Global mild solutions for a nonautonomous 2D Navier-Stokes equations with impulses at variable times

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XII ENAMA

The nonautonomous 2D Navier-Stokes equations with impulses

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$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f(t,u), & (t,x) \in (0,+\infty) \times \Omega, \\ \mathrm{div}\, u = 0, & (t,x) \in (0,+\infty) \times \Omega, \\ u = 0, & (t,x) \in (0,+\infty) \times \partial\Omega, \\ u(0,\cdot) = u_0 \in \mathbf{V}, & \\ I : M \subset \mathbf{V} \to \mathbf{V}. & \end{cases}$$

E. M. Bonotto, J. G. Mesquita, R. P. Silva, Global mild solutions for a nonautonomous 2D Navier-Stokes equations with impulses at variable times, *Journal of Mathematical Fluid Mechanics*, (2018), 801-818.

Impulses

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Many real world problems are subject to abrupt external forces which can change completely their dynamics.

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Billiard-type systems can be modeled by differential systems with impulses acting on the first derivatives of the solutions.

Medicine Intake

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$0 = t_1 < t_2 < \ldots < t_r$: instants of applications of the drug.

Impulses that vary in time are more attractive due to their complexity, applicability in real world problems, and, moreover, the impulses may occur due to conditions on the phase space and not in time.

V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

Autonomous Impulsive Systems

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S. K. Kaul, On impulsive semidynamical systems, J. Math. Anal. Appl., 150 (1990), 120- 128.

K. Ciesielski, On semicontinuity in impulsive dynamical systems, Bull. Polish Acad. Sci. Math., 52 (2004), 71-80.

• Let (X, d) be a metric space. The pair (X, π) is a dynamical system on X if the mapping $\pi : X \times \mathbb{R} \to X$ satisfies:

(i) $\pi(x, 0) = x$, for all $x \in X$;

(ii) $\pi(x, t + s) = \pi(\pi(x, t), s)$, for all $t, s \in \mathbb{R}$ and all $x \in X$;

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• An **impulsive dynamical system** (X, π, M, I) consists of a dynamical system (X, π), a nonempty closed subset $M \subseteq X$ such that for every $x \in M$ there exists $\epsilon_x > 0$ such that

$$\bigcup_{t \in (-\epsilon_x, 0)} \{\pi(t)x\} \cap M = \varnothing \quad \text{and} \quad \bigcup_{t \in (0, \epsilon_x)} \{\pi(t)x\} \cap M = \varnothing,$$

and a continuous function $I \colon M \to X$.

- Let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and consider the autonomous differential equation $\dot{x} = f(x)$ which defines a dynamical system $(\mathbb{R}^n, \pi)$.

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- Let $M \subset \mathbb{R}^n$ be a hypersurface in $\mathbb{R}^n$ of class C^k, $k \geqslant 1$, satisfying the following transversality condition:

$$\text{for each } \; p \in M \; \text{ we have } \; \langle \vec{n}_p, f(p) \rangle \neq 0,$$

where $\vec{n}_p$ denotes the normal vector of M at p, and $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^n$.

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E. M. Bonotto, M. C. Bortolan, T. Caraballo and R. Collegari, Impulsive surfaces on dynamical systems, Acta Math. Hungarica, 209-216, (2016).

The abstract nonautonomous Navier-Stokes equation

$$\begin{cases} \dfrac{du}{dt} + Au + \mathscr{B}(\sigma(t,\omega))(u,u) = \mathscr{F}(t,\sigma(t,\omega),u), \qquad t \in J, \\ u(0) = u_0 \in V. \end{cases}$$

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D. N. Cheban, Global attractors of non-autonomous dissipative dynamical systems, Interdiscip. Math. Sci., vol. 1, World Scientific Publishing, Hackensack, NJ, 2004.

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$-A$ generates an analytic semigroup $\{e^{-At}\}_{t\geq 0} \subset \mathscr{L}(H)$ satisfying

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- $(V, (\cdot,\cdot)_V)$ and $(E, (\cdot,\cdot)_E)$ are separable Hilbert spaces such that

$$V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} E,$$

$$e^{-At} \in \mathscr{L}(E, V), \quad t > 0,$$

$$\|e^{-At}\|_{\mathscr{L}(E,V)} \leq K_1 t^{-\alpha_1} e^{-at}, \quad 0 < \alpha_1 < 1,\ K_1 > 0, t > 0,$$

$$\|e^{-At}\|_{\mathscr{L}(V,V)} \leq K_2 e^{-at}, \quad K_2 > 0, \quad t > 0.$$

- $\mathscr{L}^2(V, E)$ denotes the space of all continuous bilinear operators $\mathcal{B} : V \times V \to E$ equipped with the norm

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• $(\mathcal{M}, d)$ is a metric space and $(\mathcal{M}, \sigma)$ is a dynamical system on $\mathcal{M}$, i.e., a continuous map $\sigma : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ which satisfies:

i) $\sigma(0, \omega) = \omega, \quad \omega \in \mathcal{M};$

ii) $\sigma(t + s, \omega) = \sigma(s, \sigma(t, \omega)), \quad t, s \in \mathbb{R}, \omega \in \mathcal{M}.$

- $\mathscr{B} : \mathcal{M} \to \mathscr{L}^2(V, E)$ is a continuous map such that

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$$\mathrm{Re}\,(\mathscr{B}(\omega)(u,v), w)_E = -\mathrm{Re}\,(\mathscr{B}(\omega)(u,w), v)_E,$$

which implies the orthogonality condition

$$\mathrm{Re}\,(\mathscr{B}(\omega)(u,v), v)_E = 0, \quad u, v \in V,\, \omega \in \mathcal{M}.$$

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(C3) $\exists\, M \in B(\mathbb{R}, \mathbb{R}_+)$, such that for all interval $[a, b] \subset J$,

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for all $\phi \in L^1[a, b]$, $\omega \in \mathcal{M}$ and $u \in V$.

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$$\leq \int_a^b L(s)|\phi(s)|(d(\omega_1, \omega_2) + \|u_1 - u_2\|_V) ds$$

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(C5) $\|\mathscr{F}\|_1 = \sup\{\|\mathscr{F}(t, \omega, u)\|_E : \, t \in J, \omega \in \mathcal{M}, u \in V\} < \infty$.

$$\begin{cases} \dfrac{du}{dt} + Au + \mathscr{B}(\sigma(t,\omega))(u,u) = \mathscr{F}(t,\sigma(t,\omega),u), & t \in J, \\ u(0) = u_0 \in V. \end{cases} \tag{1}$$

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A function $u : J \to V$ is a **mild solution** of (1) if u satisfies the following integral equation:

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A function $u : J \to V$ is a **mild solution** of (1) if u satisfies the following integral equation:

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u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}\mathscr{F}(s,\sigma(s,\omega),u(s))ds
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- \int_0^t e^{-A(t-s)}\mathscr{B}(\sigma(s,\omega))(u(s),u(s))ds,
$$

for all $t \in J$.

Theorem 1

Let $u_0 \in V$ *and* $r > 0$. *Then there exist positive numbers* $\delta = \delta(u_0, r) > 0$, $T = T(u_0, r) > 0$ *and a function* $\varphi : [0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M} \to V$ $(B(u_0, \delta) \subset V)$ *satisfying:*

i) $\varphi(0, u_0, \omega) = u_0$, *for all* $\omega \in \mathcal{M}$;

ii) $\|\varphi(t, u, \omega) - u_0\|_V \leq r, \quad (t, u, \omega) \in [0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M}$;

iii) $\varphi \in C([0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M}, \overline{B(u_0, r)})$.

Moreover, the function $u : [0, T] \to V$ *defined by* $u(t) = \varphi(t, u_0, \omega)$ *is the unique mild solution of the system* (1).

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$$S\varphi(t, u, \omega) = e^{-At}u + \int_0^t e^{-A(t-s)} g(s, \omega, \varphi(s))ds,$$

where

$$\varphi(s) = \varphi(s, u, \omega)$$

and

$$g(s, \omega, \varphi(s)) = -\mathscr{B}(\sigma(s, \omega))(\varphi(s), \varphi(s)) + \mathscr{F}(s, \sigma(s, \omega), \varphi(s)).$$

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- $\Gamma(\delta, T, r) = C([0, T] \times \overline{B(u_0, \delta)} \times \mathcal{M}, \overline{B(u_0, r)})$

$$S : \Gamma(\delta, T, r) \to \Gamma(\delta, T, r).$$

Lemma 2

The inequality

$$\|\varphi(t,u_0,\omega)\|_V \leq \max\left\{\|u_0\|_V, \frac{\|\mathscr{F}\|_1}{a}\right\}$$

holds for all $t \in [0, \alpha_{(u_0,\omega)})$, $\omega \in \mathcal{M}$ *and* $u_0 \in V$, *where* $[0, \alpha_{(u_0,\omega)})$ *denotes the maximal interval of existence of the solution* $\varphi(t, u_0, \omega)$ *of* (1).

Lemma 2

The inequality

$$\|\varphi(t, u_0, \omega)\|_V \leq \max\left\{\|u_0\|_V, \frac{\|\mathscr{F}\|_1}{a}\right\}$$

holds for all $t \in [0, \alpha_{(u_0,\omega)})$, $\omega \in \mathcal{M}$ *and* $u_0 \in V$, *where* $[0, \alpha_{(u_0,\omega)})$ *denotes the maximal interval of existence of the solution* $\varphi(t, u_0, \omega)$ *of* (1).

Theorem 3

If $J = \mathbb{R}_+$ *then the mild solution of system* (1) *may be prolonged on* $\mathbb{R}_+$.

- $\varphi : \mathbb{R}_+ \times V \times \mathcal{M} \to V$ is a cocycle, that is:

(*i*) $\varphi(0, u_0, \omega) = u_0$ for all $u_0 \in V$ and $\omega \in \mathcal{M}$,

(*ii*) $\varphi(t + s, u_0, \omega) = \varphi(t, \varphi(s, u_0, \omega), \sigma(s, \omega))$ for all $t, s \in \mathbb{R}_+$ and $\omega \in \mathcal{M}$,

(*iii*) the map $\mathbb{R}_+ \times V \times \mathcal{M} \ni (t, u_0, \omega) \mapsto \varphi(t, u_0, \omega) \in V$ is continuous.

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  (*iii*) the map $\mathbb{R}_+ \times V \times \mathcal{M} \ni (t, u_0, \omega) \mapsto \varphi(t, u_0, \omega) \in V$ is continuous.

- $$\lim_{t \to +\infty} \sup_{\|u_0\|_V \leq r,\, \omega \in \mathcal{M}} \|\varphi(t, u_0, \omega)\|_V \leq \frac{\|\mathscr{F}\|_1}{a},$$

for all $r > 0$.

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$$\lim_{t \to +\infty} \sup_{\|u_0\|_V \le r,\, \omega \in \mathcal{M}} \|\varphi(t, u_0, \omega)\|_V \le \frac{\|\mathscr{F}\|_1}{a},$$

for all $r > 0$.
Consequently, the set

$$B_0 = \left\{ u \in V : \|u\|_V \le \frac{\|\mathscr{F}\|_1}{a} \right\}$$

is a bounded attractor for the system (1). Hence, system (1) is bounded dissipative.

The nonautonomous 2D Navier-Stokes equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f(t, u), & (t, x) \in (0, +\infty) \times \Omega, \\ \operatorname{div} u = 0, & (t, x) \in (0, +\infty) \times \Omega, \\ u = 0, & (t, x) \in (0, +\infty) \times \partial\Omega, \\ u(0, \cdot) = u_0 \end{cases}$$

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- Ω is a bounded smooth domain in $\mathbb{R}^2$;
- $u = (u_1, u_2) = (u_1(t,x), u_2(t,x))$ denotes the velocity field of a newtonian fluid filling the domain Ω;

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(H1) $f : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2$ is a bounded function such that for each fixed $t \in \mathbb{R}_+$, $f(t, \cdot)$ is continuous on $\mathbb{R}^2$.

(H2) For each $x \in \mathbb{R}^2$, $f(\cdot, x) \in G(\mathbb{R}_+, \mathbb{R}^2)$.

(H3) There is $C > 0$ such that $|f(s,x) - f(s,y)| \leq C|x - y|$ for all $s \in \mathbb{R}_+$ and for all $x, y \in \mathbb{R}^2$.

Let $\mathbb{L}^2(\Omega) = (L^2(\Omega))^2$ and $\mathbb{H}_0^1(\Omega) = (H_0^1(\Omega))^2$.

$$\mathcal{E} = \left\{ v \in (C_0^\infty(\Omega))^2 : \ \mathrm{div}\, v = 0 \text{ in } \Omega \right\}.$$

$$\mathbf{H} = \text{closure of } \mathcal{E} \text{ in } \mathbb{L}^2(\Omega) \quad \text{and} \quad \mathbf{V} = \text{closure of } \mathcal{E} \text{ in } \mathbb{H}_0^1(\Omega).$$

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- $\left(\mathbf{H}, (\cdot,\cdot)_{\mathbb{L}^2}\right)$ and $\left(\mathbf{V}, (\cdot,\cdot)_{\mathbb{H}_0^1}\right)$ are Hilbert spaces and

$$\mathbf{V} \overset{d}{\hookrightarrow} \mathbf{H} \equiv \mathbf{H}' \overset{d}{\hookrightarrow} \mathbf{V}'.$$

We can rewrite the 2D Navier-Stokes equations as the abstract evolution equation

$$\begin{cases} \dfrac{du}{dt} + Au + B(t)(u, u) = F(t)(u), \quad \text{in} \quad \mathbf{V}', \quad t > 0, \\ u(0) = u_0 \in \mathbf{V}, \end{cases} \tag{2}$$

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where:

- $A : D(A) \subset H \to H, \quad D(A) = \{u \in \mathbb{H}^2(\Omega) \cap \mathbf{H} : \ u = 0 \text{ in } \partial\Omega\},$

$$Au = -\nu \Pi \Delta u.$$

- $F(t) : \mathbf{V} \to \mathbf{V}'$

$$F(t)(u) = \Pi f(t,u).$$

- $B(t) : \mathbf{V} \times \mathbf{V} \to \mathbf{V}'$

$$B(t)(u,u) = \Pi\big((u \cdot \nabla)u\big).$$

$$\Pi : \mathbb{L}^2(\Omega) \to \mathbf{H} \quad \text{is the Leray's ortogonal projection.}$$

- $\mathbf{E} = D(A^\gamma)$ for some $-\frac{1}{2} < \gamma < 0$.

$$B \in C(\mathbb{R}_+, \mathscr{L}^2(\mathbf{V}, \mathbf{E}))$$

and

$$F \in G(\mathbb{R}_+, C(\mathbf{V}, \mathbf{E})).$$

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- $\mathbf{Y} = C(\mathbb{R}_+, \mathscr{L}^2(\mathbf{V}, \mathbf{E})) \times G(\mathbb{R}_+, C(\mathbf{V}, \mathbf{E}))$ and $(\mathbf{Y}, \sigma)$ is the dynamical system of translations, that is,

$$\sigma(\tau, g) = g_\tau = g(\tau + \cdot), \quad g \in \mathbf{Y}, t \geq 0.$$

- $\mathbf{E} = D(A^\gamma)$ for some $-\frac{1}{2} < \gamma < 0$.

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$$\sigma(\tau, g) = g_\tau = g(\tau + \cdot), \quad g \in \mathbf{Y}, t \geq 0.$$

- $\mathcal{M} = \mathcal{H}(B, F) = \overline{\{(B_\tau, F_\tau) :\ \tau \in \mathbb{R}_+\}} \subset \mathbf{Y}$, where

$$B_\tau(t) = B(t+\tau) \quad \text{and} \quad F_\tau(t)(u) = F(t+\tau)(u), \ \forall t, \tau \in \mathbb{R}_+, u \in \mathbf{V}.$$

We set $(\mathcal{M}, \sigma|_\mathcal{M})$ the dynamical system of translations on $\mathcal{M}$,

$$\sigma|_\mathcal{M}(\tau, (\mathcal{B}, \mathcal{F})) = \sigma(\tau, (\mathcal{B}, \mathcal{F})) = (\mathcal{B}_\tau, \mathcal{F}_\tau), \ (\mathcal{B}, \mathcal{F}) \in \mathcal{M}.$$

The equation

$$\frac{du}{dt} + Au + \mathcal{B}(t)(u,u) = \mathcal{F}(t)(u),$$

where $(\mathcal{B}, \mathcal{F}) \in \mathcal{M}$, is called the $\mathcal{H}-$class along with the equation

$$\frac{du}{dt} + Au + B(t)(u,u) = F(t)(u).$$

Now, we define the mappings

$$\mathbf{B} : \mathcal{M} \to \mathscr{L}^2(\mathbf{V}, \mathbf{E})$$

by

$$\mathbf{B}(\sigma(t, \omega)) = \mathbf{B}(\mathcal{B}_t, \mathcal{F}_t) := \mathcal{B}_t(0),$$

for all $\omega = (\mathcal{B}, \mathcal{F}) \in \mathcal{M}$ and $t \geq 0$, and

$$\mathbf{F} : \mathbb{R}_+ \times \mathcal{M} \times \mathbf{V} \to \mathbf{E}$$

by

$$\mathbf{F}(t, \sigma(s, \omega), u) = \mathbf{F}(t, (\mathcal{B}_s, \mathcal{F}_s), u) := \mathcal{F}_s(0)(u),$$

for all $u \in \mathbf{V}$, $\omega = (\mathcal{B}, \mathcal{F}) \in \mathcal{M}$ and $t, s \geq 0$.

Then equation

$$\frac{du}{dt} + Au + \mathcal{B}(t)(u, u) = \mathcal{F}(t)(u),$$

can be rewritten in the form

$$\frac{du}{dt} + Au + \mathbf{B}(\sigma(t, \omega))(u, u) = \mathbf{F}(t, \sigma(t, \omega), u).$$

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\begin{cases}
\dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f(t,u), & (t,x) \in (0,+\infty) \times \Omega, \\
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\dfrac{du}{dt} + Au + \mathbf{B}(\sigma(t,\omega))(u,u) = \mathbf{F}(t,\sigma(t,\omega),u), \quad t > 0, \\
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Theorem 4

The system (3) admits a unique mild solution $\varphi(\cdot, u_0, \omega) : \mathbb{R}_+ \to \mathbf{V}$ *satisfying* $\varphi(0, u_0, \omega) = u_0$.

Theorem 5

The mild solution $\varphi(t, u_0, \omega)$ *of* (3) *satisfies the boundedness*

$$\|\varphi(t, u_0, \omega)\|_{\mathbf{V}} \leq \max\left\{\|u_0\|_{\mathbf{V}}, \frac{\|\mathbf{F}\|_1}{\alpha}\right\},$$

for all $t \geq 0$, $\omega \in \mathcal{M}$ *and* $u_0 \in \mathbf{V}$. *Moreover, system* (3) *is bounded dissipative and generates a cocycle.*

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f(t,u), & (t,x) \in (0,+\infty) \times \Omega, \\ \operatorname{div} u = 0, & (t,x) \in (0,+\infty) \times \Omega, \\ u = 0, & (t,x) \in (0,+\infty) \times \partial\Omega, \\ u(0,\cdot) = u_0 \end{cases} \tag{4}$$

Theorem 6

Assume that conditions (H1)–(H3) hold. Then there exist functions $p = p(t,x)$ and $u = u(t,x)$ on $[0,+\infty) \times \Omega$, satisfying system (4). Moreover, $[0,+\infty) \ni t \to p(t,\cdot) \in H^1(\Omega)$ and $[0,+\infty) \ni t \to u(t,\cdot) \in \mathbb{H}^1_0(\Omega)$ are continuous functions and

$$\|u(t,\cdot)\|^2_{\mathbb{H}^1_0(\Omega)} \leq \max\left\{ \|u(0,\cdot)\|^2_{\mathbb{H}^1_0(\Omega)}, \left(\frac{\eta}{\alpha}\right)^2 \right\} \quad \text{for all} \quad t \geq 0.$$

It is well known that many relevant phenomena, including some from fluid dynamics, have their behavior drastically modified somehow after an instantaneous change on their state, which may introduce in the model several discontinuities.

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Properties as velocity, density and viscosity are discontinuous at interfaces between different fluids.

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Properties as velocity, density and viscosity are discontinuous at interfaces between different fluids.

Despite of the extensive literature on NSEs and the recents progress on the impulsive dynamical systems, surprisedly models from fluid dynamics incorporating impulse effects on its structure are somewhat scarce.

The nonautonomous 2D Navier-Stokes equations with impulses

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For each $D \subseteq \mathbf{V}$, $J \subseteq \mathbb{R}_+$ and $\omega \in \mathcal{M}$ we define

$$F_\varphi(D, J, \omega) = \{u_0 \in \mathbf{V} \colon \varphi(t, u_0, \omega) \in D, \text{ for some } t \in J\}.$$

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• Impulsive set: is a nonempty closed subset $M \subset \mathbf{V}$ satisfying the property: for each $u_0 \in M$ and each $\omega \in \mathcal{M}$, $\exists\, \epsilon = \epsilon_{\omega,u_0} > 0$ with

$$\bigcup_{t\in(0,\epsilon)} F_\varphi(u_0, t, \sigma_{-t}\omega) \cap M = \emptyset \quad \text{and} \quad \{\varphi(s, u_0, \omega) \colon s \in (0, \epsilon)\} \cap M = \emptyset.$$

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• $\Phi(\cdot, \omega) \colon \mathbf{V} \to (0, +\infty]$

$$\Phi(u_0, \omega) = \begin{cases} s, & \text{if } \varphi(s, u_0, \omega) \in M \text{ and } \varphi(t, u_0, \omega) \notin M \text{ for } 0 < t < s, \\ +\infty, & \text{if } \varphi(t, u_0, \omega) \notin M \text{ for all } t > 0. \end{cases}$$

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- $M^+_\varphi(u_0, \omega) = \{\varphi(\tau, u_0, \omega) \colon \tau > 0\} \cap M$.

- If $M_{\varphi}^{+}(u_0,\omega) \neq \varnothing$ then we define $\tilde{\varphi}(\cdot, u_0, \omega)$ on $[0, \Phi(u_0,\omega)]$ by

$$\tilde{\varphi}(t, u_0, \omega) = \begin{cases} \varphi(t, u_0, \omega), & \text{if} \quad 0 \leqslant t < \Phi(u_0,\omega), \\ I(\varphi(\Phi(u_0,\omega), u_0, \omega)), & \text{if} \quad t = \Phi(u_0,\omega). \end{cases}$$

- If $M_\varphi^+(u_0,\omega) \neq \varnothing$ then we define $\tilde{\varphi}(\cdot, u_0, \omega)$ on $[0, \Phi(u_0,\omega)]$ by

$$\tilde{\varphi}(t,u_0,\omega) = \begin{cases} \varphi(t,u_0,\omega), & \text{if} \quad 0 \leqslant t < \Phi(u_0,\omega), \\ I(\varphi(\Phi(u_0,\omega),u_0,\omega)), & \text{if} \quad t = \Phi(u_0,\omega). \end{cases}$$

Let $u_0 = u_0^+$, $s_0 = \Phi(u_0^+,\omega)$, $u_1 = \varphi(s_0, u_0^+, \omega)$ and $u_1^+ = I(u_1)$.

- If $M_\varphi^+(u_0,\omega) \neq \varnothing$ then we define $\tilde{\varphi}(\cdot, u_0, \omega)$ on $[0, \Phi(u_0,\omega)]$ by

$$\tilde{\varphi}(t, u_0, \omega) = \begin{cases} \varphi(t, u_0, \omega), & \text{if} \quad 0 \leqslant t < \Phi(u_0,\omega), \\ I(\varphi(\Phi(u_0,\omega), u_0, \omega)), & \text{if} \quad t = \Phi(u_0,\omega). \end{cases}$$

Let $u_0 = u_0^+$, $s_0 = \Phi(u_0^+, \omega)$, $u_1 = \varphi(s_0, u_0^+, \omega)$ and $u_1^+ = I(u_1)$.

- If $M_\varphi^+(u_1^+, \sigma_{s_0}\omega) \neq \varnothing$ we define

$$\tilde{\varphi}(t, u_0, \omega) = \begin{cases} \varphi(t - s_0, u_1^+, \sigma_{s_0}\omega), & \text{if} \quad s_0 \leqslant t < s_0 + \Phi(u_1^+, \sigma_{s_0}\omega), \\ I(u_2), & \text{if} \quad t = \Phi(u_1^+, \sigma_{s_0}\omega). \end{cases}$$

where $u_2 = \varphi(\Phi(u_1^+, \sigma_{s_0}\omega), u_1^+, \sigma_{s_0}\omega)$.

Lemma 7

$$\tilde{\varphi}(0, u_0, \omega) = u_0$$

and

$$\tilde{\varphi}(t + s, u_0, \omega) = \tilde{\varphi}(t, \tilde{\varphi}(s, u_0, \omega), \sigma_s \omega),$$

for all $u_0 \in \mathbf{V}$, $\omega \in \mathcal{M}$ *and* $t, s \in \mathbb{R}_+$.

Theorem 8

Assume that there is $\mathcal{K} > 0$ *such that* $\|I(u)\|_{\mathbf{V}} \leq \mathcal{K}$ *for all* $u \in M$*. Then*

$$\|\tilde{\varphi}(t, u_0, \omega)\|_{\mathbf{V}} \leq \max\left\{\|u_0\|_{\mathbf{V}}, \mathcal{K}, \frac{\|\mathbf{F}\|_1}{\alpha}\right\},$$

for all $t \geq 0$, $\omega \in \mathcal{M}$ *and* $u_0 \in \mathbf{V}$*. Moreover, system* (5) *is bounded dissipative.*

$$\begin{cases} \dfrac{du}{dt} + Au + \mathbf{B}(\sigma(t,\omega))(u,u) = \mathbf{F}(t, \sigma(t,\omega), u), \quad t > 0, \\ u(0) = u_0 \in \mathbf{V}, \\ I : M \subset \mathbf{V} \to \mathbf{V}. \end{cases} \tag{5}$$

Theorem 9

Let $u_0 \in \mathbf{V} \setminus M$, $\omega \in \mathcal{M}$ *and* $\{v_n\}_{n\in\mathbb{N}} \subset \mathbf{V}$ *be a sequence such that* $\|v_n - u_0\|_{\mathbf{V}} \overset{n\to\infty}{\longrightarrow} 0$. *Given* $t \geq 0$, *there exists a sequence* $\{\eta_n\}_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *such that* $\eta_n \overset{n\to\infty}{\longrightarrow} 0$ *and*

$$\|\tilde{\varphi}(t + \eta_n, v_n, \omega) - \tilde{\varphi}(t, u_0, \omega)\|_{\mathbf{V}} \overset{n\to\infty}{\longrightarrow} 0.$$

Thank you for your attention!!