Nonlinear Diffusion and Free Boundaries

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Outline

1 **Theories of Diffusion**
- Diffusion
- Heat equation
- Linear Parabolic Equations
- Nonlinear equations

2 **Degenerate Diffusion and Free Boundaries**
- Introduction
- The basics
- Generalities

3 **Fast Diffusion Equation**
- Fast Diffusion Ranges
- Regularity through inequalities. Aronson–Caffarelli Estimates
- Local Boundedness
- Flows on manifolds

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Diffusion

Populations diffuse, substances (like particles in a solvent) diffuse, heat propagates, electrons and ions diffuse, the momentum of a viscous (Newtonian) fluid diffuses (linearly), there is diffusion in the markets, ...

- *what is diffusion anyway?* Is diffusion more or less random walk ?

- *how to explain it with mathematics? is it pure or applied mathematics ?* what would Kolmogorov say?

- *How much of it can be explained with linear models, how much is essentially nonlinear?*

- *The stationary states of diffusion belong to an important world, elliptic equations. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...*

The Laplacian Δ is the King of Differential Operators.

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Diffusion in Wikipedia

- Diffusion. The spreading of any quantity that can be described by the diffusion equation or a random walk model (e.g. concentration, heat, momentum, ideas, price) can be called diffusion.

- Some of the most important examples are listed below.

  * Atomic diffusion

  * Brownian motion, for example of a single particle in a solvent

  * Collective diffusion, the diffusion of a large number of (possibly interacting) particles * Effusion of a gas through small holes.

  * Electron diffusion, resulting in electric current

  * Facilitated diffusion, present in some organisms.

  * Gaseous diffusion, used for isotope separation

  * Heat flow * Ito- diffusion * Knudsen diffusion

  * Momentum diffusion, ex. the diffusion of the hydrodynamic velocity field

  * Osmosis is the diffusion of water through a cell membrane. * Photon diffusion

  * Reverse diffusion * Self-diffusion * Surface diffusion

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The heat equation origins

- We begin our presentation with the Heat Equation $\boxed{u_t = \Delta u}$ and the analysis proposed by Fourier, 1807, 1822 (Fourier decomposition, spectrum). The mathematical models of heat propagation and diffusion have made great progress both in theory and application.

  They have had a strong influence on 5 areas of Mathematics: PDEs, Functional Analysis, Inf. Dim. Dyn. Systems, Diff. Geometry and Probability. And on / from Physics.

- The heat flow analysis is based on two main techniques: integral representation (convolution with a Gaussian kernel) and mode separation:

$$u(x,t) = \sum T_i(t) X_i(x)$$

where the $X_i(x)$ form the spectral sequence

$$-\Delta X_i = \lambda_i X_i.$$

This is the famous linear eigenvalue problem, Spectral Theory.

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Linear heat flows

From 1822 until 1950 the heat equation has motivated
(i) Fourier analysis decomposition of functions (and set theory),
(ii) development of other linear equations
$\Longrightarrow$ Theory of Parabolic Equations

$$u_t = \sum a_{ij}\partial_i\partial_j u + \sum b_i\partial_i u + cu + f$$

Main inventions in Parabolic Theory:
(1) a_{ij}, b_i, c, f regular $\Rightarrow$ Maximum Principles, Schauder estimates,
Harnack inequalities; C^α spaces (Hölder); potential theory; generation of
semigroups.
(2) coefficients only continuous or bounded $\Rightarrow$ $W^{2,p}$ estimates,
Calderón-Zygmund theory, weak solutions; Sobolev spaces.

The probabilistic approach: Diffusion as an stochastic process: Bachelier,
Einstein, Smoluchowski; Kolmogorov, Wiener, Levy; Ito, Skorokhod, ...

$$dX = bdt + \sigma dW$$

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- In the last 50 years emphasis has shifted towards the Nonlinear World. Maths more difficult, more complex, and more realistic.
  My group works in the areas of Nonlinear Diffusion and Reaction Diffusion.
  I will talk about the theory mathematically called Nonlinear Parabolic PDEs. General formula

$$u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u)$$

- Typical nonlinear diffusion: Stefan Problem, Hele-Shaw Problem, PME: $u_t = \Delta(u^m)$, EPLE: $u_t = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$.

- Typical reaction diffusion: Fujita model $u_t = \Delta u + u^p$.

- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. *Any singularities?*

- The geometrical models: the Ricci flow, movement by curvature.

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The Nonlinear Diffusion Models

- The Stefan Problem (Lamé and Clapeyron, 1833; Stefan 1880)

$$SE : \begin{cases} u_t = k_1 \Delta u & \text{for } u > 0, \\ u_t = k_2 \Delta u & \text{for } u < 0. \end{cases} \quad TC : \begin{cases} u = 0, \\ \mathbf{v} = L(k_1 \nabla u_1 - k_2 \nabla u_2). \end{cases}$$

  Main feature: the free boundary or moving boundary where $u = 0$. TC= Transmission conditions at $u = 0$.

- The Hele-Shaw cell (Hele-Shaw, 1898; Saffman-Taylor, 1958)

$$u > 0, \ \Delta u = 0 \quad \text{in} \quad \Omega(t); \quad u = 0, \ \mathbf{v} = L \partial_n u \quad \text{on} \quad \partial \Omega(t).$$

- The Porous Medium Equation $\rightarrow$ *(hidden free boundary)*

$$u_t = \Delta u^m, \quad m > 1.$$

- The p-Laplacian Equation, $u_t = \text{div}\,(|\nabla u|^{p-2} \nabla u)$.

  Recent interest in $p = 1$ (images) or $p = \infty$ (geometry and transport)

The Reaction Diffusion Models

- The Standard Blow-Up model (Kaplan, 1963; Fujita, 1966)

$$u_t = \Delta u + u^p$$

Main feature: If $p > 1$ the norm $\|u(\cdot, t)\|_\infty$ of the solutions goes to infinity in finite time. Hint: Integrate $u_t = u^p$.
Problem: *what is the influence of diffusion / migration?*

- General scalar model

$$u_t = \mathcal{A}(u) + f(u)$$

- The system model: $\vec{u} = (u_1, \cdots, u_m) \rightarrow$ chemotaxis system.

- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. Quadratic nonlinear, Mixed type *Any singularities?*

- The geometrical models: the Ricci flow: $\partial_t g_{ij} = -R_{ij}$. This is a nonlinear heat equation. Posed in the form of PDEs by R Hamilton, 1982. Solved by G Perelman 2003.

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The Porous Medium - Fast Diffusion Equation

- If you go to Wikipedia and look for the Diffusion Equation you will find

$$\frac{\partial \phi(\vec{r}, t)}{\partial t} = \nabla \cdot (D(\phi, \vec{r}) \nabla \phi(\vec{r}, t))$$

It is not difficult from here to conclude that the simplest model of nonlinear diffusion equation is maybe

$$u_t = \Delta u^m = \nabla \cdot (c(u) \nabla u)$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = m u^{m-1} [= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$, $\implies$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \implies$ Fast Diffusion.
- But power functions are tricky:
  - $c(u) \to 0$ as $u \to \infty$ if $m > 1$ ("slow case")
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$$c(u) = mu^{m-1}[= m|u|^{m-1}]$$

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- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \implies$ Fast Diffusion.
- But power functions are tricky:
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The basics

- For for $m = 2$ the equation is re-written as

$$\tfrac{1}{2}u_t = u\Delta u + |\nabla u|^2$$

and you can see that for $u \sim 0$ it looks like the eikonal equation

$$u_t = |\nabla u|^2$$

This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.

- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$v_t = (m-1)v\Delta v + |\nabla v|^2$$

where $v = cu^{m-1}$ is the pressure; normalization $c = m/(m-1)$.
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Applied motivation for the PME

- Flow of gas in a porous medium (Leibenzon, 1930; Muskat 1933)
  $m = 1 + \gamma \geq 2$

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{v}) = 0, \\ \mathbf{v} = -\frac{k}{\mu} \nabla p, \quad p = p(\rho). \end{cases}$$

Second line left is the Darcy law for flows in porous media (Darcy, 1856). *Porous media flows are potential flows due to averaging of Navier-Stokes on the pore scales.*

To the right, put $p = p_o \, \rho^\gamma$, with $\gamma = 1$ (isothermal), $\gamma > 1$ (adiabatic flow).

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  Experimental fact: diffusivity at high temperatures is not constant as in Fourier's law, due to radiation.

$$\frac{d}{dt} \int_\Omega c\rho T \, dx = \int_{\partial\Omega} k(T) \nabla T \cdot \mathbf{n} dS.$$

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$$c\rho \frac{\partial T}{\partial t} = \text{div}(k(T)\nabla T) = c_1 \Delta T^{n+1}.$$

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These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.

- The nonlinear approach: estimates; functional spaces.

- Existence, non-existence. Uniqueness, non-uniqueness.

- Regularity of solutions: *is there a limit? C^k for some k?*

- Regularity and movement of interfaces: *C^k for some k?*.

- Asymptotic behaviour: *patterns and rates? universal?*

- The probabilistic approach. *Nonlinear process. Wasserstein estimates*

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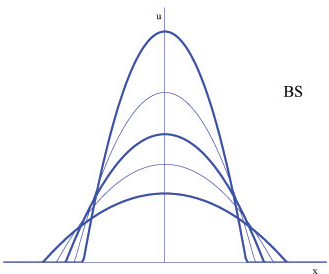
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Barenblatt profiles (ZKB)

- These profiles are the alternative to the Gaussian profiles.
  They are source solutions. *Source means that $u(x, t) \to M\,\delta(x)$ as $t \to 0$.*
- Explicit formulas (1950):

$$\mathbf{B}(x, t; M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \left(C - k\xi^2\right)_{+}^{1/(m-1)}$$

$\alpha = \frac{n}{2+n(m-1)}$

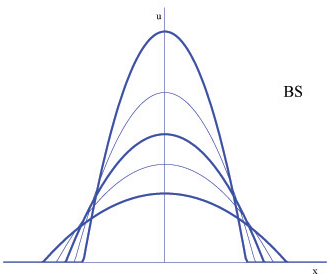$\beta = \frac{1}{2+n(m-1)} < 1/2$

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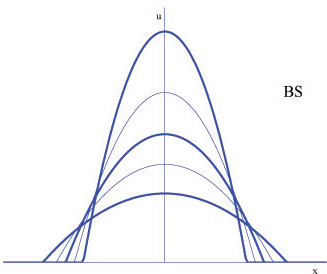
Free boundary at distance $|x| = ct^{\beta}$

Scaling law; anomalous diffusion versus Brownian motion

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Concept of solution

There are many concepts of generalized solution of the PME:

- Classical solution: only in non-degenerate situations, $u > 0$.
- Limit solution: physical, but depends on the approximation (?).
- Weak solution Test against smooth functions and eliminate derivatives on the unknown function; it is the mainstream; (Oleinik, 1958)

$$\int \int (u\, \eta_t - \nabla u^m \cdot \nabla \eta)\, dxdt + \int u_0(x)\, \eta(x, 0)\, dx = 0.$$

Very weak

$$\int \int (u\, \eta_t + u^m \Delta \eta)\, dxdt + \int u_0(x)\, \eta(x, 0)\, dx = 0.$$

More on concepts of solution

Solutions are not always, not only weak:

- **Strong solution.** More regular than weak but not classical: weak derivatives are L^p functions. *Big benefit: usual calculus is possible.*

- Semigroup solution / mild solution. The typical product of functional discretization schemes: $u = \{u_n\}_n$, $u_n = u(\cdot, t_n)$,

$$u_t = \Delta \Phi(u), \qquad \frac{u_n - u_{n-1}}{h} - \Delta \Phi(u_n) = 0$$

Now put $f := u_{n-1}$, $u := u_n$, and $v = \Phi(u)$, $u = \beta(v)$:

$$-h\Delta \Phi(u) + u = f, \qquad \boxed{-h\Delta v + \beta(v) = f.}$$

"Nonlinear elliptic equations"; Crandall-Liggett Theorems   Ambrosio, Savaré, Nochetto

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More on concepts of solution II

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Regularity results

- The universal estimate holds (Aronson-Bénilan, 79):

$$\Delta v \geq -C/t.$$

  $v \sim u^{m-1}$ is the pressure.

- (Caffarelli-Friedman, 1982) C^α regularity: there is an $\alpha \in (0, 1)$ such that a bounded solution defined in a cube is C^α continuous.

- If there is an interface Γ, it is also C^α continuous in space time.

- How far can you go?
  Free boundaries are stationary (metastable) if initial profile is quadratic near $\partial\Omega$: $u_0(x) = O(d^2)$. This is called waiting time. Characterized by JLV in 1983. *Visually interesting in thin films spreading on a table.*

  Existence of corner points possible when metastable, $\Rightarrow$ *no C^1* Aronson-Caffarelli-V. Regularity stops here in $n = 1$

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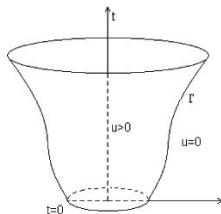
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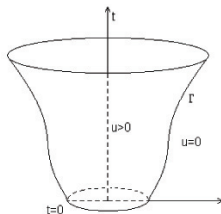
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A regular free boundary in n-D

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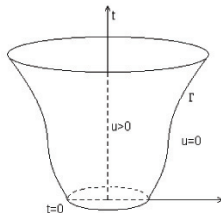
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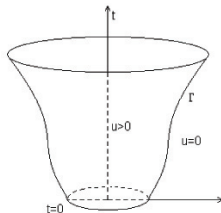
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Parabolic to Elliptic

- Semigroup solution / mild solution. The typical product of functional discretization schemes: $u = \{u_n\}_n$, $u_n = u(\cdot, t_n)$,

$$u_t = \Delta\Phi(u), \qquad \frac{u_n - u_{n-1}}{h} - \Delta\Phi(u_n) = 0$$

Now put $f := u_{n-1}$, $u := u_n$, and $v = \Phi(u)$, $u = \beta(v)$:

$$-h\Delta\Phi(u) + u = f, \qquad \boxed{-h\Delta v + \beta(v) = f.}$$

"Nonlinear elliptic equations"; Crandall-Liggett Theorems <small>Ambrosio, Savarè, Nochetto</small>

- Separation of variables. Put $u(x, t) = F(x)G(t)$. Then PME gives

$$F(x)G'(t) = G^m(t)\Delta F^m(x),$$

so that $G'(t) = -G^m(t)$, i.e., $G(t) = (m-1)t^{-1/(m-1)}$ if $m > 1$ and

$$-\Delta F^m(x) = F(x), \qquad -\Delta v(x) = v^p(x), \; p = 1/m.$$

This is more interesting for $m < 1$, specially for $m = (n-2)/(n-2)$.

Asymptotic behaviour I
Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^n$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^n)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x, t; M)$ be the Barenblatt with the asymptotic mass M; u converges to B after renormalization*

$$t^\alpha |u(x, t) - B(x, t)| \to 0$$

For every $p \geq 1$ we have

$$\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p-1).$$

Note: α *and* $\beta = \alpha/n = 1/(2 + n(m-1))$ *are the zooming exponents as in* $B(x, t)$.

- Starting result by FK takes $u_0 \geq 0$, compact support and $f = 0$.

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Calculations of entropy rates

- We rescale the function as $u(x, t) = r(t)^n \rho(y \, r(t), s)$
  where $r(t)$ is the Barenblatt radius at $t + 1$, and "new time" is
  $s = \log(1 + t)$. Equation becomes

  $$\rho_s = \operatorname{div}\left(\rho(\nabla \rho^{m-1} + \frac{c}{2} \nabla y^2)\right).$$

- Then define the entropy
  $$E(u)(t) = \int \left(\frac{1}{m} \rho^m + \frac{c}{2} \rho y^2\right) dy$$

  The minimum of entropy is identified as the Barenblatt profile.

- Calculate

  $$\frac{dE}{ds} = - \int \rho |\nabla \rho^{m-1} + cy|^2 \, dy = -D$$

  Moreover,

  $$\frac{dD}{ds} = -R, \quad R \sim \lambda D.$$

  We conclude exponential decay of D and E in new time s, which is potential in real time t.

References

References. 1903: Boussinesq, ∼1930: Liebenzon.Muskat, ∼1950: Zeldovich, Barenblatt, 1958: Oleinik,...

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Outline

FDE Barenblatt profiles

- We have well-known explicit formulas for Self-similar Barenblatt profiles with exponents less than one if $1 > m > (n-2)/n$:

$$\mathbf{B}(x, t; M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \frac{1}{(C + k\xi^2)^{1/(1-m)}}$$

The exponents are $\alpha = \frac{n}{2-n(1-m)}$ and $\beta = \frac{1}{2-n(1-m)} > 1/2$.
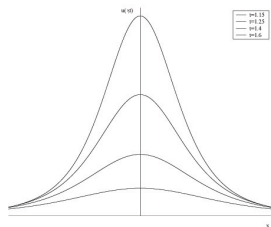
Solutions for $m > 1$ with fat tails (polynomial decay; anomalous distributions)

- Big problem: What happens for $m < (n-2)/n$?

- Main items: existence for very general data, non-existence for very fast diffusion, non-uniqueness for v.f.d., extinction, universal estimates, lack of standard Harnack.

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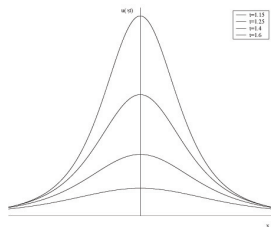The exponents are $\alpha = \frac{n}{2-n(1-m)}$ and $\beta = \frac{1}{2-n(1-m)} > 1/2$.

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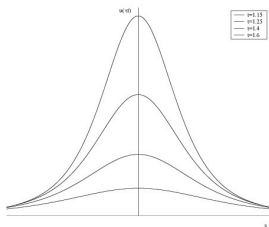The exponents are $\alpha = \frac{n}{2-n(1-m)}$ and $\beta = \frac{1}{2-n(1-m)} > 1/2$.

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Applied Motivation

Carleman model

Simple case of Diffusive limit of kinetic equations. Two types of particles in a one dimensional setting moving with speeds c and $-c$.

Densities are u and v respectively. Dynamics is

(1)
$$\begin{cases} \partial_t u + c\,\partial_x u = k(u,v)(v-u) \\ \partial_t v - c\,\partial_x v = k(u,v)(u-v), \end{cases}$$

for some interaction kernel $k(u,v) \geq 0$. Typical case $k = (u+v)^\alpha c^2$.
Write now the equations for $\rho = u + v$ and $j = c(u-v)$ and pass to the limit
$c = 1/\varepsilon \to \infty$ and you will obtain to first order in powers or $\varepsilon = 1/c$:

(2)
$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{\rho^\alpha}\frac{\partial \rho}{\partial x}\right),$$

which is the FDE with $m = 1 - \alpha$, cf. Lions Toscani, 1997. The typical value $\alpha = 1$ gives $m = 0$, a surprising equation that we will find below! The rigorous investigation of the diffusion limit of more complicated particle/kinetic models is an active area of investigation.

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Yamabe problem. Elliptic

Standard Yamabe problem . We have a Riemannian manifold (M, g_0) in space dimension $n \geq 3$, Question: of finding another metric g in the conformal class of g_0 having constant scalar curvature.

Write the conformal relation as

$$g = u^{4/(n-2)} g_0$$

locally on M for some positive smooth function u. The conformal factor is $u^{4/(n-2)}$. Denote by $R = R_g$ and R_0 the scalar curvatures of the metrics g, g_0 resp. Write Δ_0 for the Laplace-Beltrami operator of g_0, we have the formula $R = -u^{-N} L u$ on M, with $N = (n+2)/(n-2)$ and

$$Lu := \kappa \Delta_0 u - R_0 u, \quad \kappa = \frac{4(n-1)}{n-2}.$$

The Yamabe problem becomes then

$$(3) \qquad \Delta_0 u - \left(\frac{n-2}{4(n-1)} \right) R_0 u + \left(\frac{n-2}{4(n-1)} \right) R_g u^{(n+2)/(n-2)} = 0.$$

The equation should determine u (hence, g) when g_0, R_0 and R_g are known. In the standard case we take $M = \mathbb{R}^n$ and g_0 the standard metric, so that Δ_0 is the standard Laplacian, $R_0 = 0$, we take $R_g = 1$ and then we get the well-known semilinear elliptic equation with critical exponent.

Yamabe problem. Evolution

Evolution Yamabe flow is defined as an evolution equation for a family of metrics. Used as a tool to construct metrics of constant scalar curvature within a given conformal class. Wwe look for a one-parameter family $g_t(x) = g(x, t)$ of metrics solution of the evolution problem

$$(4) \qquad \partial_t g = -R\,g, \quad g(0) = g_0 \quad \text{on } M.$$

It is easy to show that this is equivalent to the equation

$$\partial_t(u^N) = Lu, \quad u(0) = 1 \quad \text{on } M.$$

after rescaling the time variable. Let now (M, g_0) be $\mathbb{R}^n$ with the standard flat metric, so that $R_0 = 0$. Put $u^N = v$, $m = 1/N = (n-2)/(n+2) \in (0, 1)$. Then

$$(5) \qquad \partial_t v = Lv^m,$$

which is a fast diffusion equation with exponent $m_y \in (0, 1)$ given by

$$m_y = \frac{n-2}{n+2}, \quad 1 - m_y = \frac{4}{n+2}.$$

If we now try separate variables solutions of the form $v(x, t) = (T - t)^\alpha f(x)$, then necessarily $\alpha = 1/(1 - m_y) = (n+2)/4$, and $F = f^m$ satisfies the semilinear elliptic equation with critical exponent that models the stationary version:

$$(6) \qquad \Delta F + \frac{n+2}{4} F^{\frac{n+2}{n-2}} = 0.$$

Logarithmic Diffusion I

- Special case: the limit case $m = 0$ of the PME/FDE in two space dimensions

$$\partial_t u = \text{div}\,(u^{-1}\nabla u) = \Delta \log(u).$$

- Application to Differential Geometry: it describes the evolution of a conformally flat metric g given by $ds^2 = u\,dr^2$ by means of its Ricci curvature:

$$\frac{\partial}{\partial t} g_{ij} = -2\,\text{Ric}_{ij} = -R\,g_{ij},$$

where Ric is the Ricci tensor and R the scalar curvature.
This flow, proposed by R. Hamilton [1] is the equivalent of the Yamabe flow in two dimensions. Remark: what we usually call the mass of the solution (thinking in diffusion terms) becomes here the total area of the surface, $A = \iint u\,dx_1\,dx_2$.

- Work on existence, nonuniqueness, extinction, and asymptotics by several authors around 1995:
  Daskalopoulos, Del Pino
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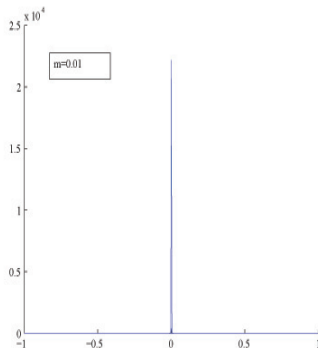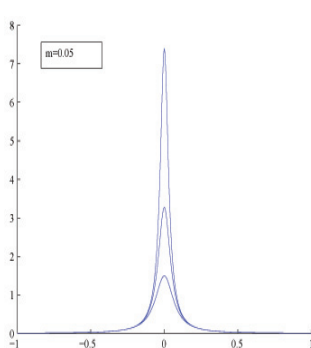 Daskalopoulos, Del Pino
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Pictures

About fast diffusion in the limit

- Evolution of the ZKB solutions; dimension $n = 2$.
  exponent near $m = 0$

Functional Analysis Program

Main facts

- Existence of an evolution semigroup.

$$u_0 \mapsto S_t(u_0) = u(t)$$

A key issue is the choice of functional space.
$X = L^1(\mathbb{R}^n)$ (Brezis, Benilan, Crandall, 1971)
$Y = L^1_{loc}(\mathbb{R}^n)$ (Herrero, Pierre 1985)
$M = $ Locally bounded measures (Pierre, 1987; Dalhberg - Kenig 1988)
$B = $ (possibly locally unbounded) Borel mesaures (Chasseigne-Vazquez ARMA 2002)

- Positivity. Nonnegative data produce positive solutions.

- "Smoothing effect": In *many cases* $L^p \to L^q$ with $q > p$. Then solutions are C^∞ smooth. In other cases, things go wrong (things=Functional Analysis)

- Theory for two signs is still poorly understood.
  Cf. Stefan Problem (Athanasopoulos, Caffarelli, Salsa)
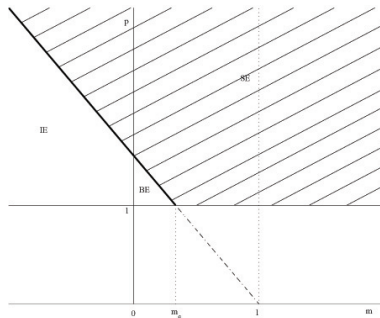
The good and bad range

Figure 1. The (m, p) diagram for the PME/FDE in dimensions $n \geq 3$.
SE: smoothing effect, BE: backwards effect, IE: instantaneous extinction
Critical line $p = n(1 - m)/2$ (in boldface)

More exponents appear. One is $m = 0$. A third exponent $m = (n - 2)/(n + 2)$ (in dimensions $n \geq 3$), which is the inverse of the famous Sobolev exponent of the elliptic theory. The relation is clear by separation of variables. Exercise

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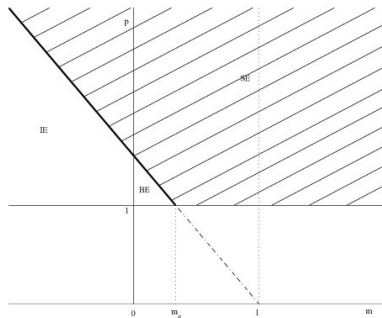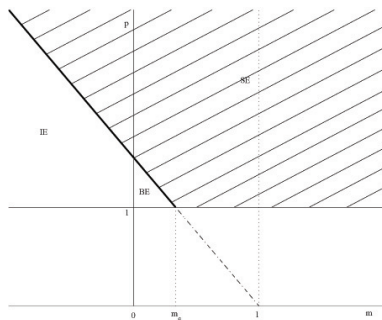
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- Backwards effect is a strange effect. Data in L^p imply $u(t) \in L^1$ for all $t > 0$ Smoothing book, 2006.

- On the green line exactly there is extinction in finite time (not at the end value $p = 1$). The correct extinction space is a Marcinkiewicz space: $X = M^{p_*}(\mathbb{R}^n)$ Smoothing book, 2006.

- For $m < 0$ below the green line there are no solutions. Vazquez 92, Daskalopoulos del Pino 97.

- For $1 > m > m_c$ we can solve the initial value problem with any nonnegative measure, even a Borel Measure because there are very good local estimates Chasseigne-Vazquez 03. For $m < m_c$ Dirac masses cannot be initial data Brezis-Friedman 83, Pierre 87.

- Harnack holds for $m > m_c$ [2], but is very difficult for $m < m_c$ work just finished [3].

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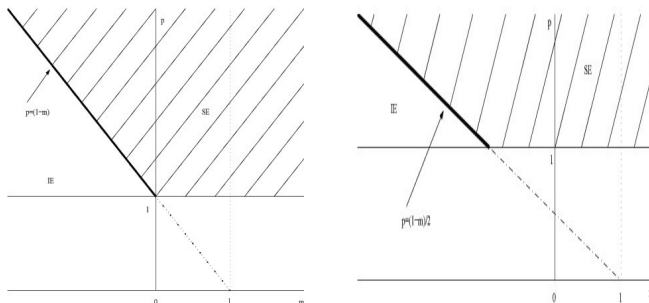
The good and bad range III

Figure 2. Left: (m, p) diagram for the PME/FDE in dimension $n = 2$
Right: (m, p) diagram for the PME/FDE in dimension $n = 1$

- There is existence and non-uniqueness if $n = 1$ and $-1 < m < 0$

The question of intrinsic regularity

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Universal Pointwise Estimates for Good Fast Diffusion

- CASE $m_c < m < 1$ This range has wonderful a priori estimates of local type. We assume that $u \geq 0$.
- If $u_0 \in L^1_{loc}(\mathbb{R}^n)$ then for all $t > 0$ we have $u(\cdot, t) \in L^\infty(\mathbb{R}^n)$, cf. Herrero-Pierre, 1985.
- There is a universal constant $C > 0$ such that if $v = u^{m-1}$

(7) $$ t|\Delta v| \leq C, \quad t|\frac{v_t}{v}| \leq C, \quad t\frac{|\nabla v|^2}{v} \leq C. $$

Estimates for the PME were original of Aronson, Crandall and Benilan. [4] [5]
Note that v satisfies the quadratic equation

$$ v_t = v\Delta v - \gamma|\nabla v|^2, \qquad \gamma = 1/(1-m). $$

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Universal Estimates continued

- Universal estimates have been found in other problems.

- Some can can be found for the heat equation. They also work for the p-Laplacian equation (fast or slow) in similar exponent ranges[6]

- Similar estimates were discovered by Yau and Li [7] for flows on manifolds and they prove that they produce continuity.

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- For $m \leq m_c$ the first estimate from below fails and the second also from below and the third from above.

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The Aronson Caffarelli Estimate for PME

- CASE $m > 1$ Aronson-Caffarelli's result [9] is a positivity estimate for the PME, $m > 1$, valid for all nonnegative weak solutions defined in the whole space.
  We take a point x_0 and a ball $B_R(x_0)$ and try to see how positive is the solution at time $t0$ if there is a "mass" $M_R(x_0) = \int_{B_R(x_0)} u_0(x)\, dx$ at $t = 0$. It says

  (8) $$\frac{M_R(x_0)}{R^d} \leq C_1\, R^{2/(m-1)}\, t^{-\frac{1}{m-1}} + C_2\, R^{-d} t^{d/2} u^{\lambda/2}(t, x_0).$$

  with $\lambda = 2 + d(m-1)$. C_1 and C_2 given positive constants depending only on m and d. Looking at the three terms we discover that there is a time t_* where the second is already less than the first one. We can calculate this intrinsic positivity time as $t_* = C(m, d)R^\lambda / M^{m-1}$.

- For $t > t_*$ the third one is positive, hence $u(x_0, t) > 0$. Hence, for all large t we have $u = O(t^{d/\lambda})$. OK!

- We go on to prove that $u \in C^\alpha$ for some $\alpha > 0$. There is no way you can get positivity for small times because of finite propagation (free boundaries).

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AC Type Estimate for Good Fast Diffusion

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- We study local solutions of the FDE in the good exponent range $m_c < m < 1$. The change in the sign of the exponent $m - 1$ implies that we get good lower estimates for $0 < t \leq t_*$ if the ideas of AC can be made to work. Moreover, we can continue these estimates for $t \geq t_*$ thanks to the fortunate circumstance that we have further differential inequalities, like $\partial_t u \geq -Cu/t$ in the case of the Cauchy problem. We get a continuation of the lower bounds with optimal decay rates in time. The final form is

$$(9) \qquad u(t,x) \geq M_R(x_0)\, H(t/t_c), \quad M_R(x_0) = R^{-d} \int_{B_R(x_0)} u_0 \, dx.$$

- The critical time is defined as before; the function $H(\eta)$ is defined as $K\eta^{1/(1-m)}$ for $\eta \leq 1$ while $H(\eta) = K\eta^{-d\vartheta}$ for $\eta \geq 1$, with $K = K(m,d)$. Note that for $0 < t < t_c$ the lower bound means

$$u(t,x_0) \geq k(m,d)(t/R^2)^{1/(1-m)}$$

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The AC Estimate for Bad Fast Diffusion

We know that for m, m_c all kinds of functional disasters may happen. In particular, extinction in finite holds for all integrable data (and some more) so that positivity for long times must be excluded. Let u be a local solution with extinction time > 0. We prove this result in M Bonforte- JL Vazquez, *Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations,* Preprint.

Theorem

Let $0 < m < 1$ and let u be the solution to the FDE under the above assumptions. Let x_0 be a point in Ω and let $d(x_0, \partial\Omega) \geq 5R$. Then the following inequality holds for all $0 < t < T$

$$(10) \qquad R^{-d} \int_{B_R(x_0)} u_0(x)\, dx \leq C_1\, R^{-2/(1-m)}\, t^{\frac{1}{1-m}} + C_2\, T^{\frac{1}{1-m}} R^{-2}\, t^{-\frac{m}{1-m}}\, u^m(t, x_0).$$

with C_1 and C_2 given positive constants depending only on d. This implies that there exists a time t_* such that for all $t \in (0, t_*]$

$$(11) \qquad\qquad u^m(t, x_0) \geq C_1'\, R^{2-d} \|u_0(x)\|_{L^1(B_R)} T^{-\frac{1}{1-m}}\, t^{\frac{m}{1-m}}.$$

where $C_1' > 0$ depends only on d; t_* depends on R and $\|u_0(x)\|_{L^1(B_R)}$ but not on T.

The local boundedness result for Fast Diffusion

- The main result of this part is the local upper bound that applies for the same type of solution and initial data, under different restrictions on p. Here is the precise formulation.

  We take $d \geq 3$. recall that $m_c = (d-2)/d$, that $p_c = d(1-m)/2$.

Theorem

Let $p \geq 1$ if $m > m_c$ or $p > p_c$ if $m \leq m_c$. Then there are positive constants C_1, C_2 such that for any $0 < R_1 < R_0$ we have

$$(12) \qquad \sup_{x \in B_{R_1}} u(t,x) \leq \frac{C_1}{t^{d \vartheta_p}} \left[\int_{B_{R_0}} |u_0(x)|^p \, dx \right]^{2 \vartheta_p} + C_2 \left[\frac{t}{R_0^2} \right]^{\frac{1}{1-m}}.$$

- We recall that $\vartheta_p = 1/(2p - d(1-m)) = 1/2(p - p_c)$. The constants C_i depend on m, d and p, R_1 and R_0 and blow up when $R_1/R_0 \to 1$; an explicit formula for C_i can be found.

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Local Boundedness II

- The proof consists in two steps: (1) The norm $\|u(\cdot, t)\|_{L^p_{loc}}$ grows with time in a controlled way in terms of its value at $t = 0$, if $p \geq 1$, $p > 1 - m$. This uses Herrero-Pierre's approach.

    (2) Solutions in $L^p_{x,t}$ locally in space/time are in fact bounded in a smaller cylinder if $p > p_c$. This uses Moser iteration.

- Local Boundedness implies existence of Large Solucions having boundary data $u = +\infty$. Such solutions form the Maximal Semigroup. A reference is E Chasseigne, JL Vazquez, Theory of extended solutions for fast-diffusion equations in optimal classes of data. Radiation from singularities. Arch. Ration. Mech. Anal. 164 (2002), no. 2, 133–187.

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Flows on manifolds

- Main result: *Let u be a positive smooth solution to PME, m > 1, on a cylinder $Q := B(\Omega, R) \times [0, T]$. Let v be the pressure and let $v_{max}^{R,T} := \max_{B(\Omega,R) \times [0,T]} v$.*
  (1) *Assume that Ricci curvature Ric ≥ 0 on $B(\Omega, R)$. Then, for any $\alpha > 1$ we have*

$$(13) \qquad \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq a\alpha^2 \left(\frac{1}{t} + \frac{v_{max}^{R,T}}{R^2} (C_1 + C_2(\alpha)) \right)$$

on $Q' := B(\Omega, R/2) \times [0, T]$. Here, $a := \frac{n(m-1)}{n(m-1)+2} = (m-1)\kappa$, and the positive constants C_1 and $C_2(\alpha)$ depend also on m and n.
(2) Assume that Ric $\geq -(n-1)K^2$ on $B(\Omega, R)$ for some $K \geq 0$. Then, for any $\alpha > 1$, we have that on Q',

$$(14) \quad \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq a\alpha^2 \left(\frac{1}{t} + C_3(\alpha)K^2 v_{max}^{R,T} \right) + a\alpha^2 \frac{v_{max}^{R,T}}{R^2} (C_2(\alpha) + C_1'(KR)) .$$
Here, a and $C_2(\alpha)$ are as before and the positive constants $C_3(\alpha)$ and $C_1'(KR)$ depend also on m and n. Acceptable values of the constants are:

$$C_1 := 40(m-1)(n+2), \qquad C_2(\alpha) := \frac{200a\alpha^2 m^2}{\alpha - 1}$$
$$C_3(\alpha) := \frac{(m-1)(n-1)}{\alpha - 1}, \qquad C_1'(KR) := 40(m-1)[3 + (n-1)(1 + KR)].$$

Flows on manifolds

Paper: "Local Aronson-Bénilan estimates and entropy formulae for porous medium and fast diffusion equations on manifolds", *Peng Lu, Lei Ni, Juan-Luis Vazquez and Cedric Villani*, JMPA, to appear online (2008)

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Thank you

♠ ♠ ♠

Gracias, Obrigado

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