Nonlinear Diffusion with Fractional Laplacian Operators

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Abstract We describe two models of flow in porous media including nonlocal (long-range) diffusion effects. The first model is based on Darcy's law and the pressure is related to the density by an inverse fractional Laplacian operator. We prove existence of solutions that propagate with finite speed. The model has the very interesting property that mass preserving self-similar solutions can be found by solving an elliptic obstacle problem with fractional Laplacian for the pair pressure-density. We use entropy methods to show that they describe the asymptotic behaviour of a wide class of solutions.
The second model is more in the spirit of fractional Laplacian flows, but nonlinear. Contrary to usual Porous Medium flows (PME in the sequel), it has infinite speed of propagation. Similarly to them, an L^1-contraction semigroup is constructed and it depends continuously on the exponent of fractional derivation and the exponent of the nonlinearity.

1 Nonlinear diffusion and fractional diffusion

Since the work by Einstein [39] and Smoluchowski [62] at the beginning of the last century (cf. also Bachelier [9]), we possess an explantation of diffusion and Brownian motion in terms of the heat equation, and in particular of the Laplace operator. This explanation has had an enormous success both in Mathematics and Physics. In the decades that followed, the Laplace operator has been often replaced by more general types of so-called elliptic operators with variable coefficients, and later by nonlinear differential operators; a huge body of theory is now available, both for the evolution equations [49] and for the stationary states, described by elliptic equations of different kinds [50, 42].

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In recent years there has been a surge of activity focused on the use of so-called fractional diffusion operators to replace the standard Laplace operator (and the other kinds of elliptic operators with variable coefficients), with the aim of further extending the theory by taking into account the presence of so-called long range interactions. The new operators do not act by pointwise differentiation but by a global integration with respect to a very singular kernel; in that way the nonlocal character of the process is represented. The paradigm of such operators is the so-called fractional Laplacian, $(-\Delta)^{\sigma/2}$, is defined as follows through Fourier transform: if g is a function in the Schwartz class and $(-\Delta)^{\sigma/2}g = h$, then

$$\widehat{h}(\xi) = |\xi|^{\sigma}\,\widehat{g}(\xi). \tag{1}$$

If $0 < \sigma < 2$ we can also use the integral representation

$$(-\Delta)^{\sigma/2}g(x) = C_{N,\sigma}\ \text{P.V.}\int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+\sigma}}\,dz, \tag{2}$$

where P.V. stands for principal value and $C_{N,\sigma} = \frac{2^{\sigma-1}\sigma\Gamma((N+\sigma)/2)}{\pi^{N/2}\Gamma(1-\sigma/2)}$ is a normalization constant, see for example [51], [63]. Note that $C_{N,\sigma} \approx \sigma$ as $\sigma \to 0$ and $C_{N,\sigma} \approx 2-\sigma$ as $\sigma \to 2$. This allows to recover in the limits respectively the identity or the standard Laplacian, The operators $(-\Delta)^{-\sigma/2}$, $0 < \sigma < 2$, are inverse the former ones and are now given by standard convolution expressions

$$(-\Delta)^{-\sigma/2}g(x) = C_{N,-\sigma}\int_{\mathbb{R}^N} \frac{g(z)}{|x-z|^{N-\sigma}}\,dz, \tag{3}$$

in terms of Riesz potentials. The basic reference for these operators are the books by Landkof [51] and Stein [63]. The interest in these operators has a long history in Probability since the fractional Laplacian operators of the form $(-\Delta)^{\sigma/2}$, $\sigma \in (0,2)$, are infinitesimal generators of stable Lévy processes [4, 14], see also [64]. Motivation from Mechanics appears in the famous Signorini problem (with $\alpha = 1/2$), cf. [59, 22]. And there are applications in Fluid Mechanics, cf. [29, 45] and the references therein. An extensive list of current applications is contained in the survey paper [37].

The systematic study of the corresponding PDE models is more recent and many of the results have arisen in the last decade. The linear or quasilinear elliptic theory has been actively studied recently in the works of Caffarelli and collaborators [6, 8, 24], Kassmann [44], Silvestre [60] and many others. The standard linear evolution equation involving fractional diffusion is

$$\frac{\partial u}{\partial t} + (-\Delta)^{\sigma/2}(u) = 0, \tag{4}$$

This is a model of so-called anomalous diffusion, a much studied topic in physics, probability and finance, see for instance [1, 47, 48, 55, 71, 72] and their references. The equation is solved with the aid of well-known Functional Analysis tools; for instance, it is proved that it generates a semigroup of ordered contractions in $L^1(\mathbb{R}^n)$. Moreover, in this setting it has the integral representation

$$u(x,t) = \int_{\mathbb{R}^N} K_\sigma(x-z,t) f(z)\, dz, \tag{5}$$

where K_σ has Fourier transform $\widehat{K}_\sigma(\xi,t) = e^{-|\xi|^\sigma t}$. This means that, for $0 < \sigma < 2$, the kernel K_σ has the form $K_\sigma(x,t) = t^{-N/\sigma} F(|x| t^{-1/\sigma})$ for some profile function F that is positive and decreasing and behaves at infinity like $F(r) \sim r^{-(N+\sigma)}$, [20]. When $\sigma = 1$, F is explicit; if $\sigma = 2$ the function K_2 is the Gaussian heat kernel.

However, an integral representation of the evolution of the form (5) is not available in the nonlinear models coming from the applications, thus motivating our work to be described below.

1.1 Nonlinear evolution models

A feature of current research in the area of PDEs is the interest in nonlinear equations and systems. The present article is devoted to presenting the progress achieved in two different models for flow in porous media including nonlocal (long-range) diffusion effects, represented by fractional operators.

• The first model is based on the usual Darcy law, with the novelty that the pressure is related to the density by an inverse fractional Laplacian operator. We prove existence of solutions that propagate with finite speed. The model has the very interesting property that mass preserving self-similar solutions can be found by solving an elliptic obstacle problem with fractional Laplacian for the pair pressure-density. We then use entropy methods to show that the asymptotic behaviour is described after renormalization by these solutions which play the role of the Barenblatt profiles of the standard porous medium model. This is a joint ongoing project with Luis Caffarelli, Univ. Texas, cf. [27, 28]. Regularity is studied in joint work with Luis Caffarelli and Fernando Soria, [26].

As a limit case of this model, we obtain a variant of the equation for the evolution of vortices in superconductivity derived heuristically by Chapman-Rubinstein-Schatzman [32] and W. E [38] as the hydrodynamic limit of Ginzburg Landau, and studied by Lin and Zhang [53], and Ambrosio and Serfaty [3]. Below I will report on progress in understanding this limit in collaboration with Sylvia Serfaty [58].

• The second model is more in the spirit of fractional Laplacian flows, but nonlinear. Contrary to standard PME flows [68] it has infinite speed of propagation. But similarly to them, an L^1-contraction semigroup is constructed and it depends continuously on the exponent of fractional derivation and the exponent of the nonlinearity. Joint work with Arturo de Pablo, Fernando Quirós and Ana Rodriguez, Madrid. Two papers contain the progress done so far, [35, 36]. On the other hand, I. Athanasopoulos and L. Caffarelli studied in [7] the continuity of the weak solutions in the framework of more general boundary heat control problems.

1.2 Traditional Porous Medium equations

Many of the concepts and techniques we will use come from the now classical theory of nonlinear diffusion. The simplest model of nonlinear diffusion equation is maybe

$$u_t = \Delta u^m = \nabla\cdot(c(u)\nabla u) \tag{6}$$

where $c(u)\ge 0$ indicates density-dependent diffusivity, in this case

$$c(u) = mu^{m-1}$$

This is valid in the typical case where $u\ge 0$. For functions u with possibly negative signs we must put $c(u)=m|u|^{m-1}$ and then the equation reads $u_t=\Delta(|u|^{m-1}u)$.

It is clear that for $m=1$ we recover the *classical Heat Equation*, while for $m>1$ the equation is degenerates at $u=0$, which is important in many applications and means slow diffusion.

A model for gases in porous media. The model arises from the consideration of a continuum, say, a fluid, represented by a *density* distribution $u(x,t)\ge 0$ that evolves with time following a *velocity field* $\mathbf{v}(\mathbf{x},\mathbf{t})$, according to the continuity equation

$$u_t + \nabla\cdot(u\,\mathbf{v}) = 0. \tag{7}$$

We assume next that $\mathbf{v}$ derives from a potential, $\mathbf{v}=-\nabla p$, as happens in fluids in porous media according to Darcy's law, and in that case p is the *pressure*. But potential velocity fields are found in many other applied instances, like Hele-Shaw cells, and other recent examples.

We still need a closure relation to relate u and p. In the case of gases in porous media, as modeled by Leibenzon and Muskat, the closure relation takes the form of a state law $p=f(u)$, where f is a nondecreasing scalar function, which is linear when the flow is isothermal, and a power, i. e., $f(u)=c\,u^{m-1}$ with $c>0$ and $m>1$, if it is adiabatic.

The linear relationship happens also in the simplified description of water infiltration in an almost horizontal soil layer according to Boussinesq. In both cases we get the standard porous medium equation, $u_t=c\Delta(u^2)$. See [68] for these and many other applications.

Fast diffusion. On the contrary, if $m<1$ the equation becomes singular at $u=0$ (i. e., $c(0)=+\infty$) which means Fast Diffusion. This equation has very different properties, like infinite speed of propagation and extinction in finite time; as m goes down to zero (or below) some quite uncommon and interesting features appear, like instantaneous extinction, [67].

General models. A more general model of nonlinear diffusion takes the divergence form

$$\partial_t H(u) = \nabla\cdot\mathscr{A}(x,u,Du) + \mathscr{B}(x,t,u,Du) \tag{8}$$

with monotonicity conditions on H and $\nabla_p\mathscr{A}(x,t,u,p)$ and structural conditions on $\mathscr{A}$ and $\mathscr{B}$. This generality includes *Stefan Problems, p-Laplacian flows* (including $p=\infty$ and total variation flow $p=1$) and many others, but this generality does not allow for a detailed theory see for instance [13].

Historical mention and references. Well-known work starting in Moscow with Zeldovich, Raizer [73] and Bartenblatt [10] around 1950 and the first systematic theory by Oleinik et al. in 1958 [56], and then Kalashnikov, Aronson, Benilan, Brezis, Caffarelli, Crandall, Di Benedetto, Friedman, Kamin, Kenig, Peletier, Vazquez, and many others.

Let us now mention some topics and authors in the new century: the group Carrillo, Toscani, Dolbeault, Del Pino, Markowich, Otto, on entropies and gradient flow and functional inequalities; Daskalopoulos, Hamilton, Lee, Vazquez on concavity. Many works on Fast Diffusion flows and logarithmic diffusion, on $p-$Laplacian flows, with recent interest on L_∞ and L_1 Laplacians. And more.

Let us finally list some convenient general references. About the PME there is a comprehensive monograph by the author, "The Porous Medium Equation. Mathematical Theory", [68]. Earlier expositions are due to Peletier [57] and Aronson [5]. About estimates and scaling we refer to the book [67] which covers also many aspects of fast diffusion. The topic of asymptotic behaviour has an enormous literature following the ideas of Lyapunov and Boltzmann. We have explained the proof of asymptotic convergence for the PME in two surveys, [65] for the Cauchy problem and [66] for the Dirichlet problem in a bounded domain. A more general survey on Nonlinear Diffusion is contained in the Proceedings of the International Congress of Mathematicians, ICM Madrid 2006 [69].

2 Nonlocal diffusion model of porous medium type

The first diffusion model with nonlocal effects we will present here uses the beginning steps of the previous derivation of the equation for gases in porous media but differs in the *closure relation* between the density and the pressure that takes the form $p=\mathscr{K}(u)$, where $\mathscr{K}$ is a linear integral operator, which we assume in practice to be the inverse of a fractional Laplacian. Hence, p es related to u through a nonlocal operator $\mathscr{K}$ which in the prototype case is the fractional potential operator, $\mathscr{K}=(-\Delta)^{-s}$ with kernel

$$k(x,y)=c|x-y|^{-(n-2s)} \tag{9}$$

(i.e., a Riesz operator). We have $(-\Delta)^s p=u$. The diffusion model with nonlocal effects is thus given by the system

$$u_t=\nabla\cdot(u\nabla p), \quad p=\mathscr{K}(u)=(-\Delta)^{-s}u. \tag{10}$$

where u is a function of the variables (x,t) to be thought of as a density or concentration, and therefore nonnegative, while p is the pressure, which is related to u via a linear operator $\mathscr{K}$.

The problem is posed for $x \in \mathbb{R}^n$, $n \geq 1$, and $t > 0$, and we give initial conditions

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{11}$$

where u_0 is a nonnegative, bounded and integrable function in $\mathbb{R}^n$.

Precedents. The interest in using *fractional Laplacians* in modeling diffusive processes has a wide literature, especially when one wants to model long-range diffusive interaction, and this interest has been activated by the recent progress in the mathematical theory as represented in [6], [8], [24], the thesis work by Silvestre [60], and many others.

A variant of the proposed model was studied by Lions and Mas-Gallic [54] They study the *regularization of the velocity field* in the standard porous medium equation by means of a convolution kernel to get a system like ours, with a difference, namely that they assume the kernel to be smooth and integrable. Since the kernel of the fractional operator $(-\Delta)^s$ is $k(x,y) = |x-y|^{-(n-2s)}$, we are far away from that case, but it may serve as a regularization step below.

Modeling dislocation dynamics as a continuum. There is a model for such dislocation phenomena proposed by A. K. Head [43] that leads to our equation in one space dimension with $s = 1/2$. It is written in an integrated version as

$$v_t = |v_x| \Lambda^\alpha(v)$$

with $\Lambda^\alpha = (-\partial^2/\partial_x^2)^\alpha$. The model applies when $\alpha = 1$, and the dislocation density is $u = v_x$. This model has been recently studied by P. Biler, G. Karch, and R. Monneau, [19]. For the integrated version they introduce viscosity solutions à la Crandall-Evans-Lions. This version has the properties of uniqueness and comparison of solutions, which makes for a simpler mathematical analysis. The study of many-dimensional models for dislocations is a widely open matter.

Limit cases. • If we take $s = 0$, then $\mathscr{K}$ = the identity operator, and we get the *standard porous medium equation*, whose behaviour is well-known, as explained above.

• In the other end of the s interval, the case $s = 1$ is novel and interesting. We take $\mathscr{K} = -\Delta$ we get

$$u_t = \nabla u \cdot \nabla p - u^2, \quad -\Delta p = u. \tag{12}$$

In one dimension this leads to

$$u_t = u_x p_x - u^2, \quad p_{xx} = -u.$$

It is then convenient to introduce the intermediate variable $v = -p_x = \int u\,dx$. We have

$$v_t = u p_x + c(t) = -v_x v + c(t),$$

For $c = 0$ this is the *Burgers equation* $v_t + vv_x = 0$ which generates shocks in finite time. Note that we may allow for u to have two signs.

Variants of this limit case in two space dimensions are used to model the evolution of vortices in superconductivity in [53] and [3], where u describes the vortex-density. The problem is sometimes posed in a bounded domain with appropriate (nonhomogeneous) boundary conditions. See Section 6 below

Summing up, the equation we study for $0 < s < 1$ may be viewed as a sort of interpolation between the extreme cases. It has better regularity properties than $s = 1$ but is different in many properties from $s = 0$.

General classes of equations. More ambitious mathematical theories are being considered. Thus, it could be assumed that $\mathscr{K}$ is an operator of integral type defined by convolution on all of $\mathbb{R}^n$, with the assumptions that is positive and symmetric. The fact the $\mathscr{K}$ is a homogeneous operator of degree $2s$, $0 < s < 1$, will be important in the proofs. An interesting variant would be $\mathscr{K} = (-\Delta + cI)^{-s}$. We are not exploring such extensions.

A formal analogue. Aggregation equations. Recent work of A. Bertozsi and collaborators has focused on aggregation models. One of them is formally the same as our porous medium equation

$$u_t = \nabla \cdot (u \nabla K \star u),$$

cf. [15, 16, 17]. However, the kernels that allow for aggregation phenomena are quite different, they are regular or in any case never very singular. A typical condition is: K radial and $\nabla K \in L^2(\mathbb{R}^n)$, and $\Delta K \in L^p(\mathbb{R}^n)$ with $p \in [2n/(n+2), 2]$, see [16]. Contrary to the theory we develop below, that model may lead to blow up in finite time. In [17] K is radially symmetric with a singularity at the origin of order $|x|^\sigma$ with $\sigma > 2 - n$, precisely outside of the fractional Laplacian range in which the nonlocal diffusion theory is set.

3 Mathematical Theory for the model of fractional porous medium equation

The work that is presented next is explained in whole detail in the following papers [27, 26, 28]. The first deals with existence and basic propagation properties, the second about boundedness and regularity in the spirit of De Giorgi [33], and the third deals with asymptotic behaviour through the associated obstacle problem and entropy dissipation methods.

3.1 Main estimates

It is convenient to write the Fractional Porous Medium Equation (10) in the more general form $\partial_t u = \nabla\cdot(u\nabla\mathscr{K}(u))$. The equation is posed in the whole space $\mathbb{R}^n$ (work on the problem posed on bounded domains is in progress). We consider $\mathscr{K} = (-\Delta)^{-s}$ for some $0<s<1$ acting on Schwartz class functions defined in the whole space. It is a positive essentially self-adjoint operator. We also let $\mathscr{H} = \mathscr{K}^{1/2} = (-\Delta)^{-s/2}$. We take a fixed $s\in(0,1)$. When necessary we indicate the dependence on s as follows: $\mathscr{K}_s$, $\mathscr{H}_s$.

We do at this stage formal calculations, assuming that $u\ge 0$ satisfies the required smoothness and integrability assumptions. This is to be justified by approximation. See whole details in [27].

- Conservation of mass

$$\frac{d}{dt}\int u(x,t)\,dx = 0. \tag{13}$$

- First energy estimate:

$$\frac{d}{dt}\int_{\mathbb{R}^n} u(x,t)\log u(x,t)\,dx = -\int_{\mathbb{R}^n}(\nabla u\cdot\nabla\mathscr{K}u)\,dx = -\int_{\mathbb{R}^n}|\nabla\mathscr{H}u|^2\,dx, \tag{14}$$

where we use the fact that $\mathscr{K}=\mathscr{H}^2$, and $\mathscr{H}$ is a positive self-adjoint operator that commutes with the gradient.

- Second energy estimate

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}|\mathscr{H}u(x,t)|^2\,dx = \int_{\mathbb{R}^n}(\mathscr{H}u)\,(\mathscr{H}u)_t\,dx = \\ &\int_{\mathbb{R}^n}(\mathscr{K}u)\,u_t\,dx = \int_{\mathbb{R}^n}(\mathscr{K}u)\,\nabla\cdot(u\nabla\mathscr{K}u)\,dx = -\int_{\mathbb{R}^n}u|\nabla\mathscr{K}u|^2\,dx.\end{aligned} \tag{15}$$

- Conservation of positivity: $u_0\ge 0$ implies that $u(t)\ge 0$ for all times.

- L^∞ estimate. We prove that the L^∞ norm does not increase in time.

Sketch of proof. At a point of maximum of u at time $t=t_0$, say $x=0$, we have

$$u_t = \nabla u\cdot\nabla P + u\,\Delta\mathscr{K}(u).$$

where $P=\mathscr{K}(u)$. The first term is zero, and for the second we have $-\Delta\mathscr{K}=L$ where $L=(-\Delta)^q$ with $q=1-s$ so that

$$\Delta\mathscr{K}u(0) = -Lu(0) = -c\int\frac{u(0)-u(y)}{|y|^{n+2(1-s)}}\,dy \le 0.$$

This concludes the proof.

- The L^p norm of the solution does not increase in time for all $1\le p\le\infty$.

- INVARIANCE AND SCALING GROUP. The equation is clearly invariant under translations in space and time. More interesting is the observation that it is also invariant under a two-parameter scaling group. Thus, if $u(x,t)$ is a weak solution so is the rescaled function

$$\widetilde{u}(x,t) := A\,u(Bx,Ct) \tag{16}$$

for arbitrary constants $A,B>0$ under the condition that $C=AB^{2-2s}$. This is based on the dimensional estimate $(\mathscr{K}\widetilde{u})(x,t)=AB^{-2s}\,(\mathscr{K}u)(Bx,Ct)$ and direct calculation on the equation.

- We did not find a clean comparison theorem, a form of the usual maximum principle is not proved, and there are counterexamples for $s>1/2$ in all space dimensions. However, comparison of solutions is established in [19] for the integrated version in dimension $n=1$ by techniques of viscosity solutions.

3.2 Finite propagation. Solutions with compact support

One of the most important features of the porous medium equation and other related degenerate parabolic equations is the property of finite propagation, whereby compactly supported initial data $u_0(x)$ gives rise to solutions $u(x,t)$ that have the same property for all positive times, i.e., the support of $u(\cdot,t)$ is contained in a ball $B_{R(t)}(0)$ for all $t>0$. One possible proof in the case of the PME is by constructing explicit weak solutions exhibiting that property (i.e., having a free boundary) and then using the comparison principle, that holds for that equation. Since we do not have such a general principle here, we have to devise a comparison method with a suitable family of "true supersolutions", which are in fact some quite excessive supersolutions. The technique has to be adapted to the peculiar form of the integral kernels involved in operator $\mathscr{K}_s$.

We begin with $n=1$ for simplicity. We assume that our solution $u(x,t)\ge 0$ has bounded initial data $u_0(x)=u(x,t_0)\le M$ with compact support and is such that

$$u_0 \quad \text{is below the parabola} \quad a(x-b)^2,\ a,b>0.$$

with graphs strictly separated. We may assume that u_0 is located under the left branch of the parabola. We take as comparison function

$$U(x,t)=a(Ct-(x-b))^2,$$

which is a traveling wave moving to the right with speed C that will be taken big enough. Then we argue at the first point and time where $u(x,t)$ touches the left branch of the parabola U from below. The key point is that if C is large enough such contact cannot exist. The formal idea is to write the equation as

$$u_t = u_x p_x + u p_{xx}$$

and observe that at the contact we have $u_t \ge U_t = 2aC(Ct-x+b)$, while $u_x=U_x=-2a(Ct-x+b)$, so the first can be made much bigger than the second by increasing C. The influence of p_x and p_{xx} as well as u is controlled, and then we conclude that the equation cannot hold if C is large enough. The argument can be translated for several dimensions. Here are the detailed results proved in [27].

Theorem 1. *Let* $0 < s < 1/2$ *and assume that* u *is a bounded solution of equation* (10) *with* $0 \le u(x,t) \le L$*, and* u_0 *lies below a function of the form*

$$U_0(x) = Ae^{-a|x|},\ A,a > 0. \tag{17}$$

If A *is large then there is a constant* $C > 0$ *that depends only on* (n,s,a,L,A) *such that for any* $T > 0$ *we will have the comparison*

$$u(x,t) \le Ae^{Ct-a|x|} \quad \text{for all } x \in \mathbb{R}^n \text{ and all } 0 < t \le T. \tag{18}$$

Theorem 2. *Let now* $1/2 \le s < 1$*. Under the assumptions of the previous theorem the stated tail estimate works locally in time. The global statement must be replaced by the following: there exists an increasing function* $C(t)$ *such that*

$$u(x,t) \le Ae^{C(t)t-a|x|} \quad \text{for all } x \in \mathbb{R}^n \text{ and all } 0 < t \le T. \tag{19}$$

3.3 Instantaneous Boundedness and regularity

• **Solutions are bounded in terms of data in L^p, $1 \le p \le \infty$.** This is a typical property of the heat semigroup and a wide class of parabolic equations with variable coefficients. The classical method of De Giorgi or Moser based on iterative techniques can be adapted to fractional diffusion in linear or nonlinear cases. This was done for instance by Caffarelli and Vasseur [29] by using the Caffarelli-Silvestre extension [25]. See also [11, 12]. Or we can use energy estimates based on the properties of the quadratic and bilinear forms associated to fractional operator, as done in [19]. For the equation and generality at hand, this is done in our paper [26] by the De Giorgi method.

Theorem. *Let* u *be a weak solution the Initial Value Problem for the Fractional Porous Medium Equation* (10) *with data* $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*, as constructed before. Then, there exists a positive constant* C *such that for every* $t > 0$

$$\sup_{x\in\mathbb{R}^n} |u(x,t)| \le C t^{-\alpha} \|u_0\|_{L^1(\mathbb{R}^n)}^{\gamma} \tag{20}$$

with $\alpha = n/(n+2-2s)$, $\gamma = (2-2s)/((n+2-2s)$*. The constant* C *depends only on* n *and* s*.*

• **Continuity.** Bounded weak solutions $u \ge 0$ of problem (10)-(11) are uniformly continuous on bounded sets of $s < 1$. Indeed, they are C^α continuous with a uniform modulus.

The proof done in [26] is lengthy and uses many techniques of the local regularity theory for elliptic and parabolic PDEs developed by Caffarelli and collaborators, and in particular some of the new ideas contained in Caffarelli-Chan-Vasseur [23]. The crucial point is to get a local version of the energy inequalities that can be iterated. It involves a delicate manipulation of the bilinear forms associated to the fractional operator, which amounts to knowing well the H^s spaces and then doing nonlinear versions of the embeddings and bounds.

4 Asymptotic behaviour for standard PME flow

In order to motivate the results for fractional diffusion, it is convenient to review the main results known for plain porous medium flow.

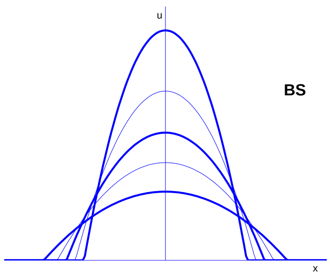
4.1. Barenblatt profiles and asymptotics. These profiles are the alternative to the Gaussian profiles that explain the asymptotic behaviour in the heat equation flow. They are called *source-type solutions*. Here source means that $u(x,t) \to M\,\delta(x)$ as $t \to 0$. There exist explicit formulas for all $m > 1$ (1950, 52) [73, 10]:

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \left(C - K\xi^2\right)_+^{1/(m-1)} \tag{21}$$

where the similarity exponents are smaller than in the Gaussian case:

$$\alpha = \frac{n}{2+n(m-1)} < \frac{n}{2}, \quad \beta = \frac{1}{2+n(m-1)} < 1/2 \tag{22}$$

and the profile looks like



The difference with the Gaussian case is striking: the solution has no tail, but a compact support limited by a clearcut *free boundary* of propagation front. The solution has height $u = Ct^{-\alpha}$ and the *free boundary* at the distance $|x| = ct^{\beta}$

We point for future reference the ideas of *Scaling law* and that of *anomalous diffusion* versus *Brownian motion* (where $\beta = 1/2$).

4.2. Nonlinear Central Limit Theorem. The standard porous medium flow has an asymptotic stabilization property that parallels the stabilization to the Gaussian profile embodied in the classical Central Limit Theorem. The choice of domain for such results is $\mathbb{R}^n$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^n)$. We write the equation we can deal with as

$$u_t = \Delta(|u|^{m-1}u) + f \tag{23}$$

We assume that $m > 1$. Let us put $f \in L^1(\mathbb{R}^n \times \mathbb{R}^+)$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$, called the total or final mass.

Theorem 3. *Let $B(x,t;M)$ be the Barenblatt solution with mass M equal to the asymptotic mass of u; u converges to B in the form*

$$\boxed{t^\alpha \|u(t) - B(t)\|_1 \to 0,} \tag{24}$$

as $t \to \infty$. Moreover, if $f = 0$ we have

$$\boxed{t^\alpha\,|u(x,t) - B(x,t)| \to 0} \tag{25}$$

uniformly in $x \in \mathbb{R}^n$ and for every $p \geq 1$ we have

$$\boxed{\|u(t) - B(t)\|_p = o\,(t^{-\alpha/p'}), \quad p' = p/(p-1).} \tag{26}$$

This is the main asymptotic theorem for the PME, proved in complete form by Vazquez in 2001, [65], expanding on the result by Friedman and Kamin, 1980, [40], where the authors took $u_0 \geq 0$, with compact support, and $f = 0$. I think it deserves the name of *Nonlinear Central Limit Theorem*. Note that the time weights are just the ones suggested by the size of the Barenblatt solutions, making the result precise.

Remarks: (1) When seeing the result from a numerical point of view, α and $\beta = \alpha/n = 1/(2 + n(m-1))$ are the *zooming exponents*, just as in $B(x,t)$.

(2) The result is still true for $m \in (0,1)$ (Fast Diffusion) if $m > (n-2)/n$, see proof in [65], but not below the critical exponent $(n-2)/n$ where the situation is quite different. It has been studied by various authors in recent times in considerable detail, and general accounts are given in [67], [21].

(3) There are a number of improvements on this theorem, that were addressed around 2000. We will mention two: eventual geometry (Lee and Vazquez (2003), [52]) and establishing convergence rates. and explain only the latter, since it motivates the work on nonlinear fractional diffusion.

4.3. Calculation of convergence rates. This is the question of speed of convergence in formulas (24)–(26). The study was initiated by Carrillo and Toscani in 2000, [30], and there many interesting contributions (by Carrillo, Del Pino, Dolbeault, Markowich, McCann, Vazquez, and many others). Using entropy functional with entropy dissipation control you can prove decay rates when $\int u_0(x)|x|^2\,dx < \infty$ (finite variance):

$$\|u(t) - B(t)\|_1 = O(t^{-\delta}), \tag{27}$$

We would like to have $\delta = 1$. This problem is still open for $m > 2$.

The entropy method. We rescale the function as $u(x,t) = r(t)^n \rho(y,s)$ with $x = y\,r(t)$ where $r(t) = c_M(t+1)^\beta$ is the Barenblatt radius at time $t+1$, and the "new time" is $s = \log(1+t)$. The PME becomes

$$\rho_s = \frac{1}{m}\Delta(\rho^m) + c\nabla(y\rho) = \text{div}\,\left(\rho\left\{\nabla\rho^{m-1} + \frac{c}{2}\nabla y^2\right\}\right). \tag{28}$$

Then we define the **entropy** as

$$E(\rho)(s) := \int (\frac{1}{m}\rho^m + \frac{c}{2}\rho y^2)\,dy \tag{29}$$

A key point is that the minimum of this entropy is identified as the entropy of the Barenblatt profile. Next, we calculate

$$\frac{dE}{ds} = -\int \rho|\nabla\rho^{m-1} + cy|^2\,dy = -D(\rho(\cdot,s))\,.$$

It is illuminating in this respect to notice that, when written in the variable ρ as a function of y and s, the self-similar Barenblatt solutions become stationary solutions $\overline{\rho}_M(y)$ of Equation (28) (with inverted parabolic shape), and then it is easy to see that the dissipation $D(\overline{\rho}) = 0$, as befits a limit of an orbit according to the theory of Dynamical Systems. Moreover, it is shown that the minimum of this entropy along an orbit is the entropy of the stationary Barenblatt profile $\overline{\rho}_M$ with the same mass M. Moreover, by another round of (not so easy) time differentiation and manipulation we get along a rescaled orbit $\rho(\cdot,s)$ the expression

$$\frac{dD(\rho)}{ds} = -R(\rho) < 0\,,$$

and moreover we can prove that $R(\rho) \sim \lambda D(\rho)$. (so-called Bakry-Emery calculation, cf. [30]). We conclude exponential decay of D, and then of $E - E_{min}$, in terms of the *new time* s, which in turn means power decay in the *real time* t.

4.4. Rates through entropies for fast diffusion. A large effort has been invested in making this machinery work for fast diffusion, $-\infty < m < 1$. The nice properties entropies have from the point of view of transport theory (cf. [70]) are lost soon, more precisely, when $m = (n-1)/n$. Indeed, the entropy of typical solutions is no more finite when the second moment is infinite, i.e. for $m = (n-1)/(n+1)$. The attractor of the evolution, i.e., the finite-mass Barenblatt solutions are lost for $m = (n-2)/n$.

The analysis for $m < (n-2)/n$ took a time to develop. A main feature is that solutions that decay reasonably at infinity will vanish completely in finite time, [67]. There is work by many authors: Blanchet, Bonforte, Carrillo, Dolbeault, Del Pino, Denzler, Grillo, McCann, Vazquez... A rather definitive account is contained in a note just appeared in Proc. Nat. Acad. Sciences USA, [21]. See previously [34]. A quite rewarding mathematical feature of those analysis is the fact that functional inequalities play a crucial role in the asymptotic analysis, they are so to say ”equivalent” to the form of asymptotic stabilization.

5 Asymptotic behaviour for the FPME

We now begin the study of the large time behaviour of the proposed model of nonlocal diffusion (i. e., the FPME) following paper [28]. The first step is constructing the self-similar solutions that will serve as attractors.

5.1 Rescaling for the FPME

Inspired by the asymptotics of the standard porous medium equation, we define the rescaled (also called renormalized) flow through the transformation

$$u(x,t) = (t+1)^{-\alpha} v(x/(t+1)^{\beta}, \tau) \tag{30}$$

with new time $\tau = \log(1+t)$. We also put $y = x/(t+1)^{\beta}$ as rescaled space variable. In order to cancel the factors including t explicitly, we get the condition on the exponents

$$\alpha + (2-2s)\beta = 1\,. \tag{31}$$

Here we use the homogeneity of $\mathscr{K}$ in the form $(\mathscr{K}u)(x,t) = t^{-\alpha+2s\beta}(\mathscr{K}v)(y,\tau)$. From physical considerations we also impose the law that states conservation of (finite) mass, which amounts to the condition $\alpha = n\beta$, and In this way we arrive at the precise value for the exponents:

$$\beta = 1/(n+2-2s), \quad \alpha = n/(n+2-2s). \tag{32}$$

Renormalized flow. We also arrive at the *nonlinear, nonlocal Fokker-Planck equation*

$$\boxed{v_\tau = \nabla_y \cdot (v\,(\nabla_y \mathscr{K}(v) + \beta y))} \tag{33}$$

The transformation formula implies a transformation for the pressure of the form

$$p(u)(x,t) = (t+1)^{-\sigma} p(v)(x/(t+1)^{\beta}, \tau), \text{ with } \sigma = \alpha - 2s\beta = 1 - 2\beta < 1\,.$$

This last formula does not play a big role below. In all the above calculations the factor $(t+1)$ can be replaced by $t+t_0$ for any $t_0 > 0$, or even by plain t.

Stationary renormalized solutions. It is important to concentrate on the stationary states of the new equation, i. e., on the solutions $V(y)$ of

$$\nabla_y \cdot (V\,\nabla_y(P + a|y|^2)) = 0, \quad \text{with} \quad P = \mathscr{K}(V). \tag{34}$$

where $a = \beta/2$, and β is defined just above. Since we are looking for asymptotic profiles of the standard solutions of the FPME we also want $V \ge 0$ and integrable. The simplest possibility is integrating once to get

$$V\,\nabla_y(P + a|y|^2)) = 0, \quad P = \mathscr{K}(V), \quad V \ge 0. \tag{35}$$

The first equation gives an alternative choice that reminds us of the complementary formulation of the obstacle problems.

5.2 *Obstacle problem. Barenblatt solutions of new type*

Indeed, if we solve the obstacle problem with fractional Laplacian we will obtain a unique solution $P(y)$ of the problem:

$$\begin{array}{l} P \ge \Phi, \quad V = (-\Delta)^s P \ge 0; \\ \text{either } \ P = \Phi \text{ or } \ V = 0. \end{array} \tag{36}$$

with $0 < s < 1$. In order for solutions of (36) to be also solutions of (35) we have to choose as obstacle

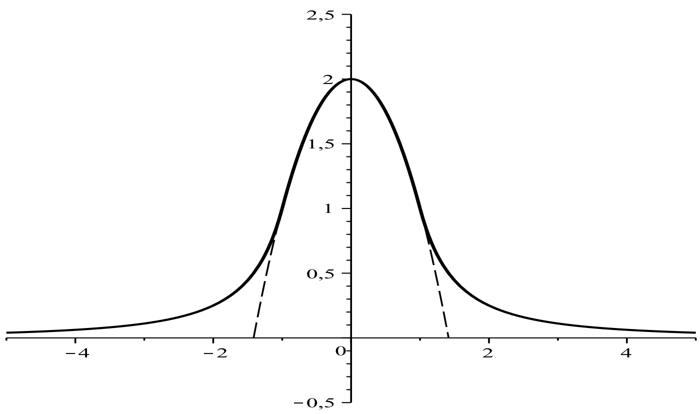
$$\Phi = C - a\,|y|^2, \tag{37}$$

where C is any positive constant and $a = \beta/2$. Note that $-\Delta\Phi = 2na = \alpha$. For uniqueness we also need the condition $P \to 0$ as $|y| \to \infty$. Fortunately, the corresponding theory had been developed by Caffarelli and collaborators, cf. [8], [24], and Silvestre's thesis [61]. The solution is unique and belongs to the space H^{-s} with pressure in H^s. Moreover, it is shown that the solutions have $P \in C^{1,s}$ and $V \in C^{1-s}$.

Note that for $C \le 0$ the solution is trivial, $P = 0$, $V = 0$, hence we choose $C > 0$. We also note the pressure is defined but for a constant, so that we could maybe take as pressure $\widehat{P} = P - C$ instead of P so that $\widehat{P} = 0$; but this does not simplify things since $P \to 0$ implies that $\widehat{P} \to -C$ as $|y| \to \infty$. Keeping thus the original proposal, we get a one parameter family of stationary profiles that we denote $V_C(y)$. These solutions of the obstacle problem produce correct weak solutions of the fractional PME equation with initial data a multiple of the Dirac delta for the density, in the form

$$U_C(x,t) = t^{-\alpha} V_C(|x| t^{-\beta}). \tag{38}$$

It is what we can call the source-type or Barenblatt solution for this problem, which is a profile $V \ge 0$. It is positive in the *contact set* of the obstacle problem, which has the form $\mathscr{C} = \{|y| \le R(C)\}$, and is zero outside, hence it has compact support. It is clear that R is smaller than the intersection of the parabola Φ with the axis $R_1 = (C/a)^{1/2}$. On the other hand, the rescaled pressure $P(|y|)$ is always positive and decays to zero as $|y| \to \infty$ according to fractional potential theory, cf. Stein [63]. The rate of decay of P as $|y| \to \infty$ turns out to be $P = O(|y|^{2s-n})$.



The solution of the obstacle problem with parabolic obstacle

Calculation of density profiles. Biler, Karch and Monneau [19] studied the existence and stability of self-similar solutions in one space dimension. Recently, Biler, Imbert and Karch [18] obtain the explicit formula for a multi-dimensional self-similar solution in the form

$$U(x,t) = c_1 t^{-\alpha}(1 - x^2 t^{-2\alpha/n})_+^{1-s} \tag{39}$$

with $\alpha = n/(n+2-2s)$ as before. The derivation uses an important identity for fractional Laplacians which is found in Getoor [41]: $(-\Delta)^{\sigma/2}(1-y^2)_+^{\sigma/2} = K_{\sigma,n}$ if $\sigma \in (0,2]$. Here we must take $\sigma = 2(1-s)$. According to our previous calculations $\Delta P = -\alpha$ on the coincidence set, hence $c_1 = \alpha/K_{\sigma,n}$. Let us work a bit more: using the scaling (16) with $A = C$ and $B = 1$ we arrive at the following one-parameter family of self-similar solutions

$$U(x,t;C_1) = t^{-\alpha}(C_1 - k_1 x^2 t^{-2\alpha/n})_+^{1-s} \tag{40}$$

where $k_1 = c_1^{1/(1-s)}$ and $C_1 > 0$ is a free parameter that can be fixed in terms of the mass of the solution $M = \int U(x,t;C_1)\,dx$. This is the family of densities that corresponds to the pressures obtained above as solutions of the obstacle problem. All is quite similar to the formulas for the standard PME, [68]; note however that in the fractional case the pressure is not compactly supported but has a power tail at infinity, which points to the long-range effects.

5.3 Estimates for the rescaled problem. Entropy dissipation

The next step is to prove that these profiles are attractors for the rescaled flow. We review the estimates of Subsection 3.1 above in order to adapt them to the rescaled equation (33).

There is no problem reproving mass conservation or positivity.
The first energy estimate becomes (recall that $\mathscr{H} = \mathscr{K}^{1/2}$)

$$\frac{d}{d\tau}\int v(y,\tau)\log v(y,\tau)\,dy = -\int |\nabla \mathscr{H} v|^2\,dy - \beta\int \nabla v\cdot y = -\int |\nabla \mathscr{H} v|^2\,dy + \alpha\int v.$$

We are going to base the proof of asymptotic behaviour on the second energy estimate after an essential change. We define the *entropy* of the rescaled flow as

$$\mathscr{E}(v(\tau)) := \frac{1}{2}\int_{\mathbb{R}^n} (v\mathscr{K}(v) + \beta y^2 v)\,dy \tag{41}$$

The entropy contains two terms. The first is

$$E_1(v(\tau)) := \int_{\mathbb{R}^n} v\mathscr{K}(v)\,dy = \int_{\mathbb{R}^n} |\mathscr{H} v|^2\,dy,$$

hence positive. The second is the moment $E_2(v(\tau)) = M_2(v(\tau)) := \int y^2 v\,dy$, also positive. By differentiation we get

$$\frac{d}{d\tau}\mathscr{E}(v) = -\mathscr{I}(v), \qquad \mathscr{I}(v) := \int \left|\nabla(\mathscr{K} v + \frac{\beta}{2}y^2)\right|^2 v\,dy. \tag{42}$$

This means that whenever the initial entropy is finite, then $\mathscr{E}(v(\tau))$ is uniformly bounded for all $\tau > 0$, $\mathscr{I}(v)$ is integrable in $(0,\infty)$ and

$$\mathscr{E}(v(\tau)) + \int_0^\tau \int \left|\nabla(\mathscr{K} v + \frac{\beta}{2}y^2)\right|^2 v\,dy\,dt \le \mathscr{E}(v_0).$$

5.4 Convergence

The standard idea is to let $\tau \to \infty$ in the renormalized flow $v(\tau) = v(\cdot,\tau)$. The estimates we have just derived will be used here in the form of uniform bounds for the rescaled orbits in different norms and this will allow us to pass to the limit. Actually, since the entropy goes down there is a limit

$$E_* = \lim_{\tau\to\infty} \mathscr{E}(v(\tau)) \ge 0.$$

Notice that the family $v(\tau)$ is bounded in L^1_y uniformly in τ, and also vy^2 is bounded in L^1_y unif. in τ, and moreover $|\nabla \mathscr{H}(v(\tau))| \in L^2_y$ unif. in τ, we have that $v(\tau)$ is a compact family in $L^1(\mathbb{R}^n)$) (since there is local compactness by Nash-Sobolev embeddings and uniform mass control at infinity). It follows that there is a subsequence $\tau_j \to \infty$ that converges in L^1_y and almost everywhere to a limit $v_* \ge 0$. The mass of v_* is the same mass of u since the tail is uniformly small (tight convergence). One consequence is that the lim inf of the component $E_2(v(\tau_j))$ is equal or larger that $M_2(v_*)$ (by Fatou).

We also have $\mathscr{H}(v) \in L^2_y$ uniformly in t. The boundedness of $\nabla\mathscr{H}(v)$ in L^2_y implies the compactness of $\mathscr{H}(v)$ in space, so that it converges along a subsequence to $\mathscr{H}(v_*)$. This allows to pass to the limit in $E_1(v(\tau_j))$ and obtain a correct limit. After some more arguments detailed in [28] we get the consequence that for every $h > 0$ fixed

$$\int_{\tau_j}^{\tau_j+h}\int \left|\nabla(\mathscr{K} v+\frac{\beta}{2}y^2)\right|^2 v\,dy\,d\tau \to 0.$$

This implies that if $w(y,\tau) = \mathscr{K} v + \frac{\beta}{2}y^2$ and $w_h(y,\tau) = w(y,\tau+h)$, then w_h converges to a constant in space wherever v is not zero, and that constant must be $\mathscr{K} v_* + \frac{\beta}{2}y^2$ along the said subsequence, hence constant also in time.

Finally, after a rather delicate analysis, it is concluded that the limit is a solution of the Barenblatt obstacle problem. The final result is stated in [28] as follows

Theorem 4. *Let $u(x,t) \ge 0$ be a weak solution of Problem* (10)–(11) *with bounded and integrable initial data such that $u_0 \ge 0$ has finite entropy in the sense defined in formula* (41). *Let $v(y,\tau)$ be the corresponding rescaled solution. As $\tau \to \infty$ we have*

$$v(\cdot,\tau) \to V_C(y) \quad in\;\; L^1(\mathbb{R}^n) \text{ and also in } L^\infty(\mathbb{R}^n). \tag{43}$$

The constant C is determined by the rule of mass equality: $\int_{\mathbb{R}^n} v(y,\tau)\,dy = \int_{\mathbb{R}^n} V_C(y)\,dy$. In terms of function u, this translates into

$$u(x,t) - U_C(x,t) \to 0 \quad in\; L^1(\mathbb{R}^n), \quad t^{\alpha}|u(x,t) - U_C(x,t)| \to 0 \quad \text{uniformly in } x, \tag{44}$$

both limits taken as $t \to \infty$.

Theorem 1.5 of [19] gives an equivalent asymptotic behaviour result in $n = 1$, though the formulation is different.

6 Limits

• **The limit** $s \to 1$**.** I recall that the work by Lin and Zhang [53] on the dynamics of the Ginzburg-Landau vortices in the hydrodynamic limit arrives at equation for the density: $u_t + \nabla(u\nabla\Delta^{-1}u) = 0$ posed in dimension 2. The authors prove existence and uniqueness of positive L^∞ solutions (they also prove existence of positive-measure valued solutions). Existence with positive initial data of finite energy is also proven (for a slightly different model) by a gradient flow approach, in bounded domains of the plane by Ambrosio et al. in [3] and in $\mathbb{R}^2$ in [2].

I report next on current work with S. Serfaty [58]. In general dimension $n \ge 2$, we obtain existence by taking the limit $s \to 1$ in the solutions for $s < 1$ constructed in [27], using of the estimates of Subsection 3.1, which are uniform in s.

Uniqueness is reflected in the following result. *There exists at most a unique solution of Equation* $u_t + \nabla(u\nabla\Delta^{-1}u) = 0$ *in* $L^\infty((0,T),L^\infty(\mathbb{R}^n))$, *i.e. if two such solutions coincide at time* 0, *they are equal for all time* $t > 0$. Note this improves the result in [53] where they require u to be in a Zygmund class.

On the other hand, the analysis of self-similarity is immediately adapted and leads to the self-similar solution

$$u(x,t) = \frac{1}{t+1} F(x/(t+1)^{1/n}), \quad F(y) = \chi_{B_C(0)} \tag{45}$$

and $C > 0$ is a free constant. We see immediately the analogy and the differences with the analysis of Subsection 5.2 for $s < 1$. Note in particular the solution is bounded, but not continuous.

A further result consists of adapting the entropy analysis to prove that general bounded solutions with compactly supported data converge to one of the self-similar profiles as $t \to \infty$ up to rescaling.

• **The limit** $s \to 0$**.** Passing that the limit $s \to 0$ in a similar way does not offer special difficulties, thus arriving at the standard PME.

7 The second fractional diffusion model

Next we turn our attention to the nonlinear heat equation with fractional diffusion

$$\frac{\partial u}{\partial t} + (-\Delta)^{\sigma/2}(u^m) = 0. \tag{46}$$

Indeed, it is a whole family of such equations with exponents $\sigma \in (0,2)$ and $m > 0$. They can be seen as fractional-diffusion versions of the PME described above, [68], [67]. The classical Heat Equation is recovered in this model in the limit $\sigma = 2$ when $m = 1$, the PME when $m > 1$, the Fast Diffusion Equation when $m < 1$.

Equations of the form (46) are a natural choice of fractional diffusion, as an alternative to the model discussed in previous chapters. We will show that the present model leads to quite different properties. Interest in studying the nonlinear model we propose is two-fold: on the one hand, experts in the mathematics of diffusion want to understand the combination of fractional operators with porous medium type propagation. On the other hand, models of this kind arise in statistical mechanics when modeling for instance heat conduction with anomalous properties and one introduces jump processes into the modeling [46], see also [48, 47]. It is mentioned in heat control by [6]. The rigorous study of such nonlinear models has been delayed by mathematical difficulties in treating at the same time the nonlinearity and fractional diffusion.

7.1 Mathematical problem and general notions

Let us present the main features and results in the theory we have developed. To be specific, the theory of existence and uniqueness as well the main properties are studied by De Pablo, Quirós, Rodríguez, and Vázquez in [35, 36] for the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + (-\Delta)^{\sigma/2}(|u|^{m-1}u) = 0, & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^n. \end{cases} \tag{47}$$

The notation $|u|^{m-1}u$ instead of u^m is used here to allow for solutions of two signs. We take initial data $f \in L^1(\mathbb{R}^n)$, which is a standard assumption in diffusion problems. As for the exponents, we consider the fractional exponent range $0 < \sigma < 2$, and take porous medium exponent $m > 0$. As we have said, in the limit $\sigma \to 2$ we want to recover the standard Porous Medium Equation (PME) $u_t - \Delta(|u|^{m-1}u) = 0$.

The papers contain a rather complete analysis of $u_t + (-\Delta)^{s/2}(|u|^{m-1}u) = 0$ for $x \in \mathbb{R}^n$, $0 < m < \infty$, $0 < s < 2$. A semigroup of weak energy solutions is constructed for every choice of m and σ, the smoothing effect C^α regularity work in most cases (if m is not near 0), and there is infinite propagation for all m and s,

The results can be viewed as a nonlinear interpolation between the extreme cases $\sigma = 2$: $u_t - \Delta(|u|^{m-1}u) = 0$, and $\sigma = 0$ which turns out to be a simple ODE: $u_t + |u|^{m-1}u = 0$. It is to be noted that the critical exponent $m_* := (n-\sigma)_+/n$ plays a role in the qualitative theory: the properties of the semigroup are more familiar when $m > m_*$. A similar exponent is well-known in the Fast Diffusion theory (putting $\sigma = 2$). Note that such exponent is not considered when $n = 1$ and $\sigma \geq 1$.

Preliminary notions

If ψ and φ belong to the Schwartz class, the definition (1) of the fractional Laplacian together with Plancherel's theorem yield

$$\int_{\mathbb{R}^n} (-\Delta)^{\sigma/2}\psi\,\varphi = \int_{\mathbb{R}^n} |\xi|^\sigma \widehat{\psi}\,\widehat{\varphi} = \int_{\mathbb{R}^n} |\xi|^{\sigma/2}\widehat{\psi}|\xi|^{\sigma/2}\,\widehat{\varphi} = \int_{\mathbb{R}^n} (-\Delta)^{\sigma/4}\psi\,(-\Delta)^{\sigma/4}\varphi.$$

Therefore, if we multiply the equation in (47) by a test function φ and integrate by parts, we obtain

$$\int_0^T \int_{\mathbb{R}^n} u\frac{\partial \varphi}{\partial t}\,dxds - \int_0^T \int_{\mathbb{R}^n} (-\Delta)^{\sigma/4}(|u|^{m-1}u)(-\Delta)^{\sigma/4}\varphi\,dxds = 0. \tag{48}$$

This identity will be the base of our definition of a weak solution. The integrals in (48) make sense if u and u^m belong to suitable spaces. The right space for u^m is the fractional Sobolev space $\dot{H}^{\sigma/2}(\mathbb{R}^n)$, defined as the completion of $C_0^\infty(\mathbb{R}^n)$ with the norm

$$\|\psi\|_{\dot{H}^{\sigma/2}} = \left(\int_{\mathbb{R}^n} |\xi|^\sigma |\widehat{\psi}|^2\,d\xi\right)^{1/2} = \|(-\Delta)^{\sigma/4}\psi\|_{L^2}.$$

Definition A function u is a *weak solution* to Problem (47) if:

- $u \in L^1(\mathbb{R}^n \times (0,T))$ for all $T > 0$, $u^m \in L^2_{\text{loc}}((0,\infty);\dot{H}^{\sigma/2}(\mathbb{R}^n))$;
- identity (48) holds for every $\varphi \in C^1_0(\mathbb{R}^n \times (0,T))$;
- $u(\cdot,t) \in L^1(\mathbb{R}^n)$ for all $t > 0$, $\lim_{t\to 0} u(\cdot,t) = f$ in $L^1(\mathbb{R}^n)$.

A drawback of this definition is that there is no formula for the fractional Laplacian of a product or of a composition of functions. Moreover, we take no advantage in using compactly supported test functions since their fractional Laplacian loses this property. To overcome these and other difficulties, we will use the fact that our solution u is the trace of the solution of a *local* problem obtained by extending u^m to a half-space whose boundary is our original space. See also the paper by Cifani and Jakobsen [31] for an alternative L^1 theory dealing with a more general class of nonlocal porous medium equations, including strong degeneration and convection.

EXTENSION METHOD. In the particular case $\sigma = 1$ studied in [35], the problem is reformulated by means of the well-known representation of the half-Laplacian in terms of the Dirichlet-Neumann operator. This allowed us to transform the nonlocal problem into a local one (i. e., involving only derivatives and not integral operators). Of course, this simplification pays a prize, namely, introducing an extra space variable. The application of such an idea is not so simple when $\sigma \neq 1$; it involves a number of difficulties that we address in [36]. We have to use the characterization of the Laplacian of order σ, $(-\Delta)^{\sigma/2}$, $0 < \sigma < 2$, recently described by Caffarelli and Silvestre [25], in terms of the so-called σ-harmonic extension, which is the solution of an elliptic problem with a degenerate or singular weight.

Let us explain this extension in some more detail. If $g = g(x)$ is a smooth bounded function defined in $\mathbb{R}^n$, its σ-harmonic extension to the upper half-space $\mathbb{R}^{n+1}_+$, $v = \mathrm{e}(g)$, is the unique smooth bounded solution $v = v(x,y)$ to

$$\begin{cases} \nabla\cdot(y^{1-\sigma}\nabla v) = 0, & x \in \mathbb{R}^n,\, y > 0, \\ v(x,0) = g(x), & x \in \mathbb{R}^n. \end{cases} \tag{49}$$

Then,

$$-\mu_\sigma \lim_{y\to 0^+} y^{1-\sigma}\frac{\partial v}{\partial y} = (-\Delta)^{\sigma/2} g(x), \tag{50}$$

where the precise constant, which does not depend on n, is $\mu_\sigma = \frac{2^{\sigma-1}\Gamma(\sigma/2)}{\Gamma(1-\sigma/2)}$, see [25]. Observe for future use that $\mu_\sigma \approx 2-\sigma$ for $\sigma \to 2^-$, $\mu_\sigma \approx 1/\sigma$ for $\sigma \to 0^+$. In (49) the operator ∇ acts in all (x,y) variables, while in (50) $(-\Delta)^{\sigma/2}$ acts only on the $x = (x_1,\cdots,x_n)$ variables. In the sequel we denote

$$L_\sigma v \equiv \nabla\cdot(y^{1-\sigma}\nabla v), \qquad \frac{\partial v}{\partial y^\sigma} \equiv \mu_\sigma \lim_{y\to 0^+} y^{1-\sigma}\frac{\partial v}{\partial y}.$$

Notation. The upper half-space, with points $\bar{x} = (x,y)$, $x \in \mathbb{R}^n$, $y > 0$, will be named Ω, and its boundary, which is identified to the original $\mathbb{R}^n$ with variable x, will be named Γ. Occasionally, Γ will be a bounded domain in $\mathbb{R}^n$ and then Ω will be the cylinder $\Gamma \times (0,\infty)$; those cases will be carefully indicated. Besides, we use the simplified notation u^m for data of any sign, instead of the actual "odd power"

$|u|^{m-1}u$, and we will also use such a notation when m is replaced by $1/m$. The convention is not applied to any other powers.

EXTENDED PROBLEM. WEAK SOLUTIONS. With the above in mind, we rewrite problem (47) for $w = u^m$ as a quasi-stationary problem with a dynamical boundary condition

$$\begin{cases} L_\sigma w = 0 & \text{for } \bar{x} \in \Omega,\, t > 0, \\ \dfrac{\partial w}{\partial y^\sigma} - \dfrac{\partial w^{1/m}}{\partial t} = 0 & \text{for } x \in \Gamma,\, t > 0, \\ w(x,0,0) = f^m(x) & \text{for } x \in \Gamma. \end{cases} \tag{51}$$

This problem has been considered by Athanasopoulos and Caffarelli [7], who prove that any bounded weak solution is Hölder continuous if $m > 1$.

To define a weak solution of this problem we multiply formally the equation in (51) by a test function φ and integrate by parts to obtain

$$\int_0^T \int_\Gamma u \frac{\partial \varphi}{\partial t}\, dx ds - \mu_\sigma \int_0^T \int_\Omega y^{1-\sigma} \langle \nabla w, \nabla \varphi \rangle\, d\bar{x} ds = 0, \tag{52}$$

where $u = (\mathrm{Tr}(w))^{1/m}$ is the trace of w on Γ to the power $1/m$. This holds on the condition that φ vanishes for $t = 0$ and $t = T$, and also for large $|x|$ and y. We then introduce the energy space $X^\sigma(\Omega)$, the completion of $C_0^\infty(\Omega)$ with the norm

$$\|v\|_{X^\sigma} = \left(\mu_\sigma \int_\Omega y^{1-\sigma} |\nabla v|^2\, d\bar{x} \right)^{1/2}. \tag{53}$$

The trace operator is well defined in this space, see below.

Definition A pair of functions (u,w) is a *weak solution* to Problem (51) if:

- $w \in L^2_{\mathrm{loc}}((0,\infty); X^\sigma(\Omega))$, $u = (\mathrm{Tr}(w))^{1/m} \in L^1(\Gamma \times (0,T))$ for all $T > 0$;
- Identity (52) holds for every $\varphi \in C_0^1(\overline{\Omega} \times (0,T))$;
- $u(\cdot,t) \in L^1(\Gamma)$ for all $t > 0$, $\lim_{t\to 0} u(\cdot,t) = f$ in $L^1(\Gamma)$.

For brevity we will refer sometimes to the solution as only u, or even only w, when no confusion arises, since it is clear how to complete the pair from one of the components, $u = (\mathrm{Tr}(w))^{1/m}$, $w = \mathrm{e}(u^m)$.

EQUIVALENCE OF WEAK FORMULATIONS. The key point of the above discussion is that the definitions of weak solution for our original nonlocal problem and for the extended local problem are equivalent. The main ingredient of the proof is that equation (50) holds in the sense of distributions for any $g \in \dot{H}^{\sigma/2}(\Gamma)$.

Proposition *A function u is a weak solution to Problem* (47) *if and only if* $(u, \mathrm{e}(u^m))$ *is a weak solution to Problem* (51).

STRONG SOLUTIONS. Weak solutions satisfy equation (47) in the sense of distributions. Hence, if the left hand side is a function, the right hand side is also a function and the equation holds almost everywhere. This fact allows to prove uniqueness and several other important properties, and hence motivates the following definition.

Definition We say that a weak solution u to Problem (47) is a strong solution if $u \in C([0,\infty): L^1(\Gamma))$ as well as $\partial_t u$ and $(-\Delta)^{\sigma/2}(|u|^{m-1}u) \in L^1_{\rm loc}(\Gamma \times (0,\infty))$.

7.2 Main results

EXISTENCE. We prove existence of a suitable concept of (weak) solution for general L^1 initial data only in the restricted range $m > m_* \equiv (n-\sigma)_+/n$, which includes as a particular case the linear fractional heat equation, case $m = 1$. If $0 < m \le m_*$ (which implies that $0 < \sigma < 1$ if $n = 1$) we need to slightly restrict the data to obtain weak solutions.

Theorem 5. *If either $f \in L^1(\mathbb{R}^n)$ and $m > m_*$, or $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ with $p > p_*(m) = (1-m)n/\sigma$ and $0 < m \le m_*$, there exists a weak solution to Problem* (47).

UNIQUENESS. We first prove uniqueness of weak solutions in the range $m \ge m_*$. If $0 < m < m_*$, we need to use the concept of strong solution, a concept that is standard in the abstract theory of evolution equations, This is no restriction in view of the next results proved in [36].

Theorem 6. *The solution given by Theorem* 5 *is a strong solution.*

We state the uniqueness result in its simplest version.

Theorem 7. *For every f and $m > 0$ there exists at most one strong solution to Problem* (47).

QUALITATIVE BEHAVIOUR. The solutions to Problem (47) have some nice properties that are summarized here.

Theorem 8. *Assume f, f_1, f_2 satisfy the hypotheses of Theorem* 5*, and let u, u_1, u_2 be the corresponding strong solutions to Problem* (47).

(i) *If $m \ge m_*$, the mass $\int_{\mathbb{R}^n} u(x,t)\,dx$ is conserved.*

(ii) *If $0 < m < m_*$, then $u(\cdot,t)$ vanishes identically in a finite time.*

(iii) *An smoothing effect holds in the form:*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le C t^{-\gamma_p} \|f\|_{L^p(\mathbb{R}^n)}^{\delta_p} \tag{54}$$

with $\gamma_p = (m-1+\sigma p/n)^{-1}$, $\delta_p = \sigma p \gamma_p/n$, and $C = C(m,p,n,\sigma)$. This holds for all $p \ge 1$ if $m > m_*$, and only for $p > p_*(m)$ if $0 < m \le m_*$.

(iv) *Any L^p-norm of the solution, $1 \le p \le \infty$, is nonincreasing in time.*

(v) *There is an L^1-order-contraction property,*

$$\int_{\mathbb{R}^n} (u_1-u_2)_+(x,t)\,dx \le \int_{\mathbb{R}^n} (u_1-u_2)_+(x,0)\,dx.$$

(vi) *If $f \ge 0$ the solution is positive for all x and all positive times if $m \ge m_*$ (resp. for all x and all $0 < t < T$ if it vanishes in finite time T when $0 < m < m_*$).*

(vii) *If either $m \ge 1$ or $f \ge 0$, then $u \in C^\alpha(\mathbb{R}^n \times (0,\infty))$ for some $0 < \alpha < 1$.*

In the linear case $m = 1$ the above properties: conservation of mass, the smoothing effect with a precise decay rate, positivity and regularity, can be derived directly from the representation formula (5) and the properties of the kernel K_σ.

CONTINUOUS DEPENDENCE. We show that the solution (i.e., the semigroup) depends continuously on the initial data and on both parameters m and σ, in particular in the nontrivial limit $\sigma \to 2$, that allows to recover the standard PME, $\partial_t u - \Delta |u|^{m-1}u = 0$, or the other end $\sigma \to 0$, for which we get the ODE : $\partial_t u + |u|^{m-1}u = 0$. Continuity will be true in general only in $L^1_{\rm loc}$, unless we stay in the region of parameters where mass is conserved.

Theorem 9. *The strong solutions depend continuously in the norm of the space* $C([0,T] : L^1_{\rm loc}(\mathbb{R}^n))$ *on the parameters* m, σ, *and the initial data* f. *If moreover* $m \ge m_*$ *and* $0 < \sigma \le 2$, *convergence also holds in* $C([0,T] : L^1(\mathbb{R}^n))$.

8 Current and future work

A number of related models, issues and perspectives on elliptic and parabolic equations involving fractional Laplacians and more general integral operators is contained in L. Caffarelli's contribution in this volume.

Let me mention some of the many questions that need investigation in the models I have presented. (1) Study the optimal regularity of the solutions, (2) Study the regularity of the free boundary, (3) Study fine asymptotic behaviour (asymptotics with rates) in the first model, or the whole asymptotic program in the second model (4) Study problems in bounded domains (current work with M. Bonforte and Y. Sire), (5) Decide conditions of uniqueness in the first model, (6) Decide conditions of comparison in the first model, (7) Write a performing numerical code, (8) Discuss the Stochastic Particle Models in the literature that involve long-range effects and anomalous diffusion parameters, (9) Study equations with more general long-range kernels in the spirit of the recent work of L. Caffarelli, L. Silvestre and collaborators, (10) Study equations and systems with convection effects, a wide and active topic involving difficult questions in Fluid Mechanics, that we will refrain from entering into since it deserves an exposition of its own.

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