Stabilization of dissipative models on manifolds

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Joint works with M. Cavalcanti, R. Fukuoka, J. Soriano and D. Toundykov

Outline

- The wave equation on compact manifolds subject to a locally distributed damping - Introduction and Literature Overview.

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- The wave equation on **compact surfaces** subject to a nonlinear locally distributed damping.

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- The wave equation on **compact surfaces** subject to a nonlinear locally distributed damping.
  - Radial Multipliers - External Vision. D.C.;CAVALCANTI,M.M.;FUKUOKA,R.;SORIANO,J.A. Uniform Stabilization of the wave equation on compact surfaces and locally distributed damping. Methods and Applications of Analysis (2008).
  - Non radial multipliers - Intrinsic Vision. D.C.;CAVALCANTI,M.M.;FUKUOKA,R.;SORIANO,J.A. Asymptotic Stability of the Wave Equation on Compact Surfaces and Locally Distributed Damping-A Sharp Result. Transactions of the American Mathematical Society (2009).

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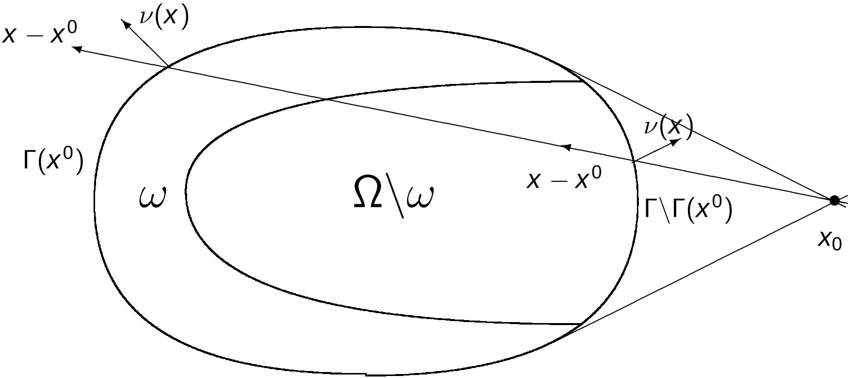
  - D.C.;CAVALCANTI,M.M.;FUKUOKA,R.;SORIANO,J.A. Asymptotic stability of the wave equation on compact manifolds and locally distributed damping - A sharp result. Archive for Rational Mechanics and Analysis (Print)(2010).

Stability for the wave equation

$$u_{tt} - \Delta u + a(x)\,g(u_t) = 0 \text{ in } \Omega \times \mathbb{R}_+, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, has been studied for long time by many authors. When the feedback term depends on the velocity in a linear way ($g(s) = s$) it has been proved by Zuazua [CPDE/90] that the energy related to the semi-linear wave equation decays exponentially if the damping region contains a neighborhood of the boundary $\partial\Omega$ of Ω or, at least, contains a neighborhood ω of the particular part given by

$$\{x \in \partial\Omega : (x - x_0) \cdot \nu(x) \geq 0\}.$$


$x - x^0$
$\nu(x)$
$\nu(x)$
$\Gamma(x^0)$
ω
$\Omega \backslash \omega$
$x - x^0$
$\Gamma \backslash \Gamma(x^0)$
x_0

Euclidian Setting

A rich body of results is currently available in the literature in what concerns the wave equation subject to a locally distributed damping in the Euclidian setting; for instance see

- Dafermos [Wisconsin/78]
- Zuazua [CPDE/90]
- Liu [SICON/97]
- Nakao [Math. Ann./96]
- Nakao [Israel J.M./96]
- Martinez[RMC/99]
- Nakao [OTAA/05]
- Nakao [Math. Nachr./05]
- Alabau-Boussouira[AMO/05]
- Toundykov [NLA/07]

Rauch and Taylor [CPAM/75] are among the pioneers in investigating the long time behaviour of weak solutions of the Cauchy problem for the linear wave equation on a compact manifold $(M, \mathbf{g})$ without boundary with a dissipative term, which is described by the equation

$$\begin{cases} u_{tt} - \Delta u + 2a(x)\, u_t = 0 & \text{in } M \times]0, \infty[\,, \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) & x \in M. \end{cases} \tag{2}$$

Assuming that a is a bounded nonnegative function on M such

that $a \in C^\infty$, we say that the *Rauch-Taylor condition* holds if there exists a time $T_0 > 0$ such that any geodesic (also called ray of the geometric optics) with length greater than T_0 meets the open set $\{x \in M; a(x) > 0\}$.

In this case it was established by Rauch and Taylor [CPAM/75] that the energy

$$E(t) = \frac{1}{2} \int_M \left(|u_t|^2 + |\nabla u|^2 \right) dx$$

decays exponentially. Analogous result was settled by Bardos, Lebeau and Rauch [SICON/99] for Riemannian manifolds with boundary. In this work the authors present sharp sufficient conditions for the observation, control and stabilization of the linear wave equation on a compact Riemannian manifold $(M, \mathbf{g})$ with boundary. In particular, when one considers the equation

$$\begin{cases} u_{tt} - \Delta u + 2a(x)\, u_t = 0 & \text{in } M \times]0, \infty[\,, \\ u = 0 & \text{on } \partial M \times]0, \infty[\,, \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) & x \in M, \end{cases}$$

$a \in C^\infty$, and $a(x) > 0$ in some nonempty open subset ω of M, they proved that the exponential decay holds if and only if a similar condition on the ray of geometric optics for Riemannian manifold with boundary is satisfied. The intuitive idea behind these kind of results is that if every ray of geometric optics remains at least a well defined proportion of time in the damping area during its traveling, then the energy decays exponentially.

A classical example, in the Euclidian setting, of an open set ω satisfying the Geometric Control Condition is a neighborhood of the boundary $\partial\Omega$ de Ω, according to the figure below.

Rays propagating inside a domain $\Omega \subset \mathbb{R}^n$ follow straight lines which are reflected on the boundary $\partial\Omega$ de Ω according to the geometric optics laws.

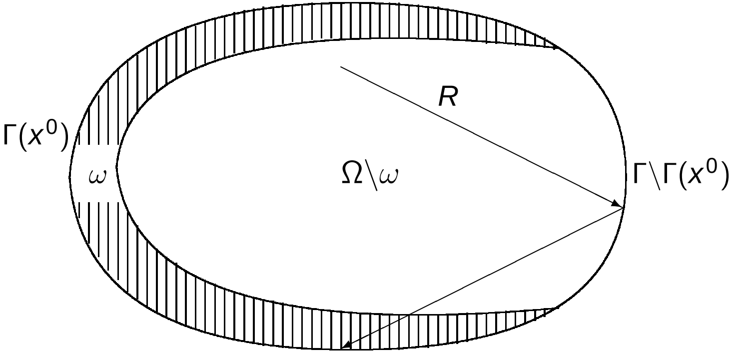


Figure: The Geometric Control Condition is satisfied for some $T_0 > 0$.

The following figure is an example of a region that does not satisfy the geometric control condition.

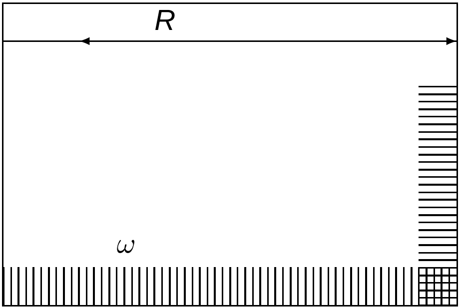


Figure: In this case there exists a ray R of the geometric optics that does not intercept the region ω for all $T > 0$.

When we consider the wave propagation on compact manifolds, or, more particularly, on compact surfaces, to determine the geometric control conditions (GCC) is a delicate problem since we need to know all the geodesics on the surface under consideration. On compact manifolds (M, g), this question is much more complex. Let's see some examples involving the *torus* or the *sphere* according to the figures below

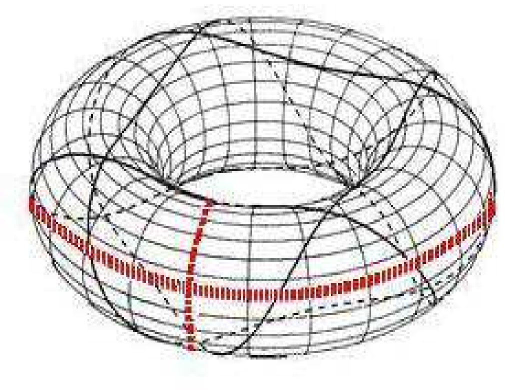


Figure: An example of a region of geometric control condition is given (in red). Note that it intercepts all the Torus's geodesics (black curves)
.

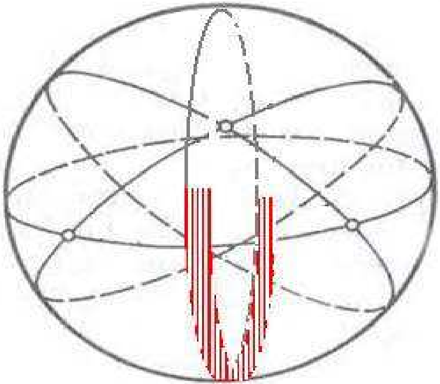


Figure: One of the regions of of geometric control condition (in red) intercepts all the geodesics on the sphere (in black).

It is worth mentioning the contribution due to Luc Miller [SICON/03] whom characterized the geodesic conditions of Bardos, Lebeau and Rauch [SICON/99] in term of *escape functions*. Roughly speaking the escape function condition provides a straightforward geometric proof that the geodesic condition holds in the situations where first order differential multiplier methods apply.

Related to Problem (2) on compact Riemannian manifolds without boundary, it is worth quoting the result due to Christianson [JFA/2007]. Assuming that $u^0 = 0$ and, in addition, that $a(x) > 0$ outside a neighbourhood of a closed hyperbolic geodesic γ, he proved the following energy estimate

$$E(t) \leq Ce^{-t^{1/2}/C} ||u^1||^2_{H^\varepsilon(M)}, \; t \geq 0,$$

for some $C > 0$ and for all ε.

The Wave propagation on a compact surface

Natural questions arise in this context:

- *(i) What does happen if the wave propagation is on a compact surface instead of a domain ?*
- *What would be the geometric impositions on the surface if we use the radial multiplier $x - x^0$ as in the Euclidian case?*
- *What would be the geometric impositions on the surface if we use an intrinsic multiplier ∇f ? (where f is a function (to be determined) defined on the surface $\mathcal{M}$)*.

The main task of this talk is to evaluate the impact of the multipliers on the geometry of surfaces (or manifolds) when stabilizing the wave propagation.

Let $\mathcal{M}$ be a smooth oriented embedded compact surface without boundary in $\mathbb{R}^3$. This talk is devoted to the study of the uniform stabilization of solutions of the following damped problem

$$\begin{cases} u_{tt} - \Delta_{\mathcal{M}} u + a(x)\, g(u_t) = 0 & \text{on } \mathcal{M} \times]0,\infty[\,, \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) & x \in \mathcal{M}, \end{cases} \tag{3}$$

where $a(x) \geq a_0 > 0$ in an open proper subset $\mathcal{M}_*$ of $\mathcal{M}$ and, in addition, g is a monotonic increasing function such that $g(s)s \geq 0$ and, moreover,

$$k|s| \leq |g(s)| \leq K|s| \text{ for all } |s| \geq 1.$$

External Vision

Initially we shall consider, as usually in the literature, $\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1$, where

$$\mathcal{M}_1 := \{x \in \mathcal{M}; m(x) \cdot \nu(x) > 0\} \text{ and } \mathcal{M}_0 = \mathcal{M} \backslash \mathcal{M}_1. \quad (4)$$

Here, $m(x) := x - x^0$, ($x^0 \in \mathbb{R}^3$ fixed) and ν is the exterior unit normal vector field of $\mathcal{M}$.

The main goal of the first part of this talk is to prove uniform decay rates of the energy when the portion of $\mathcal{M}$, where the damping is effective is strategically chosen. For $i = 1, \ldots, k$, assume that there exist open subsets $\mathcal{M}_{0i} \subset \mathcal{M}_0$ of $\mathcal{M}$ with smooth boundary $\partial \mathcal{M}_{0i}$ such that $\mathcal{M}_{0i}$ are umbilical or conical, or, more generally, that the principal curvatures k_1 and k_2 satisfy

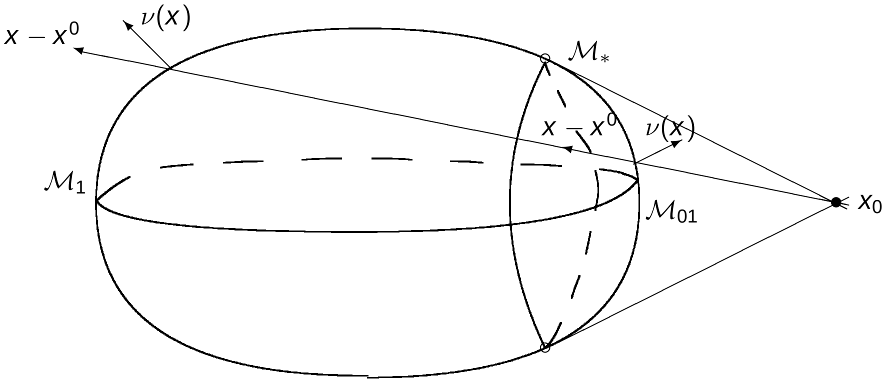
$$|k_1(x) - k_2(x)| < \varepsilon_i$$

(ε_i considered small enough) for all $x \in \mathcal{M}_{0i}$. Moreover, suppose that the *mean curvature* H of each $\mathcal{M}_{0i}$ is *non-positive* (i.e. $H \leq 0$ on $\mathcal{M}_{0i}$ for every $i = 1, \ldots, k$) and that the damping is effective on an open subset

$$\mathcal{M}_* \subset \mathcal{M}$$

that contains

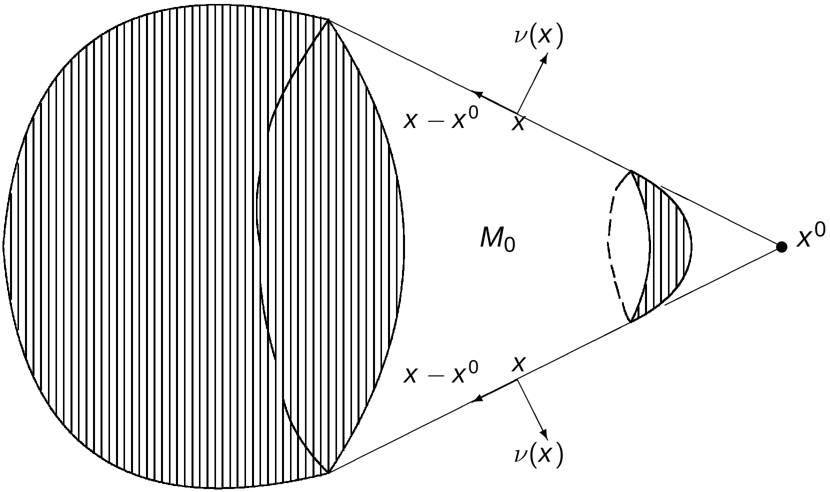
$$\mathcal{M} \backslash \cup_{i=1}^{k} \mathcal{M}_{0i}.$$



The observer is at x_0. The subset $\mathcal{M}_0$ is the “visible” part of $\mathcal{M}$ and $\mathcal{M}_1$ is its complement. The subset $\mathcal{M}_*$ is an open set that contains $\mathcal{M} \backslash \mathcal{M}_{01}$ and the damping is effective there.

Surfaces constituted by a conical part

Assume that there exists $x^0 \in \mathbb{R}^3$ such that $m(x) \cdot \nu(x) = 0$ for all $\in \mathcal{M}_0$ and, in addition, that $\mathcal{M}_*$ contains $\mathcal{M} \backslash \mathcal{M}_0$ according to the figure below,



M_0 is a non-dissipative area (in white) while the demarcated area (in black) contains dissipative effects.

Well-posedness

We set

$$W := \{v \in H^1(\mathcal{M}); \int_{\mathcal{M}} v(x)\, d\mathcal{M} = 0\},$$

which is a Hilbert space endowed with the topology given by $H^1(\mathcal{M})$. The condition $\int_{\mathcal{M}} v(x)\, d\mathcal{M} = 0$ is required in order to guarantee the validity of the Poincaré inequality,

$$||f||^2_{L^2(\mathcal{M})} \leq (\lambda_1)^{-1} ||\nabla_T f||^2_{L^2(\mathcal{M})}, \quad \text{for all } f \in W, \tag{5}$$

where λ_1 is the first eigenvalue of the Laplace-Beltrami operator. We observe that the problem (3) can be written in the following form

$$U_t + \mathcal{A}U = G(U),$$

where

$$U = \begin{pmatrix} u \\ u_t \end{pmatrix} \text{ and } \mathcal{A} = \begin{pmatrix} 0 & -I \\ -\Delta_{\mathcal{M}} & 0 \end{pmatrix}$$

is a maximal monotone operator and $G(\cdot)$ represents a locally Lipschitz perturbation. So, making use of standard semigroup arguments we have the following result:

Theorem

- (**i**) *Under the conditions above, problem* (3) *is well posed in the space* $W \times L^2(\mathcal{M})$*, i.e. for any initial data* $\{u^0, u^1\} \in W \times L^2(\mathcal{M})$*, there exists a unique weak solution of (3) in the class*

$$u \in C(\mathbb{R}_+; W) \cap C^1(\mathbb{R}_+; L^2(\mathcal{M})). \tag{6}$$

- (**ii**) *In addition, the velocity term of the solution have the following regularity:*

$$u_t \in L^2_{loc}\left(\mathbb{R}_+; L^2(\mathcal{M})\right), \tag{7}$$

(consequently, $g(u_t) \in L^2_{loc}\left(\mathbb{R}_+; L^2(\mathcal{M})\right)$*). Furthermore, if* $\left\{u^0, u^1\right\} \in \left\{W \cap H^2(\mathcal{M}) \times W\right\}$ *then the solution has the following regularity*

$$u \in L^\infty\left(\mathbb{R}_+; W \cap H^2(\mathcal{M})\right) \cap W^{1,\infty}\left(\mathbb{R}_+; W\right) \cap W^{2,\infty}\left(\mathbb{R}_+; L^2(\mathcal{M})\right).$$

Remark

It is convenient to observe that the space W may be not invariant under the flow because of the nonlinear character of the equation under consideration. In this case, it is sufficient to add an extra term αu, $(\alpha > 0)$ in the equation in order to control L_2 norms. However, for simplicity in the computations, we shall omit this term since it does not bring any additional difficulty or novelty.

Supposing that u is the unique global weak solution of problem (3), we define the corresponding energy functional by

$$E(t) = \frac{1}{2}\int_{\mathcal{M}} \left[|u_t(x,t)|^2 + |\nabla_T u(x,t)|^2 \right] d\mathcal{M}. \tag{8}$$

For every solution of (3) in the class (6) the following identity holds

$$E(t_2) - E(t_1) = -\int_{t_1}^{t_2}\int_{\mathcal{M}} a(x)\, g(u_t) u_t \, d\mathcal{M} dt, \quad \text{for all } t_2 > t_1 \geq 0, \tag{9}$$

and therefore the energy is a non increasing function of the time variable t.

Main Result

Before stating our stability result, we will define some needed functions. For this purpose, we are following the ideas firstly introduced in Lasiecka and Tataru [DIE/93]. We will repeat them briefly. Let h be a concave, strictly increasing function, with $h(0) = 0$, and such that

$$h(s\,g(s))) \geq s^2 + g^2(s), \text{ for } |s| \leq 1. \quad (10)$$

Note that such function can be straightforwardly constructed, given the hypotheses on g. With this function, we define

$$r(.) = h\left(\frac{\cdot}{meas\,(\Sigma_1)}\right). \quad (11)$$

As r is monotone increasing, then $cI + r$ is invertible for all $c \geq 0$. For L a positive constant, we set

$$p(x) = (cI + r)^{-1}(Lx), \tag{12}$$

where the function p is easily seen to be positive, continuous and strictly increasing with $p(0) = 0$. Finally, let

$$q(x) = x - (I + p)^{-1}(x). \tag{13}$$

We are now able to proceed to state our stability result.

Theorem

Let u be the weak solution of the problem (3). With the energy $E(t)$ defined as in (8), there exists a $T_0 > 0$ such that

$$E(t) \le S\left(\frac{t}{T_0} - 1\right), \quad \forall t > T_0, \tag{14}$$

with $\lim_{t\to\infty} S(t) = 0$, *where the contraction semigroup* $S(t)$ *is the solution of the differential equation*

$$\frac{d}{dt}S(t) + q(S(t)) = 0, \quad S(0) = E(0), \tag{15}$$

(where q *is given in (13)). Here, the constant* L *(from definition (12)) will depend on* $meas(\Sigma)$, *and the constant* c*(from definition (12)) is taken here to be* $c \equiv \frac{k^{-1}+K}{meas(\Sigma)(1+||a||_\infty)}$.

If the feedback is linear, e. g., $g(s) = s$, then, we have that the energy of problem (3) decays exponentially with respect to the initial energy. There exist two positive constants $C > 0$ and $k > 0$ such that

$$E(t) \leq Ce^{-kt}E(0), \quad t > 0. \tag{16}$$

As another example, we can consider $g(s) = s^p$, $p > 1$ at the origin, we obtain the following polynomial decay rate:

$$E(t) \leq C(E(0))[E(0)^{\frac{-p+1}{2}} + t(p-1)]^{\frac{2}{-p+1}}.$$

We can find more interesting explicit decay rates in D. C., I.Lasiecka and M.Cavalcanti. [JDE/07].

We observe that in the particular case when $m(x) = x - x^0$, $x \in \mathbb{R}^3$ and $x^0 \in \mathbb{R}^3$ is a fixed point in $\mathbb{R}^3$, we have

$$div\, m = 3, \quad div_T\, m_T = 2 + (m \cdot \nu) Tr\, B. \tag{17}$$

where B is the second fundamental form of $\mathcal{M}$ (the shape operator) and Tr is the trace. Let φ and m defined as above. We also have,

$$\nabla_T \varphi \cdot \nabla_T m_T \cdot \nabla_T \varphi = |\nabla_T \varphi|^2 + (m \cdot \nu)(\nabla_T \varphi \cdot B \cdot \nabla_T \varphi). \tag{18}$$

Remark

The sign of B can change in the literature. In our case, we remember that $B = -dN$, where N is the Gauss map related to ν.
The formulas (17) can be rewritten by

$$div\, m = 3, \quad div_T\, m_T = 2 + 2H\,(m \cdot \nu). \tag{19}$$

where $H = \frac{trB}{2}$ is the mean curvature of $\mathcal{M}$.

We shall work with regular solutions and by using density arguments we can extend the results for weak solutions.

Our main task is to obtain the following estimate:

$$\int_0^T E(t)\,dt \le C\left(E(T) + \int_0^T \int_{\mathcal{M}} a(x)\,\left(g(u_t)^2 + u_t^2\right)\,d\mathcal{M}dt\right),$$

for some positive constant $C > 0$. From this estimate we deduce the desired decay rates estimates following (verbatim) the ideas firstly introduced by Lasiecka and Tataru [DIE/93].

Lemma 1. *Let $\mathcal{M} \subset \mathbb{R}^3$ be oriented regular compact surface without boundary and q a regular vector field with $q = q_T + (q \cdot \nu)\nu$. Then, for every regular solution u of* (3) *we have the following identity*

The First Identity

$$\left[\int_{\mathcal{M}} u_t\, q_T \cdot \nabla_T u\, d\mathcal{M}\right]_0^T + \frac{1}{2}\int_0^T \int_{\mathcal{M}} (div_T q_T)\left\{|u_t|^2 - |\nabla_T u|^2\right\} d\mathcal{M}dt + \int_0^T \int_{\mathcal{M}} \nabla_T u \cdot \nabla_T q_T \cdot \nabla_T u\, d\mathcal{M}dt + \int_0^T \int_{\mathcal{M}} a(x)\, g(u_t)(q_T \cdot \nabla_T u) d\mathcal{M}dt = 0. \tag{20}$$

Employing (20) with $q(x) = m(x) = x - x^0$ for some $x^0 \in \mathbb{R}^3$ fixed and taking (17) and (18) into account, we infer

$$
\begin{aligned}
&\left[\int_{\mathcal{M}} u_t\, m_T \cdot \nabla_T u \, d\mathcal{M}\right]_0^T + \int_0^T \int_{\mathcal{M}} \left\{|u_t|^2 - |\nabla_T u|^2\right\} d\mathcal{M}dt \\
&+ \int_0^T \int_{\mathcal{M}} [|\nabla_T u|^2 + (m\cdot\nu)(\nabla_T u \cdot B \cdot \nabla_T u)]\, d\mathcal{M}dt \qquad (21) \\
&+ \int_0^T \int_{\mathcal{M}} (m\cdot\nu) H \left\{|u_t|^2 - |\nabla_T u|^2\right\} d\mathcal{M}dt \\
&+ \int_0^T \int_{\mathcal{M}} a(x)\, g(u_t)(m_T \cdot \nabla_T u) d\mathcal{M}dt = 0.
\end{aligned}
$$

Lemma 2 *Let u be a weak solution to problem* (3) *and* $\xi \in C^1(\mathcal{M})$. *Then*

$$
\begin{aligned}
&\left[\int_{\mathcal{M}} u_t \, \xi \, u \, d\mathcal{M}\right]_0^T \\
&= \int_0^T \int_{\mathcal{M}} \xi |u_t|^2 d\mathcal{M}dt - \int_0^T \int_{\mathcal{M}} \xi |\nabla_T u|^2 d\mathcal{M}dt \\
&- \int_0^T \int_{\mathcal{M}} (\nabla_T u \cdot \nabla_T \xi) u \, d\mathcal{M}dt \\
&- \int_0^T \int_{\mathcal{M}} a(x) \, g(u_t) \, \xi \, u \, d\mathcal{M}dt.
\end{aligned}
\tag{22}
$$

Substituting $\xi = \frac{1}{2}$ in (22) and combining the obtained result with identity (21) we deduce

$$\left[\int_{\mathcal{M}} u_t\, m_T \cdot \nabla_T u \, d\mathcal{M}\right]_0^T + \frac{1}{2}\left[\int_{\mathcal{M}} u_t\, u\, d\mathcal{M}\right]_0^T \tag{23}$$
$$+\int_0^T E(t)\,dt + \int_0^T \int_{\mathcal{M}} a(x)\,g(u_t)(m_T \cdot \nabla_T u) d\mathcal{M}dt$$
$$+\frac{1}{2}\int_0^T \int_{\mathcal{M}} a(x)\,g(u_t)u\, d\mathcal{M}dt$$
$$= -\int_0^T \int_{\mathcal{M}} (m\cdot\nu)H\left\{|u_t|^2 - |\nabla_T u|^2\right\} d\mathcal{M}dt.$$
$$-\int_0^T \int_{\mathcal{M}} (m\cdot\nu)(\nabla_T u \cdot B \cdot \nabla_T u)\, d\mathcal{M}dt.$$

Observe that some terms in (23) are easily handled by using the Cauchy Schwarz and Poincaré inequalities as well as the inequality $ab \leq \frac{1}{4\varepsilon}a^2 + \varepsilon b^2$ and exploiting the energy identity

$$E(T) - E(0) = -\int_0^T \int_{\mathcal{M}} a(x)\, g(u_t)u_t \, d\mathcal{M} dt$$

Analysis of the terms which involve the shape operator B

Let us focus our attention on the shape operator $B : T_x\mathcal{M} \to T_x\mathcal{M}$. There exist an orthonormal basis $\{e_1, e_2\}$ of $T_x\mathcal{M}$ such that $Be_1 = k_1 e_1$ and $Be_2 = k_2 e_2$ and k_1 and k_2 are the principal curvatures of $\mathcal{M}$ at x. The matrix of B with respect to the basis $\{e_1, e_2\}$ is given by

$$B := \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

Setting $\nabla_T u = (\xi, \eta)$ the coordinates of $\nabla_T u$ in the basis $\{e_1, e_2\}$, for each $x \in \mathcal{M}$, we deduce that

$$\nabla_T u \cdot B \cdot \nabla_T u = k_1 \xi^2 + k_2 \eta^2. \quad (24)$$

Then, from (40), we infer

$$(m \cdot \nu)\left[(\nabla_T u \cdot B \cdot \nabla_T u) - \frac{1}{2} Tr(B) |\nabla_T u|^2\right] \quad (25)$$
$$= (m \cdot \nu)\left[\frac{(k_1 - k_2)}{2}\xi^2 + \frac{(k_2 - k_1)}{2}\eta^2\right].$$

Remark: *Observe that this is the moment that the intrinsic properties of the manifold $\mathcal{M}$ appear, that is,*

Necessity of umbilical nondissipative region (by parts)

we strongly need that the term $-\int_0^T \int_{\mathcal{M}} (m \cdot \nu) H u_t^2 \, d\mathcal{M} \, dt$ *lies in a region where the damping term is effective. Remember that the damping term is effective on an open set* $\mathcal{M}_*$ *which contains* $\mathcal{M} \setminus \cup_{i=1}^{k} \mathcal{M}_{0i}$*. So, assuming that* $H \leq 0$ *and since* $m(x) \cdot \nu(x) \leq 0$ *on* $\mathcal{M}_0$*, we have*

$$-\int_0^T \int_{\mathcal{M}_0} (m \cdot \nu) H \left| u_t \right|^2 d\mathcal{M} dt \leq 0.$$

In addition, supposing that $\mathcal{M}_{0i}$ *is umbilical for every* $i = 1, \ldots, k$*, then, having (25) in mind, we also have that*

$$\int_0^T \int_{\mathcal{M}_{0i}} (m \cdot \nu) \left[H |\nabla_T u|^2 - (\nabla_T u \cdot B \cdot \nabla_T u) \right] d\mathcal{M} dt = 0,$$
$$i = 1, \ldots, k.$$

Observe that if $\mathcal{M}_0$ is a piece of a conical surface $\mathcal{M}$, that is, $m(x) \cdot \nu(x) = 0$, for all $x \in \mathcal{M}_0$, we also deduce that

$$-\int_0^T \int_{\mathcal{M}_0} (m \cdot \nu) H \left| u_t \right|^2 d\mathcal{M}dt = 0.$$

$$\int_0^T \int_{\mathcal{M}_0} (m \cdot \nu) \left[H |\nabla_T u|^2 - (\nabla_T u \cdot B \cdot \nabla_T u) \right] d\mathcal{M}dt = 0.$$

The general case - $|k_1 - k_2|$ small by parts

More generally, assuming that the principal curvatures k_1 and k_2 satisfy $|k_1(x) - k_2(x)| < \varepsilon_i$ (here, ε_i is assumed sufficiently small) for all $x \in \mathcal{M}_{0i}$, $i = 1, \cdots, k$, we deduce that

$$\left| \sum_{i=1}^{k} \int_0^T \int_{\mathcal{M}_{0i}} (m \cdot \nu) \left[H|\nabla_T u|^2 - (\nabla_T u \cdot B \cdot \nabla_T u) \right] d\mathcal{M} dt \right|$$

$$\leq \sum_{i=1}^{k} \int_0^T \int_{\mathcal{M}_{0i}} |(m \cdot \nu)||k_1 - k_2||\xi^2 + \eta^2| d\mathcal{M}\, dt$$

$$\leq \sum_{i=1}^{k} R_i \varepsilon_i \int_0^T \int_{\mathcal{M}_{0i}} |\nabla_T u|^2 d\mathcal{M}\, dt \leq 2 \sum_{i=1}^{k} R_i \varepsilon_i \int_0^T E(t)\, dt,$$

where $R_i = max_{x \in \overline{\mathcal{M}_{0i}}} ||x - x^0||_{\mathbb{R}^3}$.

Set $\mathcal{M}_2 = \mathcal{M} \backslash \cup_{i=1}^{k} \mathcal{M}_{0i}$. In the case where $\mathcal{M}_{0i}$ are umbilical, (or conical) and disjoint, recalling (23) taking (25) we deduce

$$
\begin{aligned}
\int_0^T E(t)\,dt \leq & -\left[\int_{\mathcal{M}} u_t\, m_T \cdot \nabla_T u\, d\mathcal{M}\right]_0^T - \frac{1}{2}\left[\int_{\mathcal{M}} u_t\, u\, d\mathcal{M}\right]_0^T \\
& + \int_0^T \int_{\mathcal{M}_2} (m \cdot \nu)\left[H|\nabla_T u|^2 - (\nabla_T u \cdot B \cdot \nabla_T u)\right] d\mathcal{M}dt \\
& - \int_0^T \int_{\mathcal{M}_2} (m \cdot \nu) H\, |u_t|^2\, d\mathcal{M}dt \qquad (26) \\
& - \int_0^T \int_{\mathcal{M}} a(x)\, g(u_t)(m_T \cdot \nabla_T u) d\mathcal{M}dt \\
& - \frac{1}{2}\int_0^T \int_{\mathcal{M}} a(x)\, g(u_t) u\, d\mathcal{M}dt.
\end{aligned}
$$

Inverse Inequality

Note that if $a = 0$, that is, if one has the linear wave equation

$$\begin{cases} u_{tt} - \Delta_{\mathcal{M}} u = 0 \ \text{ in } \mathcal{M} \times (0, \infty) \\ u(x,0) = u_0(x); \ u_t(x,0) = u_1(x), \ x \in \mathcal{M}. \end{cases}$$

then, $E(T) = E(0)$ for all $T \geq 0$ and from (26) we easily deduce the inverse inequality

$$E_0 \leq C \int_0^T \int_{\mathcal{M}_2} \left[u_t^2 + |\nabla_T u|^2 \right] d\mathcal{M}\, dt,$$

where C is a positive constant and $\mathcal{M}_2 = \mathcal{M} \backslash \cup_{i=1}^k \mathcal{M}_{0i}$.

$$\frac{1}{2}\int_0^T E(t)\,dt \leq |\chi| + C_1\int_0^T\int_{\mathcal{M}} a(x)\,(g(u_t))^2 d\mathcal{M}dt \quad (27)$$
$$+ \; C_1\int_0^T\int_{\mathcal{M}_2}[|\nabla_T u|^2 + a(x)\,u_t^2]\,d\mathcal{M}dt$$

where

$$\chi = -\left[\int_{\mathcal{M}} u_t\, m_T\cdot\nabla_T u\, d\mathcal{M}\right]_0^T - \frac{1}{2}\left[\int_{\mathcal{M}} u_t\, u\, d\mathcal{M}\right]_0^T$$
$$C_1 := \max\left\{||a||_\infty[2^{-1}\lambda_1^{-1} + 8\,R^2],\; ||B||R + |H|R,\; R\,|H|a_0^{-1}\right\},$$

$$||B|| = \sup_{x\in\mathcal{M}}|B_x|, \text{ and } |B_x| = \sup_{\{v\in T_x\mathcal{M};|v|=1\}}|B_x v|.$$

Intrinsic “cut-off”

It remains to estimate the quantity $\int_0^T \int_{\mathcal{M}_2} |\nabla_T u|^2 \, d\mathcal{M} dt$ in terms of the damping term $\int_0^T \int_{\mathcal{M}} [a(x)\,|g(u_t)|^2 + a(x)\,|u_t|^2] \, d\mathcal{M} dt$. For this purpose we have to built a “cut-off” function η_ε on a specific neighborhood of $\mathcal{M}_2$. First of all, define $\tilde{\eta} : \mathbb{R} \to \mathbb{R}$ such that

$$\tilde{\eta}(x) = \begin{cases} 1 & \text{if} \quad x \leq 0 \\ (x-1)^2 & \text{if} \quad x \in [1/2, 1] \\ 0 & \text{if} \quad x > 1 \end{cases}$$

and it is defined on $(0, 1/2)$ in such a way that $\tilde{\eta}$ is a non-increasing function of class C^1. For $\varepsilon > 0$, set $\tilde{\eta}_\varepsilon(x) := \tilde{\eta}(x/\varepsilon)$.

It is straightforward that there exists a constant M which does not depend on ε such that

$$\frac{|\tilde{\eta}_\varepsilon'(x)|^2}{\tilde{\eta}_\varepsilon(x)} \leq \frac{M}{\varepsilon^2}$$

for every $x < \varepsilon$.
Now, let $\varepsilon > 0$ be such that

$$\tilde{\omega}_\varepsilon := \{x \in \mathcal{M}; \mathrm{d}(x, \bigcup_{i=1}^{k} \partial \mathcal{M}_{0i}) < \varepsilon\}$$

is a tubular neighborhood of $\bigcup_{i=1}^{k} \partial \mathcal{M}_{0i}$ and $\omega_\varepsilon := \tilde{\omega}_\varepsilon \cup \mathcal{M}_2$ is contained in $\mathcal{M}_*$. Define $\eta_\varepsilon : \mathcal{M} \to \mathbb{R}$ as

$$\eta_\varepsilon(x) = \begin{cases} 1 & \text{if} \quad x \in \mathcal{M}_2 \\ \tilde{\eta}_\varepsilon(d(x, \mathcal{M}_2)) & \text{if} \quad x \in \omega_\varepsilon \backslash \mathcal{M}_2 \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward that η_ε is a function of class C^1 on $\mathcal{M}$ due to the smoothness of ∂M_2 and $\partial \omega_\varepsilon$. Notice also that

$$\frac{|\nabla_T \eta_\varepsilon(x)|^2}{\eta_\varepsilon(x)} = \frac{|\tilde{\eta}'_\varepsilon(d(x, \mathcal{M}_2))|^2}{\tilde{\eta}_\varepsilon(d(x, \mathcal{M}_2))} \leq \frac{M}{\varepsilon^2} \tag{28}$$

for every $x \in \omega_\varepsilon \backslash \mathcal{M}_2$. In particular, $\frac{|\nabla_T \eta_\varepsilon|^2}{\eta_\varepsilon} \in L^\infty(\omega_\varepsilon)$.

Taking $\xi = \eta_\varepsilon$ in the identity (22) we obtain

$$\begin{aligned}
&\int_0^T \int_{\omega_\varepsilon} \eta_\varepsilon |\nabla_T u|^2 d\mathcal{M} dt \qquad (29)\\
= &-\left[\int_{\omega_\varepsilon} u_t u \eta_\varepsilon \, d\mathcal{M}\right]_0^T + \int_0^T \int_{\omega_\varepsilon} \eta_\varepsilon |u_t|^2 \, d\mathcal{M}\\
- &\int_0^T \int_{\omega_\varepsilon} u(\nabla_T u \cdot \nabla_T \eta_\varepsilon) \, d\mathcal{M} dt - \int_0^T \int_{\omega_\varepsilon} a(x) \, g(u_t) u \eta_\varepsilon \, d\mathcal{M} dt.
\end{aligned}$$

After some estimates we arrive to the following inequality

$$\frac{1}{2}\int_0^T\int_{\omega_\varepsilon}\eta_\varepsilon|\nabla_T u|^2\,d\mathcal{M}dt \tag{30}$$

$$\leq |\mathcal{Y}| + \frac{\lambda_1^{-1}||a||_{L^\infty(\mathcal{M})}}{4\alpha}\int_0^T\int_{\mathcal{M}} a(x)\,|g(u_t)|^2\,d\mathcal{M}$$

$$+2\alpha\int_0^T E(t)\,dt + \frac{M}{2\varepsilon^2}\int_0^T\int_{\omega_\varepsilon}|u|^2\,d\mathcal{M}\,dt,$$

$$+a_0^{-1}\int_0^T\int_{\mathcal{M}} a(x)\,u_t^2\,d\mathcal{M}\,dt.$$

where $\alpha > 0$ is an arbitrary number and

$$\mathcal{Y} := -\left[\int_{\omega_\varepsilon} u_t u \eta_\varepsilon\,d\mathcal{M}\right]_0^T. \tag{31}$$

Thus, combining (30) and (27), having in mind that

$$\frac{1}{2}\int_0^T\int_{\mathcal{M}_2}|\nabla_T u|^2\,d\mathcal{M}dt \le \frac{1}{2}\int_0^T\int_{\omega_\varepsilon}\eta_\varepsilon|\nabla_T u|^2\,d\mathcal{M}dt$$

and choosing $\alpha = 1/16C_1$ we deduce

$$\begin{aligned}\frac{1}{4}\int_0^T E(t)\,dt &\le |\chi| + 2C_1|\mathcal{Y}| \\ &+C_2\int_0^T\int_{\mathcal{M}}[a(x)\,|g(u_t)|^2 + a(x)\,|u_t|^2]\,d\mathcal{M}dt \\ &+\frac{MC_1}{\varepsilon^2}\int_0^T\int_{\omega_\varepsilon}|u|^2\,d\mathcal{M}\,dt,\end{aligned} \tag{32}$$

where $C_2 = max\{C_1, 8C_1^2\lambda_1^{-1}||a||_{L^\infty(\mathcal{M})}, 2C_1a_0^{-1}\}$.

On the other hand, the following estimate holds

$$|\chi| + 2C_2|\mathcal{Y}| \leq C(E(0) + E(T)) \tag{33}$$
$$= C\left[2E(T) + \int_0^T \int_{\mathcal{M}} a(x)\, g(u_t)\, u_t \, d\mathcal{M}\right],$$

where C is a positive constant which depends also on R. Then,

$$T\,E(T) \leq \int_0^T E(t)\, dt \tag{34}$$
$$\leq C\,E(T) + C\left[\int_0^T \int_{\mathcal{M}} [a(x)\,|g(u_t)|^2 + a(x)\,|u_t|^2]\, d\mathcal{M} dt\right]$$
$$+ \; C \int_0^T \int_{\omega_\varepsilon} |u|^2 \, d\mathcal{M}\, dt,$$

where C is a positive constant which depends on $a_0, ||a||_\infty, \lambda_1, R, |H|, ||B||$ and $\frac{M}{\varepsilon^2}$. Our aim is to estimate the last term on the RHS of (34). In order to do this let us consider the following lemma, where T_0 is a positive constant which is sufficiently large.

Lemma

Under the hypothesis of Theorem 2, there exists a positive constant $C(E(0))$ such that if u is the solution of (3) with weak initial data, we have

$$\int_0^T \int_{\mathcal{M}} |u|^2 \, d\mathcal{M} \, dt \leq C(E(0)) \left\{ \int_0^T \int_{\mathcal{M}} \left(a(x)\, g^2(u_t) + a(x) u_t^2 \right) d\mathcal{M} \, dt \right\}, \tag{35}$$

for all $T > T_0$.

In order to prove the above lemma we argue by contradiction and it is essential to use the uniqueness result which comes from the Inverse Inequality or, more generally we can also employ Triggiani and Yao's [AMO/02] Uniqueness result in the proof.

Inequalities (34) and (35) lead us to the following result.

Proposition 5.2.2: *For* $T > 0$ *large enough, the solution* u *of* (3) *satisfies*

$$E(T) \leq C \int_0^T \int_{\mathcal{M}} \left[a(x) \left| u_t \right|^2 + a(x) \left| g\left(u_t\right) \right|^2 \right] d\mathcal{M} dt \qquad (36)$$

where the constant $C = C(T, E(0), a_0, \lambda_1, R, ||B||, \frac{M}{\varepsilon^2})$.

From this point we are able to employ Lasiecka and Tataru's method [DIE/93] in order to obtain the desired decay rates.

Invoking the second fundamental identity one more time now with $\xi = (m \cdot \nu)H$ we deduce

$$\int_0^T \int_{\mathcal{M}} (m \cdot \nu)H \left[|u_t|^2 - |\nabla_T u|^2 \right] d\mathcal{M} dt \tag{37}$$
$$= \left[\int_{\mathcal{M}} (m \cdot \nu)H \, u_t \, u \, d\mathcal{M} \right]_0^T$$
$$+ \int_0^T \int_{\mathcal{M}} (\nabla_T u \cdot \nabla_T (m \cdot \nu)H) \, u \, d\mathcal{M} dt$$
$$+ \int_0^T \int_{\mathcal{M}} a(x) \, g(u_t) \, (m \cdot \nu)H \, u \, d\mathcal{M} dt.$$

Substituting (37) in (23) we infer

$$\left[\int_{\mathcal{M}} u_t\, m_T \cdot \nabla_T u\, d\mathcal{M}\right]_0^T + \frac{1}{2}\left[\int_{\mathcal{M}} u_t\, u\, d\mathcal{M}\right]_0^T \tag{38}$$
$$+ \int_0^T E(t)\, dt + \int_0^T \int_{\mathcal{M}} a(x)\, g(u_t)(m_T \cdot \nabla_T u) d\mathcal{M} dt$$
$$+\frac{1}{2}\int_0^T \int_{\mathcal{M}} a(x)\, g(u_t) u\, d\mathcal{M} dt = -\left[\int_{\mathcal{M}} (m\cdot\nu) H\, u_t\, u\, d\mathcal{M}\right]_0^T$$
$$-\int_0^T \int_{\mathcal{M}} (\nabla_T u \cdot \nabla_T\, (m\cdot\nu) H)\, u\, d\mathcal{M} dt$$
$$-\int_0^T \int_{\mathcal{M}} a(x)\, g(u_t)\, (m\cdot\nu) H\, u\, d\mathcal{M} dt.$$
$$-\int_0^T \int_{\mathcal{M}} (m\cdot\nu)(\nabla_T u \cdot B \cdot \nabla_T u)\, d\mathcal{M} dt.$$

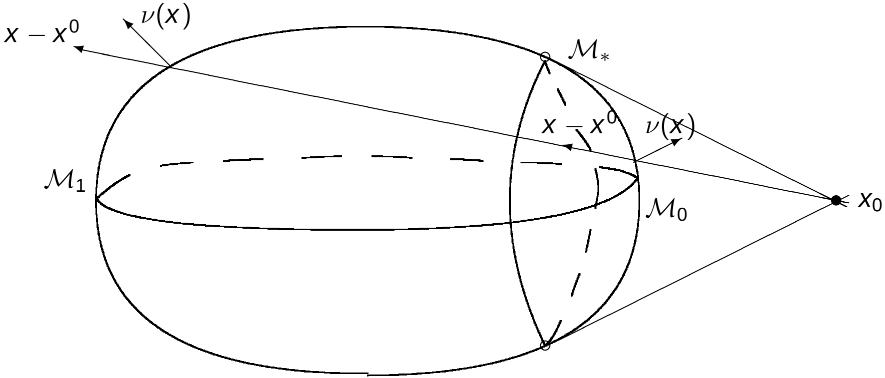
Setting $\nabla_T u = (\xi, \eta)$ the coordinates of $\nabla_T u$ in the basis $\{e_1, e_2\}$, for each $x \in \mathcal{M}$, we deduce that

$$\nabla_T u \cdot B \cdot \nabla_T u = k_1 \xi^2 + k_2 \eta^2 .$$

Geometric Conditions

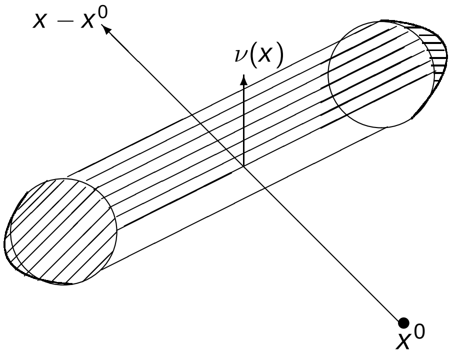
The sub-surface $\mathcal{M}_0$ without damping must have nonnegative Gaussian curvature, that is, $K = k_1 k_2 \geq 0$, with $k_1, k_2 \leq 0$ connected, and the closure of the Gauss map must be contained in an open semi-sphere (the last condition is required in order to guarantee that $x - x^0(x) \cdot \nu(x) \leq 0$ for all $x \in \mathcal{M}_0$. The above geometric condition, now in terms of the **Gaussian curvature** $K = k_1 k_2$ instead of the **Mean curvature** $H = \frac{k_1+k_2}{2}$ allow us to generalize our previous results. However, observe that *we strongly need a Unique Continuation Property based on Carleman estimates* which has been proved by Triggiani and Yao [AMO/02] for wave propagation on compact manifolds. Note that umbilical and conical sub-surfaces satisfy the above condition. In addition, we can consider new sub-surfaces without damping. See figures below:

$\mathcal{M}_0$ possesses Gaussian curvature $K > 0$

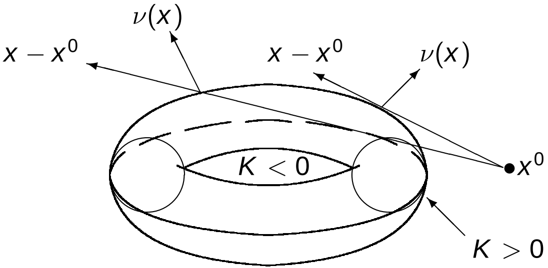


The observer is at x_0. The subset $\mathcal{M}_0$ is the "visible" part of $\mathcal{M}$ ($K > 0$ on $\mathcal{M}_0$) and $\mathcal{M}_1$ is its complement. The subset $\mathcal{M}_*$ is an open set that contains $\mathcal{M} \backslash \mathcal{M}_0$ and the damping is effective there.

Note that conical or cylindrical surfaces $K = 0$ (where $(x - x_0) \cdot \nu \leq 0$) can be also considered.



Torus - we can avoid damping where $m(x) \cdot \nu(x) \leq 0$ and $K \geq 0$



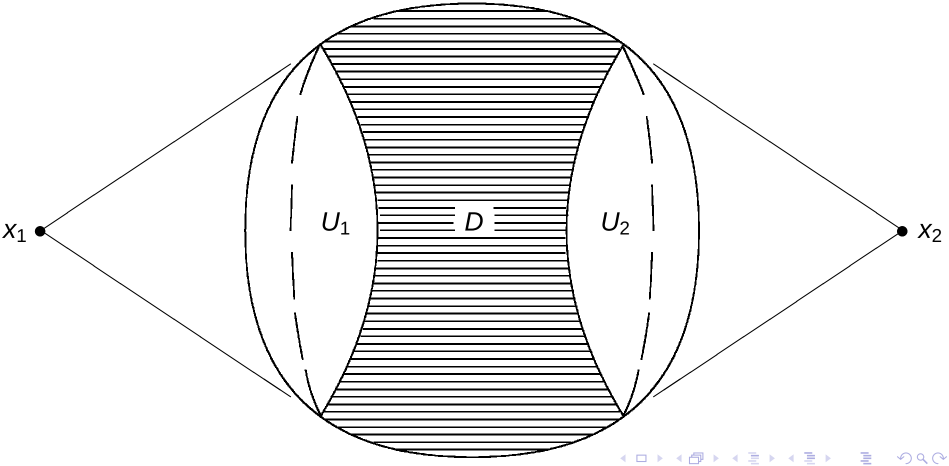
Further Remarks

It is important to mention that the techniques developed until now can be naturally extended for a finite number of observers $x_1, \cdots, x_n$ in connection with a finite number of *disjoint regions satisfying our geometrical impositions* $U_1, \cdots, U_n$. Indeed, for the sake of simplicity let us consider the simple case where we have just two observers located at x_1 and x_2 and U_1 and U_2 are umbilical. Thus, it is sufficient to make use of the multiplier $q \cdot \nabla_T u$ where q is defined by

$$q(x) := \begin{cases} x - x_i \ \text{ if } \ x \in U_i, \ i = 1, 2, \\ \text{smoothly extended in } \mathcal{M} \backslash (U_1 \cup U_2) \end{cases} \tag{39}$$

accordingly the figure below.

Note that it is necessary to put damping in D where $(x - x_i) \cdot \nu(x) \leq 0$, $i = 1, 2$.



Observe that if we consider x_1 and x_2 in the opposite side with respect to the center of the **sphere** and sufficiently far from each other, the damping can be made effective in an arbitrarily small neighborhood of the meridian. This almost reaches the sharp result for the linear case due to Bardos, Lebeau and Rauch [SICON/99]. However, note that we have a nonlinear and localized damping. In addition, we can extend our results for the semi-linear wave equation as well having in ming we need a unique continuation property based on Carleman estimates.

If A and A' are antipodal points the damping can be reduced as A and A' go to infinity

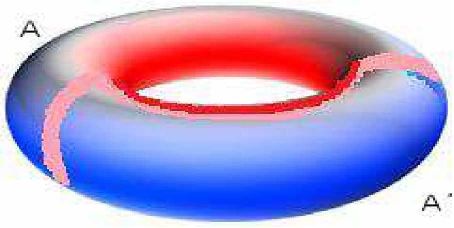
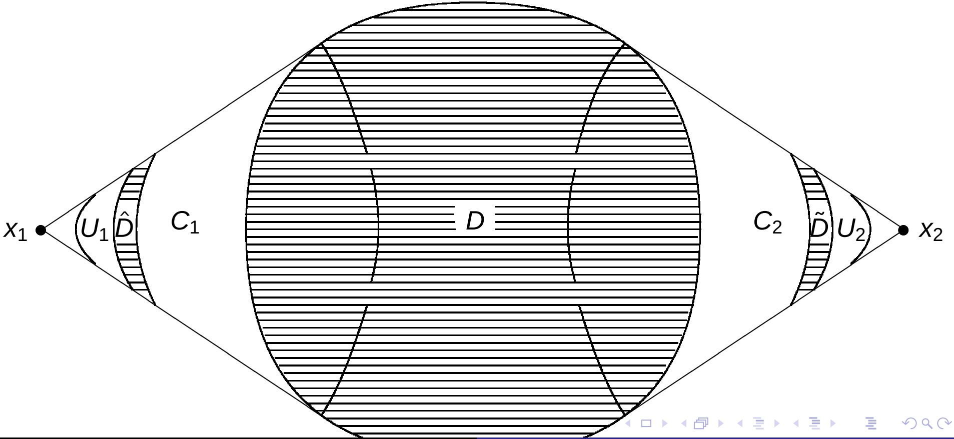


Figure: Observe that if we consider A and A' observers opposite with respect to the center of the **Torus** and sufficiently far from each other, the area without damping can be made effective in a large region (in blue)

Analogous considerations can be done for a finite number of *glued regions* according to figure below. Note that it is necessary to put damping in $\tilde{D}$, in $\hat{D}$ and, in addition, in D where (if) $(x - x_i) \cdot \nu(x) \leq 0$, $i = 1, 2$.



Intrinsic Vision - A Sharp Result

The main goal of the second part of this talk is to improve considerably our previous result *reducing arbitrarily* the volume of the region where the dissipative effect lies. Denoting by **g** the Riemannian metric induced on $\mathcal{M}$ by $\mathbb{R}^3$, we prove that for each $\epsilon > 0$, there exist an open subset $V \subset \mathcal{M}$ and a smooth function $f : \mathcal{M} \to \mathbb{R}$ such that

$$meas(V) \geq meas(\mathcal{M}) - \epsilon, \; Hessf \approx \mathbf{g}$$

on V and $\inf_{x \in V} |\nabla f(x)| > 0$. This new intrinsic multiplier $\nabla f(x)$, instead of the previous one $m(x) = x - x^0$, will play a crucial role when establishing the desired uniform decay rates of the energy.

In what follows we are going to proceed the proof of the main result. It will be done by several steps and we are going to use some identities which had been presented before.
The first step is to consider an identity we have already presented, namely.

Proposition 4.2.1. *Let* $\mathcal{M} \subset \mathbb{R}^3$ *be an oriented regular compact surface without boundary and* q *a vector field of class* C^1. *Then, for every regular solution* u *of* (3) *we have the following identity:*

$$
\left[\int_{\mathcal{M}} u_t\, q \cdot \nabla_T u \, d\mathcal{M}\right]_0^T
+\frac{1}{2}\int_0^T \int_{\mathcal{M}} (div_T q)\left\{|u_t|^2 - |\nabla_T u|^2\right\} d\mathcal{M}dt
+\int_0^T \int_{\mathcal{M}} \nabla_T u \cdot \nabla_T q \cdot \nabla_T u \, d\mathcal{M}dt
+\int_0^T \int_{\mathcal{M}} a(x)\, g(u_t)(q \cdot \nabla_T u) d\mathcal{M}dt = 0.
$$

The proof is based in multiplying the equation by $q \cdot \nabla_T u$ and integrating by parts.

Employing the above inequality with $q(x) = \nabla_T f$ where $f : \mathcal{M} \to \mathbb{R}$ is a C^3 function to be determined later, we infer

$$\left[\int_{\mathcal{M}} u_t \, \nabla_T f \cdot \nabla_T u \, d\mathcal{M}\right]_0^T$$
$$+\frac{1}{2}\int_0^T \int_{\mathcal{M}} \Delta_{\mathcal{M}} f \left\{|u_t|^2 - |\nabla_T u|^2\right\} d\mathcal{M} dt$$
$$+\int_0^T \int_{\mathcal{M}} (\nabla_T u \cdot Hess(f) \cdot \nabla_T u) \, d\mathcal{M} dt$$
$$+\int_0^T \int_{\mathcal{M}} a(x) \, g(u_t)(\nabla_T f \cdot \nabla_T u) d\mathcal{M} dt = 0.$$

Lemma 4.2.3. *Let u be a weak solution to problem* (3) *and* $\xi \in C^1(\mathcal{M})$. *Then*

$$\left[\int_{\mathcal{M}} u_t\,\xi\,u\,d\mathcal{M}\right]_0^T = \int_0^T \int_{\mathcal{M}} \xi|u_t|^2 d\mathcal{M}dt - \int_0^T \int_{\mathcal{M}} \xi|\nabla_T u|^2 d\mathcal{M}dt - \int_0^T \int_{\mathcal{M}} (\nabla_T u \cdot \nabla_T \xi) u\,d\mathcal{M}dt - \int_0^T \int_{\mathcal{M}} a(x)\,g(u_t)\,\xi\,u\,d\mathcal{M}dt.$$

The proof is based in multiplying the equation by $\xi\,u$ and integrating by parts.

Substituting $\xi = \alpha > 0$ in the last inequality and combining the obtained result with the previous identity we deduce

$$\int_0^T \int_{\mathcal{M}} (\frac{\Delta_{\mathcal{M}} f}{2} - \alpha) |u_t|^2 \, d\mathcal{M} dt.$$

$$+ \int_0^T \int_{\mathcal{M}} \left[(\nabla_T u \cdot Hess(f) \cdot \nabla_T u) + \left(\alpha - \frac{\Delta_{\mathcal{M}} f}{2} \right) |\nabla_T u|^2 \right] d\mathcal{M} dt$$

$$= - \left[\int_{\mathcal{M}} u_t \, \nabla_T f \cdot \nabla_T u \, d\mathcal{M} \right]_0^T - \alpha \left[\int_{\mathcal{M}} u_t \, u \, d\mathcal{M} \right]_0^T$$

$$-\alpha \int_0^T \int_{\mathcal{M}} a(x) \, g(u_t) u \, d\mathcal{M} dt$$

$$- \int_0^T \int_{\mathcal{M}} a(x) \, g(u_t) (\nabla_T f \cdot \nabla_T u) d\mathcal{M} dt.$$

This is the moment where the properties of function f play an important role. Note that what we just need is to find a subset V of $\mathcal{M}$ such that

$$C \int_0^T \int_V \left[u_t^2 + |\nabla_T u|^2\right] d\mathcal{M}dt$$
$$\leq \int_0^T \int_V (\frac{\Delta_{\mathcal{M}} f}{2} - \alpha) |u_t|^2 d\mathcal{M}dt$$
$$+ \int_0^T \int_V \left[(\nabla_T u \cdot Hess(f) \cdot \nabla_T u) + \left(\alpha - \frac{\Delta_{\mathcal{M}} f}{2}\right) |\nabla_T u|^2\right] d\mathcal{M}dt,$$

for some positive constant C, provided that α is suitably chosen.

Assuming, for a moment, that the last inequality holds we obtain

$$
\begin{aligned}
2C\int_0^T E(t)dt \quad \leq \quad & C\int_0^T\int_{\mathcal{M}\setminus V}\left[u_t^2+|\nabla_T u|^2\right]d\mathcal{M}dt \qquad (40)\\
+ \quad & \left|\left[\int_{\mathcal{M}} u_t\,\nabla_T f\cdot\nabla_T u\,d\mathcal{M}\right]_0^T\right| + \alpha\left|\left[\int_{\mathcal{M}} u_t\,u\,d\mathcal{M}\right]_0^T\right|\\
+ \quad & \left|\alpha\int_0^T\int_{\mathcal{M}} a(x)\,g(u_t)u\,d\mathcal{M}dt\right|\\
+ \quad & \left|\int_0^T\int_{\mathcal{M}} a(x)\,g(u_t)(\nabla_T f\cdot\nabla_T u)d\mathcal{M}dt\right|.
\end{aligned}
$$

The above inequality is controlled considering a standard procedure. The main idea behind this procedure is to consider the dissipative area, namely, $\mathcal{M}_*$, containing the set $\mathcal{M}\backslash V$. It is important to observe that $\mathcal{M}_*$ is as small as big V can be.

The next steps are devoted to the construction of a function f as well as a subset V of $\mathcal{M}$ such that the desired inequality holds.

Construction of the function f - local version

Let $\mathcal{M}$ be a compact n-dimensional Riemannian manifold (without boundary) with Riemmanian metric $\mathbf{g}$ of class C^2. Let ∇ denote the Levi-Civita connection. Fix $p \in \mathcal{M}$. Our aim is to construct a function $f : V_p \to \mathbb{R}$ such that $Hess f \approx \mathbf{g}$ and $\inf_{x \in V_p} |\nabla f(x)| > 0$, where V_p is a neighborhood of p and the Hessian of f is seen as a bilinear form defined on the tangent space $T_p\mathcal{M}$ of $\mathcal{M}$ at p.

We begin with an orthonormal basis $(e_1, \ldots, e_n)$ of T_pM. Define a normal coordinate system $(x_1, \ldots, x_n)$ in a neighborhood $\widetilde{V}_p$ of p such that $\partial/\partial x_i(p) = e_i(p)$ for every $i = 1, \ldots, n$. It is well known that in this coordinate system we have that $\Gamma_{ij}^k(p) = 0$, where Γ_{ij}^k are the Christoffel symbols with respect to $(x_1, \ldots, x_n)$.

The Hessian with respect to $(x_1, \ldots, x_n)$ is given by

$$Hessf\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{k=1}^{n} \Gamma_{ij}^{k} \frac{\partial f}{\partial x_k}.$$

The Laplacian of f is the trace of the Hessian with respect to the metric $\mathbf{g}$. If $\mathbf{g}_{ij}$ denote the components of the Riemannian metric with respect to $(x_1, \ldots, x_n)$ and $\mathbf{g}^{ij}$ are the components of the inverse matrix of $\mathbf{g}_{ij}$, then the Laplacian of f is given by

$$\Delta f = \sum_{i,j} \mathbf{g}^{ij} Hessf\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right).$$

Consider the function $f : \widetilde{V}_p \to \mathbb{R}$ defined by

$$f(x) = x_1 + \frac{1}{2}\sum_{i=1}^{n} x_i^2.$$

It is immediate that $\Delta f(p) = n$ and $|\nabla f(p)| = 1$. Moreover, $Hessf(p) = \mathbf{g}(p)$, which implies that

$$Hessf(p)(v, v) = |v|_p^2.$$

We are interested in finding a neighborhood $V_p \subset \widetilde{V}_p$ of p and a strictly positive constant C such that

$$C \int_0^T \int_{V_p} \left(|\nabla u|^2 + u_t^2 \right) d\mathcal{M}dt$$

$$\leq \int_0^T \int_{V_p} \left[Hessf(\nabla u, \nabla u) + \left(\alpha - \frac{\Delta f}{2} \right) |\nabla u|^2 + \left(\frac{\Delta f}{2} - \alpha \right) u_t^2 \right] d\mathcal{M} \tag{41}$$

for some $\alpha \in \mathbb{R}$.

We claim that if we consider $\alpha = \frac{n}{2} - \frac{1}{2}$ and $C = 1/4$ we obtain the desired inequality, what means that it is enough to prove that there exist $V_p \subset \widetilde{V}_p$ verifying

$$\int_0^T \int_{V_p} Hessf(\nabla u, \nabla u) + \left(\frac{n}{2} - \frac{3}{4} - \frac{\Delta f}{2} \right) |\nabla u|^2 d\mathcal{M}dt \geq 0$$

and

$$\int_0^T \int_{V_p} \left(\frac{\Delta f}{2} - \frac{n}{2} + \frac{1}{4} \right) u_t^2 d\mathcal{M} dt \geq 0.$$

In order to prove the existence of a subset $V_p \subset \widetilde{V}_p$ where the first inequality holds, let θ_1 be the smooth field of symmetric bilinear form on $\widetilde{V}_p$ defined as

$$\theta_1(X, Y) = Hessf(X, Y) + \left(\frac{n}{2} - \frac{3}{4} - \frac{\Delta f}{2} \right) \mathbf{g}(X, Y)$$

where X and Y are vector fields on $\widetilde{V}_p$.

It is clearly a positive definite bilinear form on p since $Hessf(p)(X,Y) = \mathbf{g}(p)(X,Y)$ and

$$\theta_1(p)(X,Y) = \frac{1}{4}\mathbf{g}(p)(X,Y).$$

Therefore, there exist a neighborhood $\widehat{V}_p$ such that θ_1 is positive definite and

$$\int_0^T \int_{\widehat{V}_p} Hessf(\nabla u, \nabla u) + \left(\frac{n}{2} - \frac{3}{4} - \frac{\Delta f}{2}\right) |\nabla u|^2 d\mathcal{M} dt \geq 0.$$

To prove the existence of $\check{V}_p \subset \widetilde{V}_p$ such that the desired inequality holds is easier. It is enough to notice that at p we have that

$$\left(\frac{\Delta f(p)}{2} - \frac{n}{2} + \frac{1}{4}\right) = \frac{1}{4}$$

and the existence of $\check{V}_p \subset \widetilde{V}_p$ is immediate. Furthermore we can eventually choose a smaller V_p such that $\inf_{x \in V_p} |\nabla f(x)| > 0$. Therefore the existence of $V_p \subset \widetilde{V}_p$ such that $\inf_{x \in V_p} |\nabla f(x)| > 0$ and (41) holds is settled.

In what follows, $\bar{V}$ denotes the closure of V and ∂V denotes the boundary of V. When $\bar{V} \subset W$ is bounded, we say that V is compactly contained in W and we denote by $V \subset\subset W$.

Theorem

Let $(\mathcal{M}, \mathbf{g})$ be a two dimensional Riemannian manifold. Then, for every $\epsilon > 0$, there exist a finite family $\{V_i\}_{i=1\ldots k}$ of open sets with smooth boundary, smooth functions $f_i : \bar{V}_i \to \mathbb{R}$ and a constant $C > 0$ such that

1. *The subsets $\bar{V}_i$ are pairwise disjoint;*
2. $\mathrm{vol}(\bigcup_{i=1}^{k} V_i) \geq \mathrm{vol}(M) - \epsilon$;
3. *Inequality (41) holds for every f_i;*
4. $\inf_{x \in V_i} |\nabla f(x)| > 0$ *for every* $i = 1, \cdots, k$.

In order to prove this Theorem let us consider the following steps:

First of all, it is possible to get open subsets $\{\widetilde{W_j}\}_{j=1,\dots,s}$ with smooth boundaries and a family of smooth functions $\{\widetilde{f_j} : \widetilde{W_j} \to \mathbb{R}\}_{j=1,\dots,s}$ such that $\{\widetilde{W_j}\}_{j=1,\dots,s}$ is a cover of $\mathcal{M}$ and each $\widetilde{f_j}$ satisfies Inequality (41). Moreover, we can choose $\widetilde{W_j}$ in such a way that their boundaries intercept themselves transversally and three or more boundaries do not intercept themselves at the same point.
Set by $A := \bigcup_{j=1}^{s} \partial \widetilde{W_j}$. Then, $\mathcal{M} \backslash A$ is a disjoint union of connected open sets $\bigcup_{i=1}^{k} W_i$ such that ∂W_i is a piecewise smooth curve.

Each W_i is contained in some $\widetilde{W}_j$. Therefore, for each W_i, choose a function $\hat{f}_i := \widetilde{f}_j|_{W_i}$.
The open subsets V_i, $i = 1, \ldots, k$, we are looking for are subsets of W_i. We can choose them in such a way that

1. $V_i \subset\subset W_i$;
2. ∂V_i is smooth;
3. $\mathrm{vol}(W_i) - \mathrm{vol}(V_i) < \epsilon/k$.

Finally, if we set $f_i = \hat{f}_i|_{\bar{V}_i}$, we prove the theorem.

Theorem

Let $(\mathcal{M}, \mathbf{g})$ be a two-dimensional Riemannian manifold. Fix $\epsilon > 0$. Then, there exist a smooth function $f : M \to \mathbb{R}$ such that inequality (41) and the condition $\inf_{x \in V_i} |\nabla f(x)| > 0$ hold in a subset V with $\mathrm{vol}(V) \geq \mathrm{vol}(\mathcal{M}) - \epsilon$.

In order to give an idea of the proof, consider Theorem 4 and the constructions made in its proof. Denote $\lambda := \min_{i \neq j} \mathrm{dist}(V_i, V_j) > 0$. Consider a tubular neighborhood V^δ of $V = \cup_{i=1}^k V_i$ of the points whose distance is less than or equal to $\delta < \lambda/4$. Then, it is possible to define a smooth (cut-off) function given by

$\eta : \mathcal{M} \to \mathbb{R}$ such that

$$\eta(x) = \begin{cases} 1 \text{ if } x \in V \\ 0 \text{ if } x \in \mathcal{M} \backslash V^{\delta} \\ \text{between } 0 \text{ and } 1 \text{ otherwise}. \end{cases}$$

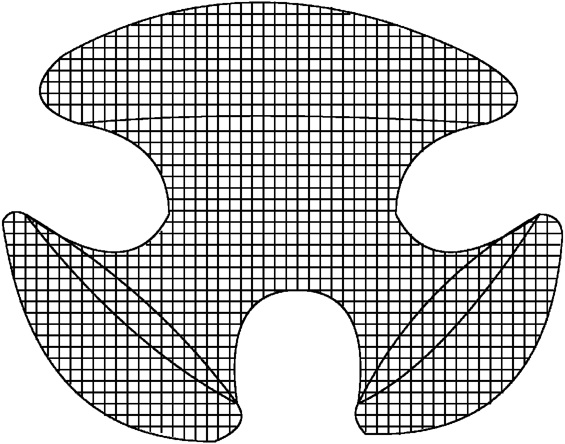
Now, notice that $f : \mathcal{M} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \hat{f}_i(x)\eta(x) \text{ if } x \in W_i; \\ 0 \text{ otherwise} \end{cases}$$

is smooth and satisfy inequality (41) and the condition $\inf_{x \in V} |\nabla f(x)| > 0$. In addition, the inequality $\text{vol}(V) \geq \text{vol}(\mathcal{M}) - \epsilon$ holds, which settles the theorem.

Final Conclusions

Although the *intrinsic result* is sharp with respect to the volume where the damping acts, we do not have any control about the regions that can be left free of damping. The connected disjoint components of V can be extremely small. See figure below.



M_0 is a non-dissipative area (in white) arbitrarily large while the demarcated area (in black) contains dissipative effects and can be considered arbitrarily small, both totally distributed on M.

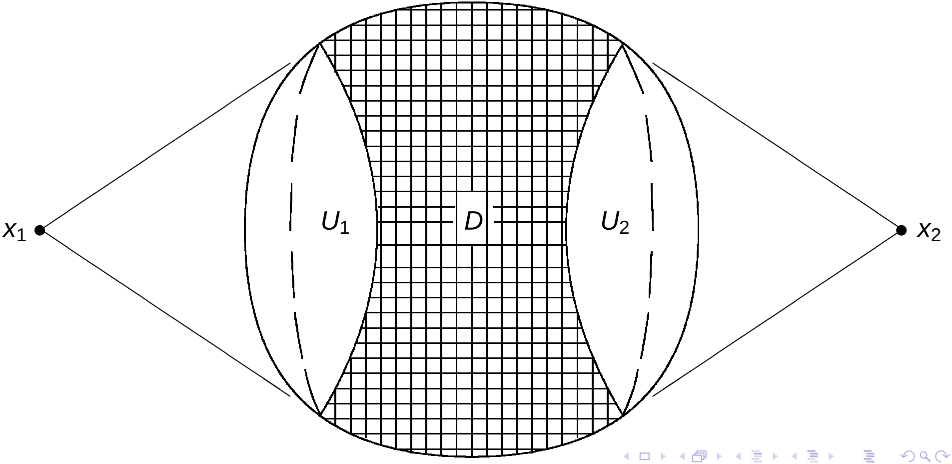
In the other direction, the *external vision* states that some umbilical domains of surfaces in $\mathbb{R}^3$ can be left free of damping. Therefore the next step is to combine the ideas of both techniques and try to put the damping in a arbitrarily small domain, but in such a way that domains with interesting properties can be left free of damping.

Finally it is worth mentioning that combining the techniques developed in [MAA/08] and [TRANS AMS/09] we can reduce arbitrarily the superficial measure of the dissipative area. Here the vector field q is defined as

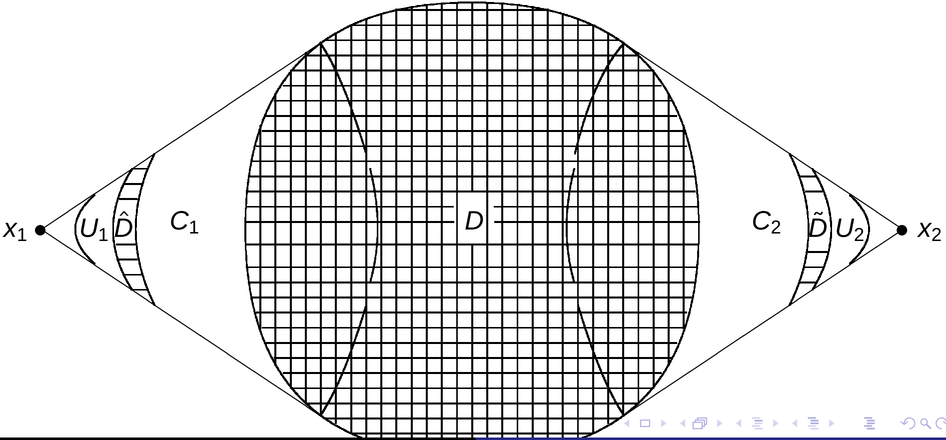
$$q(x) := \begin{cases} x - x_i \ \text{ if } x \in U_i,\, i = 1, 2, \\ \nabla f(x), Hess(f) \approx \mathbf{g},\ \text{if } x \text{ is in some small white domain of} \\ D \\ \text{smoothly extended otherwise} \end{cases}$$

where **g** is the Riemannian metric on $\mathcal{M}$ (see Figure below).

Note that it is necessary to put damping in D where $(x - x_i) \cdot \nu(x) \leq 0,\ i = 1, 2.$



Analogous considerations can be done for a finite number of ***glued regions*** **according to figure below. Note that it is necessary to put damping in** $\tilde{D}$**, in** $\hat{D}$ **and, in addition, in** D **where (if)** $(x - x_i) \cdot \nu(x) \leq 0$**,** $i = 1, 2$.



Wave Equation on Compact Manifolds

The main goal now is to generalize the results presented previously for n-dimensional compact Riemannian manifolds $(M, \mathbf{g})$ with or without boundary. We proceed as follows:

1. We prove that for every $x \in M$ (including the case $x \in \partial M$), there exist a neighborhood that can be left without damping;
2. We prove that every radially symmetric portion can be left without damping;
3. Let $\varepsilon > 0$ and $V_1, \ldots, V_k$ be domains as in (i) and (ii) which closures are pairwise disjoint. We prove that there exist a $V \supset \cup_{i=1}^{k} V_i$ that can be left without damping and such that $meas(V) \geq meas(M) - \varepsilon$ and $meas(V \cap \partial M) \geq meas(\partial M) - \varepsilon$.

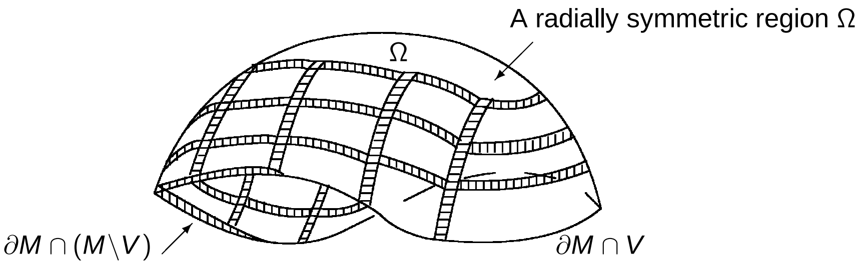


Figure: The *demarcated region* $M \backslash V$ (in black) illustrates the damped region on the compact manifold M with boundary ∂M, which can be considered *as small as desired*. Ω is a radially symmetric region without damping. The measure of $\partial M \cap (M \backslash V)$ can also be arbitrarily small

In particular several radially symmetric domains can be left without damping in a similar way as in Figure above. For this purpose, we construct an intrinsic multiplier that plays an important role when establishing the desired uniform decay rates of the energy. Fix $\epsilon > 0$. This multiplier is given by $\langle \nabla f, \nabla u \rangle$, where $f : M \to \mathbb{R}$ is a smooth function such that its Hessian $\nabla^2 f$ is closely related to **g** on an open subset $V \subset M$ that satisfies $meas(V) \geq meas(M) - \epsilon$, $meas(V \cap \partial M) \geq meas(\partial M) - \epsilon$ and $\langle \nabla f, \nu \rangle < 0$ on $V \cap \partial M$.

The complete proof of the above result can be found in [ARMA/10].

Final Conclusion

The results in terms of the ray of the geometric optics are more general than our results for the linear case. But our results also consider the nonlinear case and give explicitly examples of regions that can be left without damping, which can be a difficult task if we use the hypothesis on the ray of geometric optics on a general compact Riemannian manifold.

There are a lot to be done in this direction, I mean, about the relationship between these two different kind of hypothesis.

The last result obtained in this context was published recently: D.C.; CAVALCANTI,M. M.; FUKUOKA, R.; TOUNDYKOV,D. . Unified Approach to Stabilization of Waves on Compact Surfaces by Simultaneous Interior and Boundary Feedbacks of Unrestricted Growth. Applied Mathematics and Optimization, (2014).

THANK YOU VERY MUCH!

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