Some controllability properties for Schrödinger equations and open problems. Part II.

Jean-Pierre Puel
Ikerbasque and BCAM, Bizkaia Technology Park, Building 500, Derio, 48160, Spain
and
Laboratoire de Mathématiques de Versailles, Université de Versailles Saint-Quentin, 45 avenue des Etats Unis, 78035 Versailles Cedex, France

1. Setting of the controllability problem.

From now on we consider the case of dimension $N \leq 3$.

Proposition 1 *Let* $\mu \in H^3(\Omega)$. *For every* $\psi_0 \in H_\triangle(\Omega)$ *and every* $u \in L^2(0,T)$, *there exists a unique solution* $\psi \in C([0,T]; H_\triangle(\Omega))$ *to the Schrödinger equation*

$$\begin{aligned} & i\frac{\partial \psi}{\partial t} + \Delta\psi + u.\mu.\psi = 0 \text{ in } \Omega \times (0,T), \\ & \psi = 0 \text{ on } \Gamma \times (0,T), \\ & \psi(0) = \psi_0 \text{ in } \Omega. \end{aligned}$$

Proof. For each $u \in L^2(0,T)$ fixed, given $\epsilon > 0$ we can divide $(0,T)$ in k subintervals $(T_0 = 0, T_1)$, (T_1, T_2), ... , $(T_{k-1}, T_k = T)$ such that $||u||_{L^2(T_{j-1},T_j)} \leq \epsilon$.

We will prove the proposition for $||u||_{L^2(0,T)}$ small enough on the interval $(0,T)$ and as the argument will be independent on the size of the initial data ψ_0, the same arguments on each subinterval will give the complete result.
If $\psi \in C([0,T];H_\Delta(\Omega))$ as $\mu \in H^3(\Omega)$ we see that $\mu.\psi \in C([0,T];H^3(\Omega) \cap H_0^1(\Omega))$ with

$$||\mu.\psi||_{C([0,T];H^3(\Omega))\cap H_0^1(\Omega))} \leq C_\mu ||\psi||_{C([0,T];H_\Delta(\Omega))}.$$

Therefore $u.\mu.\psi \in L^2(0,T;H^3(\Omega) \cap H_0^1(\Omega))$ and we can define $\widehat{\psi} = \mathcal{S}(\psi)$ as the solution of the following Schrödinger equation

$$i\frac{\partial \widehat{\psi}}{\partial t} + \Delta \widehat{\psi} + u.\mu.\psi = 0 \text{ in } \Omega \times (0,T),$$
$$\widehat{\psi} = 0 \text{ on } \Gamma \times (0,T),$$
$$\widehat{\psi}(0) = \psi_0 \text{ in } \Omega.$$

From the regularity result proved in Part I we have

$$\hat{\psi} = \mathcal{S}(\psi) \in C([0,T]; H_\Delta(\Omega)).$$

Now we have

$$||\mathcal{S}(\psi^1 - \psi^2||_{C([0,T];H_\Delta(\Omega))} \leq C||u.\mu.(\psi^1 - \psi^2)||_{L^2(0,T;H^3(\Omega)\cap H_0^1(\Omega))}$$
$$\leq C.C_\mu||u||_{L^2(0,T)}||\psi^1 - \psi^2||_{C([0,T];H_\Delta(\Omega))}.$$

Taking $||u||_{L^2(0,T)}$ small enough so that $C.C_\mu.||u||_{L^2(0,T)} \leq \frac{1}{2}$ we see that $\mathcal{S}$ is a strict contraction and therefore has a unique fixed point which is solution to our problem.

Comment.

• We now have a correct functional setting for our controllability problem which avoids the negative result of Ball, Marsden and Slemrod as multiplication by μ is not (in general) a bounded linear operator in $H_\triangle(\Omega)$.

• We would like to study the local controllability problem in a neighborhood of the first eigenfunction w_1. We have already noticed that, due to the reversibility of Schrödinger equation, it is sufficient to start from the initial value $\psi_0 = w_1$ and to try to reach (in time $\frac{T}{2}$ a target ψ_1 in a (small) neighborhood of w_1.

• This problem is completely open in dimension $N \geq 2$! It has been solved in dimension 1 by Karine Beauchard and Camille Laurent (J. de Math. Pures et Appl. 2010) and we will present their result below.

2. Linearization.

We now take the problem

$$
\begin{aligned}
& i\frac{\partial \psi}{\partial t} + \Delta\psi + u.\mu.\psi = 0 \text{ in } \Omega \times (0,T), \\
& \psi = 0 \text{ on } \Gamma \times (0,T), \\
& \psi(0) = w_1 \text{ in } \Omega.
\end{aligned}
$$

We want to study the mapping $\mathcal{T}$ defined from $L^2(0,T)$ to $H_\Delta(\Omega)$ by

$$\mathcal{T}(u) = \psi(T).$$

When $u = 0$ the solution is

$$w(t) = e^{-i\lambda_1 t} w_1$$

and we have

$$\mathcal{T}(0) = w(T) = e^{-i\lambda_1 T} w_1.$$

Lemma 2 *The mapping $\mathcal{T}$ is continuously differentiable on $L^2(0,T)$ and we have for every u and v in $L^2(0,T)$*

$$D\mathcal{T}(u)[v] = z(T)$$

where z is the solution to the following equation

$$\begin{aligned} & i\frac{\partial z}{\partial t} + \Delta z + u.\mu.z + v.\mu.\psi = 0 \text{ in } \Omega \times (0,T), \\ & z = 0 \text{ on } \Gamma \times (0,T), \\ & z(0) = 0 \text{ in } \Omega. \end{aligned}$$

Proof.
Existence and uniqueness for the solution z can be done exactly with the same arguments as the ones used in Proposition 1. Let us write ψ the solution associated with u and $\widehat{\psi}$ the solution associated with $u+v$ and $\xi = \widehat{\psi} - \psi - z$. We have

$$\begin{aligned}
&i\frac{\partial \xi}{\partial t} + \Delta \xi + u.\mu.\xi + v.\mu.(\widehat{\psi} - \psi) = 0 \text{ in } \Omega \times (0,T),\\
&\xi = 0 \text{ on } \Gamma \times (0,T),\\
&\xi(0) = 0 \text{ in } \Omega.
\end{aligned}$$

Then

$$||\xi||_{C([0,T];H_\Delta(\Omega))} \leq C.C_\mu ||v||_{L^2(0,T)} ||\widehat{\psi} - \psi||_{C([0,T];H_\Delta(\Omega))}$$

and

$$||\widehat{\psi} - \psi||_{C([0,T];H_\Delta(\Omega))} \leq C.C_\mu ||v||_{L^2(0,T)} ||\widehat{\psi}||_{C([0,T];H_\Delta(\Omega))}.$$

This shows the differentiability of $\mathcal{T}$ and that $D\mathcal{T}(u)[v] = z(T)$. The continuity of $u \to D\mathcal{T}(u)$ is immediate.

The difference between the cases of dimension $N = 1$ and dimension $N \geq 2$ will appear in the fact that for $N = 1$ we will be able to find conditons on μ which are often satisfied such that $D\mathcal{T}(0)$ will be surjective (so that the linearized problem will be controllable) whereas in dimension $N \geq 2$ this will not be possible and in general, the linearized problem at $u = 0$ will not be controllable.

3. 1-dimensional case. Controllability of the linearized problem at $u = 0$.

Here we work on the interval $(0,1)$ for the x variable. We have

$$\lambda_j = j^2\pi^2, \quad w_j = \sqrt{2}\sin(\sqrt{\lambda_j}x), \quad w_1(t) = \sqrt{2}e^{-i\pi^2 t}\sin(\pi t)$$

We want to show that (under good hypotheses on μ), if we consider the problem

$$\begin{aligned}
& i\frac{\partial z}{\partial t} + i\frac{\partial^2 z}{\partial x^2} + v.\mu.w_1 = 0 \text{ on } (0,1)\times(0,T),\\
& z(0,t) = z(1,t) = 0,\\
& z(x,0) = 0 \text{ on } (0,1),
\end{aligned}$$

for every $z_1 \in H_\Delta(0,1)$, there exists a control $v \in L^2(0,T)$ (with real values) such that $z(T) = z_1$ with continuity of the mapping $z_1 \to v$.

We can write

$$z_1 = \sum_{k\geq 1} b_k w_k \text{ with } ||z_1||^2_{H_\triangle(0,1)} = \sum_{k\geq 1} k^6 |b_k|^2 < +\infty.$$

If we look for z in the form

$$z(t) = \sum_{k\geq 1} \beta_k(t) w_k,$$

we have for each $k \geq 1$

$$i\beta_k'(t) = k^2\pi^2\beta_k(t) - v(t)\left(\int_0^1 \mu(x)w_1(x)w_k(x)dx\right)e^{-i\pi^2 t},$$
$$\beta_k(0) = 0,$$

so that, writing $\mu_{1,k} = \int_0^1 \mu(x)w_1(x)w_k(x)dx$,

$$i\beta_k(t) = e^{-ik^2\pi^2 t}\int_0^t v(s)e^{i(k^2-1)\pi^2 s}\mu_{1,k}ds.$$

We now want to find v such that for every $k \geq 1$, $\beta_k(T) = b_k$ so that

$$i b_k e^{ik^2\pi^2 T} = \mu_{1,k} \int_0^T v(s) e^{i(k^2-1)\pi^2 s} ds.$$

Of course a first necessary condition on μ is $\forall k \geq 1, \ \mu_{1,k} \neq 0$.

Proposition 3 *Let us assume that $\mu \in H^3(0,1)$ and satisfies*

$$\exists C > 0, \quad \forall k \geq 1, \quad |\mu_{1,k}| \geq \frac{C}{k^3}.$$

Then there exists a constant $C > 0$ and $v \in L^2(0,T)$ with real values, such that

$$i b_k e^{ik^2\pi^2 T} = \mu_{1,k} \int_0^T v(s) e^{i(k^2-1)\pi^2 s} ds$$

$$\int_0^T |v(s)|^2 ds \leq C \sum_{k \geq 1} k^6 |b_k|^2.$$

Define

$$\tilde{b}_k = i\frac{b_k}{\mu_{1,k}}e^{ik^2\pi^2 T}.$$

The assumption on μ implies that $\sum_{k\geq 1}|\tilde{b}_k|^2 < +\infty$.
Writing $\omega_k = (k+1)^2 - 1$ for $k \geq 0$ and $\omega_k = -\omega_{-k}$ for $k < 0$, as $\omega_{k+1} - \omega_k \to +\infty$ when $k \to +\infty$, for any $T > 0$, the family $(\omega_k)_{k\in Z}$ is a Riesz basis in $L^2(0,T)$. Choosing $\tilde{b}_k = \bar{\tilde{b}}_{-k}$ for $k < 0$ we can find $v \in L^2(0,T)$ with real values and two constants $C_1 > 0$ and $C_2 > 0$ such that for every $k \in Z$

$$\int_0^T v(s)e^{i\omega_k s}ds = \tilde{b}_k$$

$$C_1 \sum_{k\geq 1} |\tilde{b}_k|^2 \leq \int_0^T |v(s)|^2 dt \leq C_2 \sum_{k\geq 1} |\tilde{b}_k|^2.$$

This proves Proposition 3.

Comment.

A Riesz basis is the image by an isomorphism of an orthonormal family. It is in fact a Riesz basis on the closure of its span.

The fact that $\omega_{k+1} - \omega_k \to +\infty$ when $k \to +\infty$ implies that the family $(\omega_k)_{k \in Z}$ is a Riesz basis in $L^2(0,T)$ for $T > 0$ comes from Ingham inequality and a version proved by A.Haraux (J. Math. Pures et Appl., 68:457465, 1989.)

A Riesz basis has a biorthogonal family which is also a Riesz basis and the solution v of

$$\int_0^T v(s) e^{i\omega_k s} ds = \tilde{b}_k$$

can be written in terms of this biorthogonal family.

Condition on μ.

By an immediate calculation (integration by parts) we can show that

$$\mu_{1,k} = \int_0^1 \mu(x)\sin(\pi x)\sin(k\pi x)dx = \frac{2\pi}{k^3\pi^3}((-1)^{k+1}\mu'(1) - \mu'(0))$$
$$+\frac{1}{k^3\pi^3}\int_0^1 \cos(k\pi x)(\mu'''(x)\sin(\pi x) + 3\pi\mu''(x)\cos(\pi x)$$
$$+3\pi^2\mu'(x)\sin(\pi x) + \pi^3\mu(x)\cos(\pi x))dx.$$

As $\cos(k\pi x)$ converges to 0 weakly in $L^2(0,1)$ when $k \to +\infty$ the integral term in the right hans side tends to 0 when $k \to +\infty$. Therefore, for k large, the main term will be the first one.

We then have the following result.

Proposition 4 *If* $\mu \in H^3(0,1)$ *satisfies*

$$\forall k \geq 1, \quad \mu_{1,k} \neq 0 \quad \textit{and} \quad \mu'(0) \pm \mu'(1) \neq 0,$$

then there exists $C > 0$ *such that*

$$|\mu_{1,k}| \geq \frac{C}{k^3}.$$

These conditions are not difficult to ensure, and they are generically satisfied by μ. For example, $\mu(x) = x^2$ satisfies the conditions.

4. 1-dimensional case. Controllability result.

Theorem 5 *Let* $\mu \in H^3(0,1)$ *satisfying*

$$\exists C > 0, \quad \forall k \geq 1, \quad |\mu_{1,k}| \geq \frac{C}{k^3}.$$

Then for every $T > 0$*, there exists* $\eta > 0$ *such that for every* $\psi_1 \in H_\Delta(0,1)$ *with* $||\psi_1 - e^{-i\pi^2 T} w_1||_{H_\Delta(0,1)} \leq \eta$*, there exists a control* $u \in L^2(0,T)$ *with real values such that the corresponding solution* ψ *of Schrödinger equation satisfies*

$$\psi(T) = \psi_1.$$

Moreover there exists a constant $C > 0$ *such that we can choose the control* u *with*

$$||u||_{L^2(0,T)} \leq C||\psi_1 - e^{-i\pi^2 T} w_1||_{H_\Delta(0,1)}.$$

As already noticed, this also implies an analogous result with initial condition $\psi(0) = \psi_0$ *with* $||\psi_0 - w_1||_{H_\Delta(0,1)} \leq \eta$*.*

The proof of Theorem 5 is now classical. We consider the mapping $\mathcal{T}$ already defined and we want to show that there exists $u \in L^2(0,T)$ such that

$$\mathcal{T}(u) = \psi_1.$$

We know that

• $\mathcal{T}(0) = e^{-i\pi^2 T} w_1$

• $\mathcal{T}$ is a C^1 mapping from $L^2(0,T)$ to $H_\triangle(0,1)$.

• The controllability of the linearized problem at $u = 0$ (with continuity of the control with respect to the target) says that $D\mathcal{T}(0)$ has a right (continuous) inverse.

Therefore, there exists a neighborhood of $e^{-i\pi^2 T} w_1$ (say a ball of radius η) in $H_\triangle(0,1)$ such that for every ψ_1 in this neighborhood, we can find a control $u \in L^2(0,T)$ (with real values) such that

$$||u||_{L^2(0,T)} \leq C||\psi_1 - e^{-i\pi^2 T} w_1||_{H_\triangle(0,1)}$$

and

$$\mathcal{T}(u) = \psi_1.$$

This finishes the proof of Theorem 5.

Open problems.

• What can we say in the case of dimension $N \geq 2$?

• What happens in a neighborhood of other eigenfunctions?

• What happens for the case of the whole real line, even with the harmonic oscillator (which has a discrete spectrum) ?