THE SPECTRAL PROBLEM FOR THE CURL OPERATOR

RODOLFO RODRÍGUEZ

Abstract. Vector fields $\boldsymbol{H}$ satisfying $\mathbf{curl}\,\boldsymbol{H} = \lambda\boldsymbol{H}$ with λ being a scalar field are called *force-free fields*. This name arises from magnetohydrodynamics, since a magnetic field of this kind induces a vanishing Lorentz force: $\boldsymbol{F} := \boldsymbol{J} \times \boldsymbol{B} = \mathbf{curl}\,\boldsymbol{H} \times (\mu\boldsymbol{H})$. In 1958 Woltjer [**?**] showed that the lowest state of magnetic energy density within a closed system is attained when λ is spatially constant. In such a case $\boldsymbol{H}$ is called a *linear* force-free field and its determination is naturally related with the spectral problem for the curl operator.

This problem has a longstanding tradition in mathematical physics. A large measure of the credit goes to Beltrami [**?**], who seems to be the first who considered it in the context of fluid dynamics and electromagnetism. This is the reason why the corresponding eigenfunctions are also called *Beltrami fields*. On bounded domains, the natural boundary condition for this problem is $\boldsymbol{H} \cdot \boldsymbol{n} = 0$, which corresponds to a field confined within the domain. Moreover, additional constraints have to been added for the spectral problem to be well posed when the domain is not topologically trivial (see [**?**, **?**]. Analytical solutions of this problem are only known under particular symmetry assumptions. The first one was obtained in 1957 by Chandrasekhar and Kendall [**?**] in the context of astrophysical plasmas arising in modeling of the solar crown.

More recently, some numerical methods have been introduced to compute force-free fields in domains without symmetry assumptions [**?**, **?**]. Following [**?**], we perform the mathematical analysis of the underlying spectral problem, which takes into account the topology of the domain. We propose a variational formulation for the spectral problem for the curl operator which allows us to obtain a thorough characterization. This formulation, after discretization, leads to a well-posed generalized eigenvalue problem. We propose a method for its numerical solution based on Nédélec finite elements of arbitrary order [**?**]. We prove spectral convergence, optimal order error estimates and that the method is free of spurious-modes. Finally we report some numerical experiments which confirm the theoretical results and allow us to assess the performance of the method.

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CI^{2}MA, Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile. E-mail: rodolfo@ing-mat.udec.cl