Some controllability properties for Schrödinger equations and open problems. Part I.

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1. Main controllability problem

Ω is a bounded open subset of $\mathbb{R}^N$ with boundary Γ.

Schrödinger equation in a bounded domain :

$$\begin{aligned} & i\frac{\partial \psi}{\partial t} + \Delta\psi + u(t)\mu(x)\psi = 0 \text{ in } \Omega \times (0,T), \\ & \psi = 0 \text{ on } \Gamma \times (0,T), \\ & \psi(0) = \psi_0 \text{ in } \Omega. \end{aligned}$$

μ is a **real** potential usually depending only on the x-variable.

u is the **real** control depending only on the t-variable (amplitude).

Solution ψ is (a priori) a function with **complex** values.

Question

For a fixed function μ with suitable properties, given a target ψ_1, can we choose a control u such that the solution ψ of the corresponding Schrödinger equation satisfy

$$\psi(T) = \psi_1 \text{ ?}$$

Immediate condition

As the potential $u(t)\mu(x)$ is real, we have conservation of the $L^2(\Omega)$-norm. To show this property, multiply the equation by $\bar{\psi}$ and integrate on Ω then take the imaginary part. We obtain

$$\frac{d}{dt}||\psi(t)||^2_{L^2(\Omega)} = 0.$$

Therefore we have an immediate necessary condition

$$||\psi_1||_{L^2(\Omega)} = ||\psi_0||_{L^2(\Omega)}.$$

2. Classical properties of Schrödinger equation.

Schrödinger equation with inital data ψ_0 and right hand side f.

$$
\begin{aligned}
&i\frac{\partial \psi}{\partial t} + \Delta \psi = f \text{ in } \Omega \times (0,T), \\
&\psi = 0 \text{ on } \Gamma \times (0,T), \\
&\psi(0) = \psi_0 \text{ in } \Omega.
\end{aligned}
$$

What are the "good" functional spaces to solve this equation?

One can use a semi-group approach or, even simpler here, the Fourier method which consists in expanding all functions on the basis of eigenfunctions for the Laplace operator.

Eigenfunctions of Laplace operator

$$\begin{aligned}
&-\Delta w_j = \lambda_j w_j \text{ in } \Omega, \ \ j = 1, \cdots, +\infty \\
&w_j = 0 \text{ on } \Gamma, \\
&(w_j, w_k)_{L^2(\Omega)} = \delta_{j,k}.
\end{aligned}$$

We then have

$$\begin{aligned}
&L^2(\Omega) = \{w = \sum_{j=1}^{+\infty} a_j w_j, \quad \sum_{j=1}^{+\infty} |a_j|^2 < +\infty\}, \\
&H_0^1(\Omega) = \{w = \sum_{j=1}^{+\infty} a_j w_j, \quad \sum_{j=1}^{+\infty} \lambda_j |a_j|^2 < +\infty\}, \\
&H^2(\Omega) \cap H_0^1(\Omega) = \{w = \sum_{j=1}^{+\infty} a_j w_j, \quad \sum_{j=1}^{+\infty} \lambda_j^2 |a_j|^2 < +\infty\}, \\
&\cdots
\end{aligned}$$

Now we expand the datas on the basis (w_j) and we define the truncated series

$$\psi_0 = \sum_{j=1}^{+\infty} \alpha_j w_j, \quad \psi_0^M = \sum_{j=1}^{M} \alpha_j w_j,$$

$$f(t) = \sum_{j=1}^{+\infty} f_j(t) w_j \quad f^M(t) = \sum_{j=1}^{M} f_j(t) w_j.$$

For fixed M we look for $\psi^M(t) = \sum_{j=1}^{M} a_j(t) w_j$ solution of the problem with datas ψ_0^M and f^M. This leads to a diagonal system

$$\begin{aligned} &i a_j'(t) = \lambda_j a_j(t) + f_j(t), \quad j = 1, \cdots, M, \\ &a_j(0) = \alpha_j, \end{aligned}$$

This gives an explicit formula

$$a_j(t) = \alpha_j e^{-i\lambda_j t} - i\int_0^t e^{-i\lambda_j(t-s)} f_j(s)ds.$$

Now, for $M > P$, we can write the equation satisfied by

$$\psi^M - \psi^P = \sum_{j=P+1}^{M} a_j w_j$$

and it is easy to obtain estimates on this quantity depending on the hypotheses on f and ψ_0. The estimates have to be obtained directly from the equation satisfied by a_j and not from the explicit formula giving a_j.

We obtain the existence and uniqueness for a solution ψ with

- If $\psi_0 \in L^2(\Omega)$ and $f \in L^1(0,T;L^2(\Omega))$ then $\psi \in C([0,T];L^2(\Omega))$.

- If $\psi_0 \in H_0^1(\Omega)$ and $f \in L^1(0,T;H_0^1(\Omega))$ then $\psi \in C([0,T];H_0^1(\Omega))$.

- If $\psi_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $f \in L^1(0,T;H^2(\Omega) \cap H_0^1(\Omega))$ then $\psi \in C([0,T];H^2(\Omega) \cap H_0^1(\Omega))$.

- If $\psi_0 \in H_\Delta(\Omega)$ and $f \in L^1(0,T;H_\Delta(\Omega))$ then $\psi \in C([0,T];H_\Delta(\Omega))$, where

$$H_\Delta(\Omega) = \{w \in H_0^1(\Omega),\ \text{such that } \Delta w \in H_0^1(\Omega)\}.$$

Notice that in that space H_Δ we have no information on the trace of the normal derivative....

3. Negative results of controllability.

Ball, Marsden, Slemrod result (SIAM J. Control and Optimization, 1982).

Ball, Marsden and Slemrod considered the problem of bilinear control

$$\frac{\partial y}{\partial t} + Ay + u(t)By = 0$$

where A generates a C^0 semi group on a Banach space X with $\dim X = +\infty$, and B is a bounded linear operator from X to X.

Then writing for solutions y the set of reachable states

$$R(y_0) = \{y(t),\ t \geq 0,\ y(0) = y_0,\ u \in L^r_{loc}(0, +\infty; \mathbb{R}), r > 1\}$$

they prove that $R(y_0)$ is contained in a countable union of compact sets, and therefore has an empty interior.

For a long time this result has prevented people from looking for positive controllability results !

In our context, take $A = -i\Delta$ with Dirichlet boundary conditions and $B\psi = -i\mu(x)\psi$.

We know that A generates a C^0 semi group on $L^2(\Omega)$, on $H_0^1(\Omega)$, on $H^2(\Omega) \cap H_0^1(\Omega)$ and also on $H_\Delta(\Omega)$.

If μ is regular enough, multiplication by μ is immediately a bounded linear operator in $L^2(\Omega)$ and also in $H_0^1(\Omega)$ and in $H^2(\Omega) \cap H_0^1(\Omega)$.

Therefore, by taking $u \in L^2(0, T)$ there is no chance to have a controllability result in these spaces. In fact due to the group property of Schrödinger equation, we can reverse the time and start from ψ_1. At any time T we would obtain a reachable set with empty interior, whatever the control is.

Situation fo $H_\Delta(\Omega)$.

For $H_\Delta(\Omega)$ the situation is more complex. Take μ regular and $\psi \in H_\Delta(\Omega)$. Then of course $\mu.\psi \in H^1(\Omega)$ and $\mu.\psi_{/\Gamma} = 0$ so that $\mu.\psi \in H^1_0(\Omega)$.

Now we have

$$\Delta(\mu.\psi) = \mu.\Delta\psi + 2\nabla\mu.\nabla\psi + \psi.\Delta\mu$$

and we easily have $\Delta(\mu.\psi) \in H^1(\Omega)$. But on the boundary we have, writing ν for the outward unit normal vector on Γ (as $\psi_{/\Gamma} = 0$)

$$\nabla\psi = (\nabla\psi.\nu)\nu$$

and therefore

$$\Delta(\mu.\psi)_{/\Gamma} = 2\frac{\partial\mu}{\partial\nu} \cdot \frac{\partial\psi}{\partial\nu}.$$

This trace will vanish for all $\psi \in H_\Delta(\Omega)$ if and only if

$$\frac{\partial \mu}{\partial \nu} = 0$$

and in that case we also have a negative result because multiplication by μ would be a bounded linear operator on $H_\Delta(\Omega)$!

But if μ is regular enough but such that

$$\frac{\partial \mu}{\partial \nu} \neq 0$$

then we have

$$\mu.\psi \in H_0^1(\Omega),\ \Delta\psi \in H^1(\Omega)$$

but in general

$$\mu.\psi \notin H_\Delta(\Omega).$$

Remark.

Despite of the negative results, Karine Beauchard has tried and succeeded in proving local controllability results for the 1 dimensional case, first of all, in an article in J. de Math. Pures et Appl. in 2005 in a very complex functional spaces framework, then, in collaboration with Camille Laurent again in J. de Math. Pures et Appl. 2010, using the above remark on $H_{\triangle}(\Omega)$. We will develop the latter method below.

4. Regularity result.

It is well known that Schrödinger equation is not regularizing.... Nevertheless, we will prove a kind of regularity result which will be essential for the controllability result and which has its own interest. This result has been proved by K.Beauchard and C.Laurent in the 1 dimensional case using arguments coming from harmonic analysis and the explicit knowledge of eigenvalues and eigenfunctions of $-\Delta$ in the 1 dimensional case.

I will present here an extension to the general case which is proved by arguments which appear to be completely different.....(J.-P.Puel, Revista Mathematica Complutense, submitted)

Theorem 1 *Let T be positive and Ω be a bounded open subset of $\mathbb{R}^N$ of class $C^{2,\alpha}$ with $\alpha > 0$. For every $\psi_0 \in H_\Delta(\Omega)$ and for every $f \in L^2(0,T;H^3(\Omega) \cap H_0^1(\Omega))$ the solution ψ of*

$$
\begin{aligned}
& i\frac{\partial \psi}{\partial t} + \Delta\psi = f \text{ in } \Omega \times (0,T), \\
& \psi = 0 \text{ on } \Gamma \times (0,T), \\
& \psi(0) = \psi_0 \text{ in } \Omega.
\end{aligned}
$$

satisfies

$$\psi \in C([0,T];H_\Delta(\Omega))$$

and there exists $C > 0$ independent of ψ_0, g and h such that

$$||\psi||_{C([0,T];H_\Delta(\Omega))} \leq C(||\psi_0||_{H_\Delta(\Omega)} + ||f||_{L^2(0,T;H^3(\Omega)\cap H_0^1(\Omega))}).$$

Remark.

In fact we can take f such that $f = g + h$ where

$$g \in L^1(0,T; H_\Delta(\Omega))$$

and

$$h \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)),\ \Delta^2 h = 0,\ \Delta h_{/\Gamma} \in L^2(0,T; L^2(\Gamma)),$$

The proof will be given with this condition on the right hand side which is more general.

Indeed, If $f \in L^2(0,T;H^3(\Omega)\cap H^1_0(\Omega))$, let us take g such that for almost every $t \in (0,T)$

$$\Delta^2 g(t) = \Delta^2 f(t) \text{ in } \Omega,$$
$$g = 0 \text{ on } \Gamma,$$
$$\Delta g = 0 \text{ on } \Gamma.$$

As $\Delta^2 f \in L^2(0,T;H^{-1}(\Omega))$, this uniquely defines g with $g \in L^2(0,T;H_\Delta(\Omega))$.

Let us now write $h = f - g$. Then $h \in L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega))$, $\Delta h \in L^2(0,T;H^1(\Omega))$ and

$$\Delta^2 h = 0 \text{ in } \Omega,$$
$$h = 0 \text{ on } \Gamma,$$
$$\Delta h_{/\Gamma} \in L^2(0,T;H^{\frac{1}{2}}(\Gamma)).$$

Sketch of the proof.

The equation for ψ is linear. Therefore we can treat separately the case $h = 0$ and the case $\psi_0 = 0$ and $g = 0$.

The case $h = 0$ goes back to classical results which give a solution in $C([0, T]; H_\Delta(\Omega))$. Therefore the only problem comes from the case where $h \neq 0$, $\psi_0 = 0$ and $g = 0$. By a density argument we can restrict ourselves to $h \in C_0^\infty((0, T); H^2(\Omega) \cap H_0^1(\Omega))$, with $\Delta h \in C_0^\infty((0, T); H^1(\Omega))$ and such that

$$\Delta^2 h = 0 \text{ in } \Omega,$$
$$h = 0 \text{ on } \Gamma,$$
$$\Delta h_{/\Gamma} \in C_0^\infty((0, T); H^{\frac{1}{2}}(\Gamma)).$$

In that case we can take the derivative of the equation with respect to time and obtain a regularity result on $\frac{\partial \psi}{\partial t}$ and therefore on $\Delta\psi$ which implies that $\Delta\psi_{/\Gamma} = 0$. The only thing to prove the estimate on $||\psi||_{C([0,T];H_\Delta(\Omega))}$.

Let us define $\xi = \Delta\psi$. Then ξ is solution to the equation, if we write $k = \Delta h$,

$$i\frac{\partial \xi}{\partial t} + \Delta\xi = k \text{ in } \Omega \times (0,T),$$
$$\xi = 0 \text{ on } \Gamma \times (0,T),$$
$$\xi(0) = 0 \text{ in } \Omega.$$

Theorem 1 will be an immediate consequence of the following result

Theorem 2 *Assume that* $k \in L^2(0,T;L^2(\Omega))$ *with* $\Delta k = 0$ *and* $k_{/\Gamma} \in L^2(0,T;L^2(\Gamma))$. *Then the solution* ξ *of the previous problem satisfies*

$$\xi \in C([0,T];H_0^1(\Omega))$$

and there exists a constant $C > 0$ *independent of* k *such that*

$$||\xi||_{C([0,T];H_0^1(\Omega))} \leq C||k_{/\Gamma}||_{L^2(0,T;L^2(\Gamma))}.$$

Again here, by a density argument, it is enough to prove the theorem for functions k which are C^∞ with compact support in the time variable.

Let us now set

$$\varphi = \Delta\xi.$$

Then φ satisfies the following equation (we recall that $\Delta k = \Delta^2 h = 0$)

$$\begin{aligned}
&i\frac{\partial \varphi}{\partial t} + \Delta\varphi = 0 \text{ in } \Omega \times (0,T),\\
&\varphi = \Delta h_{/\Gamma} \text{ on } \Gamma \times (0,T),\\
&\varphi(0) = 0 \text{ in } \Omega.
\end{aligned}$$

Then Theorem 2 will be an immediate consequence of the following result.

Theorem 3 *As* $\Delta h_{/\Gamma} \in L^2(0,T;L^2(\Gamma))$, *there exists a unique (weak) solution* φ *to the previous problem with* $\varphi \in C([0,T];H^{-1}(\Omega))$ *and there exists a constant* $C > 0$ *independent of* $\Delta h_{/\Gamma}$ *such that*

$$||\varphi||_{C([0,T];H^{-1}(\Omega))} \leq C||\Delta h_{/\Gamma}||_{L^2(0,T;L^2(\Gamma))}.$$

In order to prove Theorem 3 we argue by transposition. Let $z \in L^1(0, T; H_0^1(\Omega))$ and let us define η as the solution of

$$
\begin{aligned}
& i\frac{\partial \eta}{\partial t} + \Delta \eta = z \text{ in } \Omega \times (0, T), \\
& \eta = 0 \text{ on } \Gamma \times (0, T), \\
& \eta(0) = 0 \text{ in } \Omega.
\end{aligned}
$$

Then we know that $\eta \in C([0, T]; H_0^1(\Omega))$. Moreover we have

Lemma 4 *If Ω is of class $C^{2,\alpha}$ with $\alpha > 0$, the function η satifies $\frac{\partial \eta}{\partial \nu} \in L^2(0, T; L^2(\Gamma))$ and the mapping $z \to \frac{\partial \eta}{\partial \nu}$ is linear continuous from $L^1(0, T; H_0^1(\Omega))$ to $L^2(0, T; L^2(\Gamma))$ so that there exists a constant $C > 0$ independent of z such that*

$$
||\frac{\partial \eta}{\partial \nu}||_{L^2(0,T;L^2(\Gamma))} \leq C||z||_{L^1(0,T;H_0^1(\Omega))}.
$$

This regularity result has been proved by Elaine Maychtyngier (SIAM J. Control and Optim., 32 (1), 24-34, 1994.) but we give here a sketch of a slightly improved proof for sake of completeness by the multiplier method with a specific multiplier.

Let w_1 be the positive unitary eigenfunction of $-\Delta$ on Ω associated with the first eigenvalue λ_1. From the regularity of Ω we know that $w_1 \in C^1(\bar{\Omega})$ and from the strong maximum principle there exists $\beta > 0$ such that

$$\forall x \in \Gamma,\ -\frac{\partial w_1}{\partial \nu}(x) \geq \beta > 0$$

We now take the mutiplier $m = -\nabla w_1$ and multiply the equation for η by $m.\nabla\bar{\eta}$, then take the real part, taking into account that $\operatorname{div} m = -\Delta w_1 = \lambda_1 w_1$, and that on γ, $\nabla\eta = (\nabla\eta.\nu)\nu$.

After some standard calculations we obtain

$$\frac{1}{2}\int_0^T\int_\Gamma |\nabla\eta.\nu|^2(m.\nu)d\gamma dt = Re\int_0^T\int_\Omega zm.\nabla\bar{\eta}dxdt$$

$$-\frac{\lambda_1}{2}\int_0^T\int_\Omega(\eta m.\nabla\bar{\eta}+\eta\bar{z}w_1)dxdt + Re(\sum_{j,k=1}^N\int_0^T\int_\Omega\frac{\partial\eta}{\partial x_j}\frac{\partial m_k}{\partial x_j}\frac{\partial\bar{\eta}}{\partial x_k}dxdt).$$

From the standard estimates on solutions η, we then obtain with a constant C independent of z

$$\int_0^T\int_\Gamma |\nabla\eta.\nu|^2 d\gamma dt \le C||z||^2_{L^1(0,T;H_0^1(\Omega))}.$$

This finishes the proof of Lemma 4.

Now for every $z \in L^1(0,T;H_0^1(\Omega))$ let us define

$$\mathcal{L}(z) = \int_0^T \int_\Gamma \Delta h_{/\Gamma} \frac{\partial \bar{\eta}}{\partial \nu} d\gamma dt.$$

From Lemma 4 the mapping

$$z \to \mathcal{L}(z)$$

is an antilinear continuous form on $L^1(0,T;H_0^1(\Omega))$. Therefore, there exists a unique element $\varphi \in L^\infty(0,T;H^{-1}(\Omega))$ such that

$$\forall z \in L^1(0,T;H_0^1(\Omega)),\ \int_0^T < \varphi, \bar{z} >_{H^{-1}(\Omega),H_0^1(\Omega)} dt = \mathcal{L}(z).$$

Moreover we have

$$||\varphi||_{L^\infty(0,T;H^{-1}(\Omega))} \leq C||\Delta h_{/\Gamma}||_{L^2(0,T;L^2(\Gamma))}.$$

When the data $\Delta h_{/\Gamma}$ is taken in a dense subset of regular functions (C^∞ functions with compact support in space and time), it is well known that the solution φ is regular and we have $\varphi \in C([0,T]; H^{-1}(\Omega))$ with the same estimate. Therefore, taking a sequence of regular datas $\Delta h^n_{/\Gamma}$ converging to $\Delta h_{/\Gamma}$ we have, denoting by φ^n the corresponding solution

$$||\varphi^m - \varphi^p||_{C([0,T];H^{-1}(\Omega))} \leq C||\Delta h^m_{/\Gamma} - \Delta h^p_{/\Gamma}||_{L^2(0,T;L^2(\Gamma))}.$$

This shows that φ^n is a Cauchy sequence in $C([0,T]; H^{-1}(\Omega))$ and of course φ^n converges to φ so that $\varphi \in C([0,T]; H^{-1}(\Omega))$. This finishes the proof of Theorem 3, and therefore of Theorem 2 and Theorem 1.