The spectral problem for the curl operator

Rodolfo Rodríguez

CI^{2}MA, Departamento de Ingeniería Matemática
Universidad de Concepción, Chile

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1 Force-free fields

Let $\boldsymbol{H}$ be a magnetic field acting on a conducting fluid, whose motion is driven by the so-called ***Lorentz force***:

$$\boldsymbol{F} := \boldsymbol{J} \times \boldsymbol{B},$$

where

- $\boldsymbol{J} := \mathbf{curl}\, \boldsymbol{H}$ is the current density,
- $\boldsymbol{B} := \mu \boldsymbol{H}$ is the magnetic induction (μ being the magnetic permeability, which in an isotropic medium is a scalar).

Because of this, in magnetohydrodynamics, a magnetic field satisfying

$$\mathbf{curl}\, \boldsymbol{H} = \lambda \boldsymbol{H},$$

with λ a scalar function, is called a ***force-free field***.

This kind of fields appear in solar physics for theories on flares and coronal heating,[a,b] in fluids for the study of the static equilibrium of smectic liquid crystals, in plasma physics, superconducting materials, etc.

In 1958 Woltjer[c] showed that the lowest state of magnetic energy density within a closed magnetohydrodynamics system is attained when λ is spatially constant.

In such a case $\boldsymbol{H}$ is called a ***linear force-free field***. Its determination is naturally related with the spectral problem for the curl operator:

$$\mathbf{curl}\, \boldsymbol{H} = \lambda \boldsymbol{H}.$$

The eigenfunctions of this problem are known as ***free-decay fields*** and play an important role, for instance, in the study of turbulence in plasma physics.

[a] S. Chandrasekhar & L. Woltjer, On force-free magnetic fields, *Proc. Natl. Acad. Sci. USA*, **44** (1958) 285–289.
[b] L. Woltjer, The crab nebula, *Bull. Astron. Inst. Neth.*, **14** (1958) 39–80.
[c] L. Woltjer, A theorem on force-free magnetic fields, *Proc. Natl. Acad. Sci. USA*, **44** (1958) 489–491.

1.1 The eigenvalue problem for the curl operator

The spectral problem for the curl operator, has a longstanding tradition in mathematical physics. Enrico Beltrami seems to be the first who considered this problem in the context of fluid dynamics and electromagnetism.[a] This is the reason why the corresponding eigenfunctions are also called ***Beltrami fields***.

Analytical solutions of this problem are only known under particular symmetry assumptions. The first one was obtained in 1957 by Chandrasekhar and Kendall[b] for a sphere (the so called ***spheromak***) in the context of astrophysical plasmas arising in modeling of the solar crown.

[a] E. Beltrami, Considerazioni idrodinamiche, *Rend. Inst. Lombardo Acad. Sci. Let.*, **22** (1889) 122–131. (English translation: Considerations on hydrodynamics, *Int. J. Fusion Energy*, **3** (1985) 53–57.)
[b] S. Chandrasekhar & P.C. Kendall, On force-free magnetic fields, *Astrophys. J.*, **126** (1957) 457–460.

To reduce the problem to a bounded domain Ω, the natural boundary condition is $\boldsymbol{H}\cdot\boldsymbol{n}=0$ on $\partial\Omega$. Thus we are led to the following:

Problem 0: *Find $\lambda\in\mathbb{C}$ and $\boldsymbol{H}\not\equiv\boldsymbol{0}$ such that*

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{H}&=\lambda\boldsymbol{H} && \text{in } \Omega,\\ \operatorname{div}\boldsymbol{H}&=0 && \text{in } \Omega,\\ \boldsymbol{H}\cdot\boldsymbol{n}&=0 && \text{on } \partial\Omega. \end{aligned}$$

The second equation (which for $\lambda\neq0$ is a consequence of the first one) rules out the trivial solutions $\lambda=0,\ \boldsymbol{H}=\nabla\varphi$.

The approximation of this problem was analyzed for Ω simply connected.[a]

However, when the domain Ω is multiply connected, the set of eigenvalues of Problem 0 is the whole complex plane $\mathbb{C}$![b]

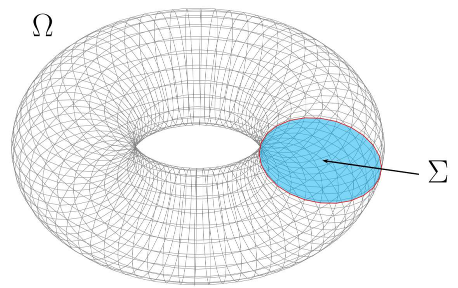
[a] R. Rodríguez & P. Venegas, Numerical approximation of the spectrum of the curl operator. *Math. Comp.* **83** (2014) 553-577.

[b] Z. Yoshida & Y. Giga, Remarks on spectra of operator rot. *Math. Z.*, **204** (1990) 235–245.

2 Geometrical preliminaries

Let $\Omega\subset\mathbb{R}^3$ be a multiply connected bounded domain with Lipschitz boundary Γ and outer unit normal $\boldsymbol{n}$. Let $\{\Sigma_j\}_{j=1}^J$ be a set of ***cutting surfaces***, namely, connected open surfaces with boundary satisfying:

- $\Sigma_j\subset\Omega$;
- $\partial\Sigma_j\subset\Gamma$;
- $\bar\Sigma_i\cap\bar\Sigma_j=\emptyset,\ i\neq j$;
- $\Omega^0:=\Omega\setminus\bigcup_{j=1}^J\Sigma_j$ is **simply connected** and pseudo-Lipschitz.



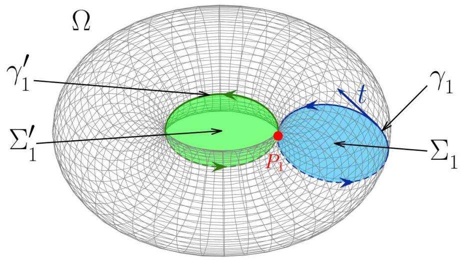
We fix a unit normal $\boldsymbol{n}_j$ on each Σ_j and denote its two faces by Σ_j^+ and Σ_j^-, with $\boldsymbol{n}_j$ being 'outer' normal to $\partial\Omega^0$ along Σ_j^+. For any $\psi\in\mathrm{H}^1(\Omega^0)$, we denote by $[\![\psi]\!]_{\Sigma_j}:=\psi|_{\Sigma_j^-}-\psi|_{\Sigma_j^+}$ the jump of ψ through Σ_j along $\boldsymbol{n}_j$.

2.1 A well posed eigenvalue problem

On a multiple connected domain, one additional constraint per cutting surface must be added to obtain a well posed eigenvalue problem for the curl operator. For each cutting surface, there are two alternatives:[a]

either $$\int_{\gamma_j}\boldsymbol{H}\cdot\boldsymbol{t}_j=0,$$

or $$\int_{\gamma_j'}\boldsymbol{H}\cdot\boldsymbol{t}_j'=0.$$



We focus on the first one which, according to the ***Stokes Theorem***, can be equivalently written as follows: $\displaystyle\int_{\Sigma_j}\mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{n}_j=0.$

[a] R. Hiptmair, P.R. Kotiuga & S. Tordeux, Self-adjoint curl operators. *Ann. Mat. Pura Appl.*, **191** (2012) 431–457.

Since for an eigenfunction of the curl operator with eigenvalue $\lambda\neq0$,

$$\int_{\Sigma_j}\boldsymbol{H}\cdot\boldsymbol{n}_j=\frac{1}{\lambda}\int_{\Sigma_j}\mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{n}_j=0.$$

we are led to the following eigenvalue problem, whose analysis and numerical approximation is our goal:

Problem 1: *Find $\lambda\in\mathbb{C}$ and $\boldsymbol{H}\in\mathrm{L}^2(\Omega)^3$, $\boldsymbol{H}\not\equiv\boldsymbol{0}$, such that*

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{H}&=\lambda\boldsymbol{H} && \text{in } \Omega,\\ \operatorname{div}\boldsymbol{H}&=0 && \text{in } \Omega,\\ \boldsymbol{H}\cdot\boldsymbol{n}&=0 && \text{on } \partial\Omega,\\ \int_{\Sigma_j}\boldsymbol{H}\cdot\boldsymbol{n}_j&=0, && 1\leq j\leq J. \end{aligned}$$

3 Function spaces

We recall the definitions of some function spaces:

$$\mathrm{L}^2(\Omega) := \left\{ v : \Omega \to \mathbb{C} : \int_\Omega |v|^2 < \infty \right\},$$
$$\mathrm{H}^1(\Omega) := \left\{ v \in \mathrm{L}^2(\Omega) : \nabla v \in \mathrm{L}^2(\Omega)^3 \right\},$$
$$\mathrm{H}(\mathrm{div}, \Omega) := \left\{ \boldsymbol{v} \in \mathrm{L}^2(\Omega)^3 : \mathrm{div}\, \boldsymbol{v} \in \mathrm{L}^2(\Omega) \right\},$$
$$\mathrm{H}(\mathrm{div}^0, \Omega) := \{ \boldsymbol{v} \in \mathrm{H}(\mathrm{div}, \Omega) : \ \mathrm{div}\, \boldsymbol{v} = 0 \text{ in } \Omega \},$$
$$\mathrm{H}_0(\mathrm{div}, \Omega) := \{ \boldsymbol{v} \in \mathrm{H}(\mathrm{div}, \Omega) : \ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega \},$$
$$\mathrm{H}_0(\mathrm{div}^0, \Omega) := \left\{ \boldsymbol{v} \in \mathrm{H}(\mathrm{div}^0, \Omega) : \ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega \right\},$$
$$\mathrm{H}(\mathbf{curl}, \Omega) := \left\{ \boldsymbol{v} \in \mathrm{L}^2(\Omega)^3 : \mathbf{curl}\, \boldsymbol{v} \in \mathrm{L}^2(\Omega)^3 \right\},$$
$$\mathrm{H}(\mathbf{curl}^0, \Omega) := \{ \boldsymbol{v} \in \mathrm{H}(\mathbf{curl}, \Omega) : \ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0} \text{ in } \Omega \},$$
$\mathrm{H}^s(\Omega)$ $(0 < s < 1)$ Sobolev space: $\mathrm{L}^2(\Omega) \overset{\text{comp.}}{\hookleftarrow} \mathrm{H}^s(\Omega) \overset{\text{comp.}}{\hookleftarrow} \mathrm{H}^1(\Omega)$,
$$\mathrm{H}^s(\mathbf{curl}, \Omega) := \left\{ \boldsymbol{v} \in \mathrm{H}^s(\Omega)^3 : \ \mathbf{curl}\, \boldsymbol{v} \in \mathrm{H}^s(\Omega)^3 \right\}.$$

3.1 Characterization of $\mathrm{H}(\mathbf{curl}^0, \Omega)$

Recall that $\Omega^0 := \Omega \setminus \bigcup_{j=1}^J \Sigma_j$, with Σ_j being the cutting surfaces.

For all $\chi \in \mathrm{H}^1(\Omega^0)$, $\nabla\chi \in \mathrm{L}^2(\Omega^0)^3$ but, in general, there is no extension $\widetilde{\chi}$ of χ to the whole Ω such that $\widetilde{\chi} \in \mathrm{H}^1(\Omega)$. Instead, any extension of $\nabla\chi$ obviously belongs to $\mathrm{L}^2(\Omega)^3$. We denote such extension $\widetilde{\nabla}\chi$. Let

$$\Theta := \left\{ \psi \in \mathrm{H}^1(\Omega^0) : \ [\![\psi]\!]_{\Sigma_j} = \text{const.}, \ 1 \le j \le J \right\}.$$

Lemma.[a] $\mathrm{H}(\mathbf{curl}^0, \Omega) = \widetilde{\nabla}\Theta.$

Lemma. *$\mathcal{C}^\infty(\Omega^0) \cap \Theta$ is dense in Θ.*

[a] C. Amrouche, C. Bernardi, M. Dauge & V. Girault, Vector potentials in three-dimensional non-smooth domains. *Math. Methods Appl. Sci.*, **21** (1998) 823–864.

3.2 A Green's formula

Let $\langle \cdot, \cdot \rangle_{\Sigma_j}$ denote the ***duality pairing*** between $\mathrm{H}^{1/2}(\Sigma_j)'$ and $\mathrm{H}^{1/2}(\Sigma_j)$.

Lemma. *For all $\boldsymbol{v} \in \mathrm{H}_0(\mathrm{div}, \Omega)$,*

$$\boldsymbol{v} \cdot \boldsymbol{n}_j|_{\Sigma_j} \in \mathrm{H}^{1/2}(\Sigma_j)', \qquad 1 \le j \le J,$$

*and the following **Green's formula** holds true:*

$$\sum_{j=1}^J \left\langle \boldsymbol{v} \cdot \boldsymbol{n}_j, [\![\psi]\!]_{\Sigma_j} \right\rangle_{\Sigma_j} = \int_{\Omega^0} \boldsymbol{v} \cdot \widetilde{\nabla}\psi + \int_{\Omega^0} (\mathrm{div}\, \boldsymbol{v})\, \psi \qquad \forall \psi \in \mathrm{H}^1(\Omega^0).$$

3.3 Harmonic Neumann fields

$$\mathcal{K}_T(\Omega) := \mathrm{H}(\mathbf{curl}^0, \Omega) \cap \mathrm{H}_0(\mathrm{div}^0, \Omega).$$

Lemma. $\mathrm{H}(\mathbf{curl}^0, \Omega) = \mathcal{K}_T \overset{\perp}{\oplus} \nabla(\mathrm{H}^1(\Omega)).$

Lemma.[a] *$\dim(\mathcal{K}_T(\Omega)) = J$ (number of cutting surfaces). A basis is given by $\left\{ \widetilde{\nabla}\phi_j \right\}_{j=1}^J$, where $\phi_j \in \Theta/\mathbb{R}$ is the unique solution of*

$$\begin{aligned} \Delta\phi_j &= 0 && \text{in } \Omega^0, \\ \partial_n \phi_j &= 0 && \text{on } \Gamma, \\ [\![\partial_n \phi_j]\!]_{\Sigma_k} &= 0, && 1 \le k \le J, \\ [\![\phi_j]\!]_{\Sigma_k} &= \delta_{j,k}, && 1 \le k \le J. \end{aligned}$$

[a] C. Foias & R. Temam, Remarques sur les équations de Navier-Stokes stationnaires et les phńomènes successifs de bifurcation. *Ann. Sc. Norm. Sup. Pisa*, **5** (1978) 29–63.

3.4 Space $\boldsymbol{\mathcal{X}}$

Helmholtz decomposition: $\mathrm{L}^2(\Omega)^3 = \mathrm{H}_0(\mathrm{div}^0,\Omega) \overset{\perp}{\oplus} \nabla(\mathrm{H}^1(\Omega))$.

Recall: $\mathcal{K}_T(\Omega) := \mathrm{H}(\mathbf{curl}^0,\Omega) \cap \mathrm{H}_0(\mathrm{div}^0,\Omega)$. Let

$$\boldsymbol{\mathcal{X}} := \mathcal{K}_T{}^{\perp_{\mathrm{H}_0(\mathrm{div}^0,\Omega)}}.$$

$$\mathrm{L}^2(\Omega)^3 = \underbrace{\boldsymbol{\mathcal{X}} \overset{\perp}{\oplus} \mathcal{K}_T}_{H_0(\mathrm{div}^0;\Omega)} \overset{\perp}{\oplus} \nabla(\mathrm{H}^1(\Omega)) = \boldsymbol{\mathcal{X}} \overset{\perp}{\oplus} \underbrace{\mathcal{K}_T \overset{\perp}{\oplus} \nabla(\mathrm{H}^1(\Omega))}_{\mathrm{H}(\mathbf{curl}^0,\Omega)}.$$

Lemma. $$\boldsymbol{u} \in \boldsymbol{\mathcal{X}} \iff \begin{cases} \mathrm{div}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 & \text{on } \Gamma, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n}_j, 1\rangle_{\Sigma_j} = 0, & 1 \le j \le J. \end{cases}$$

3.5 Space $\boldsymbol{\mathcal{Z}}$

Let

$$\boldsymbol{\mathcal{Z}} := \{\boldsymbol{v} \in \mathrm{H}(\mathbf{curl},\Omega) : \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{\mathcal{X}}\}.$$

Then,

$$\boldsymbol{v} \in \boldsymbol{\mathcal{Z}} \iff \begin{cases} \boldsymbol{v} \in \mathrm{H}(\mathbf{curl},\Omega), & \\ \mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{n} = 0 & \text{on } \Gamma, \\ \langle \mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{n}_j, 1\rangle_{\Sigma_j} = 0, & 1 \le j \le J. \end{cases}$$

Lemma. $\boldsymbol{\mathcal{Z}} \supset \mathcal{D}(\Omega)^3$.

Lemma. $\displaystyle\int_\Omega (\mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{v} - \boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}) = 0 \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{\mathcal{Z}}.$

Lemma. *$\mathcal{C}^\infty(\bar\Omega)^3 \cap \boldsymbol{\mathcal{Z}}$ is dense in $\boldsymbol{\mathcal{Z}}$.*

3.6 Space $\boldsymbol{\mathcal{V}}$

Let

$$\boldsymbol{\mathcal{V}} := \boldsymbol{\mathcal{X}} \cap \boldsymbol{\mathcal{Z}}.$$

Then,

$$\boldsymbol{v} \in \boldsymbol{\mathcal{V}} \iff \begin{cases} \boldsymbol{v} \in \mathrm{H}(\mathrm{div}^0,\Omega) \cap \mathrm{H}(\mathbf{curl},\Omega), & \\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 & \text{on } \Gamma, \\ \langle \boldsymbol{v}\cdot\boldsymbol{n}_j, 1\rangle_{\Sigma_j} = 0, & 1 \le j \le J, \\ \mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{n} = 0 & \text{on } \Gamma, \\ \langle \mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{n}_j, 1\rangle_{\Sigma_j} = 0, & 1 \le j \le J. \end{cases}$$

Lemma. $\boldsymbol{\mathcal{Z}} = \mathrm{H}(\mathbf{curl}^0,\Omega) \overset{\perp}{\oplus} \boldsymbol{\mathcal{V}}$.

4 Mixed variational formulation

Problem 1: *Find $\lambda \in \mathbb{C}$ and $\boldsymbol{u} \in \mathrm{L}^2(\Omega)^3$, $\boldsymbol{u} \not\equiv \mathbf{0}$, such that*

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{u} &= \lambda\boldsymbol{u} && \text{in } \Omega, \\ \mathrm{div}\,\boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} &= 0 && \text{on } \partial\Omega, \\ \int_{\Sigma_j} \boldsymbol{u}\cdot\boldsymbol{n}_j &= 0, && 1 \le j \le J. \end{aligned}$$

Problem 2: *Find $\lambda \in \mathbb{C}$ and $(\boldsymbol{u},\boldsymbol{\chi}) \in \boldsymbol{\mathcal{Z}} \times \mathrm{H}(\mathbf{curl}^0,\Omega)$, $(\boldsymbol{u},\boldsymbol{\chi}) \not\equiv \mathbf{0}$, such that*

$$\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\bar{\boldsymbol{v}} + \int_\Omega \boldsymbol{\chi}\cdot\bar{\boldsymbol{v}} = \lambda\int_\Omega \boldsymbol{u}\cdot\mathbf{curl}\,\bar{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{Z}},$$

$$\int_\Omega \boldsymbol{u}\cdot\bar{\boldsymbol{\eta}} = 0 \qquad \forall \boldsymbol{\eta} \in \mathrm{H}(\mathbf{curl}^0,\Omega).$$

Lemma. *Problem 1 and Problem 2 are equivalent.*

PROOF. Let $\boldsymbol{u} \in \mathrm{L}^2(\Omega)^3$.

$$\left.\begin{array}{l}\displaystyle\int_\Omega \boldsymbol{u}\cdot\bar{\boldsymbol{\eta}} = 0 \\ \quad \forall \boldsymbol{\eta} \in \mathrm{H}(\mathbf{curl}^0,\Omega)\end{array}\right\} \overset{\boldsymbol{u}\in\boldsymbol{\mathcal{X}}}{\Longleftrightarrow} \left\{\begin{array}{ll} \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 & \text{on } \Gamma, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n}_j, 1\rangle_{\Sigma_j} = 0, & 1 \le j \le J. \end{array}\right.$$

$(\lambda, \boldsymbol{u})$ solution of Problem 1 $\Longrightarrow$ $(\lambda, \boldsymbol{u}, \mathbf{0})$ solution of Problem 2.

Let $(\lambda, \boldsymbol{u}, \boldsymbol{\chi})$ be a solution of Problem 2.

Taking $\boldsymbol{v} = \boldsymbol{\chi} \in \boldsymbol{\mathcal{Z}}$ $\Longrightarrow$ ***Lagrange multiplier*** $\boldsymbol{\chi} = \mathbf{0}$.

Taking $\boldsymbol{v} \in \mathcal{D}(\Omega)^3 \subset \boldsymbol{\mathcal{Z}}$ $\Longrightarrow$ $\mathbf{curl}\,\boldsymbol{u} - \lambda\boldsymbol{u} \in \mathrm{H}(\mathbf{curl}^0,\Omega)$.

$\boldsymbol{u} \in \boldsymbol{\mathcal{Z}}$ $\Longrightarrow$ $\mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{\mathcal{X}}$.

$\boldsymbol{u} \in \boldsymbol{\mathcal{X}}$ $\Longrightarrow$ $\mathbf{curl}\,\boldsymbol{u} - \lambda\boldsymbol{u} \in \boldsymbol{\mathcal{X}} \cap \mathrm{H}(\mathbf{curl}^0,\Omega) = \{\mathbf{0}\}$ $\Longrightarrow$

$$\mathbf{curl}\,\boldsymbol{u} = \lambda\boldsymbol{u} \quad \text{in } \Omega.$$

4.1 Solution operator

$$\boldsymbol{T}: \boldsymbol{\mathcal{Z}} \longrightarrow \boldsymbol{\mathcal{Z}},$$

$$\boldsymbol{f} \longmapsto \boldsymbol{T}\boldsymbol{f} := \boldsymbol{w} \in \boldsymbol{\mathcal{Z}}: \ \exists \boldsymbol{\chi} \in \mathrm{H}(\mathbf{curl}^0,\Omega):$$

$$\left\{\begin{array}{l} \displaystyle\int_\Omega \mathbf{curl}\,\boldsymbol{w}\cdot\mathbf{curl}\,\bar{\boldsymbol{v}} + \int_\Omega \boldsymbol{\chi}\cdot\bar{\boldsymbol{v}} = \int_\Omega \boldsymbol{f}\cdot\mathbf{curl}\,\bar{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{Z}}, \\ \displaystyle\int_\Omega \boldsymbol{w}\cdot\bar{\boldsymbol{\eta}} = 0 \qquad \forall \boldsymbol{\eta} \in \mathrm{H}(\mathbf{curl}^0,\Omega). \end{array}\right.$$

$\mu \neq 0: \ \boldsymbol{T}\boldsymbol{u} = \mu\boldsymbol{u}, \ \boldsymbol{u} \not\equiv \mathbf{0}$ $\Longleftrightarrow$ $\left(\frac{1}{\mu}, \boldsymbol{u}, \mathbf{0}\right)$ solution of Problem 2.

The ***Babuška-Brezzi conditions*** hold true:

$$\exists \alpha > 0: \quad \int_\Omega |\mathbf{curl}\,\boldsymbol{v}|^2 \ge \alpha \|\boldsymbol{v}\|_{\mathbf{curl},\Omega} \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}} := \mathrm{H}(\mathbf{curl}^0,\Omega)^{\perp_{\boldsymbol{\mathcal{Z}}}},$$

$$\exists \beta > 0: \quad \sup_{\boldsymbol{v}\in\boldsymbol{\mathcal{Z}}} \frac{\int_\Omega \boldsymbol{\eta}\cdot\boldsymbol{v}}{\|\boldsymbol{v}\|_{\mathbf{curl},\Omega}} \ge \beta \|\boldsymbol{\eta}\|_{0,\Omega} \quad \forall \boldsymbol{\eta} \in \mathrm{H}(\mathbf{curl}^0,\Omega),$$

4.2 Spectral characterization

Lemma. $\exists s > \frac{1}{2}$ *and* $C > 0: \ \forall \boldsymbol{f} \in \boldsymbol{\mathcal{Z}}, \ \boldsymbol{w} = \boldsymbol{T}\boldsymbol{f} \in \mathrm{H}^s(\mathbf{curl},\Omega)$ *and*

$$\|\boldsymbol{w}\|_{s,\Omega} + \|\mathbf{curl}\,\boldsymbol{w}\|_{s,\Omega} \le C\,\|\boldsymbol{f}\|_{0,\Omega}.$$

Consequently, $\boldsymbol{T}: \boldsymbol{\mathcal{Z}} \to \boldsymbol{\mathcal{Z}}$ ***is compact.***

Lemma. $\boldsymbol{T}: \boldsymbol{\mathcal{Z}} \to \boldsymbol{\mathcal{Z}}$ ***is self-adjoint.***

Theorem. $\operatorname{sp}(\boldsymbol{T}) = \{\mu_n\}_{n\in\mathbb{N}} \cup \{0\}$.

(i) $\mu_0 = 0$ *is an infinite-multiplicity eigenvalue with associated eigenspace* $\mathrm{H}(\mathbf{curl}^0,\Omega)$;

(ii) $\{\mu_n\}_{n\in\mathbb{N}}$ *is a sequence of finite-multiplicity eigenvalues (repeated according to their respective multiplicities) and* $\mu_n \to 0$.
Moreover, there exists a ***Hilbertian basis*** $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ *of* $\boldsymbol{\mathcal{V}}$, *with* $\boldsymbol{u}_n$ *such that* $\boldsymbol{T}\boldsymbol{u}_n = \mu_n\boldsymbol{u}_n$, $n \in \mathbb{N}$.

5 Finite element spectral approximation

$\{\mathcal{T}_h\}_{h>0}$ regular family of tetrahedral partitions of a polyhedral domain $\bar{\Omega}$.

Nédélec F.E. space:

$$\boldsymbol{\mathcal{N}}_h := \left\{\boldsymbol{v}_h \in \mathrm{H}(\mathbf{curl},\Omega): \ \boldsymbol{v}_h|_T \in \boldsymbol{\mathcal{N}}^k(T) \ \ \forall T \in \mathcal{T}_h\right\}$$

with $\quad \boldsymbol{\mathcal{N}}^k(T) := \mathcal{P}_{k-1}(T)^3 \oplus \left\{\boldsymbol{p} \in \bar{\mathcal{P}}_k(T)^3: \ \boldsymbol{p}(\boldsymbol{x})\cdot\boldsymbol{x} = 0\right\}$.

$$\boldsymbol{\mathcal{Z}}_h := \boldsymbol{\mathcal{N}}_h \cap \boldsymbol{\mathcal{Z}}.$$

Stokes Theorem $\Longrightarrow$

$$\boldsymbol{\mathcal{Z}}_h = \left\{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_h: \ \mathbf{curl}\,\boldsymbol{v}_h\cdot\boldsymbol{n} = 0 \text{ on } \Gamma \text{ and } \int_{\gamma_j} \boldsymbol{v}_h\cdot\boldsymbol{t}_j = 0\right\}.$$

Nédélec interpolant: $\quad \boldsymbol{I}_h^N: \mathrm{H}^s(\mathbf{curl},\Omega) \longrightarrow \boldsymbol{\mathcal{N}}_h \qquad (s > \frac{1}{2})$.

Lemma. $\forall \boldsymbol{v} \in \mathrm{H}^s(\mathbf{curl},\Omega) \cap \boldsymbol{\mathcal{Z}}$ *with* $s > \frac{1}{2}$, $\boldsymbol{I}_h^N\boldsymbol{v} \in \boldsymbol{\mathcal{Z}}_h$.

5.1 Finite element discretization

To discretize the Lagrange multiplier $\chi \in \mathrm{H}(\mathbf{curl}^0, \Omega)$, we recall that $\mathrm{H}(\mathbf{curl}^0, \Omega) = \widetilde{\nabla}\Theta$, to write $\chi = \widetilde{\nabla}\varphi$ with $\varphi \in \Theta$, and use

$$\Theta_h := \left\{ \psi_h \in \mathcal{C}(\Omega^0) : \psi_h|_T \in P_k(T)\ \forall T \in \mathcal{T}_h \text{ and } [\![\psi_h]\!]_{\Sigma_j} = \text{const.} \right\},$$

Lemma. $\widetilde{\nabla}\Theta_h = \boldsymbol{\mathcal{N}}_h \cap \mathrm{H}(\mathbf{curl}^0, \Omega).$

Problem 2_h: *Find* $\lambda_h \in \mathbb{C}$ *and* $(\boldsymbol{u}_h, \varphi_h) \in \boldsymbol{\mathcal{Z}}_h \times \Theta_h/\mathbb{C}$, $(\boldsymbol{u}_h, \varphi_h) \not\equiv \mathbf{0}$, *such that*

$$\int_\Omega \mathbf{curl}\, \boldsymbol{u}_h \cdot \mathbf{curl}\, \bar{\boldsymbol{v}}_h + \int_\Omega \widetilde{\nabla}\varphi_h \cdot \bar{\boldsymbol{v}}_h = \lambda_h \int_\Omega \boldsymbol{u}_h \cdot \mathbf{curl}\, \bar{\boldsymbol{v}}_h \qquad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{Z}}_h,$$

$$\int_\Omega \boldsymbol{u}_h \cdot \widetilde{\nabla}\bar{\psi}_h = 0 \qquad \forall \psi_h \in \Theta_h/\mathbb{C}.$$

As in the continuous case, the Lagrange multiplier vanishes: $\varphi_h \equiv 0$.

5.2 Finite element approximation

We resort to the classical theory for finite element spectral approximation of ***mixed problems of type*** $\boldsymbol{Q_1}$.[a] With this aim, we have to prove:

- ***Babuška-Brezzi conditions for the continuous problem.*** ✓
- ***Babuška-Brezzi conditions for the discrete problem:***

$$\exists \alpha_* > 0 : \quad \int_\Omega |\mathbf{curl}\, \boldsymbol{v}_h|^2 \geq \alpha_* \|\boldsymbol{v}_h\|_{\mathbf{curl},\Omega} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{V}}_h,$$

where $\boldsymbol{\mathcal{V}}_h := \left\{ \boldsymbol{v}_h \in \boldsymbol{\mathcal{Z}}_h : \int_\Omega \boldsymbol{v}_h \cdot \widetilde{\nabla}\psi_h = 0\ \forall \psi_h \in \Theta_h \right\}$, and

$$\exists \beta_* > 0 : \quad \sup_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{Z}}_h} \frac{\int_\Omega \widetilde{\nabla}\psi_h \cdot \boldsymbol{v}_h}{\|\boldsymbol{v}_h\|_{\mathbf{curl},\Omega}} \geq \beta_* \left\|\widetilde{\nabla}\psi_h\right\|_{0,\Omega} \qquad \forall \psi_h \in \Theta_h.$$

[a] B. Mercier, J. Osborn, J. Rappaz & P.-A. Raviart, Eigenvalue approximation by mixed and hybrid methods. *Math. Comp.*, **36** (1981) 427–453.

- ***Density of the finite element spaces:*** $\forall (\boldsymbol{v}, \psi) \in \boldsymbol{\mathcal{Z}} \times \Theta$,

$$\inf_{(\boldsymbol{v}_h, \psi_h) \in \boldsymbol{\mathcal{Z}}_h \times \Theta_h} \left(\|\boldsymbol{v} - \boldsymbol{v}_h\|_{\mathbf{curl},\Omega} + \left\|\widetilde{\nabla}\psi - \widetilde{\nabla}\psi_h\right\|_{0,\Omega} \right) \xrightarrow{h \to 0} 0.$$

It follows from the density of $\mathcal{C}^\infty(\bar{\Omega})^3 \cap \boldsymbol{\mathcal{Z}}$ and $\mathcal{C}^\infty(\Omega^0) \cap \Theta$ in $\boldsymbol{\mathcal{Z}}$ and Θ, resp.

- ***Compactness of the global solution operator***

$$\boldsymbol{G} : \ \boldsymbol{\mathcal{Z}} \times \Theta/\mathbb{C} \longrightarrow \boldsymbol{\mathcal{Z}} \times \Theta/\mathbb{C},$$
$$(\boldsymbol{f}, g) \longmapsto \boldsymbol{G}(\boldsymbol{f}, g) := (\boldsymbol{w}, \xi) \in \boldsymbol{\mathcal{Z}} \times \Theta/\mathbb{C} :$$

$$\begin{cases} \displaystyle\int_\Omega \mathbf{curl}\, \boldsymbol{w} \cdot \mathbf{curl}\, \bar{\boldsymbol{v}} + \int_\Omega \nabla\xi \cdot \bar{\boldsymbol{v}} = \int_\Omega \boldsymbol{f} \cdot \mathbf{curl}\, \bar{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{Z}}, \\ \displaystyle\int_\Omega \boldsymbol{w} \cdot \nabla\bar{\psi} = 0 \qquad \forall \psi \in \Theta/\mathbb{C}. \end{cases}$$

It follows that $\boldsymbol{G}(\boldsymbol{f}, g) = (\boldsymbol{T}\boldsymbol{f}, 0)$ and, hence, $\boldsymbol{G}$ is compact.

5.3 Distance between subspaces

We recall that, given two subspaces $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{F}}$ of $\boldsymbol{\mathcal{Z}}$,

$$\delta(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{F}}) := \sup_{\substack{\boldsymbol{v} \in \boldsymbol{\mathcal{E}} \\ \|\boldsymbol{v}\|_{\mathbf{curl},\Omega} = 1}} \left(\inf_{\boldsymbol{w} \in \boldsymbol{\mathcal{F}}} \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbf{curl},\Omega} \right)$$

is the ***distance from*** $\boldsymbol{\mathcal{E}}$ ***to*** $\boldsymbol{\mathcal{F}}$ and

$$\widehat{\delta}(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{F}}) := \max \left\{ \delta(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{F}}), \delta(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{E}}) \right\}$$

is the ***gap*** (or symmetric distance) between both subspaces.

5.4 Spectral approximation results

Theorem. *Let λ be an eigenvalue of Problem 2 with finite multiplicity m and $\boldsymbol{\mathcal{E}} \times \{0\} \subset \boldsymbol{\mathcal{Z}} \times \Theta/\mathbb{C}$ the corresponding eigenspace.*
There exist exactly m eigenvalues $\lambda_h^{(1)}, \dots, \lambda_h^{(m)}$ of Problem $\mathbf{2_h}$ (repeated according to their respective multiplicities) which converge to λ as $h \to 0$.
Let $\boldsymbol{\mathcal{E}}_h \times \{0\}$ be the direct sum of the eigenspaces corresponding to $\lambda_h^{(1)}, \dots, \lambda_h^{(m)}$. Then,

$$\widehat{\delta}(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}_h) \leq C\gamma_h,$$

$$\left|\lambda - \lambda_h^{(i)}\right| \leq C\gamma_h^2, \qquad i = 1, \dots, m,$$

where

$$\gamma_h := \delta(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{Z}}_h) \leq C h^{\min\{s,k\}}.$$

6 Implementation issues

It remains to show how to impose in $\boldsymbol{\mathcal{N}}_h$ the constraints defining $\boldsymbol{\mathcal{Z}}_h$:[a]

$$\mathbf{curl}\, \boldsymbol{v}_h \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma \qquad \text{and} \qquad \int_{\gamma_j} \boldsymbol{v}_h \cdot \boldsymbol{t}_j = 0, \quad 1 \leq j \leq J.$$

$$\boldsymbol{v}_h \in \boldsymbol{\mathcal{Z}}_h \qquad \Longleftrightarrow \qquad \boldsymbol{n} \times \boldsymbol{v}_h|_{\widetilde{\Gamma}} \times \boldsymbol{n} = \widetilde{\nabla}_{\widetilde{\Gamma}} \left(\varphi_h|_{\widetilde{\Gamma}}\right) \quad \text{with } \varphi_h \in \Theta_h,$$

where $\widetilde{\Gamma} := \Gamma \setminus \bigcup_{j=1}^{J} \gamma_j$ and $\widetilde{\nabla}_{\widetilde{\Gamma}}$ is the surface gradient on $\widetilde{\Gamma}$.
Let $\mathcal{L}_h := \{\psi_h \in \mathcal{C}(\Omega) : \ \psi_h|_T \in \mathcal{P}_k(T) \ \forall T \in \mathcal{T}_h\}$.
For each Σ_j, let $\widehat{\varphi}_j \in \Theta_h$ be such that, for all nodes P,

$$\widehat{\varphi}_j(P) = \begin{cases} 1, & \text{if } P \in \Sigma_j^+, \\ 0, & \text{if } P \notin \Sigma_j^+. \end{cases} \qquad \text{Then,} \qquad \Theta_h = \mathcal{L}_h \oplus \left\langle \{\widehat{\varphi}_j\}_{j=1}^{J} \right\rangle.$$

[a] S. Meddahi and V. Selgas, A mixed-FEM and BEM coupling for a three-dimensional eddy current problem, *M^2AN Math. Model. Numer. Anal.*, **37** (2003) 291–318.

For simplicity, we assume that the boundary Γ is connected.

Let $\{\varphi_j\}_{j=1}^{L}$ be the nodal basis of $\mathcal{L}_h$. Without loss of generality we order these basis functions so that the first K of them correspond to nodal values on Γ.

Since surface gradients are determined up to an additive constant, we choose one vertex on Γ and drop out the basis function corresponding to this vertex (for instance vertex number K).

Let $\{\boldsymbol{\Phi}_m\}_{m=1}^{M}$ be the nodal basis of $\boldsymbol{\mathcal{N}}_h$; without loss of generality, we assume that $\{\boldsymbol{\Phi}_m\}_{m=M'+1}^{M}$ are those related to the faces or edges on Γ.

Theorem. $\{\boldsymbol{\Phi}_m\}_{m=1}^{M'} \cup \{\nabla\varphi_k\}_{k=1}^{K-1} \cup \{\widetilde{\nabla}\widehat{\varphi}_j\}_{j=1}^{J}$ *is a basis of $\boldsymbol{\mathcal{Z}}_h$.*

The matrices of the algebraic eigenvalue problem corresponding to the discrete mixed formulation can be easily obtained by ***static condensation*** from the matrices of the classical Nédélec and $\mathcal{P}_k$-continuous elements.

6.1 Reduction to a well posed eigenvalue problem

The resulting algebraic generalized eigenvalue problem has the form

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^{\mathrm{t}} \\ \boldsymbol{B} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{u}}_h \\ \vec{\varphi}_h \end{pmatrix} = \lambda_h \begin{pmatrix} \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{u}}_h \\ \vec{\varphi}_h \end{pmatrix},$$

where $\vec{\boldsymbol{u}}_h$ and $\vec{\varphi}_h$ are the vectors of nodal components of $\boldsymbol{u}_h$ and φ_h, resp.
Both matrices above are symmetric, but none is positive definite.

Since $\vec{\varphi}_h = \boldsymbol{0}$, the above problem is equivalent to

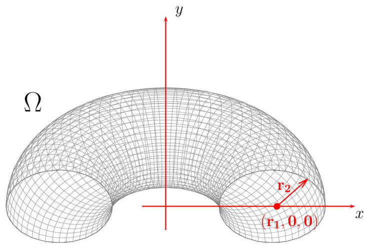
$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^{\mathrm{t}} \\ \boldsymbol{B} & -\boldsymbol{I} \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{u}}_h \\ \vec{\varphi}_h \end{pmatrix} = \lambda_h \begin{pmatrix} \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{u}}_h \\ \vec{\varphi}_h \end{pmatrix},$$

which, in turn, is equivalent to

$$\left(\boldsymbol{A} + \boldsymbol{B}^{\mathrm{t}}\boldsymbol{B}\right) \vec{\boldsymbol{u}}_h = \lambda_h \boldsymbol{C} \vec{\boldsymbol{u}}_h,$$

with symmetric and positive definite left-hand side matrix.

7 Numerical results

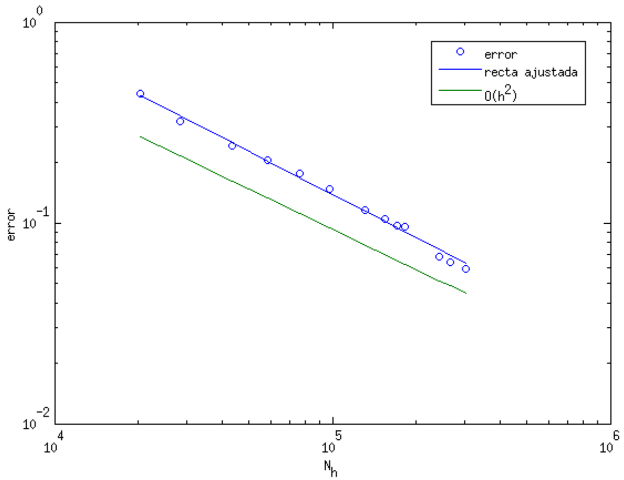


- We have solved the problem in a toroidal domain as that shown above, with $r_1 = 1$ and $r_2 = 0.5$. No analytical solution is available.
- We have used meshes $\mathcal{T}_h$ with different levels of refinement; we identify each mesh by the corresponding number N_h of tetrahedra.
- For each computed eigenvalue we have estimated the convergence order and a more accurate value by a least-squares fitting of the model $\lambda_{h,k} \approx \lambda_{\mathrm{ex}} + Ch^t$.

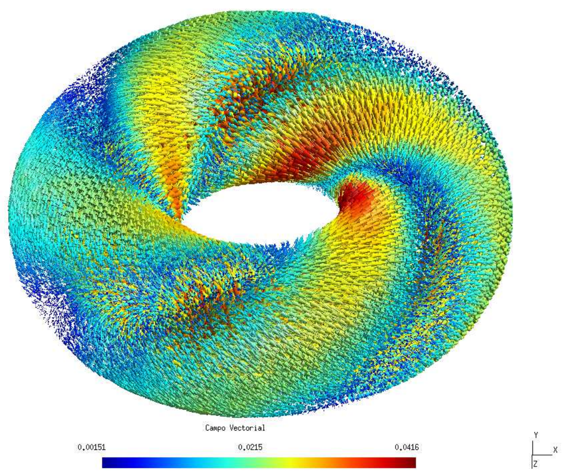
7.1 Computed eigenvalues

N_h	λ_1	λ_2	λ_3	λ_4
1259	13.7352	14.2419	14.7751	15.5001
2656	8.4648	8.6325	8.7671	8.9228
3822	7.9359	8.0224	8.1618	8.2056
7812	7.2752	7.2941	7.4303	7.4328
20384	6.6320	6.6353	6.7104	6.7118
28321	6.5199	6.5252	6.5888	6.5903
43667	6.4406	6.4444	6.5105	6.5121
58732	6.4096	6.4129	6.4723	6.4746
75999	6.3824	6.3871	6.4439	6.4448
97886	6.3565	6.3590	6.4155	6.4161
131222	6.3276	6.3285	6.3840	6.3842
154592	6.3156	6.3162	6.3726	6.3734
171127	6.3093	6.3098	6.3657	6.3662
182885	6.3075	6.3080	6.3642	6.3644
241429	6.2829	6.2833	6.3370	6.3370
264623	6.2781	6.2787	6.3323	6.3324
301862	6.2737	6.2741	6.3275	6.3277
λ_{ex}	**6.2233**	**6.2174**	**6.2714**	**6.2690**
order	**2.20**	**2.12**	**2.17**	**2.14**

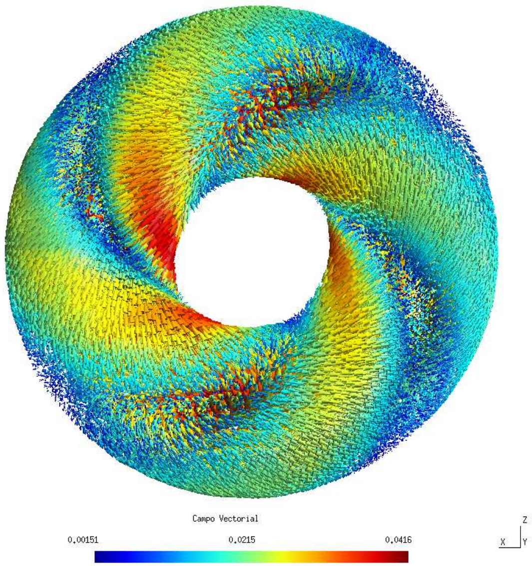
7.2 Error curve



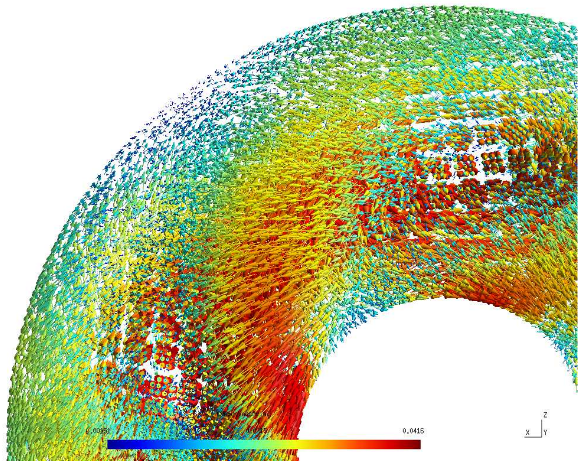
7.3 Eigenfunction corresponding to the smallest eigenvalue



Frontal view



Zoom



8 A Maxwell-like primal formulation

If $(\lambda, \boldsymbol{u})$ is a solution of Problem 1, then $\boldsymbol{u} \in \mathrm{H}(\mathbf{curl}, \Omega)$ and

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{u} &= \lambda \boldsymbol{u} && \text{in } \Omega, \\ \mathbf{curl}\,\boldsymbol{u} \cdot \boldsymbol{n} &= 0 && \text{on } \partial\Omega, \\ \langle \mathbf{curl}\,\boldsymbol{u} \cdot \boldsymbol{n}_j, 1 \rangle_{\Sigma_j} &= 0, && 1 \leq j \leq J. \end{aligned}$$

Hence, $\boldsymbol{u} \in \boldsymbol{\mathcal{Z}}$ and $\forall \boldsymbol{v} \in \boldsymbol{\mathcal{Z}}$ there holds

$$\int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\bar{\boldsymbol{v}} = \lambda \int_\Omega \boldsymbol{u} \cdot \mathbf{curl}\,\bar{\boldsymbol{v}} = \lambda \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \bar{\boldsymbol{v}} = \lambda^2 \int_\Omega \boldsymbol{u} \cdot \bar{\boldsymbol{v}}.$$

Thus, if $(\lambda, \boldsymbol{u})$ is a solution of Problem 1, then $(\lambda^2, \boldsymbol{u})$ is a solution of:

Problem 3: *Find $\lambda \in \mathbb{C}$ and $\boldsymbol{u} \in \boldsymbol{\mathcal{Z}}$, $\boldsymbol{u} \not\equiv \mathbf{0}$, such that*

$$\int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\bar{\boldsymbol{v}} = \lambda^2 \int_\Omega \boldsymbol{u} \cdot \bar{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{Z}}.$$

8.1 Solution operator

We will prove a sort of equivalence between Problem 1 and Problem 3.

$\lambda^2 = 0$ is an infinite-multiplicity eigenvalue of Problem 3 with eigenspace $\mathrm{H}(\mathbf{curl}^0, \Omega)$. Then, we consider the following equivalent problem:

$$\int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\bar{\boldsymbol{v}} + \int_\Omega \boldsymbol{u} \cdot \bar{\boldsymbol{v}} = (\lambda^2 + 1) \int_\Omega \boldsymbol{u} \cdot \bar{\boldsymbol{v}} \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{Z}}.$$

Thus, we are able to define the solution operator

$$\begin{aligned} \boldsymbol{S} : \boldsymbol{\mathcal{Z}} &\longrightarrow \boldsymbol{\mathcal{Z}}, \\ \boldsymbol{f} &\longmapsto \boldsymbol{S}\boldsymbol{f} := \boldsymbol{w} \in \boldsymbol{\mathcal{Z}} : \end{aligned}$$

$$\int_\Omega \mathbf{curl}\,\boldsymbol{w} \cdot \mathbf{curl}\,\bar{\boldsymbol{v}} + \int_\Omega \boldsymbol{w} \cdot \bar{\boldsymbol{v}} = \int_\Omega \boldsymbol{f} \cdot \bar{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{Z}}.$$

$\boldsymbol{S}$ is a well defined bounded self-adjoint linear operator. Moreover,

$(\lambda, \boldsymbol{u})$ solution of Problem 3 $\iff$ $\boldsymbol{S}\boldsymbol{u} = \mu \boldsymbol{u},\ \boldsymbol{u} \not\equiv \mathbf{0},\ \mu = \frac{1}{\lambda^2+1}$.

$S : \mathcal{Z} \longrightarrow \mathcal{Z}$ **is not compact.** In fact, $\mu = 1$ is an eigenvalue of $\boldsymbol{S}$ with infinite-dimensional eigenspace $\mathrm{H}(\mathbf{curl}^0, \Omega)$.

However, since $\boldsymbol{S}$ is self-adjoint, $\mathcal{V} := \mathrm{H}(\mathbf{curl}^0, \Omega)^{\perp_{\mathcal{Z}}}$ is an invariant subspace of $\boldsymbol{S}$ and

$$\boldsymbol{S}|_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathcal{V}$$

is compact, because of the following result:

Lemma. $\exists s > \frac{1}{2}$ *and* $C > 0 :\ \forall \boldsymbol{f} \in \mathcal{V},\ \boldsymbol{w} = \boldsymbol{S}\boldsymbol{f} \in \mathrm{H}^s(\mathbf{curl}, \Omega)$ *and*

$$\|\boldsymbol{w}\|_{s,\Omega} + \|\mathbf{curl}\,\boldsymbol{w}\|_{s,\Omega} \leq C\,\|\boldsymbol{f}\|_{0,\Omega}\,.$$

Consequently, $\boldsymbol{S}|_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ *is compact.*

Thus, we are able to write a thorough spectral characterization of $\boldsymbol{S}$.

8.2 Spectral characterization

Theorem. $\mathrm{sp}(\boldsymbol{S}) = \{\mu_n\}_{n\in\mathbb{N}} \cup \{0, 1\}.$

(i) $\mu_0 = 1$ *is an infinite-multiplicity eigenvalue with associated eigenspace* $\mathrm{H}(\mathbf{curl}^0, \Omega)$*;*

(ii) $\{\mu_n\}_{n\in\mathbb{N}}$ *is a sequence of finite-multiplicity eigenvalues (repeated according to their respective multiplicities),* $0 < \mu_n < 1$ *and* $\mu_n \to 0.$ *Moreover, there exists a Hilbertian basis* $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ *of* $\mathcal{V}$*, with* $\boldsymbol{u}_n$ *such that* $\boldsymbol{S}\boldsymbol{u}_n = \mu_n \boldsymbol{u}_n$*,* $n \in \mathbb{N}$*.*

(ii) $\mu = 0$ *is not an eigenvalue of* $\boldsymbol{S}$*.*

Theorem. *If* λ *is an eigenvalue of Problem 1 with eigenspace* $\mathcal{E}$*, then* $\mu = \frac{1}{1+\lambda^2}$ *is an eigenvalue of* $\boldsymbol{S}$ *and* $\mathcal{E}$ *an* ***invariant subspace****.* *Conversely, if* $\mu \neq 1$ *is an eigenvalue of* $\boldsymbol{S}$ *with eigenspace* $\mathcal{E}$*, then there exists at least one eigenvalue* λ *of Problem 1 such that* $\mu = \frac{1}{\lambda^2+1}$ *and* $\mathcal{E}$ *is an* ***invariant subspace*** *of this problem.*

Remark. When λ and $-\lambda$ are both eigenvalues of Problem 1, λ^2 is a multiple eigenvalue of Problem 3. In such a case, the eigenspace of λ^2 in Problem 3 is the subspace spanned by the eigenfunctions of λ and $-\lambda$.

Therefore, in this case, the eigenfunctions of Problem 3 are not necessarily eigenfunctions of Problem 1 (and hence Beltrami fields), but linear combinations of them.

This happens in particular when Ω is symmetric in the sense that there is an orthogonal coordinate system in which $\boldsymbol{x} \in \Omega \iff -\boldsymbol{x} \in \Omega$.
In fact, in such a case,

$$\left.\begin{array}{l} \mathbf{curl}\,\boldsymbol{u} = \lambda \boldsymbol{u} \quad \text{in } \Omega \\ \boldsymbol{u}'(\boldsymbol{x}) := \boldsymbol{u}(-\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \end{array}\right\} \Longrightarrow \mathbf{curl}\,\boldsymbol{u}' = -\lambda \boldsymbol{u}' \quad \text{in } \Omega.$$

Thus, $(\lambda, \boldsymbol{u})$ solves Problem 1 if and only if $(-\lambda, \boldsymbol{u}')$ solves it too.

8.3 Finite element approximation

The ***Ritz-Galerkin approximation*** of Problem 3 reads as follows:

Problem 3_h: *Find* $\lambda_h \in \mathbb{C}$ *and* $\boldsymbol{u}_h \in \mathcal{Z}_h$*,* $\boldsymbol{u}_h \neq \mathbf{0}$*, such that*

$$\int_\Omega \mathbf{curl}\,\boldsymbol{u}_h \cdot \mathbf{curl}\,\bar{\boldsymbol{v}}_h = \lambda_h^2 \int_\Omega \boldsymbol{u}_h \cdot \bar{\boldsymbol{v}}_h \qquad \forall \boldsymbol{v}_h \in \mathcal{Z}_h.$$

The resulting generalized matrix eigenvalue problem has the form

$$\boldsymbol{A}\,\vec{\boldsymbol{u}}_h = \lambda_h^2\,\boldsymbol{M}\,\vec{\boldsymbol{u}}_h,$$

where $\vec{\boldsymbol{u}}_h$ is the vector of nodal components of $\boldsymbol{u}_h$.

This is a well-posed generalized eigenvalue problem, because M is a symmetric and positive definite matrix.

As in the continuous case, $\lambda_h^2 = 0$ is an eigenvalue of Problem $\mathbf{3_h}$ with eigenspace $\mathrm{H}(\mathbf{curl}^0,\Omega)\cap\boldsymbol{\mathcal{N}}_h$. Then, we proceed as above and define the solution operator

$$\begin{aligned} \boldsymbol{S}_h : \boldsymbol{\mathcal{Z}} &\longrightarrow \boldsymbol{\mathcal{Z}}, \\ \boldsymbol{f} &\longmapsto \boldsymbol{S}_h\boldsymbol{f} := \boldsymbol{w}_h \in \boldsymbol{\mathcal{Z}}_h : \end{aligned}$$

$$\int_\Omega \mathbf{curl}\,\boldsymbol{w}_h \cdot \mathbf{curl}\,\bar{\boldsymbol{v}}_h + \int_\Omega \boldsymbol{w}_h \cdot \bar{\boldsymbol{v}}_h = \int_\Omega \boldsymbol{f}\cdot\bar{\boldsymbol{v}}_h \qquad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{Z}}_h.$$

$\boldsymbol{S}_h$ is a well defined bounded self-adjoint linear operator. Moreover,

$(\lambda_h, \boldsymbol{u}_h)$ solution of Problem $\mathbf{3_h}$

$$\iff \quad \boldsymbol{S}_h\boldsymbol{u}_h = \mu_h\boldsymbol{u}_h, \;\; \boldsymbol{u}_h \neq \mathbf{0}, \;\; \mu_h = \frac{1}{\lambda_h^2+1} \neq 0.$$

Therefore, to prove convergence of the proposed Ritz-Galerkin scheme, we will prove spectral convergence of the operators S_h to S.

8.4 Spectral convergence

For compact operators, spectral convergence typically follows from convergence in norm:

$$\|\boldsymbol{S}-\boldsymbol{S}_h\|_{\mathcal{L}(\boldsymbol{\mathcal{Z}},\boldsymbol{\mathcal{Z}})} := \sup_{\boldsymbol{f}\in\boldsymbol{\mathcal{Z}}} \frac{\|(\boldsymbol{S}-\boldsymbol{S}_h)\boldsymbol{f}\|_{\mathbf{curl},\Omega}}{\|\boldsymbol{f}\|_{\mathbf{curl},\Omega}} \xrightarrow{h\to 0} 0.$$

However, such a convergence cannot hold for a noncompact operator like $\boldsymbol{S}$. In fact, since $\boldsymbol{S}_h$ are finite-rank operators, its limit in norm should be compact.

Instead, we will resort to the ***spectral approximation theory for noncompact operators***.[a,b]

With this aim, the following two properties have to be proved:

- **P1**: $\|\boldsymbol{S}-\boldsymbol{S}_h\|_h := \sup_{\boldsymbol{f}_h\in\boldsymbol{\mathcal{Z}}_h} \dfrac{\|(\boldsymbol{S}-\boldsymbol{S}_h)\boldsymbol{f}_h\|_{\mathbf{curl},\Omega}}{\|\boldsymbol{f}_h\|_{\mathbf{curl},\Omega}} \xrightarrow{h\to 0} 0;$
- **P2**: $\forall \boldsymbol{v}\in\boldsymbol{\mathcal{Z}} \quad \inf_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{Z}}_h} \|\boldsymbol{v}-\boldsymbol{v}_h\|_{\mathbf{curl},\Omega} \xrightarrow{h\to 0} 0.$

P2 follows from the density of $\mathcal{C}^\infty(\bar{\Omega})^3 \cap \boldsymbol{\mathcal{Z}}$ in $\boldsymbol{\mathcal{Z}}$ and standard interpolation error estimates for Nédélec finite elements.

[a] J. Descloux, N. Nassif & J. Rappaz, On spectral approximation. Part I: The problem of convergence. *RAIRO Anal. Numér.*, **12** (1978) 97–112.

[b] J. Descloux, N. Nassif & J. Rappaz, On spectral approximation. Part II: Error estimates for the Galerkin method. *RAIRO Anal. Numér.*, **12** (1978) 113–119.

To prove **P1**, recall that $\boldsymbol{\mathcal{V}} = \mathrm{H}(\mathbf{curl}^0,\Omega)^{\perp_{\boldsymbol{\mathcal{Z}}}}$ and let

$$\boldsymbol{\mathcal{V}}_h := \left[\mathrm{H}(\mathbf{curl}^0,\Omega)\cap\boldsymbol{\mathcal{N}}_h\right]^{\perp_{\boldsymbol{\mathcal{Z}}_h}}.$$

Notice that $\boldsymbol{\mathcal{V}}_h \not\subset \boldsymbol{\mathcal{V}}$. However, we have the following result:

Lemma. *Given* $\boldsymbol{f}_h \in \boldsymbol{\mathcal{V}}_h \subset \boldsymbol{\mathcal{Z}}_h \subset \boldsymbol{\mathcal{Z}}$, *let* $\boldsymbol{\chi}\in\boldsymbol{\mathcal{V}}$ *and* $\boldsymbol{\eta}\in\mathrm{H}(\mathbf{curl}^0,\Omega)$ *such that* $\boldsymbol{f}_h = \boldsymbol{\chi}+\boldsymbol{\eta}$. *Then,*

a) $\boldsymbol{\chi}\in\mathrm{H}^s(\Omega)^3$ *with* $\|\boldsymbol{\chi}\|_{\mathrm{H}^s(\Omega)^3} \le C\,\|\mathbf{curl}\,\boldsymbol{f}_h\|_{\mathrm{L}^2(\Omega)^3}$,

b) $\|\boldsymbol{\eta}\|_{\mathrm{L}^2(\Omega)^3} \le C h^{\min\{s,1\}}\,\|\mathbf{curl}\,\boldsymbol{f}_h\|_{\mathrm{L}^2(\Omega)^3}$.

This lemma plays a key role in the proof of **P1**:

Lemma (P1). *There exists* $C>0$ *such that, for all* $\boldsymbol{f}_h\in\boldsymbol{\mathcal{Z}}_h$,

$$\|(\boldsymbol{S}-\boldsymbol{S}_h)\boldsymbol{f}_h\|_{\mathbf{curl},\Omega} \le C h^{\min\{s,1\}}\,\|\boldsymbol{f}_h\|_{\mathbf{curl},\Omega}.$$

Theorem. *Let F be a closed subset of $\mathbb{R}$ such that $F \cap \mathrm{sp}(\boldsymbol{S}) = \emptyset$. Then, there exists $h_0 > 0$ such that, for all $h < h_0$, $F \cap \mathrm{sp}(\boldsymbol{S}_h) = \emptyset$.*

Theorem. *Let λ be an eigenvalue of Problem 3 with finite multiplicity m and $\boldsymbol{\mathcal{E}}$ the corresponding eigenspace.*
There exist exactly m eigenvalues $\lambda_h^{(1)}, \dots, \lambda_h^{(m)}$ of Problem $\boldsymbol{3_h}$ (repeated according to their respective multiplicities) which converge to λ as $h \to 0$.
Let $\boldsymbol{\mathcal{E}}_h$ be the direct sum of the eigenspaces corresponding to $\lambda_h^{(1)}, \dots, \lambda_h^{(m)}$. Then,

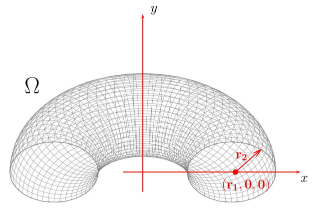
$$\widehat{\delta}\,(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}_h) \le C\gamma_h,$$

$$\left|\lambda - \lambda_h^{(i)}\right| \le C\gamma_h^2, \qquad i = 1, \dots, m,$$

where, as above, $\gamma_h := \delta(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{Z}}_h) \le Ch^{\min\{s,k\}}$.

8.5 Numerical results

Comparison of both methods on the same test problem.



$r_1 = 1, \quad r_2 = 0.5.$

λ_1^{M}, λ_3^{M} : values computed with the mixed formulation.

λ_1^{P}, λ_5^{P} : values computed with the Maxwell-like primal formulation.

N_h	λ_1^{M}	λ_1^{P}	λ_3^{M}	λ_5^{P}
1259	13.7352	5.6529	14.7751	6.2320
2656	8.4648	5.9366	8.7671	6.2194
3822	7.9359	6.1680	8.1618	6.2701
7812	7.2752	6.2100	7.4303	6.2662
20384	6.6320	6.1931	6.7104	6.2608
28321	6.5199	6.2251	6.5888	6.2785
43667	6.4406	6.2156	6.5105	6.2732
58732	6.4096	6.2087	6.4723	6.2624
75999	6.3824	6.2079	6.4439	6.2631
97886	6.3565	6.2083	6.4155	6.2578
131222	6.3276	6.2143	6.3840	6.2679
154592	6.3156	6.2145	6.3726	6.2658
171127	6.3093	6.2165	6.3657	6.2683
182885	6.3075	6.2155	6.3642	6.2662
241429	6.2829	6.2209	6.3370	6.2722
264623	6.2781	6.2214	6.3323	6.2736
301862	6.2737	6.2208	6.3275	6.2730
λ_{ex}	**6.2233**	—	**6.2714**	—
order	**2.20**	—	**2.17**	—

8.6 Conclusions for the Maxwell-like primal formulation

- The convergence order cannot be clearly estimated.
- However, it yields much more accurate results than the mixed formulation on the same meshes.
- It is significantly less expensive than the mixed formulation.
  Indeed, $\boldsymbol{A} + \boldsymbol{B}^{\mathrm{t}}\boldsymbol{B}$ is much less sparse than $\boldsymbol{A}$:
  $\boldsymbol{A}_{ij} \neq 0$, only if nodes i and j correspond to edges of a same element;
  $(\boldsymbol{B}^{\mathrm{t}}\boldsymbol{B})_{ij} \neq 0$, for nodes i and j corresponding to edges of neighboring elements which share an edge.
- It is thoroughly free of spurious modes (which typically may arise in problems like this with an infinite-multiplicity eigenvalue).
- It allows computing the (squared) eigenvalue, but not the eigenfunction (i.e., the Beltrami field).