Semilinear equations with discrete spectrum radial solutions

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XII ENAMA, Brasilia, November 2018

$$(1) \quad u''(t) + g(u(t)) = Q(t), \;\; t \in (0, \pi) \;\; u(0) = 0, \;\; u(\pi) = 0.$$

(2) $\sigma(-'') = \{1, 4, \ldots, k^2, \ldots \to +\infty\},$

All the eigenvalues are simple.

THEOREM 1.1 *If*

$$(3) \qquad \lim_{|u| \to +\infty} \frac{g(u)}{u} = +\infty \;\; (\textit{superlinear}),$$

g is differentiable, and q is bounded and continuous then (1) *has infinitely many solutions. Moreover, there exists K such that if $k \geq K$ is a positive integer then* (1) *has two solutions with k nodes (zeroes).*

Let Ω be bounded region in $\mathbb{R}^N$.

$$(4) \qquad \Delta u(x) + g(u(x)) = Q(x),\ x \in \Omega \quad u(x) = 0,\ x \in \partial\Omega.$$

$$(5) \qquad \sigma(-\Delta) = \{\lambda_1, \ldots, \lambda_k, \ldots \to +\infty\},$$

The eigenvalue λ_1 is a simple, others have finite multiplicity but need not be simple.

PROBLEM. *Suppose g is superlinear, differentiable, and Q is bounded and continuous. Does (4) have infinitely many solutions? Are there solutions with large number of nodal regions?*

Proof of Theorem 1.1. Let $u(\cdot, d)$ satisfy $u''(t) + g(u(t)) = Q(t)$ $t \in (0, \pi)$, $\quad u(0) = 0,\ u'(0) = d$. Let $G(u) = \int_0^u g(s)ds$ and

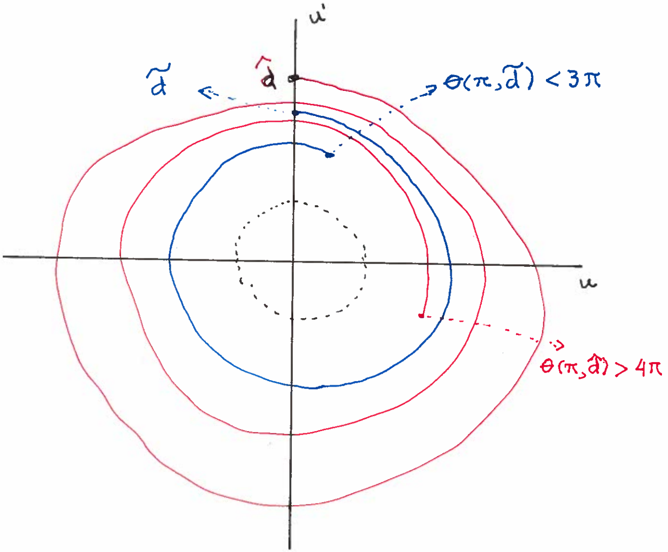
$$E(t,d) = \frac{(u'(t,d))^2}{2} + G(u(t,d)) \tag{6}$$

$$\lim_{d \to +\infty} E(t,d) = +\infty. \tag{7}$$

Hence, there exists D and a continuous function $\theta(t, d)$ such that

$$\begin{aligned} u(t,d) = \rho(t,d)\sin(\theta(t,d)),\ \rho(t,d)\sin(\theta(t,d)) \\ \lim_{d \to +\infty} \theta(\pi, d) = +\infty. \end{aligned} \tag{8}$$

for $d > D$. Here $\rho^2(t,d) = u^2(t,d) + (u'(t,d))^2$. Similarly for $d < -D$.


u'
$\hat{d}$
$\tilde{d}$
$\theta(\pi,\tilde{d}) < 3\pi$
u
$\theta(\pi,\hat{d}) > 4\pi$

From (8), there exists a sequence $d_n \to +\infty$ such that $\theta(\pi, d_n) = n\pi$. That is $u(\cdot, d_n)$ is a solution to (1) with $n-1$ zeroes in $(0, \pi)$.

ANSWER TO PROBLEM $=$ NO, S. Pohozaev (1965).

THEOREM 1.2. (Pohozaev identity) *If u be a solution to* (4) *with $Q = 0$, then*

$$(9) \quad \int_\Omega \left(NG(u) - \frac{N-2}{2} ug(u) \right) dx = \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (\eta(x) \cdot x) dS.$$

Proof. Multiply by u and integrate by parts eliminating integrals of second order derivatives. Multiply by $x \cdot \nabla u$ and integrate by parts eliminating terms that contain second order derivatives. Combining the resulting relations (9) appears by eliminating the terms that contain derivatives of u integrated on Ω .

THEOREM 1.3. *If $g(u) = |u|^{p-1}u$, $p \geq (N+2)/(N-2)$, $Q = 0$, and Ω is starlike then $u = 0$ is the only solution to (4)*

Proof. Now $NG(u) - \frac{N-2}{2}ug(u) = \gamma_p|u|^{p+1}$ with $\gamma_p = \frac{2N-(p+1)(N-2)}{2N}$. Hence $\gamma_p \leq 0$. This, (9), and $\eta(x) \cdot x \geq 0$ prove that if u is a solution to (4) then the left hand side in (9) is not positive while the right hand side is not negative. Hence $u = 0$.

Note. a) Imitating the proof of Theorem 1.1 one sees that if $0 < a < b < \infty$,

$$\Omega = \{x \in \mathbb{R}^N; a < \|x\| < b\} := A$$

then (4) has infinitely many radial solutions. See [B-C, 1988, P-1989].
b) Theorem 1.3 applies to the case

$$\Omega = \{x \in \mathbb{R}^N; \|x\| < 1\} := B.$$

RADIAL SOLUTIONS TO (4) IN B

Recall that, in spherical coordinates $(r, \theta_1, \ldots, \theta_{N-1})$,

$$\Delta u = u_{rr} + \frac{N-1}{r} u_r + \frac{1}{r^2} D(\theta_1, \ldots, \theta_{N-1}) u.$$

Thus for radial solutions in B, (4) becomes

$$u_{rr} + \frac{N-1}{r} u_r + g(u(r)) = Q(r), u'(0) = 0, u(1) = 0. \tag{10}$$

The eigenvalues of

$$-u_{rr} - \frac{N-1}{r} u_r = \lambda u, u'(0) = 0, u(1) = 0. \tag{11}$$

are a sequence of positive numbers that converge to $+\infty$ and are simple.

Trying to imitate the proof of Theorem 1.1 for (10) one needs to consider

$$u_{rr} + \frac{N-1}{r} u_r + g(u(r)) = Q(r), u'(0) = 0, u(0) = d, \tag{12}$$

and try to find d such that $u(1, d) = 0$.
This is known as **a shooting argument.** For Theorem 1.1 or the case $\Omega = A$, one is shooting from a regular point to another regular point of an ordinary differential equation. Now we consider the cases:

- Shooting form a singular point to a regular point (equation (10)).
- Shooting from a regular point to a singular point. Here we obtain singular solutions for (10)
- Shooting from a singular point to a singular point. Here we obtain rotationally invariant solutions to (4) in manifolds of revolution.

Shooting from a singular point to a regular point.

In the study of (10) we distinguish three cases.

- Under Serrin, $\lim_{u\to+\infty} \frac{g(u)}{u^p} \in (0,\infty)$ with $1 < p < \frac{N}{N-2}$.
- Between Serrin and Sobolev, $\lim_{u\to+\infty} \frac{g(u)}{u^p} \in (0,\infty)$ with $\frac{N}{N-2} < p < \frac{N+2}{N-2}$, and $\lim_{u\to-\infty} \frac{g(u)}{u^q} \in (0,\infty)$ with $1 < q < \frac{N+2}{N-2}$
- Sub-super critical case $1 < p < \frac{N+2}{N-2} < q$.

Consider

$$u_{rr} + \frac{N-1}{r} u_r + g(u(r)) = Q(r), u'(0) = 0, u(0) = d, \tag{13}$$

From now on $Q = 0$, $g(0) = 0$, and g is monotonically increasing.
Energy dissipates rapidly near 0.

$$\begin{aligned} \left(\frac{(u'(t))^2}{2} + G(u(t)) \right)' &= u'(t)u''(t) + g(u(t))u'(t) \\ &= -\frac{N-1}{r}(u'(t))^2 \\ &\leq 0. \end{aligned} \tag{14}$$

Pohozaev identity

Multiplying (13) by $r^{N-1}u$ and integrating on $[0, r]$ we have

$$(15) \quad r^{N-1}u'u - \int_0^r s^{N-1}(u')^2 ds + \int_0^r s^{N-1}u(s)g(u(s))ds = 0$$

Multiplying (13) by $r^N u'$ and integrating on $[0, r]$ we have

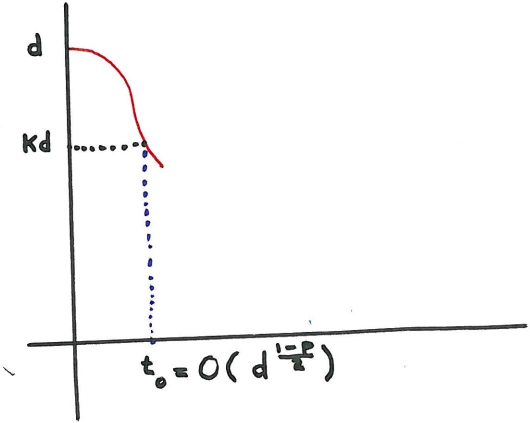
$$\begin{aligned} 0 &= r^N \frac{(u')^2}{2} + (N-1)\int_0^r s^{N-1}(u')^2 ds + \int_0^r s^N (G(u(s)))' ds \\ (16) \qquad &= r^N \frac{(u')^2}{2} + r^N G(u(r)) + (N-1)\int_0^r s^{N-1}(u')^2 ds \\ &\quad - \int_0^r N s^{N-1}(G(u(s)))ds \end{aligned}$$

Multiplying (15) by $N-2$ and adding to (16) we have:

$$\begin{aligned} P(r,d) &:= r^N \frac{(u')^2}{2} + r^N G(u(r)) + \frac{N-2}{2} r^{N-1} u(r) u'(r) \\ &= \int_0^r s^{N-1} \Big(\big(NG(u(s)) - \frac{N-2}{2} u(s) g(u(s)) \big) \Big) ds \end{aligned} \tag{17}$$

If $g(u) = u^{p-1}u$ then

$$NG(u(s)) - \frac{N-2}{2} u(s) g(u(s)) = \left(\frac{N}{p+1} - \frac{N-2}{2} \right) |u|^{p-2}. \tag{18}$$


d
Kd
t₀ = O(d^{(1-p)/2})

Under Serrin

Since $E' = -\frac{N-1}{r}(u'(r))^2 \geq -\frac{2(N-1)}{r}E(r)$, If $p < \frac{N}{N-2}$ then

$$E(t) \geq \frac{E(t_0)t_0^{2(N-1)}}{t^{2(N-1)}} \geq Kd^{p+1}d^{(1-p)(N-1)} \to +\infty$$

as $d \to +\infty$. The proof follows as in Theorem 1.1

Between Serrin and Sobolev

$$\begin{aligned} P(t_0, d) &= t_0^N E(t_0) + \frac{N-2}{2}t^{N-1}u(t_0)u'(t_0) \geq \int_0^{t_0} s^{N-1}u^{p+1}(s)ds \\ &\geq Kt_0^N d^{p+1} \geq Kd^{(N(1-p)/2)+p+1} \\ &\to +\infty \quad \text{as } d \to +\infty. \end{aligned}$$

With $p, q \in (1, (N+2)/(N-2))$, $NG(u) - \frac{N-2}{2} ug(u)$ is bounded below. Hence

$$
\begin{aligned}
E(1) &\geq t_0^N E(t_0) + \frac{N-2}{2}(t^{N-1}(u \cdot u')(t_0) - (u \cdot u')(1)) - M \\
&\geq P(t_0, d) - M - \frac{(u'(1))^2}{4} - \frac{(N-2)^2}{2} u^2(1) \\
&\to +\infty \quad \text{as} \quad d \to +\infty.
\end{aligned}
$$

Since $E(\cdot, d)$ is a decreasing function we have

$$
(19) \qquad \lim_{d \to +\infty} E(t, d) = +\infty \quad \text{uniformly for} \quad t \in [0, 1].
$$

and the existence of infinitely many solutions follows as in Theorem 1.1

The sub-super critical case: $1 < p < (N+2)/(N-2) < q$

$$t^N E(t) + \frac{N-2}{2} t^{N-1} u(t) u'(t) \geq \tag{20}$$
$$t_0^N E(t_0) + \frac{N-2}{2} t^{N-1} u(t_0) u'(t_0) - M. \quad \to +\infty$$

Lemma 1.1 For d large, there exist $t_1 < s_1 < s_2 < t_2$ such that

$$t_1 > t_0, \; u(t_1) = u'(s_1) = u(t_2) = u'(t_2) = 0, \; t_2 = O(t_0) \tag{21}$$

u decreases on (t_0, s_1), and increases on (s_1, s_2).
Lemma 1.2 For $d > 0$ large,

$$t_2 - t_1 \leq C_2 \, d^{(p+1)\left[\frac{1}{q+1} - \frac{1}{2}\right]} \ll d^{(1-p)/2}. \tag{22}$$

From Lemma 2.2 we have the following.

Lemma 1.3 For $d > 0$ large,

$$P(t,d) \geq P(t_0,d) \quad \text{for all} t > t_0. \tag{23}$$

Let k be such that $kd^{(1-p)/2} \ll 1$. Repeating this argument k times, we see that there exists $t_1 < t_2 < \cdots < t_k$ such that $u(t_i) = 0$ for $i = 1, 2, \ldots, k$ and $P(t,d) \geq P(t_0,d)$ for all $t \in [t_0, t_k]$. Hence, $\lim_{d \to +\infty} \theta(1,d) = \infty$. Thus:

THEOREM 1.4 If is g is a sub-super critical nonlinearity then (10) has infinitely many solutions.

SINGULAR SOLUTIONS

Shooting from a regular point to a singular and a regular point

Let $N/(N-2) < p < (N+2)/(N-2)$ and $q > 1$.

THEOREM 2.1. ([A-C-C-2010]) *If g is subcritical or sub-super critical then* (10) *has a countable number of non-degenerate continua of singular radial solutions.*

The case $p = q$ is found in [D-E-R-2002] for $Q = 0$.

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Let

(24)
$$\theta = \frac{2}{1-p}, \quad A = (-\theta(\theta + N - 2))^{\frac{1}{p-1}}, \quad \text{and} \quad \tau_2 \in \left(\theta, \frac{2-N}{2}\right).$$

Since $p > N/(N-2)$, there exists $c > 0$ such that

(25)
$$\frac{(\theta A + \tau_2 c)^2}{2} + \frac{(A+c)^{p+1}}{p+1} + \frac{(N-2)(A+c)(\theta A + \tau_2 c)}{2} = 0.$$

Let $\tau_1 \in (\theta, \tau_2)$ be such that

$$(26) \qquad (N - 2 + \theta)(A\theta + c\tau_2) + (A + c)^p - c\tau_1(\theta - \tau_2) \neq 0.$$

Let $\tilde{b} = b^{1/(\theta-\tau_2)}$.

LEMMA 3.1. Let $I \subset \mathbb{R}$ be a compact interval. For $b \geq 1$ and $a \in I$ there exists a unique function $u(\cdot, a, b) : (0, 1] \to \mathbb{R}$ that satisfies

$$
\begin{aligned}
& u'' + \frac{N-1}{r}u' + g(u) = 0, \quad 0 < r \leq 1, \\
(27) \qquad & u(\tilde{b}, a, b) = (A + c)\tilde{b}^{\theta} + a\tilde{b}^{\tau_1}, \text{ and} \\
& u'(\tilde{b}, a, b) = (\theta A + \tau_2 c)\tilde{b}^{\theta-1} + \tau_1 a\tilde{b}^{\tau_1 - 1}.
\end{aligned}
$$

Moreover, $u(x) = u(\|x\|, a, b) \in H^{1,1}(B)$ satisface $\Delta u + g(u) = 0$ as a distribution, i.e.

$$\int_B (\nabla u \cdot \nabla \varphi - g(u)\varphi)dx = 0$$

for all $\varphi \in C_0^{\infty}(B)$.

$$
P(r) = r^{N-1}\left(rE(r) + \frac{N-2}{2}u(r)u'(r)\right),
$$

$$
\gamma_1 = \left(\frac{N}{p+1} - \frac{N-2}{2}\right) > 0, \quad \gamma_2 = \left(\frac{N}{q+1} - \frac{N-2}{2}\right), \tag{28}
$$

$$
\text{y} \quad \Gamma(u) = \begin{cases} \gamma_1 u^{p+1} & \text{para} \quad u \geq 0 \\ \gamma_2 |u|^{q+1} & \text{para} \quad u \leq 0. \end{cases}
$$

Lema 2.2. *If g is subcritical or sub-super subcritical, then there exists $b_1 \in \mathbb{R}$ such that for $b > b_1$ the solutions to (27) are singular and have no zero in $(0, \tilde{b})$.*

Proof. It is based on the fact that there is $m \in (0,1)$ such that $P(m\tilde{b}) < 0$. Hence $P(t) < 0$ for all $t \in (0, m\tilde{b})$.

Lema 2.3. *There exist $m_1 > 1$, $b_2 > b_1$, and $K > 0$ such that*

$$P(m_1\tilde{b}) \geq Ku^{p+1}(\tilde{b})\tilde{b}^N, \tag{29}$$

for $b > b_2$, uniformly for $a \in I$. In particular, $P(m_1\tilde{b}) \to +\infty$ as $b \to +\infty$.

Lema 2.4. *There exists* $b_2 > b_1$ *such that if* $\hat{b} > b_1$, $\hat{a} \in I$, $u(\cdot, \hat{a}, \hat{b})$ *is a solution to* (27) *with* $u(1, \hat{a}, \hat{b}) = 0$, *then there exists* $\delta > 0$ *and continuous functions* $\alpha : (-\delta, \delta) \to \mathbb{R}$ *and* $\beta : (-\delta, \delta) \to \mathbb{R}$ *such that* $u(\cdot, \hat{a} + \alpha(t), \hat{b} + \beta(t))$ *satisfies* (10) *and* $u(\cdot, \hat{a} + \alpha(t), \hat{b} + \beta(t)) \neq u(\cdot, \hat{a} + \alpha(s), \hat{b} + \beta(s))$ *for* $s \neq t$. *Hence,* (10) *has a a non-degenerate continuum of radial singular solutions.*

Let $\phi(r,a,b)$ be a differentiable function such that

$$
\begin{aligned}
u(r,a,b) &= \rho(r,a,b)\cos(\phi(r,a,b)), \\
u'(r,a,b) &= -\rho(r,a,b)\sin(\phi(r,a,b)), \\
\phi(\tilde{b},a,b) &= \tan^{-1}\left(\frac{-u'(\tilde{b},a,b)}{u(\tilde{b},a,b)}\right)
\end{aligned}
\tag{30}
$$

By Lemma 2.3,

$$
\lim_{b\to+\infty}\phi(1,a,b) = +\infty, \tag{31}
$$

uniformly for $a \in I$. Hence, for each $a \in I$ there exists a positive integrer $J(a)$ and a sequence $\{b_j(a)\}_{j\geq J(a)}$ such that $\phi(1,a,b_j(a)) = j\pi + (\pi/2)$. By the continuous dependence on parameters of solutions to initial value problems, the functions b_j are continuous. By Lemma 2.4, $\{u(\cdot,a,b_j(a)); a \in I\}$ defines a non-degenerate continuum of singular solutions to (10).

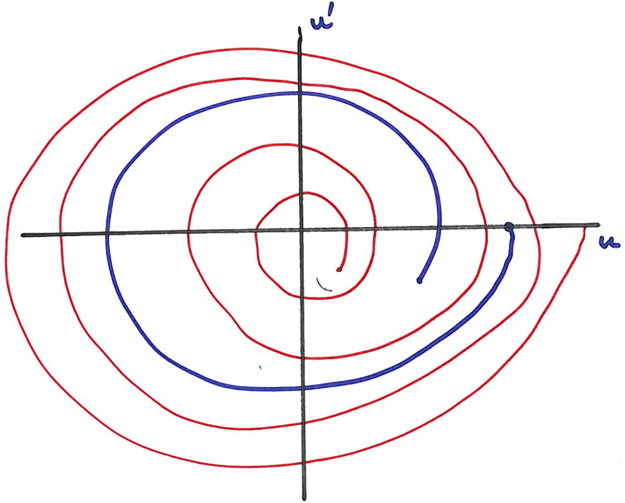
These ideas extend to quasilinear equations. For example the can be taken to Kirchhoff's equation as follows.

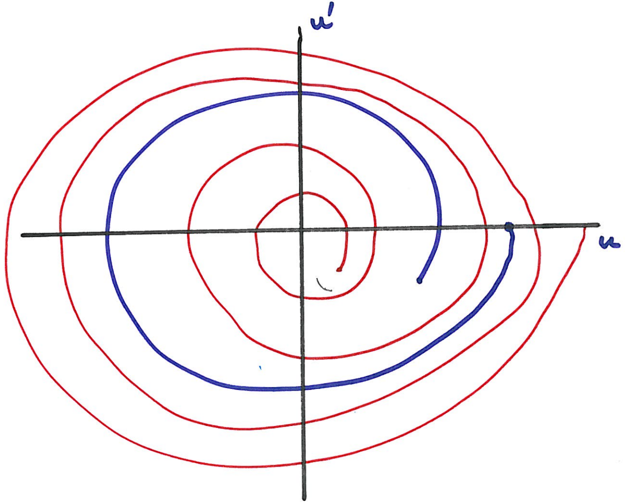
THEOREM 2.2 (Joint work with ShuZhi Song) *If g is supercubic and subcritical* $(3 < p, q < (N+2)/(N-2)\ )$ *then*

(32)

$$(a + b\int_B |\nabla u|^2)\Delta u(x) + g(u(x)) = 0,\ x \in B \quad u(x) = 0,\ x \in \partial B$$

has infinitely many radially symmetric solutions.


u'
u


u'
u

$$
\begin{aligned}
\left(\frac{-tu'}{u}\right)' &= \frac{(-tu''-u')u+t(u')^2}{u^2} \\
&= \frac{-t(-\frac{n}{t}u'-u^p)u-uu'+t(u')^2}{u^2} \\
&= \frac{(\overbrace{n-1}^{N-2})uu'+tu^{p+1}+t(u')^2}{u^2} \\
&= \frac{2t^{-n}\int_0^t s^n\gamma u^{p+1}-2\frac{tu^{p+1}}{p+1}+tu^{p+1}}{u^2} \\
&= \frac{2t^{-n}\int_0^t s^n\gamma u^{p+1}+t\left(\frac{p-1}{p+1}\right)u^{p+1}}{u^2} \\
&\geq \frac{2\gamma}{t}\left(\frac{-tu'(t)}{u(t)}\right),
\end{aligned}
\tag{33}
$$

A LAPLACE-BELTRAMI EQUATION
shooting from singularity to singularity

Joint work with I. Ventura.

Let M be a codimension 1 manifold of revolution in $\mathbb{R}^N$ of class C^3 containing its axis of rotation. We assume M to be boundaryless, connected, and compact. Without loss of generality we may assume that M revolves around the $x_N = z$ axis. Also we may denote by $P_- = (0, \dots, -1)$ and $P_+ = (0, \dots, 1)$ the points of intersection of M with its axis of revolution. We study the existence of rotationally symmetric solutions to

$$\Delta_M u + f(u) = 0 \qquad \text{on } M, \tag{34}$$

where Δ_M is the *Laplace-Beltrami operator* on M and $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that

$$\lim_{|u| \to +\infty} \frac{f(u)}{u} = +\infty. \tag{35}$$

Let $d : M \times M \to [0, \infty)$ denote the geodesic distance in M and $a = \max\{d(P_{-1}, x); x \in M\} = d(P_-, P_+)$. Hence there exist differentiable functions $G, z : [0, a] \to [0, \infty)$ such that $G(t) = 0$ if and only if $t \in \{0, a\}$, and $M = \{(\theta, z(r)); \theta = G(r), r \in [0, a]\}$. Moreover,

$$G'(0) = -G'(a) = 1, \quad z(0) = -1, \text{ and } z(a) = 1. \tag{36}$$

For $u : M \to \mathbb{R}$ rotationally symmetric the equation (34) is equivalent to

$$\begin{gathered} u_{tt} + (N-2)\frac{G'(t)}{G(t)}u_t + f(u(t)) = 0 \\ u(0),\ u(a) \in \mathbb{R}, \quad u'(0) = u'(a) = 0. \end{gathered} \tag{37}$$

We assume that there exists $m_1 > 0$ sych that

$$(38) \qquad f \quad \text{increases on} \quad (-\infty, -m_1) \cup (m_1, +\infty).$$

Also we assume that there exist $p_1, p_2 \in (1, (N+1)/(N-3))$ such that

$$(39) \qquad \begin{aligned} &\lim_{u \to +\infty} \frac{f(u)}{|u|^{p_1 - 1} u} := f_\infty \in (0, \infty), \quad \text{and} \\ &\lim_{u \to -\infty} \frac{f(u)}{|u|^{p_2 - 1} u} := f_{-\infty} \in (0, \infty). \end{aligned}$$

The main result is.
THEOREM 3.1: *The equation* (34) *has infinitely many rotationally symmetric solutions.*

In [A.C. and E.M. Fischer, *Infinitely Many Rotationally Symmetric Solutions to a Class of Semilinear Laplace-Beltrami Equations on Spheres*, Canadian Mathematical Bulletin, Vol 58 (2015), no. 4, 723-729.] when M is a sphere it was proved that (34) has infinitely many rotationally symmetric solutions also symmetric with respect to the equator. That case becomes

$$\begin{aligned} u_{rr} + (N-2)\frac{\cos(r)}{\sin(r)} u_r + f(u) = 0, \quad r \in [0, \pi/2] \\ u'(0) = 0, \quad u'(\pi/2) = 0. \end{aligned} \tag{40}$$

This equation is singular at at 0 but not at $\pi/2$. That is the problem is very similar to the problem to finding radial solutions in a ball.

Let us consider the initial value problem

$$
\begin{aligned}
&u_{tt} + (N-2)\frac{G'(t)}{G(t)}u_t + f(u(t)) = 0 \\
&u(0) = d, \;\; u'(0) = 0.
\end{aligned}
\tag{41}
$$

Multiplying (41) by $G^{N-2}(t)$ we have

$$
(G^{N-2}(t)u_t)_t + G^{N-2}(t)f(u) = 0, \quad t \in [0, a] \tag{42}
$$

For $\beta(t) = G^{N-2}(t) \int_{a/2}^{t} G^{2-N}(s)ds$,

$$\lim_{t \to 0} \beta(t) = 0. \tag{43}$$

Let $u(t, d) := u(t)$ be a solution to (41). Now the Pohozaev identity is

$$\begin{aligned} P(t,u) := {} & G^{N-2}(t)\beta(t)\left[\frac{(u'(t))^2}{2} + F(u(t))\right] \\ & - \frac{G^{N-2}(t)}{2}u(t)u'(t) \\ = {} & \int_0^t ((\beta(s)G^{N-2}(s))'F(u(s)) \\ & + \frac{G^{N-2}(s)}{2}u(s)f(u(s)))ds \end{aligned} \tag{44}$$

From (44) and (39) we deduce:

Lemma 1: *There exists $D > 0$ such that if $d > D$ then*

$$E(t,d) := \frac{(u'(t,d))^2}{2} + F(u(t,d)) > 0, \textit{ for all } \; t \in [0,a].$$

Moreover,

(45)
$$\lim_{|d| \to +\infty} E(t,d) = +\infty,$$

uniformly in t

Hence there exists a continuous function $\varphi : [0,a) \times [D,\infty) \to \mathbb{R}$ such that

$$u(t,d) = (u^2(t,d) + (u_t(t,d))^2) \cos(\varphi(t,d),$$

$$u_t(t,d) = -(u^2(t,d) + (u_t(t,d))^2) \sin(\varphi(t,d)), \textit{ and}$$

(46)
$$\lim_{d \to +\infty} \varphi(t,d) = +\infty, \textit{ for each } t \in (0,a).$$

Lemma 2: *For each* $d \in \mathbb{R}$ *there exists* $M(d)$ *such that the solution to* (41) *satisfies* $|P(r,d)| \leq M(d)$ *for all* $r \in (0,a)$.
Lemma 3: *If* v *is bounded solution to* (41) *then* $\lim_{r \to a-} v'(r) = 0$.
Lemma 4: *For each* $d \geq D$, $u(\cdot, d)$ *has finitely many zeroes in* $[0,a)$.
Lemma 5: If $\lim_{t \to a} v(t, \hat{d}) = +\infty$, the there exists $\eta > 0$ such that if $|d - \hat{d}| < \eta$ then $\lim_{t \to a^-} v(t,d) = +\infty$.

Proof of Theorem A. Let $d_0 > D$. Suppose that $u(\cdot, d_0)$ is not a solution to (37), i.e. does not define a solution to (34). By Lemmas 3 and , we may assume w.l.o.g. that $\lim_{t \to a-} u(t) = +\infty$ and, for some, there exists $\epsilon > 0$ such that $u'(t, d_0) > 0$. Let $\tilde{d} = \sup\{d \geq d_0; u(t, d)$ is monotonically increasing on $[t_1, a)$ and $\lim_{t \to a-} v(t, d) = +\infty\}$. Due to Lemma 5, $\tilde{d} > d_1$. By (46) $\tilde{d} < +\infty$. If $u(t, \tilde{d}) = 0$ for some $t \in (t_1, a)$, by continuous dependence there is a sequence $d_j \to \tilde{d}$, $d_j < \tilde{d}$ such that $u(t, d_j) \to 0$. Since this contradicts that $u(\cdot, d_j)$ increases on $[t_1, a)$, $u(\cdot, \tilde{d})$ is bounded. This and Lemma 3, prove that $u(\cdot, \tilde{d})$ is a solution to (37). Assuming that $u(\cdot, \tilde{d} + 1)$ is not a solution to (37) we find a second solution of the form $u(\cdot, \tilde{d}_1)$ with $\tilde{d}_1 \geq \tilde{+}1$. Iterating this process we have a sequence $\{\tilde{d}_k\}_k$ converging to $+\infty$.

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