exactly controllable* if for every pair $y^0, y^1 \in L^2(0,\pi)$ it exists $h = h(y^0)$ such that

$$y(T;h) = y^1.$$

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# Exact controllability?

Is it possible to control exactly the heat equation?

Given $y^0, y^1 \in L^2(0,\pi)$ does it exist $h$ such that the solution to (Hc) satisfies $y(T; h) = y^1$?

# Exact controllability?

Is it possible to control exactly the heat equation?

Given $y^0, y^1 \in L^2(0, \pi)$ does it exist $h$ such that the solution to (Hc) satisfies $y(T; h) = y^1$?

In general NO

# Exact controllability?

Regularizing effects of the heat equation.

# Exact controllability?

Let us control on the whole interval $(0, \pi)$. Let us study the set

$$\mathcal{R}(0; T) = \{y(T; h),\ y \text{ solution to (Hc) with } h \in L^2((0, T) \times (0, \pi))\}$$

That is, we want to describe the solutions at time $T$ to

$$(Hc) \qquad \begin{cases} y_t - y_{xx} = h & (t, x) \in \Omega_T, \\ y(t, 0) = y(t, \pi) = 0, & t \in (0, T), \\ y(0, x) = 0, & x \in \Omega. \end{cases}$$

when $h \in L^2(\Omega_T)$.

# Exact controllability?

$$(Hc) \qquad \begin{cases} y_t - y_{xx} = h & (t,x) \in \Omega_T, \\ y(t,0) = y(t,\pi) = 0, & t \in (0,T), \\ y(0,x) = 0, & x \in \Omega. \end{cases}$$

when $h \in L^2(\Omega_T)$.

$$y(T;h) = \sum_{k=1}^{\infty} y_k(T) \sin kx = \sum_{k=1}^{\infty} e^{-Tk^2} \int_0^T e^{k^2 t} h_k(t) dt \sin kx$$

with $h_k(t) = \int_0^\pi h(t,x) \sin kx dx$

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# Exact controllability?

$$y(T;h) = \sum_{k=1}^{\infty} y_k(T) \sin kx = \sum_{k=1}^{\infty} e^{-Tk^2} \int_0^T e^{k^2 t} h_k(t) dt \sin kx$$

with $h_k(t) = \int_0^\pi h(t,x) \sin kx dx$

$$|y_k(T)|^2 = |e^{-Tk^2} \int_0^T e^{k^2 t} h_k(t) dt|^2 \leq (\int_0^T h_k^2(t) dt) \frac{1}{2k^2}$$

so

$$\sum_{k=1}^{\infty} k^2 |y_k(T)|^2 < \infty.$$

That means that $y(T;h) \in H_0^1(0,\pi)$.

Much more regular!

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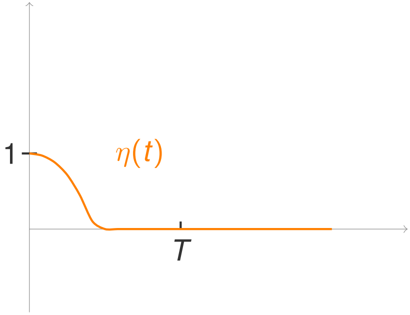
# Null controllability: $\omega = (0, \pi)$.

Given $T > 0$ we take $\eta(t) \in C^1(0, T)$ such that $\eta(0) = 1, \eta(T) = 0$.



# Explicit construction

Given $y^0 \in L^2(0,\pi)$ let $z(t,x)$ solve

$$\begin{cases} z_t - z_{xx} = 0 \quad (t,x) \in \Omega_T, \\ z(t,0) = z(t,\pi) = 0, t \in (0,T), \\ z(0,x) = y^0, x \in (0,\pi). \end{cases}$$

# Explicit construction

We define $y(x,t) = \eta(t)z(t,x)$.

Observe that $y(x,0) = y^0$, $y(x,T) = 0$ and $y$ solves

$$\begin{cases} y_t - y_{xx} = h(t,x), & (t,x) \in \Omega_T, \\ y(t,0) = y(t,\pi) = 0, & t \in (0,T), \\ y(0,x) = y^0, & x \in (0,\pi). \end{cases}$$

with $h(t,x) = \eta'(t)z(t,x)$ .

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# Heat equation: Null and approximate controllability

1. There is not exact controllability (regularizing effect).
2. Approximate controllability $\Longleftrightarrow$ null controllability.
3. There is not minimal control time, not geometric conditions on the control set.

# Approximate controllability and the adjoint equation

**Lemma**

*Consider the adjoint system*

$$(Adj)\begin{cases} v_t + v_{xx} = 0 \quad (t,x) \in \Omega_T, \\ v(t,0) = v(t,\pi) = 0, \quad t \in (0,T), \\ v(T,x) = v^T, \quad x \in (0,\pi). \end{cases}$$

*Suppose that*

$$v(t,x) = 0 \text{ a.e. in } (0,T) \times \omega$$

*implies*

$$\boxed{v^T = 0}$$

*Then (Hc) is approximately controllable at time* $T > 0$.

**Proof.**

Take $\boxed{v^T \in \mathcal{R}(0;T)^\perp}$, $v^T \neq 0$ and $v$ the corresponding solution to (Adj). Multiplying (Hc) (with initial datum $y^0 = 0$) by $v$ and integrating by parts in $(0,T) \times (0,\pi)$. We get

$$\boxed{\int_0^\pi v^T(x) y(T,x) dx = \int_0^T \int_\omega h(t,x) v(t,x).}$$

$$v^T \in \mathcal{R}(0;T)^\perp \Rightarrow \int_0^T \int_\omega h(t,x) v(t,x) dxdt = 0$$

for every $h \in L^2(0,T) \times \omega$ and then $\boxed{v(t,x) \equiv 0 \text{ in } (0,T) \times \omega.}$ Since we are assuming Unique Continuation true $v^T = 0$ we got a contradiction. □

# Unique continuation: analiticity of the solution

**Proof of the Unique Continuation Property.**

$$v(t,x)=\sum_{n=1}^{\infty} v_n^T e^{-n^2(T-t)}\sin nx,\;\; v_n^T=\int_0^{\pi} v^T(x)\sin nx dx.$$

□

$$(Adj)\begin{cases} v_t+v_{xx}=0 & (t,x)\in\Omega_T,\\ v(t,0)=v(t,\pi)=0, & t\in(0,T),\\ v(T,x)=v^T, & x\in(0,\pi).\end{cases}$$

# Unique continuation: analiticity of the solution

**Proof of the Unique Continuation Property.**

$$v(t,x) = \sum_{n=1}^{\infty} v_n^T e^{-n^2(T-t)} \sin nx, \; v_n^T = \int_0^{\pi} v^T(x) \sin nx dx.$$

We need to prove that $v = 0$ in $(0, T) \times \omega \Rightarrow v_n^T = 0, \forall n$.

$\square$

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**Proof of the Unique Continuation Property.**

Since $v$ is analytic in $t$, we take the analytic extension to $t \in (-\infty, 0)$, and $\widetilde{v}(t,x) = 0, (t,x) \in (-\infty, T) \times \omega$.

$\square$

# Unique continuation: analiticity of the solution

**Proof of the Unique Continuation Property.**

Since $v$ is analytic in $t$, we take the analytic extension to $t \in (-\infty, 0)$, and $\tilde{v}(t,x) = 0, (t,x) \in (-\infty, T) \times \omega$.

Suppose that $v_1^T \neq 0$. Then, for every $t \in (-\infty, T)$

$$-v_1^T \chi_\omega \sin x = \sum_{n=2}^{\infty} v_n^T e^{-(n^2-1)(T-t)} \sin nx \chi_\omega$$

□

# Unique continuation: analiticity of the solution

**Proof of the Unique Continuation Property.**

$$-v_1^T \chi_\omega \sin x = \sum_{n=2}^{\infty} v_n^T e^{-(n^2-1)(T-t)} \sin nx \chi_\omega \to 0, \; t \to -\infty$$

Then $v_1^T \sin x \chi_\omega = 0$ but $\boxed{\sin x \neq 0 \text{ in } \omega}$ so $v_1^T = 0$.
Inductively we get $v_n^T = 0$ for every $n$. □

# Observability inequality

**Lemma**

*Back to the adjoint equation*

$$(Adj)\begin{cases} v_t + v_{xx} = 0 \quad (t,x) \in \Omega_T, \\ v(t,0) = v(t,\pi) = 0, \quad t \in (0,T), \\ v(T,x) = v^T, \quad x \in (0,\pi). \end{cases}$$

*Then (Hc) is null controllable iff there exists $C > 0$ such that $v$ any solution to (Adj) satisfies*

$$\int_0^\pi |v(0,x)|^2 dx \leq C \int_0^T \int_\omega |v(t,x)|^2 dx dt.$$

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# Minimization

Given $y^0 \in L^2(0,\pi)$ we define

$$J(v^T) = \frac{1}{2}\int_0^T\!\!\int_\omega |v|^2 dxdt + \int_0^T y^0(x)v(0,x)dx.$$

Observability inequality $\Rightarrow$ existence of a minimum $\hat{v}^T$

$$\begin{cases} y_t - y_{xx} = \hat{v}\chi_\omega & \hat{v}_t + \hat{v}_{xx} = 0 \quad (t,x) \in \Omega_T, \\ y(t,0) = y(t,\pi) = 0, & \hat{v}(t,0) = \hat{v}(t,\pi) = 0, \quad t \in (0,T), \\ y(0,x) = y^0, \boxed{y(T) = 0} & \hat{v}(T,x) = \hat{v}^T, \quad x \in \Omega. \end{cases}$$

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# Carleman inequalities

Carleman inequalities are weighted inequalities which relate a differential operator with the local weighted norm of the solution.

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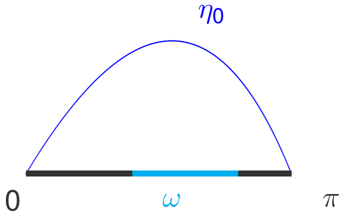
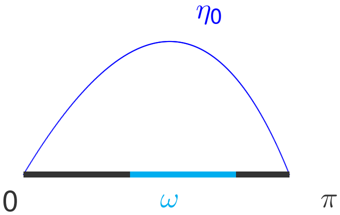
Nowadays this kind of inequalities have been generalized and are a very useful technique for inverse and control problems.

# Kind of function



# Kind of function




**Theorem (Fursikov-Imanuvilov)**

*Let $\Omega \subset \mathbb{R}^n$ be a open and bounded set of class $C^2$. Let $\omega \subset \Omega$ be a non empty open set and $B_\delta$ an open ball centered at $x_0 \in \omega$ with $B_\delta \subset \omega$. Then there exists $\eta_0 \in C^\infty(\overline{\Omega_T})$ such that $\eta_0(x) > 0$ in $\Omega$, $\frac{\partial \eta_0}{\partial \nu} < 0$ on $\partial\Omega$ , $|\nabla \eta_0| > 0$ in $\Omega \backslash B_\delta$ .*

# Carleman: weighted inequality

Given $\eta_0$ as before, we define

$$\alpha(x,t) = \frac{e^{\lambda(2\|\eta_0\|_\infty + \eta_0(x))} - e^{2\lambda\|\eta_0\|_\infty}}{t(T-t)},$$
$$\xi(x,t) = \frac{e^{\lambda(2\|\eta_0\|_\infty + \eta_0(x))}}{t(T-t)}, \qquad \rho(x,t) = e^{\alpha(x,t)}, \tag{3}$$

with $\lambda > 0$ .

Key fact $\boxed{\lim_{s\to 0^+, T^-} \rho^{-1} = 0.}$

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# Now......... Carleman inequality

$$\begin{cases} v_t + \Delta v = F_0 & \text{in } \Omega_T, \\ v = 0 & \text{on } \Sigma, \\ v(x,T) = v_T(x) & \text{in } \Omega, \end{cases}$$

with $v_T \in L^2(\Omega)$, $F_0 \in L^2(\Omega_T)$,

# Now......... Carleman inequality

$$\begin{cases} v_t + \Delta v = F_0 & \text{in } \Omega_T, \\ v = 0 & \text{on } \Sigma, \\ v(x,T) = v_T(x) & \text{in } \Omega, \end{cases}$$

with $v_T \in L^2(\Omega)$, $F_0 \in L^2(\Omega_T)$,

**Theorem**

*$\Omega \subset \mathbb{R}^n$ smooth. There exists constants $s_0$, $\lambda_0$ and $C$ such that, for every $s \geq s_0$ and $\lambda \geq \lambda_0$, the solutions satisfy*

$$\iint_{\Omega_T} \rho^{-2s}(s\lambda^2\xi|\nabla v|^2 + s^3\lambda^4\xi^3 v^2) \leq C \left( s^3\lambda^4 \int_0^T\!\!\int_\omega \rho^{-2s}\xi^3|v|^2 + \iint_{\Omega_T} \rho^{-2s}|F_0|^2 \right).$$

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# Observability inequality from Carleman

For every $s \geq s_0(T + T^2)$. We got lower and upper bounds:

$$\rho^{-2s}(x,t)\xi^3(x,t) \geq e^{-2C(1+1/T)}\frac{1}{T^6} \quad \text{in} \quad \boxed{\Omega \times (T/4, 3T/4),}$$

$$\rho^{-2s}(x,t)\xi^3(x,t) \leq M \quad \text{in} \quad \Omega \times (0,T).$$

Then,

$$\iint_{\Omega\times(T/4,3T/4)} |v|^2 \leq C \iint_{\omega\times(0,T)} |v|^2$$

Classical energy estimates give:

$$\|v(0)\|^2_{L^2(\Omega)} \leq \frac{2}{T} \iint_{\Omega\times(T/4,3T/4)} |v|^2$$

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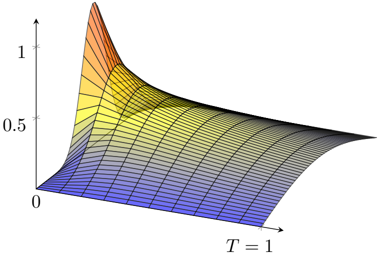
# Some numerics (Franck Boyer)

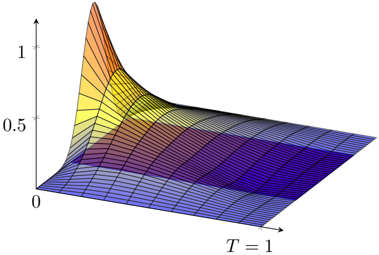
$$\begin{cases} y_t - 0.1 y_{xx} = 0 \\ y(t,0) = y(t,1) = 0 \\ y(0,x) = \sin^{10}(\pi x) \end{cases}$$




Uncontrolled heat equation

# Some numerics (Franck Boyer)

$$\begin{cases} y_t - 0.1 y_{xx} = h\chi_\omega, & \omega = (0.3, 0.8) \\ y(t,0) = y(t,1) = 0 \\ y(0,x) = \sin^{10}(\pi x) \end{cases}$$




Controlled heat equation

# Single Parabolic Equation: Boundary control

# One-dimensional boundary control

# The method of moments

We consider the operator $-\partial_{xx}$ on $(0, \pi)$ with homogeneous Dirichlet conditions. We have a Hilbert basis of $L^2(0, \pi)$ given by

$$\lambda_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad k \geq 1, \quad x \in (0, \pi) \tag{4}$$

For every $y \in L^2(0, \pi)$ there exists a sequence $\{y_k\}_{k \geq 1} \subset \mathbb{R}$ such that

$$y = \sum_{k \geq 1} y_k \phi_k \, .$$

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Control problem

$$\begin{cases} y_t - y_{xx} = 0 & \text{in } \Omega_T = (0,\pi) \times (0,T), \\ y(0,t) = v(t), \quad y(\pi,t) = 0 & t \in (0,T), \\ y(x,0) = y^0(x) & x \in (0,\pi), \end{cases}$$

with $y^0 \in H^{-1}(0,\pi)$ and $v \in L^2(0,T)$.

Given $y^0 \in H^{-1}(0,\pi)$, there exists $v \in L^2(0,T)$ such that the solution satisfies $y(x,T) = 0, x \in (0,\pi)$ iff there exists $v \in L^2(0,T)$ such that

$$-\langle y^0, e^{-\lambda_k T}\phi_k \rangle_{H^{-1}(0,\pi),H_0^1(0,\pi)} = \int_0^T v(t) e^{-k^2(T-t)} \partial_x \phi_k(0)\, dt, \quad \forall k \geq 1.$$

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By Fourier $y^0 = \sum_{k\geq 1} y_{0,k}\phi_k$, this is equivalent to the existence of $v \in L^2(0,T)$ such that

$$k\sqrt{\frac{2}{\pi}}\int_0^T e^{-k^2(T-t)}v(t)\,dt = -e^{-k^2T}y_{0,k} \quad \forall k \geq 1.$$

We define $\widetilde{v}(t) = v(T-t)$, then we have to solve

$$\boxed{\int_0^T e^{-k^2t}\widetilde{v}(t)\,dt = -\frac{\sqrt{\pi}e^{-k^2T}}{k\sqrt{2}}y_{0,k} := c_k \quad \forall k \geq 1.} \tag{5}$$

This problem is known as a *problem of moments*.

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We have:

**Theorem (Fattorini-Russell 1971.)**

*For every $y^0 \in L^2(0,\pi)$ and $T > 0$, there exists $\widetilde{v} \in L^2(0,T)$ solution to the problem of moments. That is, $v(t) = \widetilde{v}(T-t)$ is a null boundary control for the one-dimensional heat equation.*

# Idea of the proof.

We say that a family $\{p_k\}_{k\geq 1} \subset L^2(0,T)$ is biorthogonal to $\{e^{-k^2 t}\}_{k\geq 1}$ if it satisfies

$$\int_0^T e^{-k^2 t} p_l(t) = \delta_{kl}, \quad \forall (k,l) : k,l \geq 1.$$

Fattorini-Russell that there exists $\{p_k\}_{k\geq 1}$ biorthogonal to $\{e^{-k^2 t}\}_{k\geq 1}$ that has an additional property: $\forall \varepsilon > 0$ there exists a constant $C(\varepsilon, T) > 0$ such that $\|p_k\|_{L^2(0,T)} \leq C(\varepsilon, T) e^{\varepsilon k^2}$. We define

$$v(T-s) = \widetilde{v}(s) = \sum_{k\geq 1} c_k p_k(s) := \sqrt{\frac{\pi}{2}} \sum_{k\geq 1} \frac{1}{k} e^{-k^2 T} y_{0,k} p_k(s)$$

the given bounds prove the convergence in $L^2(0,T)$. □

# Coupled equations

# Models: competitive models between species

Lotka-Volterra-like equations

- $u$ and $v$ two species
- predator prey models, radiation to new habitats

$$\partial_t u - d_1 \Delta u + r_1 u = a_{11} u^2 + a_{12} uv, \text{ in } \Omega$$
$$\partial_t v - d_2 \Delta v + r_2 v = a_{22} v^2 + a_{21} uv \text{ in } \Omega$$
$$+BC \text{ on } \partial\Omega$$
$$+ID \text{ in } \Omega$$

Gives two coupled parabolic non linear equations.

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# Models: T. Hillen; K. J. Painter

Keller-Seller type (chemotaxis)

- $u$ denotes de cell or organism density
- $v$ describes the concentration of the chemical signal.

$$\begin{aligned}
&\partial_t u = \nabla(k_1(u,v)\nabla u - k_2(u,v)\nabla v) + k_3(u,v), \text{ in } \Omega \\
&\partial_t v = D_v \Delta v + k_4(u,v) - k_5(u,v)v \text{ in } \Omega \\
&+BC \text{ on } \partial\Omega \\
&+ID \text{ in } \Omega
\end{aligned}$$

Gives two coupled parabolic non linear equations.

# Models: Clair Poignard

Cell migration modelling: Patlak-Keller-Segel type.

- $u_1(t, x, y)$ the density of endothelial cells, at any point $(x, y)$ and at time $t$, that can freely move.

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- Cells that are adhering on the substrate are tracked through their density $u_2$.

# Models: Clair Poignard

Cell migration modelling: Patlak-Keller-Segel type.

- $u_1(t,x,y)$ the density of endothelial cells, at any point $(x,y)$ and at time $t$, that can freely move.
- Cells that are adhering on the substrate are tracked through their density $u_2$.
- $v$ represents the density of the chemoattractant.

The equations governing the endothelial cell migration are

$$\begin{aligned}
&\partial_t u_1 = d_1 \Delta u_1 - \lambda 1_{\tilde{\Omega}} u_1(1-u_2) - \nabla\cdot(\xi(u_1,v)u_1\nabla v), \text{ in } \Omega\\
&\partial_t u_2 = d_2 \Delta u_2 - \lambda 1_{\tilde{\Omega}} u_1(1-u_2) \text{ in } \tilde{\Omega}\\
&\partial_t v = \Delta v - \eta v + \gamma_1 u_1 + \gamma_2 u_2 \text{ in } \Omega\\
&\partial_\nu u_1 = \partial_\nu u_2 = \partial_\nu v = 0 \text{ on } \partial\Omega\\
&u_1(0,x,y) = u_1^0,\ u_2(0,x,y) = u_2^0,\ v(0,x,y) = 0 \text{ in } \Omega
\end{aligned}$$

# Model: Clair Poignard

Three nonlinear parabolic coupled equations.

# LINEARIZED MODELS

# Linearized Models

$$
(S)\begin{cases}
\partial_t y = (D\Delta + A)y & \text{in } \Omega_T = \Omega \times (0,T), \\
y = Bv(x,t)\chi_\gamma & \text{on } \Sigma = \partial\Omega \times (0,T), \quad \gamma \subset \partial\Omega \\
y(\cdot,0) = y^0 & \text{in } \Omega,
\end{cases}
$$

$$
y(x,t) = (y_1(x,t), \cdots, y_n(x,t))
$$

$$
D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & d_n \end{pmatrix}
$$

$$
A \in \mathcal{M}_{n\times n}, B \in \mathcal{M}_{n\times m}
$$

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# INTERNAL CONTROLLABILITY OF TWO COUPLED PARABOLIC EQUATIONS

# Preliminaries

Let $\Omega \subset \mathbb{R}^n$ open and smooth set. Let $\omega, \mathcal{O} \subset \Omega$ be a nonempty subset and $\Omega_T = \Omega \times (0,T)$; $\Sigma = \partial\Omega \times (0,T)$ We consider

$$\begin{cases} y_t - \Delta y + f(y,u) = h\chi_\omega; & u_t - \alpha\Delta u + g(u) = y\chi_{\mathcal{O}} & \text{in } \Omega_T, \\ y = 0; & u = 0 & \text{on } \Sigma, \\ y(0) = y^0; & u(0) = u^0 & \text{in } \Omega \end{cases}$$

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Control problem: For every $y^0, u^0 \in L^2(\Omega)$ and $T > 0$

does there exists $h \in L^2(\Omega_T)$ such that simultaneously $y(T) = u(T) = 0$?

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González-Burgos; de T. Yes;

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When $\mathcal{O} \cap \omega \neq \emptyset$

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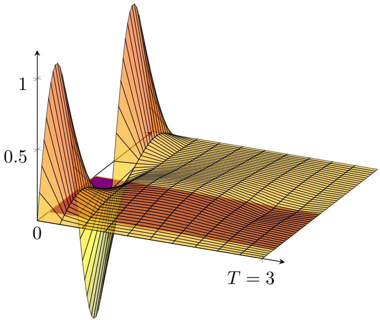
When $\mathcal{O} \cap \omega \neq \emptyset$

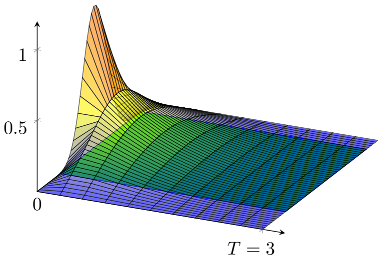
Techniques used: Carleman inequalities for the adjoint system, local energy estimates, fixed point arguments.

# Numerical example (Franck Boyer)

$$\begin{cases} y_t - (0.1)y_{xx} = h\chi_\omega & u_t - (0.1)u_{xx} = y\chi_{\mathcal{O}} & \text{in } \Omega_T\,, \\ y(t,0) = y(t,\pi) = 0 & u(t,0) = u(t,\pi) = 0 & \text{in } (0,T)\,, \\ y(\cdot,0) = \sin(3\pi x) & u(\cdot,0) = \sin^{10}(\pi x) & \text{in } (0,1)\,, \\ \omega = (0.1, 0.5) & \mathcal{O} = (0.2, 0.9) & \omega \cap \mathcal{O} \neq \emptyset \end{cases}$$




First component $y$




Second component $u$

# NEW CONTROL QUESTION: BOUNDARY CONTROLLABILITY OF TWO COUPLED PARABOLIC EQUATIONS

# Coupled systems problem:boundary

Let us consider for $z = (y, q)$, the system

$$\left\{\begin{array}{lll} y_t - \alpha y_{xx} = 0 & u_t - u_{xx} = y & \text{in } \ \Omega_T \,, \\ y(t,0) = v(t) & u(t,0) = 0 & t \in (0,T)\,, \\ y(t,\pi) = 0 & u(t,\pi) = 0 & t \in (0,T) \\ y(0,x) = y^0(x) & u(0,x) = u^0(x) & x \in (0,\pi)\,, \end{array}\right.$$

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Approximate controllability is equivalent to a unique continuation property for the adjoint problem:

$$\begin{cases} -\tilde{\varphi}_t - \alpha\tilde{\varphi}_{xx} = \tilde{\psi} & -\tilde{\psi}_t - \tilde{\psi}_{xx} = 0 & \text{in } \Omega_T, \\ \tilde{\varphi}(t,0) = \tilde{\varphi}(t,\pi) = 0 & \tilde{\psi}(t,0) = \tilde{\psi}(t,\pi) = 0 & t \in (0,T), \\ \tilde{\varphi}(T,x) = \tilde{\varphi}^0(x) & \tilde{\psi}(T,x) = \tilde{\psi}^0(x) & x \in (0,\pi)\,, \end{cases}$$

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$\tilde{\varphi}_x|_{x=0} = 0$ implies $\tilde{\psi} \equiv \tilde{\varphi} \equiv 0$?

# Coupled systems problem:boundary

**Theorem (Fernández-Cara, González-Burgos, deT)**

*Suppose that* $\alpha \neq 1$ *then unique continuation property is true if and only if* $\sqrt{\alpha} \notin \mathbb{Q}$*. In other words if* $\alpha \neq 1$*, system is approximately controllable at time* $T > 0$ *if and only if* $\sqrt{\alpha} \notin \mathbb{Q}$*.*

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# Proof

Let $w_j(x) = \sin(jx)$ the eigenfunctions of the Dirichlet Laplacian in $(0,\pi)$, for the eigenvalue $j^2$.

Then

$$\tilde{\varphi}(x, T-t) =$$
$$\varphi(x,t) = \sum_{j\geq 1}(a_j - \frac{b_j}{(\alpha-1)j^2})e^{-\alpha j^2 t} w_j(x) + \sum_{j\geq 1} \frac{b_j}{(\alpha-1)j^2} e^{-j^2 t} w_j(x),$$

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$$\tilde{\psi}(T-t,x) = \psi(t,x) = \sum_{j\geq 1} \frac{b_j}{(\alpha-1)j^2} e^{-j^2 t} w_j(x),$$

with $b_j = \int_0^\pi \tilde{\psi}^0(x)\sin(jx)dx, \quad a_j = \int_0^\pi \tilde{\varphi}^0(x)\sin(jx)dx.$

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Let $w_j(x) = \sin(jx)$ the eigenfunctions of the Dirichlet Laplacian in $(0, \pi)$, for the eigenvalue $j^2$.

Then

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# Proof

$$\varphi_x(t,0) = \sum_{j\geq 1} j\left( (a_j - \frac{b_j}{(\alpha-1)j^2})e^{-\alpha j^2 t} + \frac{b_j}{(\alpha-1)j^2}e^{-j^2 t}\right)$$

# Proof

$$\varphi_x(t,0) = \sum_{j\geq 1} j\left((a_j - \frac{b_j}{(\alpha-1)j^2})e^{-\alpha j^2 t} + \frac{b_j}{(\alpha-1)j^2}e^{-j^2 t}\right)$$

Suppose that $\sqrt{\alpha} \in \mathbb{Q}$. That means that $\boxed{\alpha = \frac{k_0^2}{i_0^2}}$ and then

$$\alpha i_0^2 = k_0^2.$$

Choose $b_j = a_j = 0$ for $j \neq k_0, i_0$, $b_{i_0} = 0, b_{k_0} = 1$ and

$$a_{k_0} = \frac{1}{(\alpha-1){k_0}^2}, \quad a_{i_0} = \frac{-1}{(\alpha-1){k_0}^2}.$$

Then, $\varphi_x(0,t) = 0$ in $(0,T)$ but $\varphi \neq 0, \psi \neq 0$.

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# Approximate controllability

Given $\alpha$ such that $\sqrt{\alpha} \notin \mathbb{Q}$. Take sequences $\alpha j^2$ and $j^2$. We can reorder and write an increasing sequence

$$0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots$$

and

$$\varphi_x(0,t) = \sum_{j=1} A_j e^{-\mu_j t}.$$

We observe that $e^{-\mu_j t}$ is a family of linearly independent functions in $(0, T)$ and then

$$\varphi_x(0,t) = 0 \Rightarrow A_j = 0.$$

That implies $b_j = 0, \forall j$ and then that $a_j = 0, \forall j$. In particular, $\tilde{\psi}^0 = \tilde{\varphi}^0 = 0$ and the unique continuation property holds true .

# Non trivial example: null controllability

**Theorem (Fernández-Cara, González-Burgos, deT)**

*Suppose that* $\alpha = 1$*. Then system*

$$\begin{cases} y_t - y_{xx} = 0 & u_t - u_{xx} = y & \text{in } \Omega_T, \\ y(t,0) = h(t) & u(t,0) = 0 & t \in (0,T), \\ y(t,\pi) = 0 & u(t,\pi) = 0 & t \in (0,T) \\ y(0,x) = y^0(x) & u(0,x) = u^0(x) & x \in (0,\pi), \end{cases}$$

*is null controllable at time* $T$ *for any* $T > 0$*.*

# Non trivial example: null controllability

**Theorem (Fernández-Cara, González-Burgos, deT)**

*Suppose that $\alpha = 1$. Then, there exists a constant $C > 0$ such that the solution to the adjoint system*

$$\begin{cases} -\tilde{\varphi}_t - \tilde{\varphi}_{xx} = \tilde{\psi} & -\tilde{\psi}_t - \tilde{\psi}_{xx} = 0 & \text{in } \Omega_T, \\ \tilde{\varphi}(t,0) = \tilde{\varphi}(t,\pi) = 0 & \tilde{\psi}(t,0) = \tilde{\psi}(t,\pi) = 0 & t \in (0,T), \\ \tilde{\varphi}(T,x) = \tilde{\varphi}^0(x) & \tilde{\psi}(T,x) = \tilde{\psi}^0(x) & x \in (0,\pi), \end{cases}$$

*satisfies*

$$\int_0^\pi |\tilde{\psi}(0,x)|^2 dx + \int_0^\pi |\tilde{\varphi}(0,x)|^2 dx \leq C \int_0^T |\tilde{\varphi}_x(t,0)|^2 dt.$$

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# Null controllability

What happens if $\sqrt{\alpha} \notin \mathbb{Q}$?

**Theorem (Luca-deT (2013))**

*Boundary control: There exist values of $\alpha$ such that $\sqrt{\alpha} \notin \mathbb{Q}$ and there is not NULL controllability.*

# Null controllability

What happens if $\sqrt{\alpha} \notin \mathbb{Q}$?

**Proof.**

There exists $\sqrt{\alpha} \notin \mathbb{Q}$, such that the solution to the system

$$\begin{cases} -\tilde{\varphi}_t - \alpha\tilde{\varphi}_{xx} = \tilde{\psi} & -\tilde{\psi}_t - \tilde{\psi}_{xx} = 0 & \text{in } \ \Omega_T, \\ \tilde{\varphi}(t,0) = \tilde{\varphi}(t,\pi) = 0 & \tilde{\psi}(t,0) = \tilde{\psi}(t,\pi) = 0 & t \in (0,T), \\ \tilde{\varphi}(T,x) = \tilde{\varphi}^0(x) & \tilde{\psi}(T,x) = \tilde{\psi}^0(x) & x \in (0,\pi)\,, \end{cases}$$

does not satisfy inequality

$$\int_0^\pi |\tilde{\psi}(0,x)|^2 dx + \int_0^\pi |\tilde{\varphi}(0,x)|^2 dx \leq C \int_0^T |\tilde{\varphi}_x(t,0)|^2 dt$$

for any $C > 0$ and $T > 0$.

Construction of $\alpha$ using Diophantine approximations of real numbers. □

# Generalization

**Theorem (F. Ammar Khodja, A. Benabdallah, M. González–Burgos, L. deT, 2014)**

*Let $\alpha \neq 1$*

1. $\forall T > 0$ : *System is approximately controllable iff* $\sqrt{\alpha} \notin \mathbb{Q}$
2. $\exists T_0 = c(\Lambda) \in [0, +\infty]$ *such that*
   - *System is null controllable at time $T$ if $\sqrt{\alpha} \notin \mathbb{Q}$ and $T > T_0$*
   - *Even when $\sqrt{\alpha} \notin \mathbb{Q}$, if $T < T_0$, system is not null controllable at time $T$*

$c(\Lambda)$ is the condensation index of the sequence $\Lambda = (k^2, dk^2)_{k \geq 1}$.

- The *condensation index* of a sequence $\Lambda = (\lambda_k) \subset \mathbb{C}$ is a real number

$$c(\Lambda) \in [0, +\infty]$$

associated to the sequence and "measures" the condensation at infinity.

- The notion was introduced by:
  - V.I. Bernstein in 1933:
    Leçons sur les progrès récents de la théorie des séries de Dirichlet
    for real sequences,
  - extended by J. R. Shackell in 1967 for complex sequences.

# Condensation Index

**Definition**

The condensation index of $\Lambda = \{\lambda_k\}$ is:

$$c(\Lambda) = \limsup_{k\to\infty} \frac{-\ln|E'(\lambda_k)|}{\Re(\lambda_k)} \in [0, +\infty]\,.$$

$$E'(\lambda_k) = -\frac{2}{\lambda_k}\prod_{j\neq k}^{\infty}\left(1-\frac{\lambda_k^2}{\lambda_j^2}\right)$$

# More results

- In $\mathbb{R}^n$ the boundary control problem is almost open.

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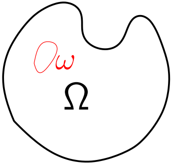
- In $\mathbb{R}^n$ the boundary control problem is almost open.
- Techniques do not allow to treat the non linear boundary control problem.

# More results

- Other problem: internal controllability

$$
\begin{cases}
\partial_t y = (D\Delta + A)\, y + \chi_\omega B v, & \text{in } (0,T)\times\Omega,\\
y = 0 & \text{on } (0,T)\times\partial\Omega,\\
y(0,x) = y^0(x) & x\in\Omega,\\
v \in L^2(\Omega\times(0,T))^m,\ \omega \Subset \Omega.
\end{cases}
$$



# More results

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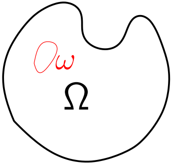


- $D$ diagonal, $A$ independent of $x$ well understood.

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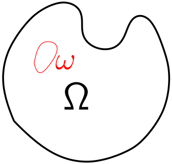


- $D$ diagonal, $A$ independent of $x$ well understood.
- $D$ non diagonal. Results related with the Jordan decomposition of $A$! May be technical....(Fernández-Cara, González-Burgos, deT (COCV:2015) )

$$\begin{cases} y_t - y_{xx} + \alpha(x)p = 0 & p_t - p_{xx} = h\chi_{(a,b)} & \text{in } \Omega_T\,, \\ y(t,0) = y(t,\pi) = 0 & p(t,0) = p(t,\pi) = 0 & \text{in } (0,T)\,, \\ y(0,x) = y^0(x) & p(0,x) = p^0(x) & \text{in } (0,\pi)\,, \end{cases}$$

### Theorem

1. *Let* $I_{1,k}(\alpha) := \int_0^a \alpha(x)|\sin kx|^2\,dx$, $I_{2,k}(q) := \int_b^\pi \alpha(x)|\sin kx|^2\,dx$, *system is approximately controllable at time* $T > 0$ *if and only if* $I_{1,k}(\alpha) + I_{2,k}(\alpha) = I_k(\alpha) \neq 0 \quad \forall k \geq 1$.
2. *Assume that system is app.controllable. Define*

$$\widetilde{T}_0(\alpha) := \limsup \frac{-\log|I_k(\alpha)|}{k^2} \in [0,\infty)\,. \tag{6}$$

*Then, if* $T > \widetilde{T}_0(\alpha)$ *system is null controllable at time* $T$*. On the other hand, if* $T < \widetilde{T}_0(\alpha)$ *system is not null controllable at time* $T$*.*

# Internal control: Nonlinear case

Coron-Guilleron

$$\begin{cases} \alpha_t - \Delta\alpha = \beta^3, & \text{in } \Omega_T, \\ \beta_t - \Delta\beta = \gamma^3, & \text{in } \Omega_T \\ \gamma_t - \Delta\gamma = u\chi_\omega, & \text{in } \Omega_T \\ \alpha = \beta = \gamma = 0, & (t,x) \in (0,T) \times \partial\Omega \\ \alpha(0,x) = \alpha^0(x); \beta(0,x) = \beta^0(x); \gamma(0,x) = \gamma^0(x); & \text{in } \Omega, \end{cases}$$

Return method: SIAM "W.T. and Idalia Reid Prize" (J.M. Coron)

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nothing is in control

¡Gracias!

Muito Obrigada !

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