SPECTRAL APPROXIMATION OF THE CURL OPERATOR IN MULTIPLY CONNECTED DOMAINS

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Abstract. A numerical scheme based on Nédélec finite elements has been recently introduced to solve the eigenvalue problem for the curl operator in simply connected domains. This topological assumption is not just a technicality, since the eigenvalue problem is ill-posed on multiply connected domains, in the sense that its spectrum is the whole complex plane. However, additional constraints can be added to the eigenvalue problem in order to recover a well-posed problem with a discrete spectrum. Vanishing circulations on each non-bounding cycle of the domain have been chosen as additional constraints in this paper. A mixed weak formulation including a Lagrange multiplier (that turns out to vanish) is introduced and shown to be well-posed. This formulations is discretized by Nédélec elements, while standard finite elements are used for the Lagrange multiplier. Spectral convergence is proved as well as a priori error estimates. It is also shown how to implement this finite element discretization taking care of these additional constraints. Finally, a numerical test to assess the performance of the proposed methods is reported.

1. **Introduction.** Let $\boldsymbol{H}$ be a magnetic field acting on a conducting fluid, whose motion is driven by the so-called *Lorentz force*:

$$\boldsymbol{F} := \boldsymbol{J} \times \boldsymbol{B},$$

where $\boldsymbol{J} := \mathbf{curl}\,\boldsymbol{H}$ is the current density and $\boldsymbol{B} := \mu\boldsymbol{H}$ is the magnetic induction (μ being the magnetic permeability, which in an isotropic medium is a scalar).

Because of this, a magnetic field satisfying

$$\mathbf{curl}\,\boldsymbol{H} = \lambda\boldsymbol{H},$$

with λ a scalar function, is called a *force-free field* [15] or a *Beltrami field* [2]. This kind of fields appear in solar physics for theories on flares and coronal heating [5]

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(see also [6, 15, 16]), in fluids for the study of the static equilibrium of smectic liquid crystals, in plasma physics, superconducting materials, etc.

A magnetic field satisfying the above equation with a constant λ is called a *linear force-free field* or also a *free-decay field*. In the theory of fusion plasma, for instance, such a field is the final state that makes the energy a minimum in order to leave the plasma in equilibrium.

From the mathematical point of view, to find a linear force-free field corresponds to solving an eigenvalue problem for the curl operator:

$$\mathbf{curl}\, \boldsymbol{u} = \lambda \boldsymbol{u}.$$

In a bounded domain Ω, the natural boundary conditions are either $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ or $\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}$. However, because of the Stokes theorem, the latter implies $\mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ and, hence, for $\lambda \neq 0$, $\boldsymbol{u} \cdot \boldsymbol{n} = \frac{1}{\lambda} \mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{n} = 0$ too. Therefore, $\boldsymbol{u}$ would vanish identically on $\partial\Omega$ and, hence, it is possible to prove that $\boldsymbol{u}$ should actually vanish on the whole domain Ω (see [17, Lemma 3]).

Thus we are in principle led to the following spectral problem for the curl operator: Find $\lambda \in \mathbb{C}$ and $\boldsymbol{u} \not\equiv \mathbf{0}$ such that

$$\begin{aligned} \mathbf{curl}\, \boldsymbol{u} &= \lambda \boldsymbol{u} \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The second equation (which for $\lambda \neq 0$ is a consequence of the first one) rules out the trivial solutions $\lambda = 0$, $\boldsymbol{u} = \nabla\varphi$. The numerical approximation of this problem was analyzed in [14] for a simply connected domain Ω. However, when Ω is multiply connected, the set of eigenvalues of this problem is the whole complex plane (see [18, Theorem 2]).

The eigenvalue problem for the curl operator in multiply connected domains was recently analyzed in [9], where it was shown that additional constraints related to the homology of the domain have to be added for the problem to have a discrete spectrum. We consider in this paper one of the choices proposed in that reference. We introduce a variational mixed formulation of the resulting problem and a finite element discretization. We prove that the numerical approximation provides an optimal-order spectral approximation. We also discuss how to implement this numerical method and report some results for a numerical test which allows us to assess its performance.

2. **A well posed eigenvalue problem on a multiply connected domain.**

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz boundary Γ and outer unit normal $\boldsymbol{n}$. We assume that Γ is either smooth or polyhedral.

We restrict our attention to a multiply connected domain and we assume that there exist *cutting surfaces* Σ_j, $j = 1, \dots, J$, such that the *cut domain*

$$\Omega^0 := \Omega \setminus \bigcup_{j=1}^{J} \Sigma_j$$

becomes simply connected. More precisely, we assume that there exist a set $\{\Sigma_j\}_{j=1}^{J}$ of connected open subsets of smooth manifolds satisfying:

- $\Sigma_j \subset \Omega$;
- $\partial\Sigma_j \subset \Gamma$;
- $\bar{\Sigma}_i \cap \bar{\Sigma}_j = \emptyset$, $i \neq j$;

- $\Omega^0 := \Omega \setminus \bigcup_{j=1}^J \Sigma_j$ is simply connected and pseudo-Lipschitz.

We fix a unit normal $\boldsymbol{n}_j$ on each Σ_j and denote its two faces by Σ_j^+ and Σ_j^-, with $\boldsymbol{n}_j$ being the 'outer' normal to $\partial\Omega^0$ along Σ_j^+. For any $\psi \in \mathrm{H}^1(\Omega^0)$, we denote by $[\![\psi]\!]_{\Sigma_j} := \psi|_{\Sigma_j^-} - \psi|_{\Sigma_j^+}$ the jump of ψ through Σ_j along $\boldsymbol{n}_j$. We denote by γ_j the curves $\partial\Sigma_j$ and by $\boldsymbol{t}_j$ the corresponding tangent unit vector oriented counterclockwise with respect to Σ_j^+.

The set $\{\gamma_j\}_{j=1}^J$ is a family of independent non-bounding cycles of Ω (i.e., the union of the cycles of any non-empty subfamily cannot be the boundary of a surface contained in Ω). A similar family of independent non-bounding cycles $\{\gamma_j'\}_{j=1}^J$ can be given for the complement Ω' of Ω, each cycle γ_j' being the boundary of a cutting surface Σ_j' of Ω'. We denote by $\boldsymbol{t}_j'$ a corresponding tangent unit vector (see Figure 1).

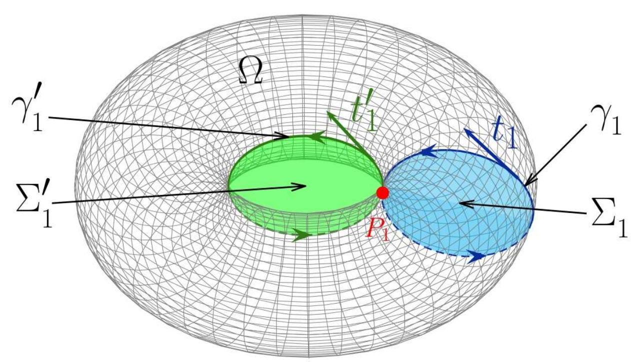

FIGURE 1. A multiply connected domain. Notation.

In order to obtain a well posed eigenvalue problem for the curl operator on a multiple connected domain, one additional constraint per cutting surface must be added. Whenever one can associate one cutting surface Σ_j' of Ω' with one cutting surface Σ_j of Ω (as in Figure 1), there are two alternatives for this additional constraint (see [9]): either $\int_{\gamma_j} \boldsymbol{u}\cdot\boldsymbol{t}_j = 0$ or $\int_{\gamma_j'} \boldsymbol{u}\cdot\boldsymbol{t}_j' = 0$. We focus on the first one which, according to the Stokes Theorem, can be equivalently written as follows:

$$\int_{\Sigma_j} \mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n}_j = 0, \quad 1 \le j \le J.$$

Since for an eigenfunction of the curl operator with eigenvalue $\lambda \neq 0$, $\boldsymbol{u} = \frac{1}{\lambda}\,\mathbf{curl}\,\boldsymbol{u}$, we also have

$$\int_{\Sigma_j} \boldsymbol{u}\cdot\boldsymbol{n}_j = 0, \quad 1 \le j \le J,$$

which lead us to the following eigenvalue problem, whose analysis and numerical approximation is our goal:

Problem 1. *Find* $\lambda \in \mathbb{C}$ *and* $\boldsymbol{u} \in \mathrm{L}^2(\Omega)^3$, $\boldsymbol{u} \not\equiv \mathbf{0}$, *such that*

$$\begin{aligned} \mathbf{curl}\, \boldsymbol{u} &= \lambda \boldsymbol{u} \quad &&\text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 \quad &&\text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} &= 0 \quad &&\text{on } \partial\Omega, \\ \int_{\Sigma_j} \boldsymbol{u} \cdot \boldsymbol{n}_j &= 0, \quad &&1 \leq j \leq J. \end{aligned}$$

Let us remark that the last equation above makes sense since, as will be shown below (cf. (2)), the first three equations imply that $\boldsymbol{u} \cdot \boldsymbol{n}_j \in \mathrm{L}^2(\Sigma_j)$.

3. **Function spaces.** In this section, we introduce some function spaces appropriate for setting and analyzing a convenient variational formulation of Problem 1. First, we recall the definitions of some classical spaces that will be used in the sequel:

$$\begin{aligned} &\mathrm{L}^2(\Omega) := \left\{ v : \Omega \to \mathbb{C} : \ \textstyle\int_\Omega |v|^2 < \infty \right\}, \\ &\mathrm{H}^1(\Omega) := \left\{ v \in \mathrm{L}^2(\Omega) : \nabla v \in \mathrm{L}^2(\Omega)^3 \right\}, \\ &\mathrm{H}(\operatorname{div}, \Omega) := \left\{ \boldsymbol{v} \in \mathrm{L}^2(\Omega)^3 : \ \operatorname{div} \boldsymbol{v} \in \mathrm{L}^2(\Omega) \right\}, \\ &\mathrm{H}(\operatorname{div}^0, \Omega) := \left\{ \boldsymbol{v} \in \mathrm{H}(\operatorname{div}, \Omega) : \ \operatorname{div} \boldsymbol{v} = 0 \ \text{ in } \ \Omega \right\}, \\ &\mathrm{H}_0(\operatorname{div}, \Omega) := \left\{ \boldsymbol{v} \in \mathrm{H}(\operatorname{div}, \Omega) : \ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \ \text{ on } \ \partial\Omega \right\}, \\ &\mathrm{H}_0(\operatorname{div}^0, \Omega) := \left\{ \boldsymbol{v} \in \mathrm{H}(\operatorname{div}^0, \Omega) : \ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \ \text{ on } \ \partial\Omega \right\}, \\ &\mathrm{H}(\mathbf{curl}, \Omega) := \left\{ \boldsymbol{v} \in \mathrm{L}^2(\Omega)^3 : \mathbf{curl}\, \boldsymbol{v} \in \mathrm{L}^2(\Omega)^3 \right\}, \\ &\mathrm{H}(\mathbf{curl}^0, \Omega) := \left\{ \boldsymbol{v} \in \mathrm{H}(\mathbf{curl}, \Omega) : \ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0} \ \text{ in } \ \Omega \right\}, \\ &\mathrm{H}_0(\mathbf{curl}, \Omega) := \left\{ \boldsymbol{v} \in \mathrm{H}(\mathbf{curl}, \Omega) : \ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \ \text{ on } \ \partial\Omega \right\}. \end{aligned}$$

The spaces $\mathrm{H}(\operatorname{div}, \Omega)$ and $\mathrm{H}(\mathbf{curl}, \Omega)$ are respectively endowed with the norms defined by

$$\|\boldsymbol{v}\|_{\operatorname{div},\Omega}^2 := \|\boldsymbol{v}\|_{0,\Omega}^2 + \|\operatorname{div} \boldsymbol{v}\|_{0,\Omega}^2 \qquad \text{and} \qquad \|\boldsymbol{v}\|_{\mathbf{curl},\Omega}^2 := \|\boldsymbol{v}\|_{0,\Omega}^2 + \|\mathbf{curl}\, \boldsymbol{v}\|_{0,\Omega}^2 .$$

We recall the classical Helmholtz decomposition (cf. [8, Theorem I.2.7]):

$$\mathrm{L}^2(\Omega)^3 = \mathrm{H}_0(\operatorname{div}^0, \Omega) \overset{\perp}{\oplus} \nabla(\mathrm{H}^1(\Omega)). \tag{1}$$

Here and thereafter, the symbol $\perp$ is used to denote $\mathrm{L}^2(\Omega)^3$-orthogonality.

We will also use the fractional Sobolev spaces $\mathrm{H}^s(\Omega)$ $(0 < s < 1)$ endowed with the norms $\|\cdot\|_{s,\Omega}$, which are well known to satisfy

$$\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{H}^s(\Omega) \hookrightarrow \mathrm{L}^2(\Omega),$$

both inclusions being compact (see, for instance, [8, Section I.1.1]), and the space

$$\mathrm{H}^s(\mathbf{curl}, \Omega) := \left\{ \boldsymbol{v} \in \mathrm{H}^s(\Omega)^3 : \ \mathbf{curl}\, \boldsymbol{v} \in \mathrm{H}^s(\Omega)^3 \right\}.$$

Let us remark that, according to [1, Proposition 3.7], there exists $s > \frac{1}{2}$ such that

$$\mathrm{H}(\mathbf{curl}, \Omega) \cap \mathrm{H}_0(\operatorname{div}^0, \Omega) \hookrightarrow \mathrm{H}^s(\Omega)^3, \tag{2}$$

the inclusion being continuous.

Let $\Omega^0 := \Omega \backslash \bigcup_{j=1}^J \Sigma_j$ be the cut domain, with Σ_j the cutting surfaces and $\boldsymbol{n}_j$ the corresponding unit normal vectors as defined above. Let $\langle \cdot, \cdot \rangle_{\Sigma_j}$ denote the duality pairing between $\mathrm{H}^{1/2}(\Sigma_j)'$ and $\mathrm{H}^{1/2}(\Sigma_j)$. The following Green's identity has been proved in [1, Lemma 3.10].

Lemma 3.1. *For all* $\boldsymbol{v} \in \mathrm{H}_0(\mathrm{div}, \Omega)$, $\boldsymbol{v} \cdot \boldsymbol{n}_j|_{\Sigma_j} \in \mathrm{H}^{1/2}(\Sigma_j)'$, $1 \leq j \leq J$, *and the following Green's formula holds true:*

$$\sum_{j=1}^{J} \left\langle \boldsymbol{v} \cdot \boldsymbol{n}_j, [\![\psi]\!]_{\Sigma_j} \right\rangle_{\Sigma_j} = \int_{\Omega^0} \boldsymbol{v} \cdot \nabla\psi + \int_{\Omega^0} (\mathrm{div}\, \boldsymbol{v})\, \psi \qquad \forall \psi \in \mathrm{H}^1(\Omega^0).$$

In general, the functions $\psi \in \mathrm{H}^1(\Omega^0)$ do not admit an extension to the whole Ω that lies in the space $\mathrm{H}^1(\Omega)$. However, any extension of $\nabla\psi$ obviously belongs to $\mathrm{L}^2(\Omega)^3$. We denote such an extension $\widetilde{\nabla}\psi$. Let

$$\Theta := \left\{ \psi \in \mathrm{H}^1(\Omega^0) : \ [\![\psi]\!]_{\Sigma_j} = \text{constant}, \ 1 \leq j \leq J \right\}. \tag{3}$$

It has been proved in [1, Lemma 3.11] that, for all $\psi \in \mathrm{H}^1(\Omega^0)$, $\widetilde{\nabla}\psi \in \mathrm{H}(\mathbf{curl}, \Omega)$ if and only if $\psi \in \Theta$, in which case $\mathbf{curl}(\widetilde{\nabla}\psi) = \mathbf{0}$. Thus, we have the following characterization for the space $\mathrm{H}(\mathbf{curl}^0, \Omega)$.

Lemma 3.2. *There holds*

$$\mathrm{H}(\mathbf{curl}^0, \Omega) = \widetilde{\nabla}\Theta.$$

Next, let us consider the space of the so-called *harmonic Neumann fields*:

$$\boldsymbol{\mathcal{K}}_T(\Omega) := \mathrm{H}(\mathbf{curl}^0, \Omega) \cap \mathrm{H}_0(\mathrm{div}^0, \Omega). \tag{4}$$

This is a finite-dimensional space, its dimension being equal to the number of cutting surfaces, as shown in the following lemma whose proof is essentially contained in [7, Lemma 1.3].

Lemma 3.3. *A basis of the space* $\boldsymbol{\mathcal{K}}_T(\Omega)$ *is given by* $\left\{\widetilde{\nabla}\phi_j\right\}_{j=1}^{J}$, *where* $\phi_j \in \Theta/\mathbb{R}$ *is the unique solution of*

$$\begin{aligned} \Delta\phi_j &= 0 \quad \text{in } \Omega^0, \\ \partial_n\phi_j &= 0 \quad \text{on } \Gamma, \\ [\![\partial_n\phi_j]\!]_{\Sigma_k} &= 0, \quad 1 \leq k \leq J, \\ [\![\phi_j]\!]_{\Sigma_k} &= \delta_{j,k}, \quad 1 \leq k \leq J. \end{aligned}$$

Consequently, $\dim(\boldsymbol{\mathcal{K}}_T(\Omega)) = J$.

We can use the space of the harmonic Neumann fields to write a convenient direct decomposition of $\mathrm{H}(\mathbf{curl}^0, \Omega)$.

Lemma 3.4. *There holds*

$$\mathrm{H}(\mathbf{curl}^0, \Omega) = \boldsymbol{\mathcal{K}}_T \overset{\perp}{\oplus} \nabla(\mathrm{H}^1(\Omega)).$$

Proof. Since $\nabla(\mathrm{H}^1(\Omega)) \subset \mathrm{H}(\mathbf{curl}^0, \Omega)$, from the Helmholtz decomposition (1) and the definition (4) of $\boldsymbol{\mathcal{K}}_T$, we have that

$$\mathrm{H}(\mathbf{curl}^0, \Omega) = \mathrm{H}(\mathbf{curl}^0, \Omega) \cap \left[\mathrm{H}_0(\mathrm{div}^0, \Omega) \overset{\perp}{\oplus} \nabla(\mathrm{H}^1(\Omega))\right] = \boldsymbol{\mathcal{K}}_T \overset{\perp}{\oplus} \nabla(\mathrm{H}^1(\Omega)),$$

as claimed. $\square$

Next step is to define another three function spaces which will play a central role in the forthcoming analysis. Let

$$\boldsymbol{\mathcal{X}} := \boldsymbol{\mathcal{K}}_T^{\perp_{\mathrm{H}_0(\mathrm{div}^0,\Omega)}}, \text{ endowed with } \|\cdot\|_{\mathrm{div},\Omega} = \|\cdot\|_{0,\Omega}\,, \tag{5}$$

$$\boldsymbol{\mathcal{Z}} := \{\boldsymbol{v} \in \mathrm{H}(\mathbf{curl},\Omega) : \ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{\mathcal{X}}\}\,, \text{ endowed with } \|\cdot\|_{\mathbf{curl},\Omega} \tag{6}$$

and

$$\boldsymbol{\mathcal{V}} := \boldsymbol{\mathcal{X}} \cap \boldsymbol{\mathcal{Z}}, \text{ endowed with } \|\cdot\|_{\mathbf{curl},\Omega}\,. \tag{7}$$

The following result follows immediately from the Helmholtz decomposition (1), the definition (5) of $\boldsymbol{\mathcal{X}}$ and Lemma 3.4.

Lemma 3.5. *The following decomposition holds true:*

$$\mathrm{L}^2(\Omega)^3 = \underbrace{\boldsymbol{\mathcal{X}} \overset{\perp}{\oplus} \boldsymbol{\mathcal{K}}_T}_{H_0(\mathrm{div}^0;\Omega)} \overset{\perp}{\oplus} \nabla(\mathrm{H}^1(\Omega)) = \boldsymbol{\mathcal{X}} \overset{\perp}{\oplus} \underbrace{\boldsymbol{\mathcal{K}}_T \overset{\perp}{\oplus} \nabla(\mathrm{H}^1(\Omega))}_{\mathrm{H}(\mathbf{curl}^0,\Omega)},$$

the three subspaces being mutually $\mathrm{L}^2(\Omega)^3$*-orthogonal.*

Moreover, we have the following characterization of the functions in $\boldsymbol{\mathcal{X}}$.

Lemma 3.6. *For* $\boldsymbol{v} \in \mathrm{L}^2(\Omega)^3$*,* $\boldsymbol{v} \in \boldsymbol{\mathcal{X}}$ *if and only if*

$$\begin{aligned}
\mathrm{div}\,\boldsymbol{v} &= 0 \quad \text{in } \Omega,\\
\boldsymbol{v}\cdot\boldsymbol{n} &= 0 \quad \text{on } \Gamma,\\
\langle \boldsymbol{v}\cdot\boldsymbol{n}_j, 1\rangle_{\Sigma_j} &= 0, \quad 1 \le j \le J.
\end{aligned}$$

Proof. According to the definition (5), $\boldsymbol{v} \in \boldsymbol{\mathcal{X}}$ if and only if $\boldsymbol{v} \in \mathrm{H}_0(\mathrm{div}^0,\Omega)$ (i.e., $\boldsymbol{v}$ satisfies the first two equations of the lemma) and $\int_\Omega \boldsymbol{v}\cdot\boldsymbol{w} = 0$ for all $\boldsymbol{w} \in \boldsymbol{\mathcal{K}}_T$. Now, from Lemma 3.3, the latter is true if and only if $\int_\Omega \boldsymbol{v}\cdot\nabla\phi_j = 0$, $1 \le j \le J$, which in turn, by virtue of Lemma 3.1, is equivalent to

$$0 = \int_\Omega \boldsymbol{v}\cdot\nabla\phi_j = \sum_{k=1}^{J} \left\langle \boldsymbol{v}\cdot\boldsymbol{n}_k, [\![\phi_j]\!]_{\Sigma_k} \right\rangle_{\Sigma_k} - \int_{\Omega^0} (\mathrm{div}\,\boldsymbol{v})\,\phi_j = \left\langle \boldsymbol{v}\cdot\boldsymbol{n}_j, 1\right\rangle_{\Sigma_j}, \quad 1 \le j \le J.$$

Thus, we conclude the claimed equivalence. □

Characterizations of the functions in $\boldsymbol{\mathcal{Z}}$ and $\boldsymbol{\mathcal{V}}$ follow immediately from this lemma.

Corollary 3.7. *For* $\boldsymbol{v} \in \mathrm{H}(\mathbf{curl},\Omega)$*,* $\boldsymbol{v} \in \boldsymbol{\mathcal{Z}}$ *if and only if*

$$\begin{aligned}
\mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{n} &= 0 \quad \text{on } \Gamma,\\
\langle \mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{n}_j, 1\rangle_{\Sigma_j} &= 0, \quad 1 \le j \le J.
\end{aligned}$$

Corollary 3.8. *For* $\boldsymbol{v} \in \mathrm{H}(\mathbf{curl},\Omega)$*,* $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}$ *if and only if*

$$\begin{aligned}
\mathrm{div}\,\boldsymbol{v} &= 0 \quad \text{in } \Omega,\\
\boldsymbol{v}\cdot\boldsymbol{n} &= 0 \quad \text{on } \Gamma,\\
\langle \boldsymbol{v}\cdot\boldsymbol{n}_j, 1\rangle_{\Sigma_j} &= 0, \quad 1 \le j \le J,\\
\mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{n} &= 0 \quad \text{on } \Gamma,\\
\langle \mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{n}_j, 1\rangle_{\Sigma_j} &= 0, \quad 1 \le j \le J.
\end{aligned}$$

In the following section we will introduce a variational formulation of Problem 1, which will be used for the theoretical analysis of this problem as well as for its finite element discretization. In this formulation, the eigenfunctions will be sought in the space $\boldsymbol{\mathcal{Z}}$. In what follows, we will establish several additional properties of this space that will be used in the sequel.

Lemma 3.9. *The following decomposition holds true:*

$$\boldsymbol{\mathcal{Z}} = \boldsymbol{\mathcal{V}} \overset{\perp}{\oplus} \mathrm{H}(\mathbf{curl}^0, \Omega).$$

Proof. The result follows from the facts that $\mathrm{L}^2(\Omega)^3 = \boldsymbol{\mathcal{X}} \overset{\perp}{\oplus} \mathrm{H}(\mathbf{curl}^0, \Omega)$ (cf. Lemma 3.5), $\mathrm{H}(\mathbf{curl}^0, \Omega) \subset \boldsymbol{\mathcal{Z}}$ (cf. Corollary 3.7) and $\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{X}} \cap \boldsymbol{\mathcal{Z}}$ (cf. (7)). □

Lemma 3.10. *There holds*

$$\mathcal{D}(\Omega)^3 \subset \boldsymbol{\mathcal{Z}}.$$

Proof. Let $\boldsymbol{v} \in \mathcal{D}(\Omega)^3$. According to Corollary 3.7, we only have to prove that $\langle \mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n}_j, 1\rangle_{\Sigma_j} = 0$, $1 \leq j \leq J$, and this is a consequence of the Stokes theorem and the fact that, since $\boldsymbol{v}$ vanishes on $\Gamma \supset \gamma_j := \partial\Sigma_j$,

$$\int_{\Sigma_j} \mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n}_j = \int_{\gamma_j} \boldsymbol{v} \cdot \boldsymbol{t}_j = 0.$$

□

The following commuting property will be the basis for the spectral characterization of the problem. It has been proved in [18, Theorem 1] in a more general context (see also [11, Prop. 2.3]). For the sake of completeness, we include an elementary proof.

Lemma 3.11. *For all $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{\mathcal{Z}}$,*

$$\int_\Omega (\mathbf{curl}\, \boldsymbol{v} \cdot \bar{\boldsymbol{w}} - \boldsymbol{v} \cdot \mathbf{curl}\, \bar{\boldsymbol{w}}) = 0.$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{\mathcal{Z}}$. Then, $\mathbf{curl}\, \boldsymbol{v} \in \mathrm{H}_0(\mathrm{div}^0, \Omega)$ and $\langle \mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n}_j, 1\rangle_{\Sigma_j} = 0$, $1 \leq j \leq J$ (cf. Corollary 3.7). Hence, as has been proved in [1, Theorem 3.17], there exists $\boldsymbol{\zeta} \in \mathrm{H}_0(\mathbf{curl}, \Omega) \cap \mathrm{H}(\mathrm{div}^0, \Omega)$ such that

$$\mathbf{curl}\, \boldsymbol{\zeta} = \mathbf{curl}\, \boldsymbol{v} \quad \text{in } \Omega.$$

Therefore, we have that

$$\int_\Omega (\mathbf{curl}\, \boldsymbol{v} \cdot \bar{\boldsymbol{w}} - \boldsymbol{v} \cdot \mathbf{curl}\, \bar{\boldsymbol{w}}) = \int_\Omega (\mathbf{curl}\, \boldsymbol{\zeta} \cdot \bar{\boldsymbol{w}} - \boldsymbol{\zeta} \cdot \mathbf{curl}\, \bar{\boldsymbol{w}}) + \int_\Omega (\boldsymbol{v} - \boldsymbol{\zeta}) \cdot \mathbf{curl}\, \bar{\boldsymbol{w}}$$

for all $\boldsymbol{w} \in \boldsymbol{\mathcal{Z}}$. Now, since $\boldsymbol{\zeta} \in \mathrm{H}_0(\mathbf{curl}, \Omega)$, for all $\boldsymbol{w} \in \mathrm{H}^1(\Omega)^3$,

$$\int_\Omega (\mathbf{curl}\, \boldsymbol{\zeta} \cdot \bar{\boldsymbol{w}} - \boldsymbol{\zeta} \cdot \mathbf{curl}\, \bar{\boldsymbol{w}}) = \langle \boldsymbol{\zeta} \times \boldsymbol{n}, \bar{\boldsymbol{w}} \rangle_\Gamma = 0.$$

Moreover, because of the density of $\mathrm{H}^1(\Omega)^3$ in $\mathrm{H}(\mathbf{curl}, \Omega)$, the above equation holds also true for all $\boldsymbol{w} \in \boldsymbol{\mathcal{Z}}$. Finally, $\int_\Omega (\boldsymbol{v} - \boldsymbol{\zeta}) \cdot \mathbf{curl}\, \bar{\boldsymbol{w}}$ also vanishes for all $\boldsymbol{w} \in \boldsymbol{\mathcal{Z}}$, because $\mathbf{curl}\, \boldsymbol{w} \in \boldsymbol{\mathcal{X}} = \mathrm{H}(\mathbf{curl}^0, \Omega)^{\perp_{\mathrm{L}^2(\Omega)^3}}$ (cf. definition (6) and Lemma 3.5) and $(\boldsymbol{v} - \boldsymbol{\zeta}) \in \mathrm{H}(\mathbf{curl}^0, \Omega)$. □

Lemma 3.12. *The subspace $\mathcal{C}^\infty(\bar{\Omega})^3 \cap \boldsymbol{\mathcal{Z}}$ is dense in $\boldsymbol{\mathcal{Z}}$.*

Proof. The proof is based in a classical property, that in our case reads as follows: $\mathcal{C}^\infty(\bar{\Omega})^3 \cap \boldsymbol{\mathcal{Z}}$ is dense in $\boldsymbol{\mathcal{Z}}$ if and only if every element in $\boldsymbol{\mathcal{Z}}'$ that vanishes in $\mathcal{C}^\infty(\bar{\Omega})^3 \cap \boldsymbol{\mathcal{Z}}$ also vanishes in $\boldsymbol{\mathcal{Z}}$ (see, for instance, [8, (I.2.14)]).

Let $L \in \boldsymbol{\mathcal{Z}}'$. Since $\boldsymbol{\mathcal{Z}}$ is a Hilbert space, there exists $\boldsymbol{l} \in \boldsymbol{\mathcal{Z}}$ such that

$$\int_\Omega \left(\boldsymbol{l} \cdot \bar{\boldsymbol{v}} + \tilde{\boldsymbol{l}} \cdot \mathbf{curl}\, \bar{\boldsymbol{v}}\right) = L\boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{Z}},$$

where $\tilde{\boldsymbol{l}} := \mathbf{curl}\, \boldsymbol{l}$. Now, let us assume that L vanishes in $\mathcal{C}^\infty(\bar{\Omega})^3 \cap \boldsymbol{\mathcal{Z}}$, namely,

$$\int_\Omega \left(\boldsymbol{l} \cdot \bar{\boldsymbol{w}} + \tilde{\boldsymbol{l}} \cdot \mathbf{curl}\, \bar{\boldsymbol{w}}\right) = 0 \qquad \forall \boldsymbol{w} \in \mathcal{C}^\infty(\bar{\Omega})^3 \cap \boldsymbol{\mathcal{Z}}. \tag{8}$$

We have to prove that L also vanishes in $\boldsymbol{\mathcal{Z}}$. With this end, notice $\mathcal{D}(\Omega)^3 \subset \mathcal{C}^\infty(\bar{\Omega})^3 \cap \boldsymbol{\mathcal{Z}}$ (cf. Lemma 3.10). Hence,

$$\int_\Omega \boldsymbol{l} \cdot \bar{\boldsymbol{w}} + \int_\Omega \tilde{\boldsymbol{l}} \cdot \mathbf{curl}\, \bar{\boldsymbol{w}} = 0 \qquad \forall \boldsymbol{w} \in \mathcal{D}(\Omega)^3.$$

Therefore, $\boldsymbol{l} = -\,\mathbf{curl}\, \tilde{\boldsymbol{l}}$, so that by virtue of Lemma 3.11 it would be enough to show that $\tilde{\boldsymbol{l}} \in \boldsymbol{\mathcal{Z}}$ to conclude the proof.

To prove this, we will check that $\boldsymbol{v} = \tilde{\boldsymbol{l}}$ satisfies the two properties from Corollary 3.7. First, since $\nabla\left(\mathcal{C}^\infty(\bar{\Omega})\right) \subset \mathcal{C}^\infty(\bar{\Omega})^3 \cap \boldsymbol{\mathcal{Z}}$, we obtain from equation (8) that

$$\int_\Omega \boldsymbol{l} \cdot \nabla\bar{\psi} = 0 \qquad \forall \psi \in \mathcal{C}^\infty(\bar{\Omega}).$$

Then $\mathbf{curl}\, \tilde{\boldsymbol{l}} = -\boldsymbol{l} \in \mathrm{H}_0(\mathrm{div}^0, \Omega)$, and the first property from Corollary 3.7 is checked.

For the second one, for each cutting surface Σ_j, let U be an open connected set of $\mathbb{R}^3$ containing $\bar{\Sigma}_j$, not intersecting $\bar{\Sigma}_k$, $k \neq j$, and such that $U \cap (\Omega \setminus \Sigma_j)$ has two connected components, $\Omega_{\Sigma_j^+}$ and $\Omega_{\Sigma_j^-}$, one at each side of Σ_j as shown in Figure 2.

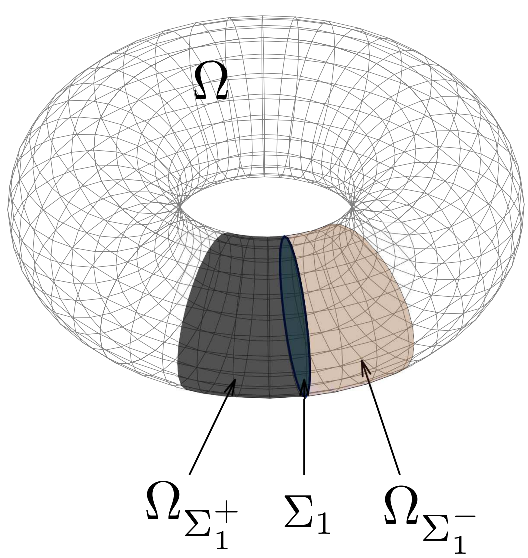

FIGURE 2. Toroidal domain for the proof of Lemma 3.12.

Let $\chi \in \mathcal{C}^\infty(\Omega \setminus \Sigma_j)$ be any smooth function satisfying $\chi \equiv 1$ in $\Omega_{\Sigma_j^+}$ and $\chi \equiv 0$ in $\Omega_{\Sigma_j^-}$. Notice that $\widetilde{\nabla}\chi \in \mathcal{C}^\infty(\bar{\Omega})^3 \cap \boldsymbol{\mathcal{Z}}$. Then, because of Lemma 3.1 and equation (8) with $\boldsymbol{w} = \widetilde{\nabla}\chi$, we have that

$$\left\langle \boldsymbol{l} \cdot \boldsymbol{n}_j, [\![\chi]\!]_{\Sigma_j} \right\rangle_{\Sigma_j} = \int_{\Omega^0} \boldsymbol{l} \cdot \nabla\chi + \int_{\Omega^0} (\operatorname{div} \boldsymbol{l})\, \chi = 0.$$

Consequently, $\langle \mathbf{curl}\, \tilde{\boldsymbol{l}} \cdot \boldsymbol{n}_j, 1\rangle_{\Sigma_j} = 0$, so that both properties of Corollary 3.7 are checked and, thus, $\tilde{\boldsymbol{l}} \in \boldsymbol{\mathcal{Z}}$. As stated above, this allows us to conclude the proof. □

To end this section, we will establish another density result that will be also used in the sequel. Let Φ be the subspace of smooth functions from the space Θ defined in (3), which are $\mathcal{C}^\infty$ up to its boundary. In general, these functions do not belong to $\mathcal{C}^\infty(\bar{\Omega})$ because they may have (constant) jumps on each cutting surface.

Lemma 3.13. *The space $\Phi/\mathbb{C}$ is dense in $\Theta/\mathbb{C}$ endowed with the $\mathrm{H}^1(\Omega^0)$-seminorm.*

Proof. We apply again the classical property used in the previous lemma. Let L be a bounded linear functional in $\Theta/\mathbb{C}$ that vanishes on $\Phi/\mathbb{C}$. To conclude the density claimed in the lemma, it is enough to show that L vanishes on the whole $\Theta/\mathbb{C}$ (cf. [8, (I.2.14)]).

Since $\Theta/\mathbb{C}$ is a Hilbert space, there exists $l \in \Theta/\mathbb{C}$ such that

$$\int_{\Omega^0} \widetilde{\nabla} l \cdot \widetilde{\nabla}\psi = L\psi \qquad \forall \psi \in \Theta/\mathbb{C}.$$

Since L vanishes on $\Phi \supset \mathcal{C}^\infty(\bar{\Omega})$,

$$\int_{\Omega^0} \widetilde{\nabla} l \cdot \nabla\psi = 0 \qquad \forall \psi \in \mathcal{C}^\infty(\bar{\Omega})/\mathbb{C}.$$

Hence, because of the Helmholtz decomposition (1), we have that $\widetilde{\nabla} l \in \mathrm{H}_0(\operatorname{div}^0, \Omega)$. Moreover, for each Σ_j, let χ be as defined in the proof of the previous lemma and note that, by construction, $\chi \in \Phi$. Then, Lemma 3.1 yields

$$\left\langle \widetilde{\nabla} l \cdot \boldsymbol{n}_j, [\![\chi]\!]_{\Sigma_j} \right\rangle_{\Sigma_j} = \int_{\Omega^0} \widetilde{\nabla} l \cdot \nabla\chi = 0.$$

Therefore, by virtue of Lemma 3.6, $\widetilde{\nabla} l \in \boldsymbol{\mathcal{X}}$ and hence is $\mathrm{L}^2(\Omega)^3$-orthogonal to $\mathrm{H}(\mathbf{curl}^0, \Omega) = \widetilde{\nabla}\Theta$ (cf. Lemmas 3.5 and 3.2). Thus,

$$L\psi = \int_{\Omega^0} \widetilde{\nabla} l \cdot \widetilde{\nabla}\psi = 0$$

for all $\psi \in \Theta/\mathbb{C}$ and we conclude the proof. □

4. **Mixed variational formulation.** The next step is to introduce a variational formulation of Problem 1. With this aim, thanks to Lemmas 3.6 and 3.5, we will impose the constraints in this eigenvalue problem by means of a Lagrange multiplier $\boldsymbol{\chi} \in \mathrm{H}(\mathbf{curl}^0, \Omega)$. Moreover, the eigenfunction $\boldsymbol{u}$ will be sought in the space $\boldsymbol{\mathcal{Z}}$. This will ensure that the sesquilinear form on the right-hand side will be Hermitian, which will allow us to prove that the corresponding solution operator is self-adjoint. Thus, we are led to the following variational formulation.

Problem 2. *Find* $\lambda \in \mathbb{C}$ *and* $(\boldsymbol{u}, \boldsymbol{\chi}) \in \boldsymbol{\mathcal{Z}} \times \mathrm{H}(\mathbf{curl}^0, \Omega)$, $(\boldsymbol{u}, \boldsymbol{\chi}) \not\equiv \mathbf{0}$, *such that*

$$\int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\bar{\boldsymbol{v}} + \int_\Omega \boldsymbol{\chi} \cdot \bar{\boldsymbol{v}} = \lambda \int_\Omega \boldsymbol{u} \cdot \mathbf{curl}\,\bar{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{Z}},$$
$$\int_\Omega \boldsymbol{u} \cdot \bar{\boldsymbol{\eta}} = 0 \qquad \forall \boldsymbol{\eta} \in \mathrm{H}(\mathbf{curl}^0, \Omega).$$

The first step is to prove that this formulation is actually equivalent to Problem 1.

Lemma 4.1. *If* $(\lambda, \boldsymbol{u})$ *is a solution to Problem 1, then* $(\lambda, \boldsymbol{u}, \mathbf{0})$ *is a solution to Problem 2. Conversely, if* $(\lambda, \boldsymbol{u}, \boldsymbol{\chi})$ *is a solution to Problem 2, then* $\boldsymbol{\chi} = \mathbf{0}$ *and* $(\lambda, \boldsymbol{u})$ *is a solution to Problem 1.*

Proof. First, as stated above, thanks to Lemmas 3.5 and 3.6, for $\boldsymbol{u} \in \mathrm{L}^2(\Omega)^3$,

$$\int_\Omega \boldsymbol{u} \cdot \bar{\boldsymbol{\eta}} = 0 \quad \forall \boldsymbol{\eta} \in \mathrm{H}(\mathbf{curl}^0, \Omega) \quad \overset{\boldsymbol{u} \in \boldsymbol{\mathcal{X}}}{\Longleftrightarrow} \quad \begin{cases} \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}_j, 1 \rangle_{\Sigma_j} = 0, & 1 \leq j \leq J. \end{cases}$$

Next, if $(\lambda, \boldsymbol{u})$ is a solution to Problem 1, clearly $(\lambda, \boldsymbol{u}, \mathbf{0})$ solves Problem 2.

Conversely, let $(\lambda, \boldsymbol{u}, \boldsymbol{\chi})$ be a solution to Problem 2. By taking $\boldsymbol{v} = \boldsymbol{\chi} \in \mathrm{H}(\mathbf{curl}^0, \Omega) \subset \boldsymbol{\mathcal{Z}}$, we have that $\boldsymbol{\chi} \equiv \mathbf{0}$. Then, by taking $\boldsymbol{v} \in \mathcal{D}(\Omega)^3 \subset \boldsymbol{\mathcal{Z}}$, we have that $\mathbf{curl}(\mathbf{curl}\,\boldsymbol{u} - \lambda \boldsymbol{u}) = \mathbf{0}$ in Ω and, hence, $\mathbf{curl}\,\boldsymbol{u} - \lambda \boldsymbol{u} \in \mathrm{H}(\mathbf{curl}^0, \Omega)$. Moreover, $\mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{\mathcal{X}}$ (because $\boldsymbol{u} \in \boldsymbol{\mathcal{Z}}$) and $\boldsymbol{u} \in \boldsymbol{\mathcal{X}}$ too (as shown above). Thus, $\mathbf{curl}\,\boldsymbol{u} - \lambda \boldsymbol{u} \in \boldsymbol{\mathcal{X}} \cap \mathrm{H}(\mathbf{curl}^0, \Omega) = \{\mathbf{0}\}$ (cf. Lemma 3.5). So, $\mathbf{curl}\,\boldsymbol{u} = \lambda \boldsymbol{u}$ in Ω. □

The *Babuška-Brezzi conditions* for this mixed problem are easy to check. To do this, first note that thanks to Lemma 3.5 and (7), we have that

$$\left\{\boldsymbol{v} \in \boldsymbol{\mathcal{Z}} : \ \int_\Omega \boldsymbol{v} \cdot \bar{\boldsymbol{\eta}} = 0 \ \ \forall \boldsymbol{\eta} \in \mathrm{H}(\mathbf{curl}^0, \Omega)\right\} = \boldsymbol{\mathcal{Z}} \cap \boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{V}}.$$

Lemma 4.2. *The Babuška-Brezzi conditions for Problem 2 hold true, namely:*

- *(ellipticity in the kernel) there exists* $\alpha > 0$ *such that*

$$\int_\Omega |\mathbf{curl}\,\boldsymbol{v}|^2 \geq \alpha \left\|\boldsymbol{v}\right\|_{\mathbf{curl},\Omega} \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}};$$

- *(inf-sup condition) there exists* $\beta > 0$ *such that*

$$\sup_{\boldsymbol{v} \in \boldsymbol{\mathcal{Z}}} \frac{\int_\Omega \boldsymbol{\eta} \cdot \boldsymbol{v}}{\left\|\boldsymbol{v}\right\|_{\mathbf{curl},\Omega}} \geq \beta \left\|\boldsymbol{\eta}\right\|_{0,\Omega} \quad \forall \boldsymbol{\eta} \in \mathrm{H}(\mathbf{curl}^0, \Omega).$$

Proof. The ellipticity in the kernel follows from the equivalence between $\left\|\cdot\right\|_{\mathbf{curl},\Omega}$ and $\left\|\mathbf{curl}\,\cdot\right\|_{0,\Omega}$ in the space $\boldsymbol{\mathcal{V}} \subset \boldsymbol{\mathcal{X}} \cap \mathrm{H}(\mathbf{curl}, \Omega)$, which in turn follows from [1, Corollary 3.16] and Corollary 3.8. The inf-sup condition follows by taking $\boldsymbol{v} = \boldsymbol{\eta} \in \mathrm{H}(\mathbf{curl}^0, \Omega) \subset \boldsymbol{\mathcal{Z}}$. □

Thanks to this lemma, we are in a position to define the solution operator corresponding to Problem 2:

$$\begin{aligned} \boldsymbol{T} : \ \boldsymbol{\mathcal{Z}} &\longrightarrow \boldsymbol{\mathcal{Z}}, \\ \boldsymbol{f} &\longmapsto \boldsymbol{T}\boldsymbol{f} := \boldsymbol{w}, \end{aligned}$$

with $\boldsymbol{w} \in \boldsymbol{\mathcal{Z}}$ such that there exists $\boldsymbol{\chi} \in \mathrm{H}(\mathbf{curl}^0, \Omega)$ satisfying

$$\int_\Omega \mathbf{curl}\, \boldsymbol{w} \cdot \mathbf{curl}\, \bar{\boldsymbol{v}} + \int_\Omega \boldsymbol{\chi} \cdot \bar{\boldsymbol{v}} = \int_\Omega \boldsymbol{f} \cdot \mathbf{curl}\, \bar{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{Z}},$$
$$\int_\Omega \boldsymbol{w} \cdot \bar{\boldsymbol{\eta}} = 0 \qquad \forall \boldsymbol{\eta} \in \mathrm{H}(\mathbf{curl}^0, \Omega).$$

By virtue of Lemma 4.2, this is a well posed problem (see, for instance, [8, Corollary I.4.1]). Moreover, clearly, $\boldsymbol{T}\boldsymbol{u} = \mu \boldsymbol{u}$, with $\boldsymbol{u} \not\equiv \mathbf{0}$ and $\mu \neq 0$ if and only if $(\lambda, \boldsymbol{u}, \mathbf{0})$ is a solution of Problem 2 with $\lambda = 1/\mu \neq 0$. Therefore, our next step will be to obtain a spectral characterization of the operator $\boldsymbol{T}$. With this end, first we establish the following additional regularity result.

Lemma 4.3. *The image of $\boldsymbol{T}$ satisfies $\boldsymbol{T}(\boldsymbol{\mathcal{Z}}) \subset \boldsymbol{\mathcal{V}}$. Moreover, there exists $s > \frac{1}{2}$ and $C > 0$ such that, for all $\boldsymbol{f} \in \boldsymbol{\mathcal{Z}}$, $\boldsymbol{w} := \boldsymbol{T}\boldsymbol{f} \in \mathrm{H}^s(\mathbf{curl}, \Omega)$ and*

$$\|\boldsymbol{w}\|_{s,\Omega} + \|\mathbf{curl}\, \boldsymbol{w}\|_{s,\Omega} \leq C \|\boldsymbol{f}\|_{\mathbf{curl},\Omega}\,.$$

Consequently, $\boldsymbol{T} : \boldsymbol{\mathcal{Z}} \to \boldsymbol{\mathcal{Z}}$ is compact.

Proof. Let $\boldsymbol{f} \in \boldsymbol{\mathcal{Z}}$ and $\boldsymbol{w} := \boldsymbol{T}\boldsymbol{f} \in \boldsymbol{\mathcal{Z}}$, too. The same arguments used in the proof of Lemma 4.1 apply to the problem defining $\boldsymbol{T}$ and allow us to show that $\boldsymbol{w} \in \boldsymbol{\mathcal{X}}$, $\boldsymbol{\chi} \equiv \mathbf{0}$ and $\mathbf{curl}(\mathbf{curl}\, \boldsymbol{w} - \boldsymbol{f}) = \mathbf{0}$ in Ω. Then, on one side, $\boldsymbol{w} \in \boldsymbol{\mathcal{Z}} \cap \boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{V}}$ (cf. (7)), so that $\boldsymbol{T}(\boldsymbol{\mathcal{Z}}) \subset \boldsymbol{\mathcal{V}}$.

On the other hand, $\boldsymbol{w} \in \boldsymbol{\mathcal{Z}} \cap \boldsymbol{\mathcal{X}} \subset \mathrm{H}(\mathbf{curl}, \Omega) \cap \mathrm{H}_0(\mathrm{div}^0, \Omega)$ and $\mathbf{curl}\, \boldsymbol{w} \in \mathrm{H}(\mathbf{curl}, \Omega) \cap \mathrm{H}_0(\mathrm{div}^0, \Omega)$, too. Then, according to (2), there exists $s > \frac{1}{2}$ such that $\mathrm{H}(\mathbf{curl}, \Omega) \cap \mathrm{H}_0(\mathrm{div}^0, \Omega)$ is continuously imbedded in $\mathrm{H}^s(\Omega)^3$. Hence, $\boldsymbol{w} \in \mathrm{H}^s(\mathbf{curl}, \Omega)$ and the estimate of the lemma holds true. Finally, we end the lemma from the compactness of the inclusion $\mathrm{H}^s(\mathbf{curl}, \Omega) \cap \boldsymbol{\mathcal{Z}} \hookrightarrow \boldsymbol{\mathcal{Z}}$, which in turn is a consequence of the fact that the inclusion $\mathrm{H}^s(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$ is compact. □

Lemma 4.4. *The operator $\boldsymbol{T} : \boldsymbol{\mathcal{Z}} \to \boldsymbol{\mathcal{Z}}$ is self-adjoint.*

Proof. Let $\boldsymbol{f}, \boldsymbol{g} \in \boldsymbol{\mathcal{Z}}$, $\boldsymbol{w} := \boldsymbol{T}\boldsymbol{f}$ and $\boldsymbol{v} := \boldsymbol{T}\boldsymbol{g}$. As shown in the proof of the previous lemma, $\boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{\mathcal{Z}} \cap \boldsymbol{\mathcal{X}}$, the corresponding Lagrange multipliers vanish and both, $(\mathbf{curl}\, \boldsymbol{w} - \boldsymbol{f})$ and $(\mathbf{curl}\, \boldsymbol{v} - \boldsymbol{g})$, belong to $\mathrm{H}(\mathbf{curl}^0, \Omega)$. Consequently, using that $\mathrm{H}(\mathbf{curl}^0, \Omega) \perp \boldsymbol{\mathcal{X}}$ (cf. Lemma 3.5) and Lemma 3.11, we have that

$$\int_\Omega \boldsymbol{w} \cdot \bar{\boldsymbol{g}} = \int_\Omega \boldsymbol{w} \cdot \mathbf{curl}\, \bar{\boldsymbol{v}} = \int_\Omega \mathbf{curl}\, \boldsymbol{w} \cdot \bar{\boldsymbol{v}} = \int_\Omega \boldsymbol{f} \cdot \bar{\boldsymbol{v}}.$$

On the other hand, from the first equation of the problem defining $\boldsymbol{T}$ and Lemma 3.11 again, we have that

$$\int_\Omega \mathbf{curl}\, \boldsymbol{w} \cdot \mathbf{curl}\, \bar{\boldsymbol{g}} = \int_\Omega \boldsymbol{f} \cdot \mathbf{curl}\, \bar{\boldsymbol{g}} = \int_\Omega \mathbf{curl}\, \boldsymbol{f} \cdot \bar{\boldsymbol{g}} = \int_\Omega \mathbf{curl}\, \boldsymbol{f} \cdot \mathbf{curl}\, \bar{\boldsymbol{v}},$$

which allow us to conclude that $\boldsymbol{T} : \boldsymbol{\mathcal{Z}} \to \boldsymbol{\mathcal{Z}}$ is self-adjoint. □

Theorem 4.5. *The spectrum of $\boldsymbol{T}$ decomposes as follows:* $\mathrm{sp}(\boldsymbol{T}) = \{\mu_n\}_{n \in \mathbb{N}} \cup \{0\}$. *Moreover,*

- *$\mu_0 = 0$ is an infinite-multiplicity eigenvalue and its associated eigenspace is $\mathrm{H}(\mathbf{curl}^0, \Omega)$;*
- *$\{\mu_n\}_{n \in \mathbb{N}}$ is a sequence of finite-multiplicity eigenvalues (repeated accordingly to their respective multiplicities) which converges to 0 and there exists a Hilbertian basis of associated eigenfunctions $\{\boldsymbol{u}_n\}_{n \in \mathbb{N}}$ of $\boldsymbol{\mathcal{V}}$ (i.e., such that $\boldsymbol{T}\boldsymbol{u}_n = \mu_n \boldsymbol{u}_n$, $n \in \mathbb{N}$).*

Proof. From the definition of the operator $\boldsymbol{T}$, its kernel is given by

$$\begin{aligned}\ker \boldsymbol{T} &:= \left\{ \boldsymbol{f} \in \boldsymbol{\mathcal{Z}} : \ \int_\Omega \boldsymbol{f} \cdot \mathbf{curl}\, \boldsymbol{v} = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{Z}} \right\} \\ &= \left\{ \boldsymbol{f} \in \boldsymbol{\mathcal{Z}} : \ \int_\Omega \mathbf{curl}\, \boldsymbol{f} \cdot \boldsymbol{v} = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{Z}} \right\} = \mathrm{H}(\mathbf{curl}^0, \Omega),\end{aligned}$$

the last two equalities because of Lemmas 3.11 and 3.10, respectively. Then, since $\boldsymbol{T}(\boldsymbol{\mathcal{Z}}) \subset \boldsymbol{\mathcal{V}}$ (cf. Lemma 4.3) and $\boldsymbol{\mathcal{Z}} = \boldsymbol{\mathcal{V}} \oplus \mathrm{H}(\mathbf{curl}^0, \Omega)$ (cf. Lemma 3.9), the theorem follows from the two previous lemmas and the classical theory for self-adjoint compact operators (see, for instance, [4, Section 6.4]). □

Taking into account the relation between the solutions of Problem 2 and the spectrum of $\boldsymbol{T}$, we also have the following characterization of the solutions of Problem 2 and, because of Lemma 4.1, of Problem 1 too.

Corollary 4.6. *Problem 1 has a countable number of solutions* $(\lambda_n, \boldsymbol{u}_n)$, $n \in \mathbb{N}$, $\lambda_n \to \infty$ *and* $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ *is a Hilbertian basis of* $\boldsymbol{\mathcal{V}}$*.*

In the following section we will introduce a finite element discretization of Problem 2. In order to prove the convergence of the proposed numerical scheme we will use the following operator:

$$\begin{aligned} \boldsymbol{G} : \ \boldsymbol{\mathcal{Z}} \times \Theta/\mathbb{C} &\longrightarrow \boldsymbol{\mathcal{Z}} \times \Theta/\mathbb{C}, \\ (\boldsymbol{f}, g) &\longmapsto \boldsymbol{G}(\boldsymbol{f}, g) := (\boldsymbol{w}, \xi), \end{aligned} \tag{9}$$

where $(\boldsymbol{w}, \xi) \in \boldsymbol{\mathcal{Z}} \times \Theta/\mathbb{C}$ is the solution of the following problem:

$$\begin{aligned} &\int_\Omega \mathbf{curl}\, \boldsymbol{w} \cdot \mathbf{curl}\, \bar{\boldsymbol{v}} + \int_\Omega \widetilde{\nabla}\xi \cdot \bar{\boldsymbol{v}} = \int_\Omega \boldsymbol{f} \cdot \mathbf{curl}\, \bar{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{Z}}, \\ &\int_\Omega \boldsymbol{w} \cdot \widetilde{\nabla}\bar{\psi} = 0 \qquad \forall \psi \in \Theta/\mathbb{C}. \end{aligned}$$

By virtue of Lemma 3.2, $\mathrm{H}(\mathbf{curl}^0, \Omega) = \widetilde{\nabla}\Theta$. Therefore, this problem is equivalent to the one used to define the operator $\boldsymbol{T}$ with $\boldsymbol{\chi} = \widetilde{\nabla}\xi$. Hence, $\boldsymbol{G}(\boldsymbol{f}, g) = (\boldsymbol{T}\boldsymbol{f}, 0)$ and we have the following result.

Lemma 4.7. *The operator* $\boldsymbol{G}$ *is compact and self-adjoint.*

Proof. Since $\boldsymbol{G}(\boldsymbol{f}, g) = (\boldsymbol{T}\boldsymbol{f}, 0)$, the result follows immediately from the compactness and the self-adjointness of $\boldsymbol{T}$ (Lemmas 4.3 and 4.4, respectively). □

Note that $(\mu, (\boldsymbol{u}, 0))$ is an eigenpair of the operator $\boldsymbol{G}$ with $\mu \neq 0$ if and only if $(\frac{1}{\mu}, \boldsymbol{u}, 0)$ is a solution of Problem 2 or, equivalently, $(\frac{1}{\mu}, \boldsymbol{u})$ a solution of Problem 1.

5. **Finite element spectral approximation.** In this section, we introduce a Galerkin approximation of Problem 2 and prove convergence and error estimates for the approximate eigenvalues and eigenfunctions.

With this end, we assume that Ω is a polyhedron and choose the cutting surfaces Σ_j, $1 \leq j \leq J$, also polyhedral. Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of tetrahedral partitions of $\bar{\Omega}$ compatible with the cutting surfaces in the sense that each Σ_j is a union of faces of tetrahedra $T \in \mathcal{T}_h$. Therefore, each $\mathcal{T}_h$ can also be seen as a mesh of the cut domain Ω^0. The mesh parameter h denotes the maximum diameter of all the tetrahedra of the mesh $\mathcal{T}_h$. From now on we will denote by C a generic constant, not necessarily the same at each occurrence but always independent of the mesh parameter h.

We use classical curl-conforming Nédélec finite elements:

$$\boldsymbol{\mathcal{N}}_h := \left\{ \boldsymbol{v}_h \in \mathrm{H}(\mathbf{curl}, \Omega) : \; \boldsymbol{v}_h|_T \in \boldsymbol{\mathcal{N}}^k(T) \;\; \forall T \in \mathcal{T}_h \right\},$$

with

$$\boldsymbol{\mathcal{N}}^k(T) := \mathcal{P}_{k-1}(T)^3 \oplus \left\{ \boldsymbol{p} \in \bar{\mathcal{P}}_k(T)^3 : \; \boldsymbol{p}(\boldsymbol{x}) \cdot \boldsymbol{x} = 0 \right\},$$

where, for $k \in \mathbb{N}$, $\mathcal{P}_k$ is the space of polynomials of degree not greater than k and $\bar{\mathcal{P}}_k$ the subspace of homogeneous polynomials of degree k. Let

$$\boldsymbol{\mathcal{Z}}_h := \boldsymbol{\mathcal{N}}_h \cap \boldsymbol{\mathcal{Z}},$$

which by virtue of Corollary 3.7 can be written as follows:

$$\boldsymbol{\mathcal{Z}}_h = \left\{ \boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_h : \; \mathbf{curl}\, \boldsymbol{v}_h \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \text{ and } \textstyle\int_{\Sigma_j} \mathbf{curl}\, \boldsymbol{v}_h \cdot \boldsymbol{n}_j = 0, \; 1 \le j \le J \right\}.$$

We denote by $\boldsymbol{I}_h^N$ the so-called *Nédélec interpolation* operator. We refer to [13, Section 5.5] for its precise definition and the properties of this interpolant that we will use in the sequel. This interpolant is well defined for functions in $\mathrm{H}^s(\mathbf{curl}, \Omega)$ provided $s > \frac{1}{2}$, so that $\boldsymbol{I}_h^N : \; \mathrm{H}^s(\mathbf{curl}, \Omega) \to \boldsymbol{\mathcal{N}}_h$ is a bounded linear operator. Moreover, as it is shown in the following lemma, the Nédélec interpolant of functions from $\boldsymbol{\mathcal{Z}}$ remains in this space.

Lemma 5.1. *For all $\boldsymbol{v} \in \mathrm{H}^s(\mathbf{curl}, \Omega) \cap \boldsymbol{\mathcal{Z}}$ with $s > \frac{1}{2}$, $\boldsymbol{I}_h^N \boldsymbol{v} \in \boldsymbol{\mathcal{Z}}_h$.*

Proof. Let $\boldsymbol{I}_h^R$ be the divergence-conforming Raviart-Thomas interpolation operator (see, for instance, [13, Section 5.4] for its definition and properties). This interpolant is well defined for functions in $\mathrm{H}^s(\Omega)^3$ with $s > \frac{1}{2}$ ([13, Lemma 5.15]).

Let $\boldsymbol{v} \in \mathrm{H}^s(\mathbf{curl}, \Omega) \cap \boldsymbol{\mathcal{Z}}$ with $s > \frac{1}{2}$. According to Corollary 3.7, $\mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ and $\int_{\Sigma_j} \mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n} = 0$, $1 \le j \le J$ (note that the integral makes sense because $\mathbf{curl}\, \boldsymbol{v} \in \mathrm{H}^s(\Omega)^3$ with $s > \frac{1}{2}$). Therefore,

$$\mathbf{curl}\left(\boldsymbol{I}_h^N \boldsymbol{v}\right) \cdot \boldsymbol{n} = \left(\boldsymbol{I}_h^R \,\mathbf{curl}\, \boldsymbol{v}\right) \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma,$$

where the first equality follows from [13, Lemma 5.40] and the second one from the fact that the Raviart-Thomas interpolant preserves vanishing normal components on the faces of the tetrahedra of the mesh (which in turn follows from the definition of this interpolant). Analogously, it also preserves the integral of these normal components, so that we have

$$\int_{\Sigma_j} \mathbf{curl}\left(\boldsymbol{I}_h^N \boldsymbol{v}\right) \cdot \boldsymbol{n} = \int_{\Sigma_j} \left(\boldsymbol{I}_h^R \,\mathbf{curl}\, \boldsymbol{v}\right) \cdot \boldsymbol{n} = \int_{\Sigma_j} \mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n} = 0, \quad 1 \le j \le J.$$

Thus, $\boldsymbol{I}_h^N \boldsymbol{v} \in \boldsymbol{\mathcal{Z}}_h$. □

To discretize the Lagrange multiplier $\boldsymbol{\chi} \in \mathrm{H}(\mathbf{curl}^0, \Omega)$, we use Lemma 3.2 to write $\boldsymbol{\chi} = \widetilde{\nabla} \varphi$ with $\varphi \in \Theta$, and approximate this space by

$$\Theta_h := \left\{ \psi_h \in \mathcal{C}(\Omega^0) : \; \psi_h|_T \in \mathcal{P}_k(T) \; \forall T \in \mathcal{T}_h \text{ and } [\![\psi_h]\!]_{\Sigma_j} = \text{constant} \right\}.$$

The following discrete version of Lemma 3.2 has been proved in [3, Lemma 5.5].

Lemma 5.2. *There holds*

$$\widetilde{\nabla} \Theta_h = \boldsymbol{\mathcal{N}}_h \cap \mathrm{H}(\mathbf{curl}^0, \Omega).$$

Now we are in a position to introduce a finite element discretization of Problem 2.

Problem 3. *Find $\lambda_h \in \mathbb{C}$ and $(\boldsymbol{u}_h, \varphi_h) \in \boldsymbol{\mathcal{Z}}_h \times \Theta_h/\mathbb{C}$, $(\boldsymbol{u}_h, \varphi_h) \not\equiv \mathbf{0}$, such that*

$$\int_\Omega \mathbf{curl}\, \boldsymbol{u}_h \cdot \mathbf{curl}\, \bar{\boldsymbol{v}}_h + \int_\Omega \widetilde{\nabla} \varphi_h \cdot \bar{\boldsymbol{v}}_h = \lambda_h \int_\Omega \boldsymbol{u}_h \cdot \mathbf{curl}\, \bar{\boldsymbol{v}}_h \qquad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{Z}}_h,$$

$$\int_\Omega \boldsymbol{u}_h \cdot \widetilde{\nabla} \bar{\psi}_h = 0 \qquad \forall \psi_h \in \Theta_h/\mathbb{C}.$$

First of all note that, as in the continuous problem, for any solution $(\lambda_h, \boldsymbol{u}_h, \varphi_h)$ of Problem 3, φ_h vanishes. In fact, since $\widetilde{\nabla}\Theta_h \subset \boldsymbol{\mathcal{Z}}_h$ (cf. Lemma 5.2), this follows by taking above $\boldsymbol{v}_h = \widetilde{\nabla}\varphi_h$.

Taking arbitrary bases of the finite dimensional spaces $\boldsymbol{\mathcal{Z}}_h$ and $\Theta_h/\mathbb{C}$, Problem 3 can be equivalently written as a degenerate generalized matrix eigenvalue problem in which both matrices are Hermitian but none is positive definite. However, we will show in the next section that it is also equivalent to another generalized matrix eigenvalue problem of smaller dimension which will be proved to be well posed.

In order to prove that the eigenvalues and eigenfunctions of Problem 2 are well approximated by those of Problem 3, we resort to the classical theory for mixed eigenvalue problems of the so-called type (Q1) reported in [12, Section 3]. With this aim, we have to check the following properties, which correspond to assumptions (3.12)–(3.16) from this reference.

- Babuška-Brezzi conditions for the continuous problem; they have been already established in Lemma 4.2.
- Babuška-Brezzi conditions for the discrete problem, namely:
  - (ellipticity in the discrete kernel) there exists $\alpha_* > 0$ such that

  $$\int_\Omega |\mathbf{curl}\, \boldsymbol{v}_h|^2 \geq \alpha_* \|\boldsymbol{v}_h\|_{\mathbf{curl},\Omega} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{V}}_h,$$

  where $\boldsymbol{\mathcal{V}}_h := \left\{\boldsymbol{v}_h \in \boldsymbol{\mathcal{Z}}_h : \int_\Omega \boldsymbol{v}_h \cdot \widetilde{\nabla}\psi_h = 0 \ \ \forall \psi_h \in \Theta_h\right\}$; it has been proved in [1, Proposition 4.6].
  - (discrete inf-sup condition) there exists $\beta_* > 0$ such that

  $$\sup_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{Z}}_h} \frac{\int_\Omega \widetilde{\nabla}\psi_h \cdot \boldsymbol{v}_h}{\|\boldsymbol{v}_h\|_{\mathbf{curl},\Omega}} \geq \beta_* \left\|\widetilde{\nabla}\psi_h\right\|_{0,\Omega} \qquad \forall \psi_h \in \Theta_h;$$

  it follows by taking $\boldsymbol{v}_h = \widetilde{\nabla}\psi_h$, which according to Lemma 5.2 lies in $\boldsymbol{\mathcal{Z}}_h$.
- Density of the finite element spaces: for all $(\boldsymbol{v}, \psi) \in \boldsymbol{\mathcal{Z}} \times \Theta$,

  $$\inf_{(\boldsymbol{v}_h, \psi_h) \in \boldsymbol{\mathcal{Z}}_h \times \Theta_h} \left(\|\boldsymbol{v} - \boldsymbol{v}_h\|_{\mathbf{curl},\Omega} + \left\|\widetilde{\nabla}\psi - \widetilde{\nabla}\psi_h\right\|_{0,\Omega} \right) \to 0 \quad \text{as } h \to 0;$$

  it follows from the densities of $\mathcal{C}^\infty(\bar{\Omega})^3 \cap \boldsymbol{\mathcal{Z}}$ in $\boldsymbol{\mathcal{Z}}$ (cf. Lemma 3.12) and $\Phi/\mathbb{C}$ in $\Theta/\mathbb{C}$ (cf. Lemma 3.13), the fact that for a smooth $v \in \boldsymbol{\mathcal{Z}}$ its Nédélec interpolant $\boldsymbol{I}_h^N$ lies in $\boldsymbol{\mathcal{Z}}_h$ (cf. Lemma 5.1) and standard approximation properties of the Nédélec and the Lagrange interpolants (cf. [13, Theorem 5.41(1)] and [8, Lemma I.A.2], for instance).

An additional hypothesis assumed in the spectral approximation theory from [12, Section 3] is the compactness of the solution operator $\boldsymbol{G}$ defined in (9), which in our case has been already established in Lemma 4.7. Moreover, this theory also involves a formal adjoint operator $\boldsymbol{G}_*$, which in the present case is defined for $(\boldsymbol{f}, g) \in \boldsymbol{\mathcal{Z}} \times \Theta/\mathbb{C}$ by $\boldsymbol{G}_*(\boldsymbol{f}, g) := (\boldsymbol{w}_*, \xi_*)$, with $(\boldsymbol{w}_*, \xi_*) \in \boldsymbol{\mathcal{Z}} \times \Theta/\mathbb{C}$ being the solution of the adjoint problem:

$$\int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\bar{\boldsymbol{w}}_* + \int_\Omega \boldsymbol{v}\cdot\nabla\bar{\xi}_* = \int_\Omega \boldsymbol{v}\cdot\mathbf{curl}\,\bar{\boldsymbol{f}} \qquad \forall\boldsymbol{v}\in\boldsymbol{\mathcal{Z}},$$
$$\int_\Omega \nabla\psi\cdot\bar{\boldsymbol{w}}_* = 0 \qquad \forall\psi\in\Theta/\mathbb{C}.$$

However, according to Lemma 3.11, for $\boldsymbol{f},\boldsymbol{v}\in\boldsymbol{\mathcal{Z}}$, $\int_\Omega \boldsymbol{v}\cdot\mathbf{curl}\,\bar{\boldsymbol{f}} = \int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\bar{\boldsymbol{f}}$ and, then, in this case, the formal adjoint $\boldsymbol{G}_*$ coincides with $\boldsymbol{G}$.

Therefore, all the hypotheses needed to apply [12, Theorem 3.1] hold true, which allows us to establish the main result of this paper.

Theorem 5.3. *Let λ be an eigenvalue of Problem 2 with multiplicity m and $\boldsymbol{\mathcal{E}}\times\{0\}\subset\boldsymbol{\mathcal{Z}}\times\Theta/\mathbb{C}$ the corresponding eigenspace.*

There exist exactly m eigenvalues $\lambda_h^{(1)},\dots,\lambda_h^{(m)}$ of Problem 3 (repeated accordingly to their respective multiplicities) which converge to λ as $h\to 0$.

Let $\boldsymbol{\mathcal{E}}_h\times\{0\}$ be the direct sum of the eigenspaces corresponding to $\lambda_h^{(1)},\dots,\lambda_h^{(m)}$. Then,

$$\widehat{\delta}\,(\boldsymbol{\mathcal{E}},\boldsymbol{\mathcal{E}}_h)\le C\gamma_h,$$

and

$$\left|\lambda-\lambda_h^{(i)}\right|\le C\gamma_h^2,\quad i=1,\dots,m,$$

where

$$\gamma_h := \delta(\boldsymbol{\mathcal{E}},\boldsymbol{\mathcal{Z}}_h) := \sup_{\substack{\boldsymbol{v}\in\boldsymbol{\mathcal{E}}\\ \|\boldsymbol{v}\|_{\mathbf{curl},\Omega}=1}} \inf_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{Z}}_h}\|\boldsymbol{v}-\boldsymbol{v}_h\|_{\mathbf{curl},\Omega}$$

is the distance from the continuous eigenspace $\boldsymbol{\mathcal{E}}$ to the discrete space $\boldsymbol{\mathcal{Z}}_h$ and

$$\widehat{\delta}\,(\boldsymbol{\mathcal{E}},\boldsymbol{\mathcal{E}}_h) := \max\left\{\delta\left(\boldsymbol{\mathcal{E}},\boldsymbol{\mathcal{E}}_h\right),\delta\left(\boldsymbol{\mathcal{E}}_h,\boldsymbol{\mathcal{E}}\right)\right\}$$

is the so-called gap between the continuous and the discrete eigenspaces.

To end this section we establish an appropriate estimate for the term γ_h. With this end, recall that $k\ge 1$ is the degree of the Nédélec finite elements and let $s>\frac{1}{2}$ be a Sobolev exponent such that the inclusion (2) holds and is continuous.

Theorem 5.4. *Let γ_h be as in Theorem 5.3. Then, there exists $C>0$ such that*

$$\gamma_h\le C\lambda h^{\min\{s,k\}}.$$

Proof. Let $\boldsymbol{v}\in\boldsymbol{\mathcal{E}}$ be such that $\|\boldsymbol{v}\|_{\mathbf{curl},\Omega}=1$ and $\mu:=\frac{1}{\lambda}$, so that $\boldsymbol{T}\boldsymbol{v}=\mu\boldsymbol{v}$. Then, from Lemma 4.3 it follows that $\boldsymbol{v}\in\mathrm{H}^s(\mathbf{curl},\Omega)$ and

$$\|\boldsymbol{v}\|_{s,\Omega}+\|\mathbf{curl}\,\boldsymbol{v}\|_{s,\Omega}\le\frac{C}{\mu}\|\boldsymbol{v}\|_{\mathbf{curl},\Omega}\le C\lambda.$$

Therefore, using again the standard error estimate for the Nédélec interpolant $\boldsymbol{I}_h^N\boldsymbol{v}$ (cf. [13, Theorem 5.41(1)]) and taking into account that according to Lemma 5.1 $\boldsymbol{I}_h^N\boldsymbol{v}\in\boldsymbol{\mathcal{Z}}_h$, we obtain

$$\delta\left(\boldsymbol{\mathcal{E}},\boldsymbol{\mathcal{Z}}_h\right)\le\sup_{\substack{\boldsymbol{v}\in\boldsymbol{\mathcal{E}}\\ \|\boldsymbol{v}\|_{\mathbf{curl},\Omega}=1}}\left\|\boldsymbol{v}-\boldsymbol{I}_h^N\boldsymbol{v}\right\|_{\mathbf{curl},\Omega}\le Ch^{\min\{s,k\}}\left(\|\boldsymbol{v}\|_{s,\Omega}+\|\mathbf{curl}\,\boldsymbol{v}\|_{s,\Omega}\right).$$

The result follows from the definition of γ_h and these two inequalities. □

As a consequence of the two previous theorems we conclude that the eigenvalues and eigenfunctions of Problem 3 converge with optimal order to those of Problem 2.

6. **Implementation issues.** In this section we will briefly describe how to impose in $\boldsymbol{\mathcal{N}}_h$ the constraints defining $\boldsymbol{\mathcal{Z}}_h$:

$$\mathbf{curl}\,\boldsymbol{v}_h \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma \qquad \text{and} \qquad \int_{\Sigma_j} \mathbf{curl}\,\boldsymbol{v}_h \cdot \boldsymbol{n}_j = 0, \quad 1 \le j \le J.$$

With this end, the results from [11, Section 5] and [14, Section 5] have been extended to multiply connected domains in [10, Section 5] yielding the following result.

Proposition 6.1. *Let $\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_h$. Then, $\boldsymbol{v}_h \in \boldsymbol{\mathcal{Z}}_h$ if and only if there exists $\varphi_h \in \Theta_h$ such that*

$$\boldsymbol{n} \times \boldsymbol{v}_h|_{\widetilde{\Gamma}} \times \boldsymbol{n} = \widetilde{\nabla}_{\widetilde{\Gamma}} \left(\varphi_h|_{\widetilde{\Gamma}}\right),$$

where $\widetilde{\Gamma} := \Gamma \setminus \bigcup_{j=1}^{J} \partial\Sigma_j$ and $\widetilde{\nabla}_{\widetilde{\Gamma}}$ denotes the surface gradient on $\widetilde{\Gamma}$.

The next step is to introduce a convenient basis of the space Θ_h. Consider the standard finite element discretization of $\mathrm{H}^1(\Omega)$:

$$\mathcal{L}_h := \{\psi_h \in \mathcal{C}(\Omega) : \ \psi_h|_T \in \mathcal{P}_k(T) \ \ \forall T \in \mathcal{T}_h\}.$$

On the other hand, for $j = 1, \ldots, J$, let $\widehat{\varphi}_j \in \Theta_h$ be such that, for all nodes P,

$$\widehat{\varphi}_j(P) = \begin{cases} 1, & \text{if } P \in \Sigma_j^+, \\ 0, & \text{if } P \notin \Sigma_j^+. \end{cases}$$

It is easy to check that $\Theta_h = \mathcal{L}_h \oplus \left\langle \{\widehat{\varphi}_j\}_{j=1}^J \right\rangle$. Hence, if $\{\varphi_j\}_{j=1}^L$ is the nodal basis of $\mathcal{L}_h$, then, $\{\varphi_j\}_{j=1}^L \cup \{\widehat{\varphi}_j\}_{j=1}^J$ is a basis of Θ_h.

The final step is to introduce a basis of $\boldsymbol{\mathcal{Z}}_h$. With this aim, for simplicity, we assume that the boundary Γ is connected (otherwise, the same procedure should be repeated for each of its connected components).

Without loss of generality we order the nodal basis functions $\{\varphi_j\}_{j=1}^L$ of $\mathcal{L}_h$ so that the first K of them correspond to nodal values on Γ. Then, it is easy to check that $\{\varphi_k|_\Gamma\}_{k=1}^K \cup \{\widehat{\varphi}_j|_{\widetilde{\Gamma}}\}_{j=1}^J$is a basis of $\Theta_h^{\widetilde{\Gamma}} := \left\{\psi_h|_{\widetilde{\Gamma}} : \ \psi_h \in \Theta_h\right\}$. Since surface gradients are determined up to an additive constant, we choose one vertex on Γ and drop out the basis function corresponding to this vertex (e.g., vertex number K).

Finally, let $\{\boldsymbol{\Phi}_m\}_{m=1}^M$ be the nodal basis of $\boldsymbol{\mathcal{N}}_h$; without loss of generality, we assume that $\{\boldsymbol{\Phi}_m\}_{m=M'+1}^M$ are those related to the faces or edges on Γ. Then we have the following result whose proof can be found in [10, Proposition 5.1].

Proposition 6.2. *$\{\boldsymbol{\Phi}_m\}_{m=1}^{M'} \cup \{\nabla\varphi_k\}_{k=1}^{K-1} \cup \left\{\widetilde{\nabla}\widehat{\varphi}_j\right\}_{j=1}^J$ is a basis of $\boldsymbol{\mathcal{Z}}_h$.*

Let us remark that the matrices of the algebraic eigenvalue problem corresponding to the discrete mixed formulation can be easily obtained by static condensation from the matrices of the classical Nédélec and $\mathcal{P}_k$-continuous elements (see [10, Section 5.1] for more details). The resulting algebraic generalized eigenvalue problem has the form

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^{\mathrm{t}} \\ \boldsymbol{B} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{u}} \\ \vec{\varphi} \end{pmatrix} = \lambda_h \begin{pmatrix} \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{u}} \\ \vec{\varphi} \end{pmatrix},$$

where the entries of $\vec{\boldsymbol{u}}$ and $\vec{\varphi}$ are the components in the above given bases of $\boldsymbol{u}_h$ and φ_h, respectively. Both matrices above are Hermitian (actually, real symmetric), but none is positive definite.

Since $\vec{\varphi} = \vec{0}$, the above problem is equivalent to

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^{\mathrm{t}} \\ \boldsymbol{B} & -\boldsymbol{I} \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{u}} \\ \vec{\varphi} \end{pmatrix} = \lambda_h \begin{pmatrix} \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{u}} \\ \vec{\varphi} \end{pmatrix},$$

which, in turn, is equivalent to

$$\left(\boldsymbol{A} + \boldsymbol{B}^{\mathrm{t}}\boldsymbol{B}\right)\vec{\boldsymbol{u}} = \lambda_h \boldsymbol{C}\vec{\boldsymbol{u}},$$

with a real symmetric and positive definite left-hand side matrix. This allows us to conclude that Problem 3 is well posed and that it has $\dim(\boldsymbol{\mathcal{Z}}_h) - \dim(\ker(\boldsymbol{C}))$ non-zero eigenvalues (repeated accordingly to their respective multiplicities).

7. Numerical results.

We have implemented the method analyzed above for the lowest-possible order ($k = 1$) in a MATLAB code. We have applied it to compute the smallest positive eigenvalues in a toroidal domain as that shown in Figure 3, with $r_1 = 1$ and $r_2 = 0.5$, for which no analytical solution is available.

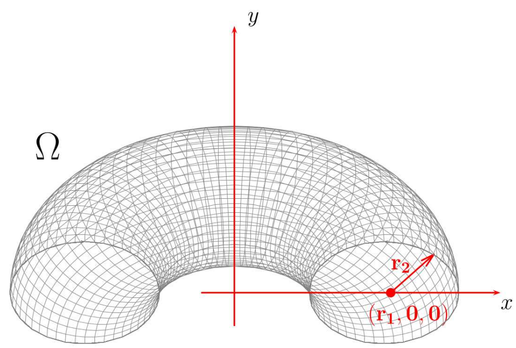

FIGURE 3. Half of the toroidal domain for the numerical test.

We have used meshes $\mathcal{T}_h$ with different levels of refinement; we identify each mesh by the corresponding number N_h of tetrahedra. For each computed eigenvalue we have estimated the order of convergence and a more accurate value by means of a least-squares fitting of the model $\lambda_{h,k} \approx \lambda_{\mathrm{ex}} + Ch^t$ with $h = N_h^{-1/3}$. Table 1 shows the obtained results for the four smallest eigenvalues. Note that the first two converge to a double-multiplicity eigenvalue of the continuous problem and the other two to another eigenvalue of multiplicity two.

A quadratic order of convergence can be seen from Table 1, which agrees with the theoretical results for the finite elements used ($k = 1$) and a smooth domain ($s \geq 1$). Figure 4 shows a log-log plot of the computed errors for the eigenvalue λ_1 versus the number of tetrahedra. Once more, the quadratic order of convergence can be appreciated.

Finally, Figure 5 shows a plot of the eigenfunction (i.e., a Beltrami field in the torus) corresponding also to the smallest positive eigenvalue λ_1.

REFERENCES

[1] C. Amrouche, C. Bernardi, M. Dauge and V. Girault, Vector potentials in three-dimensional non-smooth domains, *Math. Methods Appl. Sci.*, **21** (1998), 823–864.

[2] E. Beltrami, Considerazioni idrodinamiche, *Rend. Inst. Lombardo Acad. Sci. Let.*, **22** (1889), 122–131. English translation: Considerations on hydrodynamics, *Int. J. Fusion Energy*, **3** (1985), 53–57.

[3] A. Bermúdez, R. Rodríguez and P. Salgado, A finite element method with Lagrange multipliers for low-frequency harmonic Maxwell equations, *SIAM J. Numer. Anal.*, **40** (2002), 1823–1849.

TABLE 1. Smallest positive eigenvalues computed on different meshes.

N_h	$\lambda_{h,1}$	$\lambda_{h,2}$	$\lambda_{h,3}$	$\lambda_{h,4}$
20384	6.6320	6.6353	6.7104	6.7118
28321	6.5199	6.5252	6.5888	6.5903
43667	6.4406	6.4444	6.5105	6.5121
58732	6.4096	6.4129	6.4723	6.4746
75999	6.3824	6.3871	6.4439	6.4448
97886	6.3565	6.3590	6.4155	6.4161
131222	6.3276	6.3285	6.3840	6.3842
154592	6.3156	6.3162	6.3726	6.3734
171127	6.3093	6.3098	6.3657	6.3662
182885	6.3075	6.3080	6.3642	6.3644
241429	6.2829	6.2833	6.3370	6.3370
264623	6.2781	6.2787	6.3323	6.3324
301862	6.2737	6.2741	6.3275	6.3277
λ_{ex}	6.2233	6.2174	6.2714	6.2690
order	2.20	2.12	2.17	2.14

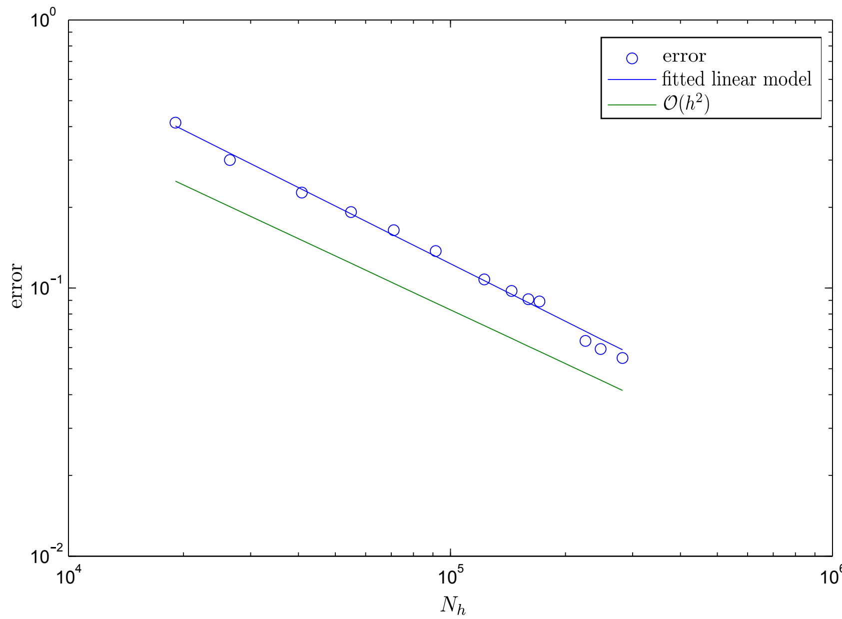

FIGURE 4. Error curve for the smallest positive eigenvalue: log-log plot of the computed error $|\lambda_{h,1} - \lambda_{\text{ex}}|$ versus the number of tetrahedra N_h.

[4] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.

[5] S. Chandrasekhar and P. C. Kendall, On force-free magnetic fields, *Astrophys. J.*, **126** (1957), 457–460.

[6] S. Chandrasekhar and L. Woltjer, On force-free magnetic fields, *Proc. Nat. Acad. Sci. USA*, **44** (1958), 285–289.

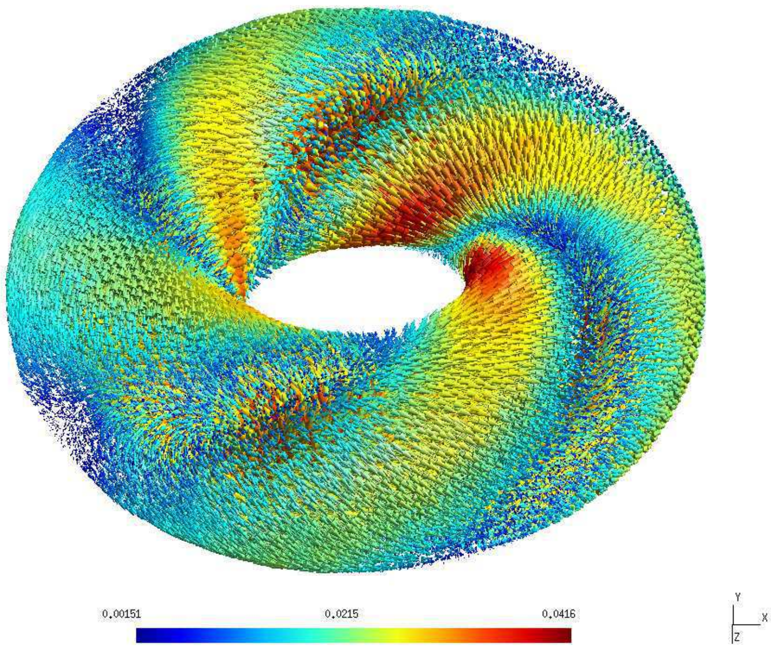

FIGURE 5. Beltrami field corresponding to the smallest positive eigenvalue.

[7] C. Foias and R. Temam, Remarques sur les équations de Navier-Stokes stationnaires et les phńomènes successifs de bifurcation, *Ann. Sc. Norm. Sup. Pisa*, **5** (1978), 29–63.

[8] V. Girault and P.-A Raviart, *Finite Element Approximations of the Navier-Stokes Equations, Theory and Algorithms*, Springer, Berlin, 1986.

[9] R. Hiptmair, P. R. Kotiuga and S. Tordeux, Self-adjoint curl operators, *Ann. Mat. Pura Appl.*, **191** (2012), 431–457.

[10] E. Lara, *Espectro del operador rotacional en dominios no simplemente conexos*, Mathematical Engineering thesis, Universidad de Concepción, Chile, 2013. Available from: http://www.ing-mat.udec.cl/~rodolfo/Tesis/Memoria_Titulo_Eduardo_Lara.pdf.

[11] S. Meddahi and V. Selgas, A mixed-FEM and BEM coupling for a three-dimensional eddy current problem, *M^2AN, Math. Model. Numer. Anal.*, **37** (2003), 291–318.

[12] B. Mercier, J. Osborn, J. Rappaz and P.-A. Raviart, Eigenvalue approximation by mixed and hybrid methods, *Math. Comp.*, **36** (1981), 427–453.

[13] P. Monk, *Finite Element Methods fro Maxwell's Equations*, Clarendon Press, Oxford, 2003.

[14] R. Rodríguez and P. Venegas, Numerical approximation of the spectrum of the curl operator, *Math. Comp.*, **83** (2014), 553–577.

[15] L. Woltjer, A theorem on force-free magnetic fields, *Proc. Natl. Acad. Sci. USA*, **44** (1958), 489–491.

[16] ———, The crab nebula, *Bull. Astron. Inst. Neth.*, **14** (1958), 39–80.

[17] J. Xiao and Q. Hu An iterative method for computing Beltrami fields on bounded domains, *Institute of Computational Mathematics and Scientific/Engineering Computing, Chinese Academy of Sciences*, Report No. ICMSEC-12-15 2012. Available from: http://www.cc.ac.cn/2012reserchreport/201215.pdf.

[18] Z. Yoshida and Y. Giga, Remarks on spectra of operator rot, *Math. Z.*, **204** (1990), 235–245.

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