A PRIORI BOUNDS and CRITICALITY

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AIM OF THIS TALK

We present some of my results motivated by the work of Brezis. The talk consists of three parts:

(i) Some history with facts that influenced some of my research.

(ii) Results on a priori estimates of solutions of Semilinear Elliptic problems, starting with the work of Brezis-Turner using Hardy inequalities, and continuing with the use of blow-up by Gidas-Spruck and moving planes by deF-Lions-Nussbaum.

(iii) The theorem of Brezis-Nirenberg on a critical problem, obtaining its solvability by a linear perturbation.
And a recent result of deF-Gossez-Quorin-Ubilla using a gradient term for perturbation, also in the case of the p-Laplacian.

SYMPOSIA ON NONLINEAR FUNCTIONAL ANALYSIS

In 1970 in a meeting of the American Mathematical Society in Chicago many results in the theory of nonlinear PDE, as well as in Functional Analysis were presented and collected in a Proceedings edited by Felix Browder.

Haim Brezis participated presenting the paper
On Some Degenerate Nonlinear Parabolic Equations.

I also participated presenting the paper
On the Extension of Contractions on Normed Spaces
a joint work with Les Karlovitz.

MY RESEARCH until 1970

(i) Results with Felix Browder and Chaitan Gupta on Monotone Nonlinear Operators.

(ii) Results with Les Karlovitz on the Geometry of Normed Spaces, including the following:

Theorem

Let E be a normed space with dimension ≥ 3. Consider the radial projection $T : E \to E$ on the unit ball, i.e.

$$Tu = u \text{ if } \|u\| \leq 1, \quad Tu = \frac{u}{\| u \|} \text{ if } \|u\| > 1 \tag{1}$$

Assume that

$$\| Tu - Tv \| \leq \| u - v \|, \forall u, v \in E.$$

Then E is a Hilbert space.

CHARACTERIZATIONS OF HILBERT SPACES

Brezis presents 4 characterizations of Hilbert spaces in his book *FUNCTIONAL ANALYSIS, SOBOLEV SPACES and PARTIAL DIFFERENTIAL EQUATIONS.*
By *characterization* it is meant that for a given normed space E with norm $\|.\|$ it is possible to define an inner product $(,)$ s.t.

$$\|u\| = (u, u)^{\frac{1}{2}}, \forall u \in E$$

Besides the one above he presents 3 additional ones, namely
(i) (Frechet-von Neumann- Jordan) Assume the norm in E satisfies the parallellogram law. Then E is a Hilbert space.
(ii) (Kakutani) Assume E is a normed space of dimension $\geqslant 3$. Assume that every subspace F of dimension 2 has a projection operator $P : E \to F$ with norm $\leqslant 1$. Then E is a Hilbert space.
(iii)(Lindenstrauss-Tzafriri, 1971). Assume E is a Banach Space s.t. every closed subspace has a complement. Then E is Hilbertizable, i.e., there is an equivalent Hilbert norm.

TWO BOOKS of BREZIS

(i) Analyse fonctionnelle-Theorie et applications , Masson 1983.

(ii)Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer 2011.

At Rutgers, 2012

A PRIORI BOUNDS

The solvability of the equation

$$-\Delta u = f(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{2}$$

as well as more general elliptic equations and systems has been the object of intensive research.
Two methods have been used depending on the special case: Variational and Topological.

We discuss here the Topological method, via Leray-Schauder Degree, through Krasnoselskii Theorem.

KRASNOSELSKII THEOREM

Theorem

Let $\mathcal{C}$ be a cone in a Banach space X and $T : \mathcal{C} \to \mathcal{C}$ a compact mapping such that $T(0) = 0$. Assume that there are real numbers $0 < r < R$ and $t > 0$ such that
(i) $x \neq tTx$ for $0 \leq t \leq 1$ and $x \in \mathcal{C}, ||x|| = r$, *and*
(ii) There exists a compact mapping $H : \overline{B}_R \times [0, \infty) \to \mathcal{C}$ *(where* $B_\rho = \{x \in \mathcal{C} : ||x|| < \rho\}$*) such that*
(a) $H(x, 0) = Tx$ for $||x|| = R$,
(b) $H(x, t) \neq x$ for $||x|| = R$ *and* $t \geq 0$
(c) $H(x, t) = x$ *has no solution* $x \in \overline{B}_R$ *for* $t \geq t_0$
Then

$$i_c(T, B_r) = 1,\ i_c(T, B_R) = 0,\ i_c(T, U) = -1,$$

where $U = \{x \in \mathcal{C} : r < ||x|| < R\}$*, and* i_c *denotes the Leray-Schauder index. As a consequence **T has a fixed point in U.***

Applying KRASNOSELSKII

When applying this result, the main difficulty arises in the verification of condition
(b) $H(x,t) \neq x \ \text{ for } \ ||x|| = R$ and $t \geq 0$
which is nothing more than an *a priori bound* on the solutions of the equation. It is well known that the existence of a priori bounds depends on the growth of the function f as u goes to infinity.
Let us comment on it.

A PRIORI BOUNDS-THREE TECHNIQUES

A priori bounds for positive solutions of superlinear elliptic equations (the scalar case), namely

$$-\Delta u = f(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{3}$$

was first considered by Brézis-Turner in 1977 using an inequality due to Hardy.
The same technique was used by Clement-deF-Mitidieri in 1996 to obtain a *priori bounds for solutions of systems.*
Two other methods have been used to get the a priori bound:
(i) Blow-up leading to Liouville theorems by Gidas-Spruck in 1981.
(ii) Moving Planes by deF-Lions-Nussbaum in 1982, obtaining bounds for nonlinearities $f(x,u)$ with faster growth at infinity.

BREZIS-TURNER THEOREM

Theorem

Let $f(x,u)$ be a continuous nonnegative function defined on $\overline{\Omega} \times [0,\infty)$ and suppose

$$lim_{u\to\infty} \frac{f(x,u)}{u} > \lambda_1, \text{ uniformly for } x \in \overline{\Omega}$$

$$lim_{u\to\infty} \frac{f(x,u)}{u^\beta} = 0, \text{uniformly for } x \in \overline{\Omega}, \text{ where } \beta = \frac{N+1}{N-1}.$$

Then there is a constant K, independent of $t \geq 0$, s.t.

$$\|u\|_{L^\infty} \leq K,$$

for all nonnegative solution $u \in H_0^1$ of

$$-\Delta u = f(x,u) + t \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{4}$$

HARDY INEQUALITY

As said above, the proof relies on an inequality of Hardy, namely

$$\|\frac{u}{\varphi_1}\|_{L^q} \leq C\|Du\|_{L^q}, \ \forall \ u \in W_0^{1,q}.$$

Here $q > 1$ and a φ_1 is the eigenfunction associated to the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. In Brezis-Turner they proceed with an interpolation of the Hardy inequality ($q = 2$) with the Sobolev inequality

$$\|u\|_{2^*} \leq C\|Du\|_{L^2}, \ \forall \ u \in H_0^1,$$

obtaining the inequality below

$$\|\frac{u}{\varphi_1^\tau}\|_{L^q} \leq C\|Du\|_{L^2}, \ \forall \ u \in H_0^1,$$

where $\frac{1}{q} = \frac{1}{2} - \frac{1-\tau}{N}$.

On the Proof of Brezis-Turner

For the purpose of proving the estimate of Brezis-Turner one needs the following result which follows from the inequalities above.

Proposition

Let $r_0 \in (1, \infty]$, $r_1 \in [1, \infty)$ *and* $u \in L^{r_0}(\Omega) \bigcap W_0^{1,r_1}$. *Then for all* $\tau \in [0, 1]$. *we have*

$$\frac{u}{\varphi_1^\tau} \in L^r(\Omega), \text{ where } \frac{1}{r} = \frac{1-\tau}{r_0} + \frac{\tau}{r_1}.$$

Moreover,

$$\|\frac{u}{\varphi_1^\tau}\|_{L^r} \leq C\|u\|_{L^{r_0}}^{1-\tau}\|u\|_{W^{1,r_1}}^{\tau},$$

where the constant C *depends only on* τ, r_0 *and* r_1.

BLOW-UP METHOD

The blow-up technique, used by Gidas-Spruck, consists in assuming, by contradiction, that there is no apriori bound for the solutions of the differential equation. This implies the existence of a sequence of solutions

$$u_n(x) s.t. \|u_n\|_{L^\infty} \to \infty$$

Using this sequence one constructs a sequence of functions $v_n(x)$ defined in the whole R^N that converges to a function v, solution of a differential equation in R^N.
And here comes the interesting point: prove (under the original hypotheses) that v cannot be the solution of this problem.
Some problems will be solved by this method, if there are Liouville Theorems available.

LIOUVILLE THEOREMS

The classical Liouville Theorem from Function Theory says that every bounded entire function is constant. In terms of a differential equation one has: if $(\partial/\partial\overline{z})f(z) = 0$ and $|f(z)| \leq C$ for all $z \in \mathbb{C}$ then $f(z) = const$. Hence, results with similar contents are nowadays called Liouville Theorems. For instance, a superharmonic function defined in the whole plane R^2, which is bounded below, is constant. Let us see the case of the equation

$$-\Delta u = u^p \tag{5}$$

If the equation is considered in R^2, then a non-negative solution is necessarily identically zero. The case when R^N, $N \geq 3$ is quite distinct.

LIOUVILLE THEOREMS, cont.

Theorem

Let u be a non-negative C^2 function defined in the whole of R^N, $N \geq 3$, such that

$$-\Delta u = u^p \tag{6}$$

holds in R^N. If $0 < p < (N+2)/(N-2)$, then $u \equiv 0$.

This result was proved by Gidas-Spruck for $1 < p < (N+2)/(N-2)$. A simpler proof using the method of moving parallel planes was given by Chen-Li.

LIOUVILLE - in half space

Theorem

Let $u \in C^2(R^N_+) \cap C^0(\overline{R^N_+})$ *be a non- negative function such that*

$$\begin{cases} -\Delta u = u^p \ \text{ in } R^N_+ \\ u(x', 0) = 0 \end{cases} \tag{7}$$

If $1 < p \leq (N+2)/(N-2)$ *then* $u \equiv 0$*.*

It is remarkable that in the case of the half-space the exponent $(N+2)/(N-2)$ is not the right one for theorems of Liouville type. Indeed, Dancer has proved the following result:

Theorem

Let $u \in C^2(R^N_+) \cap C^0(\overline{R^N_+})$ *be a non- negative bounded solution of the above problem. If* $1 < p < (N+1)/(N-3)$ *for* $N \geq 4$ *and* $1 < p < \infty$ *for* $N = 3$*, then* $u \equiv 0$*.*

INSTANTONS

If $p = (N + 2)/(N - 2), N \geq 3,$ then the equation

$$-\Delta u = u^p \tag{8}$$

has a two-parameter family of bounded positive solutions:

$$U_{\varepsilon,x_0}(x) = \left[\frac{\varepsilon\sqrt{N(N-2)}}{\varepsilon^2 + |x - x_0|^2}\right]^{\frac{N-2}{2}},$$

which are called instantons.

On the BREZIS-NIRENBERG CRITICAL PROBLEM

The problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

for $N \geq 3$ has no solution in a starshaped bounded domain $\Omega \subset R^N$.

This follows from Pohozaev identity.

POHOZAEV IDENTITY

Consider the equation

$$-\Delta u = f(x,u) \ \text{ in } \ \Omega \tag{9}$$

Using the multiplier $x.\nabla u$, one obtains

$$-\frac{1}{2}\int_{\partial\Omega} |\nabla u|^2(x.\nu) \ -(\frac{N}{2}-1)\int_{\Omega} |\nabla u|^2 =$$

$$\int_{\partial\Omega} F(x,u)(x,\nu) \ -N\int_{\Omega} F(x,u) - \int_{\Omega}\sum F_{x_j}.x_j$$

where $F(x,u) = \int_0^u f(x,u)du$

Applying POHOZAEV IDENTITY

Using the boundary condition
$u = 0 \text{ on } \partial\Omega$ and supposing Ω is star-shaped one obtains

$$N \int_\Omega F(x,u) + \int \sum F_{x_j}.x_j - (\frac{N}{2} - 1) \int_\Omega |\nabla u|^2 > 0$$

Applying it to $f(x,u) = |u|^p$ one concludes that

$$p < \frac{N+2}{N-2}$$

On BREZIS-NIRENBERG CRITICAL PROBLEM-cont

Brezis and Nirenberg proved that existence could be recovered if one adds a "perturbation", namely, the problem below has a solution if $\lambda \in (0, \lambda_1)$ in case of $N \geq 4$

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} + \lambda u & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

In the case $N = 3$, there is $\lambda_0 \in (0, \lambda_1)$ s.t. the above problem has a solution if $\lambda \in (\lambda_0, \lambda_1)$.

THE CASE OF THE p-LAPLACIAN CRITICAL

Recall $\Delta_p := div(|\nabla u|^{p-2}|\nabla u|)$
$p* = \dfrac{pN}{N-p}$
Garcia-Peral, Egnell and Guedda-Veron proved a similar result of the case of the p-Laplacian.

In deF-Gossez-Quorin-Ubilla we have another type of perturbation that still gives the existence of solution in the critical case. Namely, the following problem has a solution

$$(P) \qquad \begin{cases} -\Delta_p u = |\nabla u|^p + u^{p*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This is consequence of Theorem A below.

EQUATIONS WITH NATURAL GRADIENT

Consider the problem

$$(P) \qquad \begin{cases} -\Delta_p u = g(u)|\nabla u|^p + f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $1 < p < \infty$ and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. In the case $p = 2$ and $g \equiv 1$, Kazdan-Kramer used the change of variables $v = e^u - 1$ and transformed the quasilinear problem (P) into the semilinear one

$$\begin{cases} -\Delta v = (1+v)f(x, \log(1+v)) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

This can be extended to the general case of (P) in the way we explain next.

CHANGE OF VARIABLE

Consider the change of variable

$$v = A(u)$$

where $A : [0, \infty) \to [0, \infty)$ is a $\mathcal{C}^2$ diffeomorphism with $A(0) = 0$, $A(\infty) = \infty$ and $A' > 0$.

Clearly $u \in C^1(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$ if and only if $v \in C^1(\overline{\Omega})$ with $v = 0$ on $\partial\Omega$.

So, it follows that u solves (P) if and only if v satisfies

$$\Delta_p v = \left[(p-1)A'(u)^{p-2}A''(u) - g(u)A'(u)^{p-1}\right]|\nabla u|^p - A'(u)^{p-1}f(x,u).$$

The gradient term will disappear if A satisfies

$$(p-1)A''(u) = g(u)A'(u)$$

The change of variable, cont.

As said the gradient term will disappear if A satisfies

$$(p-1)A''(u) = g(u)A'(u).$$

Integrating we obtain

$$A(s) := \int_0^s e^{\frac{G(t)}{p-1}}\, dt \tag{10}$$

where $G(s) = \int_0^s g(t)\, dt$.

THE EQUIVALENT EQUATION WITHOUT GRADIENT

With this choice for A, problem (P) for u is equivalent to problem (Q) for v:

$$(Q) \qquad \begin{cases} -\Delta_p v = h(x, v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$h(x, s) := e^{G(A^{-1}(s))} f(x, A^{-1}(s)). \tag{11}$$

Next we present a result on the solvability of equation (Q), and consequently of (P).

SUBCRITICALITY and SUPERLINEARITY

Theorem A (dF-Gossez-Quorin-Ubilla) Assume (H_{SC}), (H_{λ_1}), (H_m) and (H_∞). Then problem (P) has at least one solution.

The first condition is a *subcriticality condition* and the second one is a *superlinearity at* 0.

(H_{SC}) There exists $r < p^*$ such that

$$\lim_{s \to \infty} \frac{f(x,s)e^{G(s)}}{\left(\int_0^s e^{\frac{G(t)}{p-1}}\,dt\right)^{r-1}} = 0$$

uniformly with respect to $x \in \Omega$.

(H_{λ_1}) $\limsup_{s \to 0} \dfrac{f(x,s)}{s^{p-1}} < \lambda_1$ uniformly with respect to $x \in \Omega$.

EXAMPLES OF EQUATIONS COVERED by THEOREM A

(i) $-\Delta_p u = \frac{p-1}{u+1}|\nabla u|^p + u^{p-1}(\log(u+1))^q$, where $q > 0$;

(ii) $-\Delta_p u = C|\nabla u|^p + u^{r-1}$, where $C > 0$, and $p < r$;

(iii) $-\Delta_p u = C|\nabla u|^p + (\log(u+1))^{r-1}$, where $C > 0$, and $p < r$.