Greedy Control

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[2] Joint work with M.Choulli and M. Lazar

Table of Contents

1. In past decades controllability theory for PDE has evolved significantly.
2. Some of the most paradigmatic models are by now well understood: Wave and heat equations, in particular.
3. But theory lacks of unity. Often times rather different analytical tools are required to tackle different models/problems.
4. Practical applications need of **robust** control theoretical results and **fast numerical solvers**.
5. One of the key issues to be addressed in that direction is the controllability of **PDE models depending on parameters**, that represent uncertain or unknown quantities.
6. In this lecture we present some basic elements of the implementation of the **greedy methods** in this context and formulate some challenging open problems.
7. This leads to a new class of **Inverse Problems**.

Table of Contents

Consider the finite dimensional linear control system (possibly obtained from a PDE control problem after space discretisation)

$$\begin{cases} x'(t) = A(\nu)x(t) + Bu(t),\ 0 < t < T, \\ x(0) = x^0. \end{cases} \tag{1}$$

In (1):

- The (column) vector valued function $x(t,\nu) = \big(x_1(t,\nu), \ldots, x_N(t,\nu)\big) \in \mathbb{R}^N$ is the state of the system,
- ν is a multi-parameter living in a compact set K of $\mathbb{R}^d$,
- $A(\nu)$ is a $N \times N-$matrix,
- $u = u(t)$ is a M-component control vector in $\mathbb{R}^M$, $M \leq N$.

Given a control time $T > 0$ and a final target $x^1 \in \mathbb{R}^N$ we look for a control u such that the solution of (1) satisfies the averaged control property:

$$\int_K x(T,\nu)d\nu = x^1. \tag{2}$$

Theorem

[a] *Averaged controllability holds if and only the following rank condition is satisfied:*

$$rank\Big[B, \int_0^1 [A(\nu)]d\nu B, \int_0^1 [A(\nu)]^2 d\nu B, \dots\Big] = N. \tag{3}$$

[a] E. Zuazua, Automatica, 2014.

Drawbacks:

1. Nothing is said about the efficiency of the control for specific realisations of ν.
2. Complex (**and interesting !**) in the PDE setting. [3]
   Consider the transport equation with unknown velocity v,

$$f_t + vf_x = 0,$$

   and take averages with respect to v. Then

$$g(x,t) = \int f(x,t;v)\rho(v)dv$$

   then, for the Gaussian density ρ:

$$\rho(v) = (4\pi)^{-1/2}\exp(-v^2/4)$$

$$g(x,t) = h(x,t^2); \quad h_t - h_{xx} = 0.$$

   One can then employ parabolic techniques based on Carleman inequalities.

[3] Q. Lü & E. Z. Average Controllability for Random Evolution Equations, IMPA, 2016. Linked to averaging Lemmas (Golse - Lions - Perthame - Sentis)

Table of Contents

[4] Assume that the system depends on a parameter $\nu \in K \subset \mathbf{R}^d$, $d \geq 1$, K being a compact set, and controllability being fulfilled for all values of ν.

$$\begin{cases} x'(t) = A(\nu)x(t) + Bu(t,\nu),\ 0 < t < T, \\ x(0) = x^0. \end{cases} \tag{4}$$

Controls $u(t,\nu)$ are chosen to be of minimal norm satisfying the controllability condition:

$$x(T,\nu) = x^1, \tag{5}$$

and lead to a manifold of dimension d in $[L^2(0,T)]^M$:

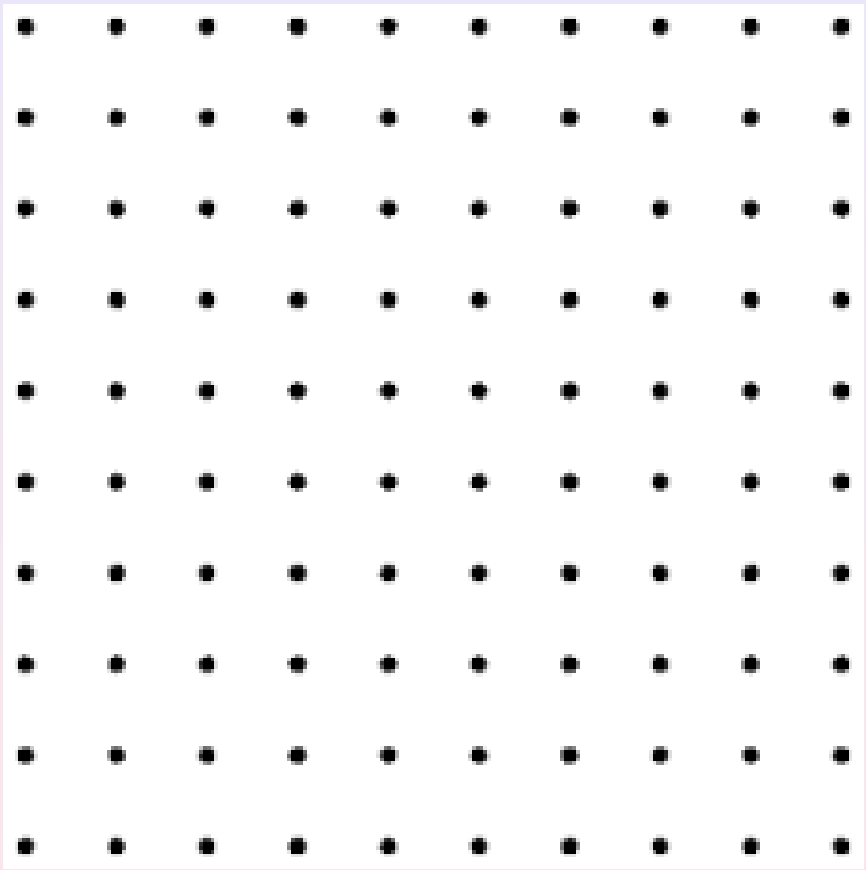
$$\nu \in K \subset \mathbf{R}^d \to u(t,\nu) \in [L^2(0,T)]^M.$$

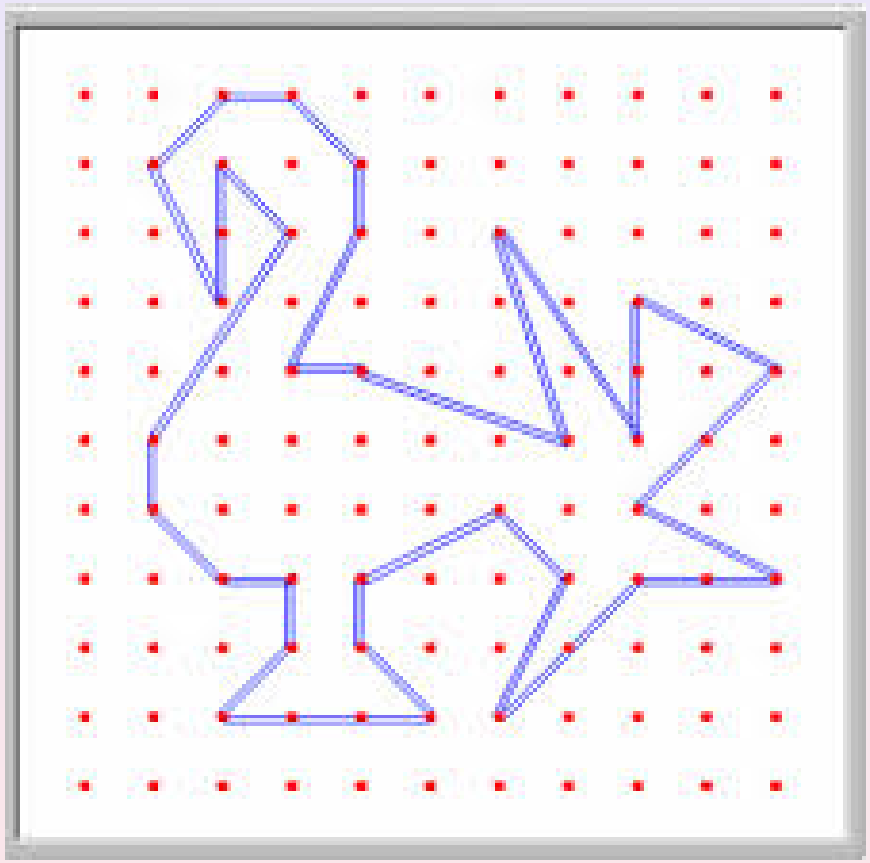
This manifold inherits the regularity of the mapping $\nu \to A(\nu)$.

To diminish the computational cost we look for the very distinguished values of ν that yield the best possible approximation of this manifold.

[4] M. Lazar & E. Zuazua, Greedy controllability of finite dimensional linear systems, Automatica, 2016.

Naive versus smart sampling of K





Our work relies on recent ones on greedy algorithms and reduced bases methods:

A. Cohen, R. DeVore, *Kolmogorov widths under holomorphic mappings*, IMA Journal on Numerical Analysis, to appear

A. Cohen, R. DeVore, *Approximation of high-dimensional parametric PDEs*, arXiv preprint, 2015.

Y. Maday, O. Mula, A. T. Patera, M. Yano, *The generalized Empirical Interpolation Method: stability theory on Hilbert spaces with an application to the Stokes equation*, submitted

M. A. Grepl, M Kärche, *Reduced basis a posteriori error bounds for parametrized linear-quadratic elliptic optimal control problems*, CRAS Paris, 2011.

S. Volkwein, PDE-Constrained Multiobjective Optimal Control by Reduced-Order Modeling, IFAC CPDE2016, Bertinoro.

Description of the Method

We look for the realisations of the parameter ν ensuring the best possible approximation of the manifold of controls

$$\nu \in K \subset \mathbf{R}^d \to u(t,\nu) \in [L^2(0,T)]^M$$

(of dimension d in $[L^2(0,T)]^M$) in the sense of the Kolmogorov width.[5]

Greedy algorithms **search for the values of ν leading to the most distinguished controls** $u(t,\nu)$, those that are farther away one from each other.

Given an error ε, the goal is to find $\nu_1, ..., \nu_{n(\epsilon)}$, so that for all parameter values ν the corresponding control $u(t,\nu)$ can be approximated by a linear combination of $u(t,\nu_1), ..., u(t,\nu_{n(\epsilon)})$ with an error $\leq \epsilon$.

An of course to do it with a minimum number $n(\epsilon)$.

[5] Ensure the optimal rate of approximation by means of all possible finite-dimensional subspaces:

$$d_n(K) := \inf_{\dim Y = n} \sup_{x \in K} \inf_{x \in Y} ||x - y||_X .$$

Step 1. *Characterization of minimal norm controls by adjoints*

The adjoint system depends also on the parameter ν:

$$-\varphi'(t) = A^*(\nu)\varphi(t), t \in (0,\ T);\ \varphi(T) = \varphi^0. \tag{6}$$

The control is

$$u(t,\nu) = B^*\varphi(t,\nu),$$

where $\varphi(t,\nu)$ is the solution of the adjoint system associated to the minimizer of the following quadratic functional in $\mathbf{R^N}$:

$$J_\nu\left(\varphi^0(\nu)\right) = \frac{1}{2}\int_0^T |B^*\varphi(t,\nu)|^2\, dt - < x^1, \varphi^0 > + < x^0, \varphi(0,\nu) > .$$

The functional is continuous and convex, and its coercivity is guaranteed by the Kalman rank condition that we assume to be satisfied for all ν.

Step 2. *Controllability distance*

Given two parameter values ν_1 and ν_2, how can we measure the distance between $u(t,\nu_1)$ and $u(t,\nu_2)$?

Of course the issue relies on the fact that these two controls are unknown! We need a surrogate!

Roughly: Compute the residual

$$||x(T,\nu_2) - x^1||$$

for the solution of the state equation ν_2 achieved by the control $u(t,\nu_1)$.

More precisely: Solve the Optimality System (OS):

$$-\varphi'(t) = A^*(\nu_2)\varphi(t)\ t \in (0,\ T);\ \varphi(T) = \varphi_1^0.$$

$$x'(t) = A(\nu_2)x(t) + BB^*\varphi(t,\nu_2),\ 0 < t < T,\ x(0) = x^0.$$

Then

$$\left|\nabla J_{\nu_2}(\varphi_1^0)\right| = ||x(T,\nu_2) - x^1|| \sim ||\varphi_1^0 - \varphi_2^0|| \sim ||u(t,\nu_1) - u(t,\nu_2)||.$$

Offline algorithm

Step 3. *Initialisation of the weak-greedy algorithm.* Choose any ν in K, $\nu = \nu_1$, and compute the minimizer of J_{ν_1}. This leads to φ_1^0.

Step 4. *Recursive choice of $\nu's$.*
Assuming we have $\nu_1, ..., \nu_p$, we choose ν_{p+1} as the maximiser of

$$\max_{\nu \in K} \min_{\phi \in span[\varphi_j^0, j=1,...,p]} |\nabla J_\nu(\phi)|$$

We take ν_{p+1} as the one realizing this maximum.
Note that

$$|\nabla J_\nu(\phi)| = ||x(T, \nu) - x^1||.$$

$x(T, \nu)$ being the solution obtained by means of the control $u = B^*\phi(t, \nu)$, ϕ being the solution of the adjoint problem associated to the initial datum ϕ^0 in $span[\varphi_j^0, j = 1, ..., p]$.

Step 5. *Stopping criterion.* Stop if the max $\leq \epsilon$.

Online part

Step 6. For a specific realisation of ν solve the finite-dimensional reduced minimisation problem:

$$\min_{\phi \in span[\varphi_j^0, j=1,\dots,p]} |\nabla J_\nu(\phi)|.$$

This minimiser yields:

$$u(t,\nu) = B^* \varphi(t,\nu),$$

$\varphi(t,\nu)$ being the solution of the adjoint problem with datum ϕ at $t = T$.

The same applies for infinite-dimensional systems when A and B are bounded operators.

Theorem

The weak-greedy algorithm above leads to an optimal approximation method.
More precisely, if the set of parametres K is finite-dimensional, and the map $\nu \to A(\nu)$ is analytic, for all $\alpha > 0$ there exists $C_\alpha > 0$ such that for all other values of ν the control $u(\cdot, \nu)$ can be approximated by linear combinations of the weak-greedy ones as follows:

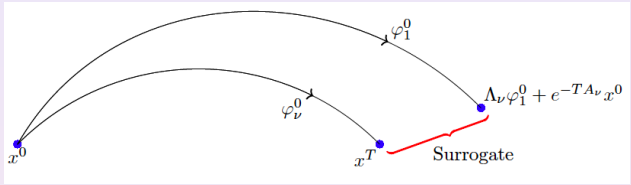
$$dist(u(\cdot,\nu); span[u(\cdot;\nu_j) : j = 1, ..., k]) \leq C_\alpha k^{-\alpha}.$$

6

[6] The approximation of the controls has to be understood in the sense above: Taking the control given by the corresponding adjoint solution, achieved by minimising the functional J over the finite-dimensional subspace generated by the adjoints for the distinguished parameter-values.

Potential improvements

1. **Find cheaper surrogates.** Is there a reduced model leading to lower bounds on controllability distances without solving the full Optimality System?



$$||x(T,\nu) - x_1|| \geq ??????$$

2. All this depends on the initial and final data: x_0, x_1.
   Can the search of the most relevant parameter-values ν be done independent of x_0, x_1?
   In other words, get **lower bounds on the controllability distances** between (A_1, B_1) and (A_2, B_2).

As we shall see this leads to Inverse Problems of a non-standard form

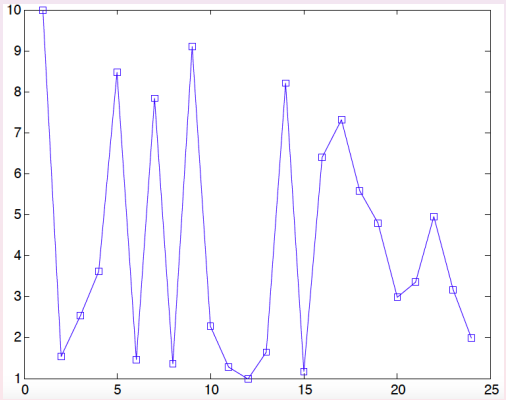
Table of Contents

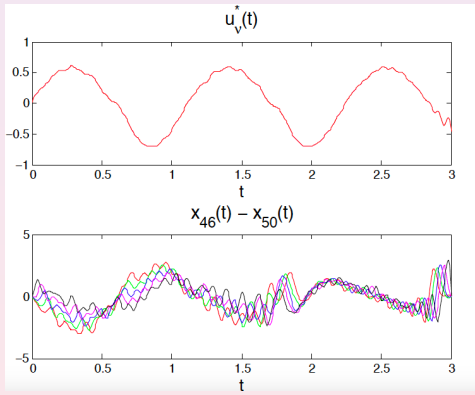
Semi-discrete wave equation

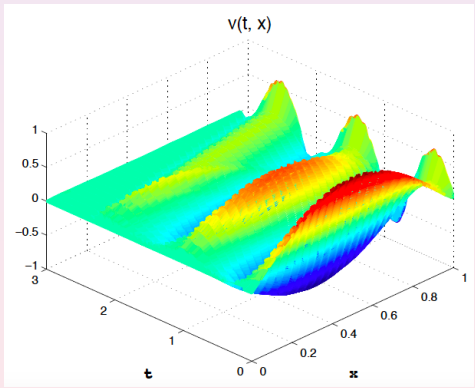
1. Finite difference approximation of the $1-d$ wave equation with 50 nodes in the space-mesh.
2. Unknown velocity v ranging within $[1, \sqrt{10}]$.
3. Discrete parameters taken over an equi-distributed set of 100 values
4. Boundary control
5. Sinusoidal initial data given: $y_0 = \sin(\pi x); y_1 \equiv 0$. Null final target.
6. Time of control $T = 3$.
7. Approximate control with error 0.5 in the energy.
8. Weak-greedy requires 24 snapshots $(\nu_1, ..., \nu_{24})$.
9. Offline time: 2.312 seconds (personal notebook with a 2.7 GHz processor and DDR3 RAM with 8 GB and 1,6 GHz).
10. Online time for one realisation ν: 7 seconds
11. Computational time for one single parameter value with standard methods: 51 seconds.

Choose a number at random within $[1, 10]$ For instance $v = \pi$!

The greedy algo leads to:

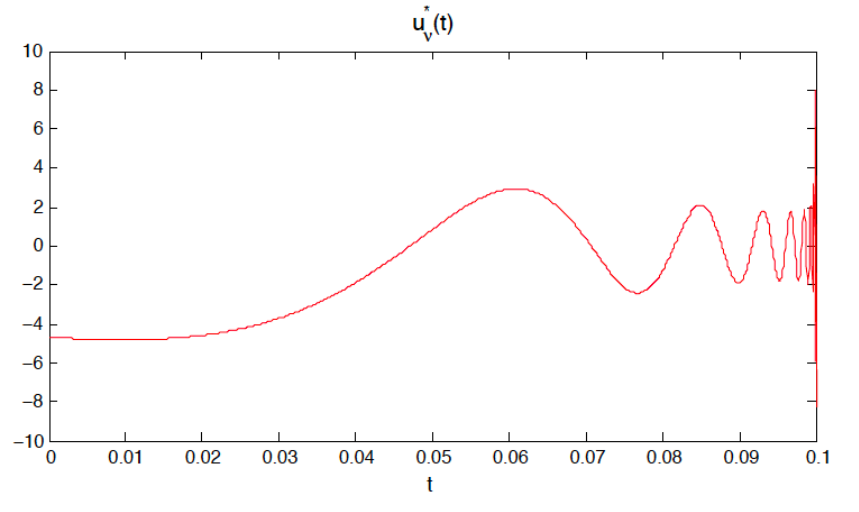


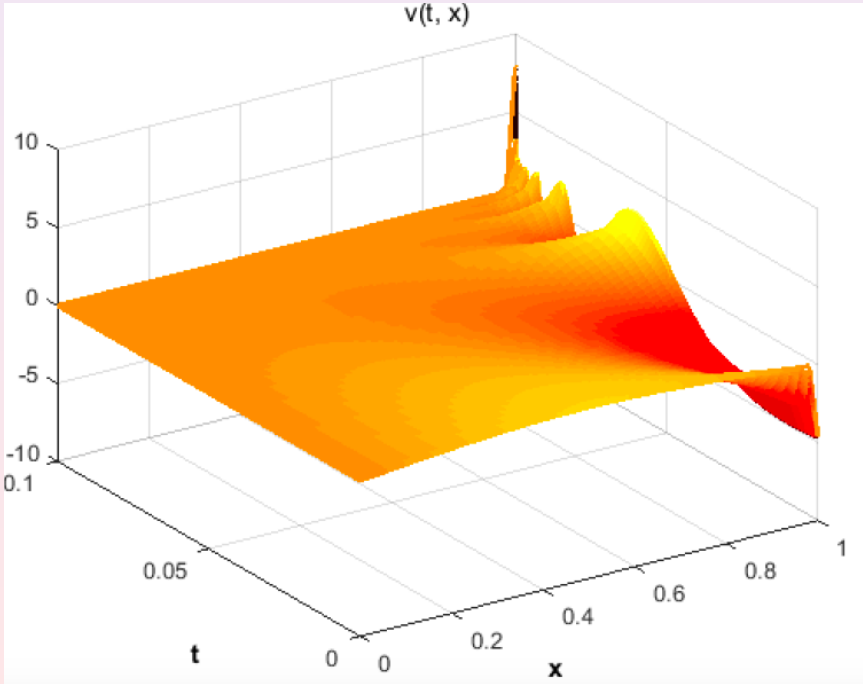




Semi-discrete heat equation

1. Finite difference approximation of the $1-d$ heat equation with 50 nodes in the space-mesh.
2. Unknown diffusivity v ranging within $[1,2]$.
3. Discrete parameters taken over an equi-distributed set of 100 values
4. Boundary control
5. Sinusoidal initial data given: $y_0 = \sin(\pi x)$. Null final target.
6. Time of control $T = 0.1$.
7. Weak-greedy requires 20 snapshots.
8. Approximate control with error 10^{-4} in each component.
9. The algo stops after 3 iterations: $\nu = 1.00, 1.18, 1.45$.
10. Offline time: 213 seconds.
11. Online time for one realisation $\nu = \sqrt{2}$: 1.5 seconds
12. Computational time for one single parameter value with standard methods: 37 seconds.


$u_v^*(t)$
10
8
6
4
2
0
−2
−4
−6
−8
−10
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
t


v(t, x)
10
5
0
-5
-10
0.1
0.05
0
t
0
0.2
0.4
0.6
0.8
1
x

Open problems and perspectives

- The method be extended to PDE. But **analyticity of controls with respect to parameters** has to be ensured to guarantee optimal Kolmogorov widths. This typically holds for elliptic and parabolic equations. But not for wave-like equations.
  Indeed, solutions of
  $$y_{tt} - v^2 y_{xx} = 0$$
  do not depend analytically on the coefficient v.
  One expects this to be true for heat equations in the context of null-controllability. But this needs to be rigorously proved.
- **Cheaper surrogates** need to be found so to make the recursive choice process of the various $\nu's$ faster.
  1. For wave equations in terms of distances between the dynamics of the Hamiltonian systems of bicharacteristic rays?
  2. For $1-d$ wave equations in terms of spectral distances?
  3. For heat equations?

Table of Contents

Outline

Problem formulation

Based on joint work with M. Choulli, C. R. Acad. Sci. Paris, to appear.

Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 1$. Fix $0 < \sigma_- < \sigma_+$ and consider the class of scalar diffusivity coefficients

$$\Sigma = \{\sigma \in L^\infty(\Omega);\ \sigma_- \leq \sigma \leq \sigma_+ \text{ a.e. in } \Omega\}.$$

For $\sigma \in \Sigma$, let $A_\sigma : H_0^1(\Omega) \to H^{-1}(\Omega)$ be the bounded operator given by

$$A_\sigma u = -\text{div}(\sigma \nabla u),$$

and the inverse or resolvent operator $R_\sigma : H^{-1}(\Omega) \to H_0^1(\Omega)$.

Motivation: Distance between two resolvents

Our goal is to show that

$$\left|R_{\sigma_1} - R_{\sigma_2}\right|_{-1,1} \sim \left|\sigma_1 - \sigma_2\right|_{L^\infty(\Omega)}.$$

Estimates are straightforward in one sense: *For any* $\sigma_1, \sigma_2 \in L^\infty(\Omega)$ *satisfying* $\sigma_- \leq \sigma_1, \sigma_2 \leq \sigma_+$,

$$\left|R_{\sigma_1} - R_{\sigma_2}\right|_{-1,1} \leq \sigma_-^{-2}\left|\sigma_1 - \sigma_2\right|_{L^\infty(\Omega)}.$$

This is nothing but the Lipschitz dependence of the resolvents (as linear and bounded operators from $H^{-1}(\Omega)$ into $H_0^1(\Omega)$) on the diffusivity coefficient in $L^\infty(\Omega)$.

Proof.

We have

$$\int_\Omega \sigma_1 \nabla u_1 \cdot \nabla v dx = \int_\Omega \sigma_2 \nabla u_2 \cdot \nabla v dx \text{ for any } v \in H_0^1(\Omega).$$

Hence

$$\int_\Omega \sigma_1 \nabla (u_1 - u_2) \cdot \nabla v dx = \int_\Omega (\sigma_2 - \sigma_1) \nabla u_2 \cdot \nabla v dx.$$

Taking the test function $v = u_1 - u_2$, we get

$$\sigma_- \int_\Omega |\nabla (u_1 - u_2)|^2 dx \leq \sigma_-^{-1} \Big|\sigma_1 - \sigma_2\Big|_{L^\infty(\Omega)} \Big|f\Big|_{H^{-1}(\Omega)} \Big|\nabla (u_1 - u_2)\Big|_{L^2(\Omega)^n}.$$

For the reverse estimates we need to show that the diffusivity coefficients can be identified in a Lipschitz manner in terms of the resolvents.
For, we use the elementary identity[7]

$$A_1 - A_2 = A_1(R_2 - R_1)A_2 \implies \Big|A_1 - A_2\Big| \leq \sigma_+^2 \Big|R_1 - R_2\Big|.$$

Furthermore

$$\langle (A_1 - A_2)u, u\rangle_{-1,1} = \int_\Omega (\sigma_1 - \sigma_2)|\nabla u|^2 dx,$$

$$\int_\Omega (\sigma_1 - \sigma_2)|\nabla u|^2 dx \leq \Big|A_1 - A_2\Big|\Big|u\Big|^2_{H_0^1(\Omega)} \leq \sigma_1^2 \Big|R_1 - R_2\Big|\Big|u\Big|^2_{H_0^1(\Omega)}.$$

Now taking $u = u_\epsilon$ so that $|\nabla u_\epsilon|^2$ **constitutes an approximation of the identity** (for each $x_0 \in \Omega$) we get

$$||\sigma_1 - \sigma_2||_\infty \leq \sigma_+^2 \Big|R_1 - R_2\Big|.$$

[7] Consequence of the elementary one $1/a - 1/b = (b-a)/(ab)$

1. The result above can be understood as the solution to an inverse problem: *Identifying the diffusivity in terms of the resolvent.*
2. And it is particularly easy to solve since we are given a lot of information on the PDE under consideration: The full resolvent!
3. Of course, roughly, all existing results on inverse problems could be used, since they use less information (Dirichlet-to-Neumann map, for instance). But this is not really needed.
4. As we shall see this problem is motivated by the need of developing robust and fast methods for solving parameter-dependent PDEs. And actually this is just the first one in a class of more complex problems.

Extensions

1. The result can be easily extended to diagonal diffusivities

$$\sum_{j=1}^{n} \partial_j(\sigma_j(x)\partial_j u).$$

   It suffices to scale the sequence of test functions to accentuate the largest diffusivity.

2. The extension to general diffusivity matrices $\sigma_{ij}(x)$ with bounded measurable coefficients is work in progress by A. Cohen.

3. Work needs to be done in the context of elliptic systems, such as the system of elasticity.

- Consider the heat equation:

$$\begin{cases} u_t - \operatorname{div}(\sigma \nabla u) = 0 \text{ in } \Omega \times (0,\infty) \\ u = 0 \text{ on } \Gamma \times (0,\infty) \\ u(x,0) = f(x) \text{ in } \Omega. \end{cases} \tag{7}$$

The same questions we have addressed in the elliptic context arise also in the parabolic one. In this case, the question can be formulated as follows: *Does the resolvent* $f \in L^2(\Omega) \to C([0,\infty); L^2(\Omega))$ *determine the diffusivity coefficient in an unique manner? Is the map from resolvent to diffusion coefficient Lipschitz in suitable norms?*
It suffices to observe that

$$v(x) = \int_0^\infty u(x,t)dt,$$

solves the elliptic equation

$$-\operatorname{div}(\sigma \nabla v) = f \text{ in } \Omega, \quad v = 0 \text{ on } \Gamma,$$

and apply the previous results.

Outline

The proptotypical question addressed so far is as follows. Consider a class of elliptic equations, whose diffusivity coefficients depend on one or various parameters that we denote by ν:

$$-\text{div}(\sigma(x,\nu)\nabla u) = f.$$

For any right hand side term $f \in H^{-1}(\Omega)$ and parameter value ν there exists an unique solution $u_\nu \in H_0^1(\Omega)$. When addressing the practical resolution of these problems the naive approach consists on, for each realisation of the parameter ν, solving the corresponding PDE in an approximated manner by using a FEM. But:

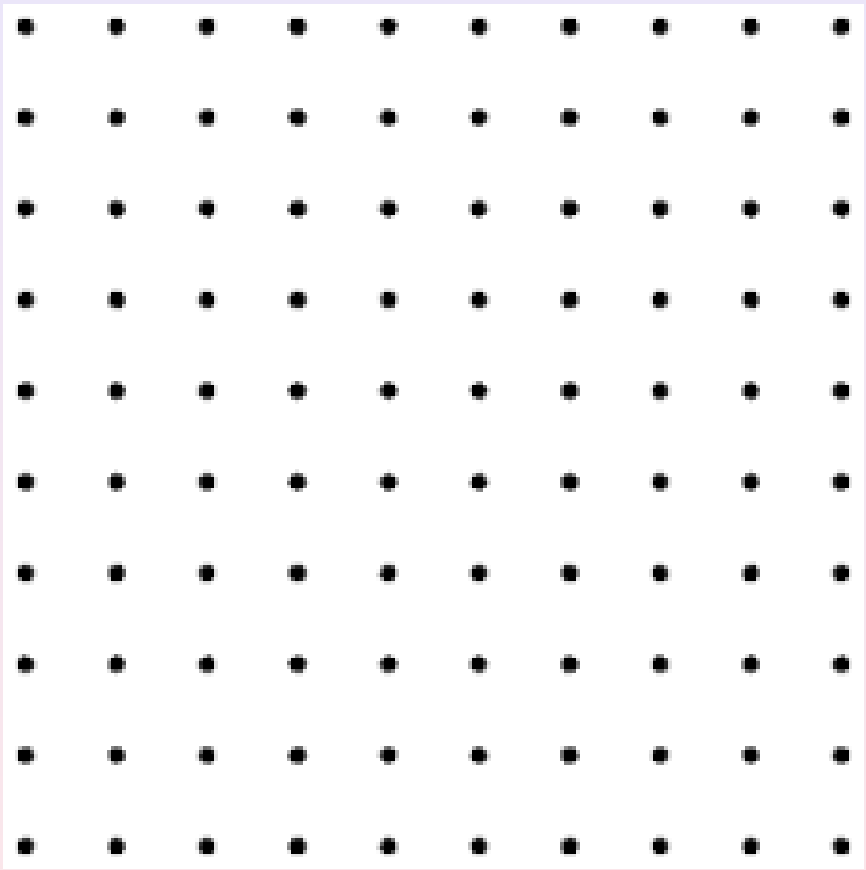
1. This approach is expensive, and unaffordable when the set of parameters ν is too large.
2. In this way, whenever we have to solve the PDE for a new parameter-value, we ignore all the work done previously.

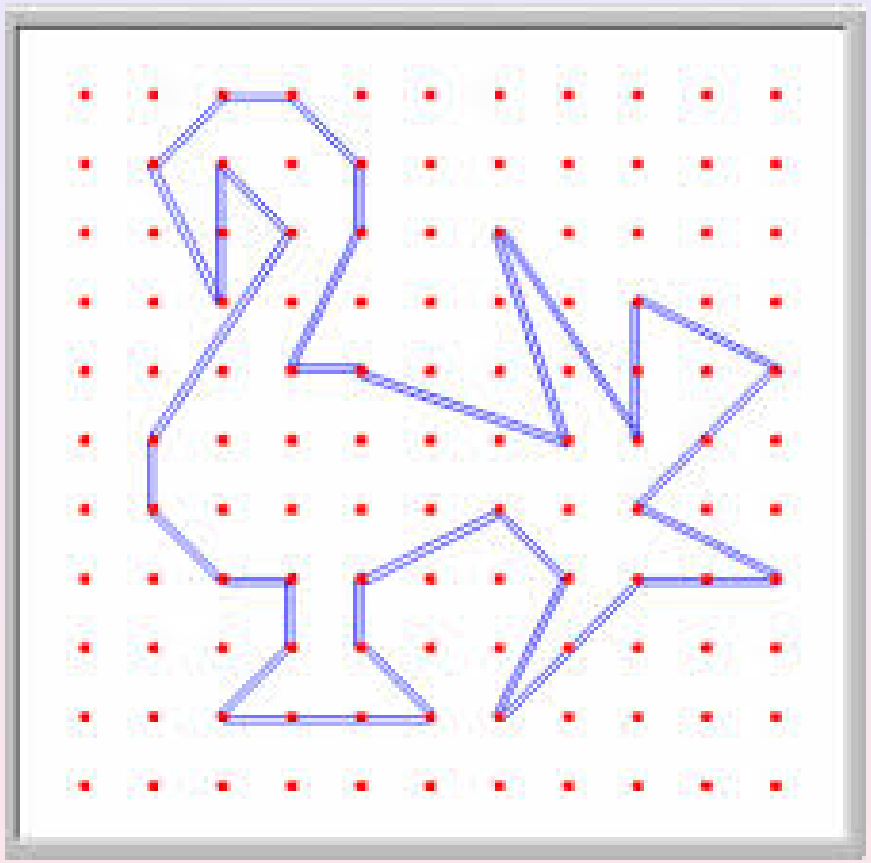
The goal of greedy methods is to find cheaper (optimal) computational methods to do it.

The viewpoint and essential aspects of the greedy approach are as follows: Assume that the system depends on a parameter $\nu \in K \subset \mathbf{R}^d$, $d \geq 1$, K being a compact set. This leads to a manifold of solutions $u(x, \nu)$ of dimension d in $H_0^1(\Omega)$. This manifold inherits the regularity of the mapping $\nu \to \sigma(x, \nu)$.

To diminish the computational cost we look for the very distinguished values of ν that yield the best possible approximation of this manifold.

Naive versus smart sampling of K





Description of the Method

We look for the realisations of the parameter ν ensuring the best possible approximation of the manifold of solutions

$$\nu \in K \subset \mathbf{R}^d \to u(x,\nu) \in H_0^1(\Omega)$$

(of dimension d in $H_0^1(\Omega)$) in the sense of the Kolmogorov width.[8]

Greedy algorithms **search for the values of ν leading to the most distinguished solutions** $u(x,\nu)$, those that are farther away one from each other.

Given an error ε, the goal is to find $\nu_1, ..., \nu_{n(\epsilon)}$, so that for all parameter values ν the corresponding solution $u(x,\nu)$ can be approximated by a linear combination of $u(x,\nu_1), ..., u(x,\nu_{n(\epsilon)})$ with an error $\leq \epsilon$.

An, of course, to do it with a minimum number $n(\epsilon)$.

[8] Ensure the optimal rate of approximation by means of all possible finite-dimensional subspaces.

Step 2. *Computing distances*

Given two parameter values ν_1 and ν_2, how can we measure the distance between $u(x,\nu_1)$ and $u(x,\nu_2)$?

Of course the difficulty relies on the fact that these two solutions (or at least one of them) are unknown!!!

Roughly: Compute the norm of the residual

$$A(\nu_2)u(\nu_1) - f = \epsilon$$

in $H^{-1}(\Omega)$.
Since

$$A(\nu_2)u(\nu_2) - f = 0$$

We have

$$A(\nu_2)(u(\nu_1) - u(\nu_2)) = \epsilon.$$

And, because of the coercivity of $A(\nu_2)$,

$$||u(\nu_1) - u(\nu_2)||_{H_0^1(\Omega)} \leq \sigma_-^{-1}||\epsilon||_{H^{-1}(\Omega)}.$$

Offline algorithm

Step 3. *Initialisation of the weak-greedy algorithm.* Choose any ν in K, $\nu = \nu_1$, and compute the solution corresponding to $A(\nu_1)$. This leads to u_1.

Step 4. *Recursive choice of $\nu's$.*
Assuming we have $\nu_1, ..., \nu_p$, we choose ν_{p+1} as the maximiser of

$$\max_{\nu \in K} \min_{v \in span[u_j, j=1,...,p]} |A(\nu)v - f|$$

We take ν_{p+1} as the one realizing this maximum. It is somehow the one that, from the viewpoint of the residual, generates the solution that is the most distant one from the span generated by all previous ones.

Step 5. *Stopping criterion.* Stop if the max $\leq \epsilon$.

Online part

Step 6. For a specific realisation of ν perform a Galerkin approximation in the finite-dimensional space generated by u_j, $j = 1, ..., p]$.

Thanks to this method:

1. The optimality is guaranteed from the viewpoint of the Kolmogorov thickness.[9]
2. In combination with FEM it drastically reduces the computational cost.

The only "drawback" of this method is that the choice of the snapshots ν_j depends on the right hand side term f.

The question arises naturally: Build a greedy strategy, independent of the right hand side term f.

9

$$d_n(K) := \inf_{\dim Y = n} \sup_{x \in K} \inf_{x \in Y} ||x - y||_X .$$

Theorem

The weak-greedy algorithm above leads to an optimal approximation method.

In particular, if the set of parameters K is finite-dimensional, and the map $\nu \to A(\nu)$ is analytic, for all $\alpha > 0$ there exists $C_\alpha > 0$ such that for all other values of ν the control $u(\cdot, \nu)$ can be approximated by linear combinations of the weak-greedy ones as follows:

$$dist(u(\cdot,\nu); span[u(\cdot;\nu_j) : j = 1, ..., k]) \leq C_\alpha k^{-\alpha}.$$

Outline

The implementation of the greedy method to approximate the manifold of resolvents needs estimating the distance from a resolvent to the subspace generated by finitely-many others:

$$R_1 - \sum_{j=1}^{k} \alpha_j R_j.$$

Can this be linked to the distance between some linear combinations of the corresponding diffusivities?

In $1-d$ the problem can be solved, thanks to the explicit representation of solutions[10]

$$-(\sigma(x)u_x)_x = f \text{ in } (0,1), \quad u_x(0) = 0 \text{ and } u(1) = 0. \tag{8}$$

$$u_x(x) = -\frac{1}{\sigma(x)} \int_0^x f(t)dt = -T_\sigma f \text{ a.e. } (0,1). \tag{9}$$

[10] Very much as in the context of homogenisation

$$||R_\sigma - R_{\widetilde{\sigma}}||_* = \left| \frac{1}{\widetilde{\sigma}(x)} - \frac{1}{\sigma(x)} \right|_{L^\infty((0,1))}.$$

$$\left(R_\tau f - \sum_{i=1}^{N} a_i R_i f \right)_x = \left(\sum_{i=1}^{N} \frac{a_i}{\sigma_i(x)} - \frac{1}{\tau(x)} \right) \int_0^x f(t)dt \text{ a.e. } (0,1) \quad (10)$$

$$\left| R_\tau - \sum_{i=1}^{N} a_i R_i \right|_* = \left| \sum_{i=1}^{N} \frac{a_i}{\sigma_i(x)} - \frac{1}{\tau(x)} \right|_{L^\infty((0,1))}. \quad (11)$$

This means that, in this $1d$ context, it suffices to apply the greedy algo in L^∞ within the class of coefficients $1/\sigma(x)$.

Multi-dimensional extension?

Work in progress extending the $1d$ result to the multi-dimensional case by means of sharp properties of Green functions.