CONVERGENCE OF VECTOR-VALUED DIRICHLET SERIES AND THE FOURIER TYPE OF A BANACH SPACE

DANIEL CARANDO

Abstract. We present an informal account on the relationship between vector-valued Dirichlet series and Fourier series. We show that the Fourier type of a Banach space X is equivalent to a summability property of the coefficients of certain X-valued Dirichlet series. To do this, we give a simplified approach to Bohr's original ideas, which he used to study absolute convergence of ordinary Dirichlet series. In order to keep things simple, we restrict most of our results to Dirichlet polynomials (i.e., finite Dirichlet series).

Introduction

The aim of this note is to present the relationship between the concept of Fourier type of a Banach space and the convergence of vector-valued Dirichlet series. Rather than giving the sharpest results, we prefer to emphasize on the connections between the different concepts involved, giving a rather informal treatment of the subject.

In the beginnings of the 20th century, the great Danish mathematician and football player (great in both activities) Harald Bohr gave important steps in the understanding of Dirichlet series and their regions of convergence, uniform convergence and absolute convergence [4, 5]. Problems of this kind attracted the attention of many important mathematics of the time, as Landau, Toeplitz, Hardy, Littlewood, Hille and, of course, Bohr.

One of the problems Bohr was interested in, named afterwards as the *Bohr's absolute convergence problem*, can be loosely stated as follows: to establish the maximum size of the region where a Dirichlet series converges uniformly but not absolutely. Among his many deep contributions, a brilliant (and beautiful) idea was to connect the problem of convergence of Dirichlet series to a question of convergence of holomorphic functions in infinitely many variables. Considering that in those years Banach space theory had not made their entrance yet, to attack a one variable problem by translating it to a infinite dimensional one was a courageous idea. And the most important: it proved to be a very useful one, and many years later it regained importance for its applications to different problems in functional analysis, harmonic analysis and number theory.

In recent years, the study of vector-valued Dirichlet series have attracted the interest of many researchers, as well as vector-valued holomorphic functions on infinite dimensional Banach spaces. Variants of Bohr's absolute convergence problem have been studied in different spaces of Dirichlet series and the original ideas of Bohr were adapted to these new frameworks [6, 7, 9, 11, 12].

We present here a simplified approach to the convergence of vector-valued Dirichlet series, which makes no use of the theory of holomorphic functions in infinitely many variables nor Banach space theory. Instead, we will relate Dirichlet polynomials (i.e., finite Dirichlet series) with trigonometric polynomials in finitely many variables. Since we will use only some basic theory, some of the presented results will be weaker than what can be obtained with tools from local theory of Banach spaces, infinite dimensional complex analysis, etc. In this way, we can present some self contained results and, most important, we can highlight the relationship between Dirichlet series and Fourier series (or trigonometric polynomials) in many variables, which our main goal.

In Section 1 we give a very brief introduction to Dirichlet series and Bohr's absolute convergence problem. Section 2 intends to be a motivation to the study of vector-valued Hausdorff-Young inequalities as a tool to understand the absolute convergence of Dirichlet series with coefficients in a Banach space. In Section 3 we recall two versions of Hausdorff-Young inequalities and define Fourier type as a way to extend these ideas to the study of Fourier series with coefficients in Banach spaces. In Section 4 we define some Hardy spaces of Dirichlet series and show that Fourier type is equivalent to some summability property of the coefficients of Dirichlet series belonging to these families.

A deep and complete treatment of Fourier type and vector-valued Hausdorff-Young inequalities can be found in [13]. We refer the reader to [2] for a nice survey on Bohr's work on Dirichlet series and to [8] for an account of results on their convergence for both the scalar and vector-valued case. Finally, the forthcoming book [10] will be a fundamental reference for the subject.

1. Some words on the convergence of Dirichlet series

A (scalar valued) Dirichlet series is a formal series $D = D(s)$ of the form

$$D = \sum_n a_n \frac{1}{n^s}$$

with coefficients $a_n \in \mathbb{C}$ and variable s in some region of $\mathbb{C}$. Of course, the most famous Dirichlet series is the Riemann zeta function $\zeta(s) = \sum_n \frac{1}{n^s}$.

The convergence of power series is a very well understood issue and is part of the background knowledge of every mathematitian. If a power series converges (or converges absolutely) at some $z_0 \in \mathbb{C}$, then it converges absolutely for every $z \in \mathbb{C}$ with $|z| < |z_0|$. Then the natural domains for convergence of power series are disks, and it makes sense to think of radius of convergence and radius of absolute convergence. As we know, these two radii coincide, and it turns out to be the supremum of all radii of uniform convergence. Moreover, in the open disk of convergence the power series defines a holomorphic function which is bounded in any smaller disk (since it is the uniform limit of polynomials).

The situation for Dirichlet series turns out to be quite different. If a Dirichlet series D converges (or converges absolutely) at some $s_0 \in \mathbb{C}$, then it converges (or converges absolutely) at every $s \in \mathbb{C}$ with $\operatorname{Re} s > \operatorname{Re} s_0$. This means that, while disk are the regions of convergence of power series, half-planes of the form

$$[\operatorname{Re} s > \sigma] := \{s \in \mathbb{C} : \operatorname{Re} s > \sigma\}$$

are the regions of convergence of Dirichlet series. Another important difference with power series is that the regions of convergence and absolute convergence do not necessarily coincide: consider $D = \sum_n (-1)^n \frac{1}{n^s}$. The largest open half-plane where D converges absolutely is $[\operatorname{Re} s > 1]$ while, by Leibniz's criterion for alternate series, the series converges in $[\operatorname{Re} s > 0]$. Then, for this series, the region of non-absolute convergence is a vertial strip of width 1. We can show that this is the extreme case: indeed, if $\sum_n a_n \frac{1}{n^s}$ converges at some s_0 then, necessarily, the sequence $\frac{|a_n|}{n^{\operatorname{Re} s_0}} \to 0$ is bounded (it actually goes to 0). Then we have

$$\sum_n \frac{|a_n|}{|n^{s_0+1+\varepsilon}|} = \sum_n \frac{|a_n|}{n^{\operatorname{Re} s_0}} \frac{1}{n^{1+\varepsilon}} < \infty.$$

Hence, if D converges in $[\operatorname{Re} s > \sigma]$, it must converge absolutely in $[\operatorname{Re} s > \sigma + 1 + \varepsilon]$. Since this holds for any $\varepsilon > 0$, we conclude that the maximum width of the strip where a Dirichlet series converges but does not converge absolutely is 1.

On its region of convergence, a Dirichlet series D defines a holomorphic function. The main interest of Bohr was to be able to determine the region of uniform convergence of D (which, in general, is different to both regions of convergence and absolute convergence) from the analytic properties of this function [4, 5].

Bohr showed that the width of the strip where a Dirichlet series converges uniformly but not converges absolutely was at most 1/2, and Bohnenblust and Hille [3] showed that this width can be attained. As a consequence, 1/2 is the maximum width of the strip of uniform but not absolute convergence for Dirichlet series.

Bohr was able to reformulate the problem on uniform, non absolute convergence of Dirichlet series into a problem of a more functional analytic flavor. Let $\mathcal{H}^\infty$ be the vector space of all Dirichlet series $D = \sum_n a_n n^{-s}$ that converge in $[\operatorname{Re} s > 0]$ and that define a bounded function there. It can be seen that $\mathcal{H}^\infty$ is a Banach space with the supremum norm given by

$$\Big\| \sum_n a_n n^{-s} \Big\|_{\mathcal{H}_\infty} = \sup_{\operatorname{Re} s > 0} \Big| \sum_{n=1}^{\infty} a_n \frac{1}{n^s} \Big|.$$

What Bohr showed is that finding the maximum width of the strip of uniform but not absolute convergence for Dirichlet series is equivalent to finding the least number S such that D converges absolutely on $[\operatorname{Re} s > S]$ for every $D \in \mathcal{H}^\infty$. Then, Bohr, Bonhenblust and Hille showed was that $S = 1/2$.

We remark the following: to say that every $D \in \mathcal{H}^\infty$ converges absolutely in $[\operatorname{Re} s > 1/2]$ is equivalent to say that, for every $D = \sum_n a_n n^{-s} \in \mathcal{H}^\infty$ we have

$$\sum_{n=1}^{\infty} |a_n| \frac{1}{n^{1/2+\varepsilon}} < \infty. \tag{1}$$

Using Hölder inequality (see, for example, (2) below), one can easily obtain (1) for any Dirichlet series whose coefficients satisfy

$$\sum_{n=1}^{\infty} |a_n|^q < \infty$$

for every $q > 2$. This kind of questions which we will investigate in the vector-valued case: if we know that a series belongs to a certain class, what can we say about the summability of its coefficients?

2. From Dirichlet series to Hausdorff-Young inequalities

The expression of a Dirichlet series

$$D = \sum_n a_n \frac{1}{n^s}$$

makes sense if we take the coefficients $(a_n)_n$ in a Banach space, as well as different questions about convergence. Let us consider a family $\mathcal{D}(X)$ of Dirichlet series with coefficients in the Banach space X, defined on the right half-plane

$$[\operatorname{Re} s > 0] = \{s \in \mathbb{C} : \operatorname{Re}(s) > 0\}.$$

One may wonder which is the largest half-plane (if it exists) where every series in $\mathcal{D}(X)$ converges absolutely. A very important case is when we take $\mathcal{D}(X) = \mathcal{H}^\infty(X)$, the space of all Dirichlet series defining a bounded function on $[\operatorname{Re} z > 0]$. The space $\mathcal{H}^\infty(X)$ is a Banach space with the supremum norm.

We then want to find $\sigma > 0$ such that every $D \in \mathcal{D}(X)$ converges absolutely in $[\operatorname{Re} z > \sigma]$. A first approach, which does not work in general but motivates what comes next), is the following: take $D = \sum_n a_n \frac{1}{n^s}$. Then, for p, q conjugate exponents (i.e., $\frac{1}{q} + \frac{1}{p} = 1$), we have

$$\sum_n \left\| a_n \frac{1}{n^s} \right\| = \sum_n \|a_n\| \frac{1}{n^{\operatorname{Re} s}} \overset{\text{Hölder}}{\leq} \left(\sum_n \|a_n\|^q \right)^{1/q} \left(\sum_n \frac{1}{n^{p \operatorname{Re} s}} \right)^{1/p}. \tag{2}$$

The last series converges if and only if $\operatorname{Re} s > 1/p$. Therefore, a result like

The coefficients of any Dirichlet series in $\mathcal{D}(X)$ are q-summable

would imply that every series in $\mathcal{D}(X)$ converges absolutely in $[\operatorname{Re} s > 1/p]$. Note that, if $\mathcal{D}(X)$ is a Banach space with some norm $\|\cdot\|_{\mathcal{D}(X)}$, such a result is usually equivalent to the existence of a constant $C > 0$ such that

$$\left(\sum_n \|a_n\|^q\right)^{1/q} \leq C\, \|D\|_{\mathcal{D}(X)}, \tag{3}$$

for every $D = \sum_n a_n \frac{1}{n^s} \in \mathcal{D}(X)$.

We consider the space $\mathcal{H}^\infty(X)$ whith the norm

$$\|D\|_{H^\infty(X)} = \sup_{\operatorname{Re} s>0} \|D(s)\|.$$

It can be seen (using the maximum modulus principle in a tricky way, see [10, Lemma 2.7]) that if $D = \sum_n a_n \frac{1}{n^s}$ is a finite sum, then we actually have

$$\|D\|_{\mathcal{H}^\infty(X)} = \sup_{\operatorname{Re} s>0} \|D(s)\| = \sup_{\operatorname{Re} s=0} \|D(s)\| = \sup_{t\in\mathbb{R}} \Big\|\sum_n a_n \frac{1}{n^{it}}\Big\|.$$

Given a sequence $(x_k)_k \subset X$, we define for each $N \in \mathbb{N}$ the Dirichlet series D_N with coefficients

$$a_n = \begin{cases} x_k & \text{if } n = 2^k \text{ for } 1 \leq k \leq N \\ 0 & \text{otherwise} \end{cases}.$$

If inequality (3) holds for $\mathcal{D}(X) = \mathcal{H}^\infty(X)$, then we have

$$\begin{aligned}\left(\sum_{k=1}^N \|x_k\|^q\right)^{1/q} &= \left(\sum_n \|a_n\|^q\right)^{1/q} \leq C\, \|D\|_{\mathcal{H}^\infty(X)} = \sup_{t\in\mathbb{R}} \Big\|\sum_n a_n \frac{1}{n^{it}}\Big\| \\ &= \sup_{t\in\mathbb{R}} \Big\|\sum_{k=1}^N x_k \frac{1}{2^{ikt}}\Big\| = \sup_{t\in\mathbb{R}} \Big\|\sum_{k=1}^N x_k e^{-i\log(2)kt}\Big\| \\ &= \sup_{t\in\mathbb{R}} \Big\|\sum_{k=1}^N x_k\; e^{2\pi ikt}\Big\|,\end{aligned}$$

where in the last step we used that, as t goes over $\mathbb{R}$, both $-\log(2)t$ and $2\pi t$ cover $\mathbb{R}$. Since the last expression is clearly 1-periodic, what we get is

$$\left(\sum_{k=1}^N \|x_k\|^q\right)^{1/q} \leq C\, \Big\|\sum_{k=1}^N x_k\; e^{2\pi ikt}\Big\|_{L^\infty([0,1],X)}, \tag{4}$$

where $L^\infty([0,1], X)$ stands the space of essentially bounded functions on $[0,1]$ with values in X. This resembles a vector-valued version of the Hausdorf-Young inequality (see next section):

$$\left(\sum_{k=1}^N \|x_k\|^q\right)^{1/q} \leq C\, \Big\|\sum_{k=1}^N x_k\; e^{2\pi ikt}\Big\|_{L^p([0,1],X)}\;, \tag{5}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $1 \leq p \leq 2$. In opposition to the scalar-valued version, the vector-valued Hausdorff-Young inequality (5) does not necessarily hold for a Banach space X if $p \neq 1$. Banach spaces satisfying it (i.e., that there exists some C for which (5) holds for every N and every sequence $(x_k)_k \subset X$) are those of Fourier type p, which we will define in next section. Of course, if a Banach space satisfies (5), then it satisfies (4). What is not so obvious is that a Banach space satisfies (4) if and only if it satisfies (3) with $\mathcal{D}(X)=\mathcal{H}^\infty(X)$. This means that the validity of Hausdorff-Young inequalities in a Banach space X will give us information about the sumability of the coefficients of Dirichlet series in $\mathcal{H}^\infty(X)$ and, as a consequence, about the absolute convergence of these series.

3. Fourier type of Banach spaces

Let us recall a version of the Hausdorff-Young inequality. In the sequel, $\ell^p=\ell^p(\mathbb{Z})$ will stand for the Banach space of sequences $a=(a_k)_{k\in\mathbb{Z}}$ such that

$$\|a\|_{\ell^p} := \left(\sum_{k\in\mathbb{Z}} |a_k|^p\right)^{1/p} < \infty.$$

For $f \in L^p[0,1]$ and $k \in \mathbb{Z}$ we define the kth Fourier coefficient of f as

$$\hat{f}(k) = \int_{[0,1]} f(t)e^{-2\pi i k t}\, dt.$$

Note that, if f belongs to $L^1[0,1]$, then for each k we have

$$|\hat{f}(k)| \leq \int_{[0,1]} |f(t)e^{-2\pi i k t}|\, dt = \int_{[0,1]} |f(t)|\, dt = \|f\|_{L^1}.$$

As a consequence, we have

$$\|(\hat{f}(k))_{k\in\mathbb{Z}}\|_{\ell^\infty} \leq \|f\|_{L^1}. \tag{6}$$

On the other hand, Parseval's identity gives

$$\|(\hat{f}(k))_{k\in\mathbb{Z}}\|_{\ell^2} = \left(\sum_k |\langle f, e^{2\pi i k \cdot}\rangle|^2\right)^{1/2} = \|f\|_{L^2}. \tag{7}$$

By the Riesz-Thorin interpolation theorem, we obtain from (6) and (7)

$$\|(\hat{f}(k))_{k\in\mathbb{Z}}\|_{\ell^q} \leq \|f\|_{L^p} \tag{8}$$

for $1 \leq p \leq 2$ where, as before, $\frac{1}{p}+\frac{1}{q}=1$. If we take a sequence $(c_k)_{k\in\mathbb{Z}}$ with only finite nonzero entries, then the function $f(t)=\sum_k c_k e^{2\pi i k t}$ belongs to $L^p[0,1]$, we have $\hat{f}(k)=c_k$ and, in this case, (8) reads as

$$\left(\sum_{k\in\mathbb{Z}} |c_k|^q\right)^{1/q} \leq \left(\int_{[0,1]} \left|\sum_k c_k e^{2\pi i k t}\right|^p dt\right)^{1/p}, \tag{9}$$

with the usual modification if $p = 1$ and $q = \infty$. Note that (9) is a scalar-valued version of (5). While (9) holds for $1 \le p \le 2$, the validity (5) for some p is a property of the Banach space X, the name of which is given in Definition 1.

A similar reasoning leads to another Hausdoff-Young inequality. Given $(c_k)_{k\in\mathbb{Z}} \in \ell_1$, the series $\sum_k c_k e^{2\pi ikt}$ converges absolute and uniformly and

$$\Big\| \sum_k c_k e^{2\pi ikt} \Big\|_{L^\infty([0,1])} \le \|c_k\|_{\ell_1}.$$

From this and Parseval's identity (7), we obtain by interpolation:

$$\left(\int_{[0,1]} \Big| \sum_k c_k e^{2\pi ikt} \Big|^q dt \right)^{1/q} \le \left(\sum_{k\in\mathbb{Z}} |c_k|^p \right)^{1/p} \tag{10}$$

for every sequence $(c_k)_{k\in\mathbb{Z}}$ with only finite nonzero entries and $1 \le p \le 2$.

It follows from (9) that, if the series $\sum_k c_k e^{2\pi ikt}$ converges in $L^p([0,1])$, then the coefficients $(c_k)_{k\in\mathbb{Z}}$ must belong to ℓ^q. Indeed, we apply (9) to see that, if

$$\Big(\sum_{-m\le k\le n} c_k e^{2\pi ikt} \Big)_{m,n}$$

is a Cauchy bi-indexed sequence in $L^p([0,1])$, then $\big((\sum_{-m\le k\le n} |c_k|^q)^{1/q} \big)_{m,n}$ is also a Cauchy sequence (in $\mathbb{R}$). Then, $(c_k)_{k\in\mathbb{Z}}$ belongs to ℓ^q.

Analogously, we can see that (10) implies that, if $(c_k)_k$ belongs to ℓ^p, then the series $\sum_k c_k e^{2\pi ikt}$ converges in $L^q([0,1])$.

In the vector-valued case (i.e., when we take $(x_k)_{k\in\mathbb{Z}}$ in some Banach space instead $(c_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$), inequalities like (9) and (10) do not necessarily hold. But it is interesting that one of them holds (with some appropriate constant) for some Banach space X and some $1 \le p \le 2$ if and only the other holds (for the same X and p) [13].

Definition 1. For $1 \le p \le 2$, we say that the Banach space has Fourier type p if there exists a constant $B > 0$ such that, for any finite sequence $(x_k)_{k\in\mathbb{Z}}) \subset X$ we have

$$\left(\int_{[0,1]} \Big\| \sum_k x_k e^{2\pi ikt} \Big\|^q dt \right)^{1/q} \le B \left(\sum_{k\in\mathbb{Z}} \|x_k\|^p \right)^{1/p}. \tag{11}$$

Equivalently, X has Fourier type p if there exists a constant $C > 0$ such that, for any finite sequence $(x_k)_{k\in\mathbb{Z}}) \subset X$ we have

$$\left(\sum_{k\in\mathbb{Z}} \|x_k\|^q \right)^{1/q} \le C \left(\int_{[0,1]} \Big\| \sum_k x_k e^{2\pi ikt} \Big\|^p dt \right)^{1/p}. \tag{12}$$

We will not see the equivalence between these two definitions. For most part of these notes, we can just take (12) as the definition of Fourier type. Let us mention that Peetre's original definition of Fourier type was neither (11) nor (12): a Banach space X has Fourier type p if the X-valued Fourier transform defines a bounded operator from $L^p(\mathbb{R}, X)$ to $L^q(\mathbb{R}, X)$ [18]. The definition of the vector-valued Fourier transform uses integration in Banach spaces (namely, the Bochner integral) and will not be treated here. We just mention that to show the equivalence between (11) and (12) one must go through the definition of Fourier type by Fourier transform.

It is easy to see that every Banach space has Fourier type 1. Using a vector-valued version of the Riesz-Thorin interpolation theorem, it can be shown that if X has Fourier type p_0, then it has Fourier type p for $1 \leq p \leq p_0$. The Hausdorff-Young inequality (9) shows that $\mathbb{C}$ has Fourier type 2. Moreover, it is not hard to see that Hilbert spaces have also Fourier type 2. The converse is much harder: a very deep result of Kwapien [16] asserts that a Banach spaces with Fourier type 2 is isomorphic to a Hilbert space. This means that Hilbert spaces and Banach spaces isomorphic to them are precisely the Banach spaces where vector-valued Hausdorff-Young inequalities hold for $1 \leq p \leq 2$. Note that, in particular, finite dimensional Banach spaces have Fourier type 2.

If a Banach space has Fourier type p, then so does all of its subspaces. More interesting: a Banach space has Fourier type p if and only if its dual has Fourier type p [13].

It is a nice exercise to show that ℓ^r has Fourier type p if and only if $1 \leq p \leq \min(r, r')$. In fat, this holds for every infinite dimensional $L^r(\Omega, \Sigma, \mu)$ (with essentially the same proof). This means, for example, that L^1 and L^∞ spaces have only Fourier type 1 (we say that they have only trivial Fourier type). It is known that Schatten classes $\mathcal{S}_p$ have the same Fourier type as the spaces L^p (see [14, Theorem 1.6] or [15, Theroem 6.8]).

In the previous section, we have seen how a Dirichlet series whose only non-zero coefficients correspond to powers of 2 was related to a Fourier series. In next section we will see that an arbitrary (finite) Dirichlet series is related to a Fourier series in many variables. This means that we will need inequalities similar to (5) or (12) for multivariate Fourier series. For this, we will consider multi-indexed families $(x_\alpha)_{\alpha \in \mathbb{N}_0^N} \subset X$. Note that each $\alpha \in \mathbb{N}_0^N$ is a multi-index of the form

$$\alpha = (\alpha_1, \ldots, \alpha_N), \quad \text{with} \quad \alpha_j \in \mathbb{N} \cup \{0\}.$$

Proposition 2. *A Banach space X has Fourier type p if and only if there exists $C > 0$ such that for every $N \in \mathbb{N}$ and every finite family $(x_\alpha)_{\alpha \in \mathbb{N}_0^N}$ we have*

$$\left(\sum_\alpha \|x_\alpha\|^q \right)^{1/q} \leq C \left(\int_{[0,1]^N} \Big\| \sum_\alpha x_\alpha e^{2\pi i \alpha_1 t_1} \cdots e^{2\pi i \alpha_N t_N} \Big\|^p dt_1 \ldots dt_N \right)^{1/p}. \tag{13}$$

Proof. If X satisfies (13), then taking $N=1$ we almost have that X has Fourier type p: we obtain (12) only for sums with nonnegative indexes, and Fourier type demands it for sums with arbitrary integer indexes. Fixing this problem is left as an easy exercise for the reader.

For the reverse implication, let m be the maximum of all α_j's such that x_α is not zero. Since the exponentials involved in the integral are 1-periodic, fixed $t_1 \in [0,1]$ we have:

$$\begin{aligned}
&\int_{[0,1]^{N-1}} \Big\| \sum_\alpha x_\alpha e^{2\pi i \alpha_1 t_1} e^{2\pi i \alpha_2 t_2} \cdots e^{2\pi i \alpha_N t_N} \Big\|^p dt_2 \cdots dt_N \\
&= \int_{[0,1]^{N-1}} \Big\| \sum_\alpha x_\alpha e^{2\pi i \alpha_1 t_1} e^{2\pi i \alpha_2 (t_2 + (m+1)t_1} \cdots e^{2\pi i \alpha_N t_N + (m+1)^{N+1} t_1} \Big\|^p dt_2 \cdots dt_N \\
&= \int_{[0,1]^{N-1}} \Big\| \sum_\alpha x_\alpha e^{2\pi i t_1 (\alpha_1 + (m+1)\alpha_2 + \cdots + (m+1)^{N-1}\alpha_N)} e^{2\pi i \alpha_2 t_2} \cdots e^{2\pi i \alpha_N t_N} \Big\|^p dt_2 \cdots dt_N .
\end{aligned}$$

As a consequence, a change in the order of integration gives

$$\begin{aligned}
&\int_{[0,1]^N} \Big\| \sum_\alpha x_\alpha e^{2\pi i \alpha_1 t_1} \cdots e^{2\pi i \alpha_N t_N} \Big\|^p dt_1 \dots dt_N \\
(14)\quad &= \int_{[0,1]^{N-1}} \left(\int_{[0,1]} \Big\| \sum_\alpha \left(x_\alpha e^{2\pi i \alpha_2 t_2} \cdots e^{2\pi i \alpha_N t_N} \right) e^{2\pi i t_1 (\alpha_1 + (m+1)\alpha_2 + \cdots + (m+1)^{N-1}\alpha_N)} \Big\|^p dt_1 \right) dt_2 \cdots dt_N .
\end{aligned}$$

For every α for which x_α is not zero we have $0 \le \alpha_j \le m$, $j = 1, \dots, N$. Also, if a multi index β satisfies $0 \le \beta_j \le m$, $j = 1, \dots, N$ and

$$\alpha_1 + (m+1)\alpha_2 + \cdots + (m+1)^{N-1}\alpha_N = \beta_1 + (m+1)\beta_2 + \cdots + (m+1)^{N-1}\beta_N,$$

then we must have $\alpha = \beta$ (this is just the uniqueness of the expansion of a natural number in base $m+1$). Therefore, the integer multiples of $2\pi i t_1$ in (14) are all different. We can then apply (12) to the inner integral in the right-hand side of (14) for each fixed $t_2, \dots, t_N$. This gives that the whole expression in (14) is bounded from below by

$$\frac{1}{C^p} \int_{\mathbb{T}^{N-1}} \left(\sum_\alpha \big\| x_\alpha e^{2\pi i \alpha_2 t_2} \cdots e^{2\pi i \alpha_N t_N} \big\|^q \right)^{p/q} dt_2 \cdots dt_N = \frac{1}{C^p} \left(\sum_\alpha \|x_\alpha\|^q \right)^{p/q} .$$

So (14) is bounded from below by this last expression, which is the result we were looking for. □

It is clear that we could have used the equivalent definition of Fourier type (11) to prove an analogous result: a Banach space X has Fourier type p if and only if there exists a constant $B > 0$ such that for every $N \in \mathbb{N}$ and every finite family $(x_\alpha)_{\alpha \in \mathbb{N}_0^N}$ we have

$$(15)\qquad \left(\int_{[0,1]^N} \Big\| \sum_\alpha x_\alpha e^{2\pi i \alpha_1 t_1} \cdots e^{2\pi i \alpha_N t_N} \Big\|^q dt_1 \dots dt_N \right)^{1/q} \le B \left(\sum_\alpha \|x_\alpha\|^p \right)^{1/p} .$$

4. Hardy spaces of Dirichlet series

Let $p = (p_1, p_2, p_3 \dots)$ be the sequence of prime numbers. For each multi-index $\alpha = (\alpha_1, \dots, \alpha_N, 0, \dots) \in \mathbb{N}_0^{(\mathbb{N})}$ we set

$$p^\alpha = p_1^{\alpha_1} \times \dots \times p_N^{\alpha_N} .$$

We have a one-to-one correspondence

$$\alpha \in \mathbb{N}_0^{(\mathbb{N})} \leftrightsquigarrow n \in \mathbb{N} \qquad \text{where } p^\alpha = n .$$

For $n \in \mathbb{N}$ we write $\alpha(n)$ for the (unique) multi-index such that $p^\alpha(n) = n$.

Recall that we have defined $\mathcal{H}^\infty(X)$ as the space of all Dirichlet series defining a bounded function on $[\operatorname{Re} z > 0]$, which is a Banach space with the supremum norm. Now we describe a great idea of Bohr, adapted to our notation. Fix n and take $\alpha = \alpha(n)$. Then we can write

$$\frac{1}{n^s} = \frac{1}{p^{\alpha s}} = \frac{1}{p_1^{\alpha_1 s}} \cdots \frac{1}{p_1^{\alpha_N s}}.$$

In particular, if $s = it$, we have

$$\frac{1}{n^{it}} = \frac{1}{p_1^{i\alpha_1 t}} \cdots \frac{1}{p_1^{i\alpha_N t}} = e^{-\log(p_1)\alpha_1 t} \cdots e^{-\log(p_N)\alpha_N t}.$$

For $D = \sum_{n=1}^M a_n \frac{1}{n^s}$ a finite Dirichlet series, take N such that for each $n = 1, \dots, M$ we can write

$$n = p_1^{\alpha_1} \cdots p_N^{\alpha_N}$$

for some $\alpha \in \mathbb{N}_0^N$. In other words, we take N such that all primes involved in the factorization of all of thesse n's are contained in the set $\{p_1, \dots, p_N\}$. A not so direct consequence of the maximum modulus principle (see [10, Lemma 2.7]) gives

$$\|D\|_{\mathcal{H}^\infty(X)} = \sup_{t \in \mathbb{R}} \Big\| \sum_{n=1}^M a_n \frac{1}{n^{it}} \Big\|.$$

By the the previous remarks we have

$$\begin{aligned}
\|D\|_{\mathcal{H}^\infty(X)} &= \sup_{t \in \mathbb{R}} \Big\| \sum_{n=1}^M a_n \frac{1}{n^{it}} \Big\| \\
&= \sup_{t \in \mathbb{R}} \Big\| \sum_{\alpha \in \mathbb{N}_0^N} a_{p^\alpha} e^{-\log(p_1)\alpha_1 t} \cdots e^{-\log(p_N)\alpha_N t} \Big\| \\
&= \sup_{t \in \mathbb{R}} \Big\| \sum_{\alpha \in \mathbb{N}_0^N} a_{p^\alpha} (e^{-\log(p_1)t})^{\alpha_1} \cdots (e^{\log(p_N)t})^{\alpha_N} \Big\| \qquad (16)
\end{aligned}$$

By a classical result of Kronecker, the set

$$\{(e^{-\log(p_1)t}, \dots, e^{-\log(p_N)t}) : t \in \mathbb{R}\}$$

is dense in the N-th dimensional torus

$$\{(e^{2\pi i t_1}, \dots, e^{2\pi i t_N}) : t_1, \dots, t_N \in \mathbb{R}\} = \{(e^{2\pi i t_1}, \dots, e^{2\pi i t_N}) : t_1, \dots, t_N \in [0,1]\} .$$

Combining this with (16) we obtain

$$\begin{aligned} \|D\|_{\mathcal{H}^\infty(X)} &= \sup_{(t_1,\dots,t_N)\in[0,1]^N} \Big\| \sum_{\alpha\in\mathbb{N}_0^N} a_{p^\alpha} (e^{2\pi i t_1})^{\alpha_1} \cdots (e^{2\pi i t_N})^{\alpha_N} \Big\| \\ &= \sup_{(t_1,\dots,t_N)\in[0,1]^N} \Big\| \sum_{\alpha\in\mathbb{N}_0^N} a_{p^\alpha} e^{2\pi i \alpha_1 t_1} \cdots e^{2\pi i \alpha_N t_N} \Big\| \\ &= \Big\| \sum_{\alpha\in\mathbb{N}_0^N} a_{p^\alpha} e^{2\pi i \alpha_1 t_1} \cdots e^{2\pi i \alpha_N t_N} \Big\|_{L^\infty([0,1]^N, X)}. \end{aligned}$$

We have obtained the following (weak) version of a fundamental theorem of Bohr [5] (see also [17])

Theorem 3. *Let $D = \sum_{n=1}^M a_n \frac{1}{n^s}$ be a Dirichlet polynomial with coefficients in X and let N be as before. Then*

$$\|D\|_{\mathcal{H}^\infty(X)} = \Big\| \sum_{\alpha\in\mathbb{N}_0^N} a_{p^\alpha} e^{2\pi i \alpha_1 t_1} \cdots e^{2\pi i \alpha_N t_N} \Big\|_{L^\infty([0,1]^N, X)}. \tag{17}$$

We can apply the previous theorem to scalar-valued Dirichlet polynomials. Take $D = \sum_{n=1}^M a_n \frac{1}{n^s}$ with complex coefficients. Observe that the exponentials $\{e^{2\pi i \alpha_1 t_1} \dots e^{2\pi i \alpha_N t_N}\}_{\alpha\in\mathbb{N}_0^N}$ form an orthonormal family in $L^2([0,1]^N$. Then, we have

$$\begin{aligned} \left(\sum_{n=1}^M |a_n|^2 \right)^{1/2} &= \left(\sum_{\alpha\in\mathbb{N}_0^N} |a_{p^\alpha}|^2 \right)^{1/2} \\ &= \Big\| \sum_{\alpha\in\mathbb{N}_0^N} a_{p^\alpha} e^{2\pi i \alpha_1 t_1} \cdots e^{2\pi i \alpha_N t_N} \Big\|_{L^2([0,1]^N)} \\ &\le \Big\| \sum_{\alpha\in\mathbb{N}_0^N} a_{p^\alpha} e^{2\pi i \alpha_1 t_1} \cdots e^{2\pi i \alpha_N t_N} \Big\|_{L^\infty([0,1]^N)} \\ &= \|D\|_{\mathcal{H}^\infty(\mathbb{C})} , \end{aligned}$$

where the first step is just a rearrangement of the sum, in the second we use orthogonality, the third is the relationship between L^2- and L^∞-norms and the last follows from Theorem 3. It is an exercise to check that the same holds if we take Dirichlet polynomials with coefficients in a Hilbert space, which gives the following.

Corollary 4. *Let $D = \sum_{n=1}^M a_n \frac{1}{n^s}$ be a Dirichlet polynomial with coefficients in a Hilbert space H. Then,*

$$\left(\sum_{n=1}^M \|a_n\|^2 \right)^{1/2} \le \|D\|_{\mathcal{H}^\infty(H)}.$$

Some comments on Bohr's transform. We have seen that the subspace of $\mathcal{H}^\infty(X)$ consisting of Dirichlet polynomials of length M is isometrically isomorphic to a subspace of $L^\infty([0,1]^N, X)$, the space of essentially bounded X-valued functions on $[0,1]^N$. The isomorphism is based in the one to one correspondence

$$\alpha \in \mathbb{N}_0^{(\mathbb{N})} \leftrightsquigarrow n \in \mathbb{N} \qquad \text{where } p^\alpha = n\,. \tag{18}$$

Given a sequence $\mathbf{t} = (t_1, t_2, t_3, \dots) \in [0,1]^{\mathbb{N}}$ and a multi-index $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ we can write

$$e^{2\pi i \alpha \mathbf{t}} = e^{2\pi i \alpha_1 t_1} \cdots e^{2\pi i \alpha_N t_N}\,,$$

where N is the length of α.

We denote by $\mathfrak{D}_P$ the set of all Dirichlet polynomials and by $\mathcal{T}_P$ the set of all trigonometric polynomials in any number of variables. Then the relation (18) defines a mapping

$$\begin{array}{cccc} \mathfrak{B}_1\colon & \mathfrak{D}_P & \longrightarrow & \mathcal{T}_P \\ & \sum_n a_n \frac{1}{n^s} & \xrightarrow{c_\alpha = a_{p^\alpha}} & \sum_\alpha c_\alpha e^{2\pi i \alpha \mathbf{t}} \end{array}.$$

The image of this mapping is the set $\mathcal{T}_P^+$ of trigonometric polynomials whose coefficients c_α are nonzero only for those α's whose coordinates are nonnegative. Since all α_j's in every term in the above sum are nonnegative, they actually define polynomials in many variables: for a sequence of complex numbers $z = (z_1, z_2, z_3, \dots)$ and a multi-index $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ we write

$$z^\alpha = z_1^{\alpha_1} \times \cdots \times z_N^{\alpha_N}\,.$$

Then,

$$\sum_\alpha c_\alpha e^{2\pi i \alpha \mathbf{t}} \longleftrightarrow \sum_\alpha c_\alpha z^\alpha$$

is a one-to-one correspondence between $\mathcal{T}_P^+$ and the set of all polynomials of arbitrary number of variables. What Bohr actually showed is that the space $\mathcal{H}^\infty(\mathbb{C})$ is in fact isometrically isomorphic to a Banach space of holomorphic functions in infinitely many variables. The so-called *Bohr transform* is the one-to-one mapping

$$\begin{array}{cccc} \mathfrak{B}\colon & \mathfrak{P} & \longrightarrow & \mathfrak{D} \\ & \sum_\alpha c_\alpha z^\alpha & \xrightarrow{c_\alpha = a_{p^\alpha}} & \sum_n a_n \frac{1}{n^s} \end{array}$$

where $\mathfrak{P}$ is the set of all formal power series and $\mathfrak{D}$ is the set of all formal Dirichlet series (here we are not assuming any kind of convergence whatosever). This is a beautiful connection between two theories, which have regained interest in the last years. We will not go deeper in these questions, and will keep our focus in Dirichlet polynomials and Fourier series in many (but not infinitely many) variables. Note that $\mathfrak{B}$ and $\mathfrak{B}_1$ are in some sense inverse to each other.

A last comment: if we consider the infinite product $[0,1]^{\mathbb{N}}$ with the product measure, it can be seen that $\mathfrak{B}_1$ extends to an isometry from $\mathcal{H}^\infty(X)$ to a subspace of $L^\infty([0,1]^{\mathbb{N}}, X)$.

Definition of Hardy spaces of Dirichlet series. Let us now define, for $1 \leq p < \infty$, the Hardy space $\mathcal{H}^p(X)$ of X-valued Dirichlet series. For a Dirichlet polynomial $D = \sum_{n=1}^{M} a_k n^{-s}$, we define its $\mathcal{H}^p(X)$-norm as:

$$\|D\|_{\mathcal{H}^p(X)} = \lim_{T\to\infty} \Big(\frac{1}{2T}\int_{-T}^{T} \Big\| \sum_{n=1}^{M} a_n \frac{1}{n^{it}} \Big\|^p dt\Big)^{1/p}.$$

This definition does not look very natural, but the Birkhoff-Khinchine ergodic theorem, together with the prime factorization performed above, give that

$$\lim_{T\to\infty} \Big(\frac{1}{2T}\int_{-T}^{T} \Big\| \sum_{n=1}^{M} a_n \frac{1}{n^{it}} \Big\|^p dt\Big)^{1/p} = \Big(\int_{[0,1]^N} \Big\| \sum_{\alpha\in\mathbb{N}_0^N} a_{p^\alpha} e^{2\pi i\alpha_1 t_1} \dots e^{2\pi i \alpha_N t_N} \Big\|^p dt\Big)^{1/p}.$$

See e.g. Bayart [1] for the scalar case, the vector-valued case follows exactly the same way. Equivalently, we have

$$\|D\|_{\mathcal{H}^p(X)} = \Big\| \sum_{\alpha\in\mathbb{N}_0^N} a_{p^\alpha} e^{2\pi i\alpha_1 t_1} \dots e^{2\pi i \alpha_N t_N} \Big\|_{L^p([0,1]^N, X)} \tag{19}$$

In fact, we can take (19) as the definition of the $\mathcal{H}^p(X)$-norm, as most authors do. Note that (17) corresponds to the case $p = \infty$ in (19). The completion of the space of Dirichlet polynomials with this norm is, by definition, the Hardy space $\mathcal{H}_p(X)$. We now state as a theorem the relation between Fourier type and summability of coefficients of Dirichlet series.

Theorem 5. *For a Banach space X and $1 \leq p \leq 2$, the following are equivalent.*
(i) The Banach space X has Fourier type p.
(ii) There exists $C > 0$ such that for any M and any $a_1, \dots, a_M \subset X$ we have

$$\left(\sum_{n=1}^{M} \|a_n\|^q\right)^{1/q} \leq C \,\Big\| \sum_{n=1}^{M} a_n \frac{1}{n^{it}} \Big\|_{\mathcal{H}^p(X)}. \tag{20}$$

(iii) There exists $B > 0$ such that for any M and any $a_1, \dots, a_M \subset X$ we have

$$\Big\| \sum_{n=1}^{M} a_n \frac{1}{n^{it}} \Big\|_{\mathcal{H}^q(X)} \leq B \left(\sum_{n=1}^{M} \|a_n\|^p\right)^{1/p}.$$

Proof. We have done almost all the work. With the previous notation, we have

$$\Big\| \sum_{n=1}^{M} a_n \frac{1}{n^{it}} \Big\|_{\mathcal{H}^p(X)} = \Big\| \sum_{\alpha\in\mathbb{N}_0^N} a_{p^\alpha} e^{2\pi i\alpha_1 t_1} \dots e^{2\pi i \alpha_N t_N} \Big\|_{L^p([0,1]^N, X)}.$$

On the other hand, we clearly have

$$\left(\sum_{n=1}^{N} \|a_n\|^q\right)^{1/q} = \left(\sum_{\alpha\in\mathbb{N}_0^N} \|a_{p^\alpha}\|^q\right)^{1/q},$$

since we are just rearranging the sum. Therefore, the equivalence between (i) and (ii) follows from Proposition 2. The equivalence between (i) with (iii) follows follows analogously, using (11) and (15). $\square$

If X has Fourier type p and the series $\sum_{n=1}^{\infty} a_n \frac{1}{n^{it}}$ converges in $\mathcal{H}^p(X)$, we deduce from Theorem 5 that

$$\left(\sum_{n=1}^{\infty} \|a_n\|^q\right)^{1/q} < \infty.$$

This follows as we did in Section 3 for Fourier series, just applying (20) to differences of partial sums. We can, then, proceed as in the first section (see (2)) to get that every such Dirichlet series converges absolutely in the half-plane $[\operatorname{Re} s > 1/p]$. Note also that, if $p \leq r \leq \infty$, then $\mathcal{H}^r(X) \subset \mathcal{H}^p(X)$ continuously. As a consequence, we obtain the following result (which is weaker that what is known, see the comments below, but which demands no work to prove at this point).

Corollary 6. *If X has Fourier type p, then for every $r \geq p$, every Dirichlet series converging in $\mathcal{H}^r(X)$, converges absolutely in the half-plane* $[\operatorname{Re} s > 1/p]$.

We mention that this last result holds under much more general assumptions: if X have *cotype* q, then every Dirichlet series in $\mathcal{H}^r(X)$ converges absolutely in the half-plane $[\operatorname{Re} s > 1/p]$ for every $1 \leq r \leq \infty$ (see [9] for the case $r = \infty$ and [6] for $1 \leq r < \infty$). This involves concepts and results from Banach space theory and infinite dimensional holomorphy.

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Departamento de Matemática, FCEyN, Universidad de Buenos Aires - IMAS, CONICET (Argentina)

E-mail address: dcarando@dm.uba.ar