The Benjamin-Ono equation in weighted Sobolev spaces

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IMPA

V ENAMA

São Carlos 2011

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Joint work with German Fonseca (UNC) and Gustavo Ponce (UCSB)

Benjamin-Ono Equation

Consider the IVP associated to the Benjamin-Ono (BO) equation

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u + u\partial_x u = 0, & t, x \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

with $\mathcal{H}$ denoting the Hilbert transform

$$\mathcal{H}f(x) = \frac{1}{\pi}\,\text{p.v.}\Big(\frac{1}{x} * f\Big)(x) = \frac{1}{\pi}\lim_{\epsilon\downarrow 0}\int_{|y|\leq\epsilon}\frac{f(x-y)}{y}dy$$
$$= -i\,(\text{sgn}(\xi)\,\widehat{f}(\xi))^{\vee}(x).$$

The BO equation was deduced by Benjamin and Ono as a model for long internal gravity waves in deep stratified fluids.

It was also shown that it is a completely integrable system as the KdV, mKdv, cubic NLS, . . .

PROBLEM: find the minimal regularity, in the Sobolev scale

$$H^s(\mathbb{R}) = \left(1 - \partial_x^2\right)^{-s/2} L^2(\mathbb{R}), \quad s \in \mathbb{R},$$

which guarantees that the IVP for the BO is locally wellposed (LWP) i.e. existence and uniqueness hold in a space embedded in

$$C([0,T] : H^s(\mathbb{R}))$$

$+$ map data-solution

$$H^s(\mathbb{R}) \;\; \longrightarrow \;\; C([0,T] : H^s(\mathbb{R}))$$

is locally continuous.

- Iorio, Abdelouhab-Bona-Felland-Saut : $s > 3/2$
- Ponce: $s \geq 3/2$
- Koch-Tzvetkov: $s > 5/4$
- Kenig-Koenig: $s > 9/8$
- Tao: $s \geq 1$
- Burq-Planchon: $s > 1/4$
- Ionescu-Kenig: $s \geq 0$
- Molinet-Pilod: uniqueness $s \geq 0$
- Molinet-Tzvetkov-Saut showed that for any $s \in \mathbb{R}$ the map data-solution from $H^s(\mathbb{R})$ to $C([0,T] : H^s(\mathbb{R}))$ is not C^3

  It cannot be solve by using only a contraction principle argument!!!

Real valued solutions of the BO satisfy infinitely many conservation laws

$$
\begin{aligned}
I_1(u) &= \int_{-\infty}^{\infty} u(x,t)dx, \qquad I_2(u) = \int_{-\infty}^{\infty} u^2(x,t)dx, \\
I_3(u) &= \int_{-\infty}^{\infty} (|D_x^{1/2}u|^2 - \frac{u^3}{3})(x,t)dx,
\end{aligned} \tag{2}
$$

where

$$D_x = \mathcal{H}\,\partial_x.$$

The k-conservation law I_k provides an a priori estimate of

$$\|D_x^{(k-2)/2}u(t)\|_2, \quad k \geq 2.$$

This allows one to deduce GWP from LWP results.

The BO has traveling wave solution $\phi_c(x+t),\ c>0$

$$\phi(x) = \frac{-4}{1+x^2}, \qquad \phi_c(x+t) = c\,\phi(c(x+ct)),$$

very mild decay at infinity.

The traveling wave is negative and travels to the left.

To get a positive traveling wave moving to the right consider the equation

$$\partial_t v - \mathcal{H}\partial_x^2 v + v\partial_x v = 0, \qquad t, x \in \mathbb{R}, \tag{3}$$

and uses that if $u(x,t)$ is a solution of the BO, then

$$v(x,t) = -u(x,-t),$$

solves the (3)

Goal: to study real valued solutions of the IVP for the BO in weighted Sobolev spaces:

$$Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx), \qquad s,\, r \in \mathbb{R}. \tag{4}$$

The operator $L = \partial_t + \mathcal{H}\partial_x^2$ commutes with $\Gamma = x - 2t\mathcal{H}\partial_x$, i.e.

$$[L;\Gamma] = L\Gamma - \Gamma L = 0.$$

so to get persistence in $Z_{s,r}$ one should have $s \geq r$.

Theorem 1 (Iorio (1986/2003)) *(i) If* $r = 1, 2$ *and* $r \leq s$*, then the IVP associated to the BO equation is globally well posed in* $Z_{s,r}$*.*

In particular, if $u_0 \in Z_{s,r}$*, then* $u \in C(\mathbb{R} : Z_{s,r})$*.*

(ii) If $r = 3$ *and* $3 \leq s$*, then the IVP associated to the BO eq. is globally well posed in* $\dot{Z}_{s,3}$*.*

In particular, if $u_0 \in \dot{Z}_{s,3}$*, then* $u \in C(\mathbb{R} : \dot{Z}_{s,3})$*, where* $\dot{Z}_{s,r} = \dot{Z}_{s,r} \cap \{ \widehat{f}(0) = 0\}$

(iii) If $u \in C(\mathbb{R} : \dot{Z}_{4,3})$ *is a solution of the BO eq. such that*

$$u(\cdot, t_j) \in Z_{4,4}, \quad j = 1, 2, 3,$$

then $u \equiv 0$*.*

Fonseca and Ponce removed the condition $r \in \mathbb{Z}^+$, no $\Gamma = x - 2t\mathcal{H}\partial_x$ operator

Theorem 2 (Fonseca-Ponce (2010))

(i) If $r < 5/2$ and $r \leq s$, then the IVP associated to the BO eq. is globally well posed in $Z_{s,r}$.

(ii) If $5/2 \leq r < 7/2$ and $r \leq s$, then the IVP associated to the BO eq. is globally well posed in $\dot{Z}_{s,r}$.

Theorem 3 (Fonseca-Ponce (2010)) *Let* $u \in C([0,T] : Z_{2,2})$ *be a solution of the IVP associated to the BO eq. If there exist* $t_1, t_2 \in [0,T]$ *such that*

$$u(\cdot, t_j) \in Z_{5/2,5/2}, \quad j = 1, 2,$$

then

$$\widehat{u}_0(0) = 0 \qquad (\, u(\cdot, t) \in \dot{Z}_{5/2,5/2}, \quad t \in [0,T]).$$

Theorem 4 (Fonseca-Ponce (2010)) *Let* $u \in C([0,T] : \dot{Z}_{3,3})$ *be a solution of the IVP associated to the BO eq. If there are*

$$t_1, t_2, t_3 \in [0,T],$$

such that

$$u(\cdot, t_j) \in Z_{7/2,7/2}, \quad j = 1,2,3,$$

then

$$u(x,t) \equiv 0.$$

So, the condition $\widehat{u}_0(0) = 0$ is necessary for the persistence of the solution in $Z_{s,5/2}$, with $s \geq 5/2$.

The maximum possible decay in L^2 of the solution is

$$|x|^{7/2^-} u(\cdot, t) \in L^\infty([0,T] : L^2(\mathbb{R})),$$

with

$$|x|^{7/2} u(\cdot, t) \notin L^\infty([0,T] : L^2(\mathbb{R})),$$

for all $T > 0$ regardless of the regularity and decay of the initial data u_0.

Ingredients of the proofs

(i) The A_p condition (Muckenhoupt)

(ii) Commutators for the Hilbert transform (Calderón)

(iii) fractional derivatives (Stein)

(i) A_p condition introduced by Muckenhoupt:

Definition 1 *A non-negative function $w \in L^1_{loc}(\mathbb{R})$ satisfies the A_p inequality with $1 < p < \infty$ if*

$$\sup_{Q \text{ interval}} \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\right)^{p-1} = c(w) < \infty,$$

where $1/p + 1/p' = 1$.

Hunt-Muckenhoupt-Wheeden Theorem:

“The Hilbert transform $\mathcal{H}$ is bounded in $L^p(w(x)dx)$, i.e.

$$(\int_{-\infty}^{\infty} |\mathcal{H}f|^p w(x)dx)^{1/p} \leq c^* (\int_{-\infty}^{\infty} |f|^p w(x)dx)^{1/p},$$

if and only if

$$w \in A_p.\text{”}$$

In particular, in $\mathbb{R}$

$$|x|^\alpha \in A_p \quad \Leftrightarrow \quad \alpha \in (-1, p-1).$$

To justify some arguments in the proof we need some further continuity properties of the Hilbert transform.

The constant c^* should depend only on $c(w)$ and on p (in fact, this is only needed for the case $p = 2$).

S. Petermichl (2007)

“For $p \in [2, \infty)$ the continuity holds with $c^* \leq c(p)c(w)$, with $c(p)$ depending only on p and $c(w)$. Moreover, for $p = 2$ this estimate is sharp."

(ii) Calderón commutator theorem:

"For any $p \in (1,\infty)$ exists $c = c(p) > 0$ such that

$$\| \, [\mathcal{H};\, a]\partial_x f \, \|_p + \| \, \partial_x[\mathcal{H};\, a] f \, \|_p \le c\|\partial_x a\|_\infty \|f\|_p."$$

Dawson-McGahagan-Ponce (2009)

For any $p \in (1,\infty)$, $l,\, m \in \mathbb{Z}^+ \cup \{0\}$, $l + m \ge 1$ there exists $c = c(p;l;m) > 0$ such that

$$\|\partial_x^l[\mathcal{H};\, a]\partial_x^m f\|_p \le c\|\partial_x^{l+m} a\|_\infty \|f\|_p.$$

(iii) Stein's characterization of the $L^p_s(\mathbb{R}^n) = (1-\Delta)^{-s/2}L^p(\mathbb{R}^n)$ spaces:

"Let $b \in (0,1)$ and $2n/(n+2b) < p < \infty$. Then $f \in L^p_b(\mathbb{R}^n)$ if and only if

$$(a) \quad f \in L^p(\mathbb{R}^n),$$

$$(b) \qquad \mathcal{D}^b f(x) = \Big(\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2b}} dy\Big)^{1/2} \in L^p(\mathbb{R}^n),$$

with

$$\|f\|_{b,p} \equiv \|(1-\Delta)^{b/2} f\|_p = \|J^b f\|_p \simeq \|f\|_p + \|D^b f\|_p \simeq \|f\|_p + \|\mathcal{D}^b f\|_p."$$

In particular for $p = 2$

$$\|\,\mathcal{D}^b(fg)\|_2 \leq c\,(\|f\mathcal{D}^b g\|_2 + \|g\mathcal{D}^b f\|_2).$$

Questions:

(Q1) Is it possible to extend the result for any pair of solutions $u_1,\ u_2$ of the BO eq. with $u_1 \neq 0,\ \ u_2 \neq 0$?

(Q2) Is the condition involving three different times $t_1,\ t_2,\ t_3$ necessary?

For (Q1) we recall that the uniqueness results of Escauriaza-Kenig-Ponce-Vega for the k-gKdV equation and for the semi-linear Schrödinger equation involves two arbitrary solutions.

Theorem 5 (Escauriaza-Kenig-Ponce-Vega-(2007))

There exists $c_0 > 0$ *such that for any pair*

$$u_1,\ u_2 \in C([0,1] : H^4(\mathbb{R}) \cap L^2(|x|^2 dx))$$

of solutions of the k-gKdV eq.

$$\partial_t u + \partial_x^3 u + u\,\partial_x u = 0, \quad k \in \mathbb{Z}^+,$$

such that if

$$u_1(\cdot,0) - u_2(\cdot,0),\ \ u_1(\cdot,1) - u_2(\cdot,1) \in L^2(e^{c_0 x_+^{3/2}} dx),$$

then $u_1 \equiv u_2$.

$x_+ = max\{x;\, 0\}$.

Q1: Is it possible to extend the result for any pair of solutions $u_1,\ u_2$ of the BO eq. with $u_1 \neq 0,\ \ u_2 \neq 0$?

NO.

Theorem 6 (Fonseca-L-Ponce (2011))

Exist $u_1,\ u_2 \in C([0,T] : \dot{Z}_{4,2})$ *solutions of the IVP for the BO eq. such that*

$$u_1 - u_2 \in L^\infty([-T,T] : Z_{4,4}).$$

with

$$u_1(x,t) \neq u_2(x,t).$$

Q2: Can the assumption involving three times be reduced to two different times $t_1 < t_2$?

This is the case of the gKdV, NLS, Camassa-Holm,.......

NO.

Exists

$$u \in C(\mathbb{R} : \dot{Z}_{5,7/2-}), \ u \neq 0$$

solution of the BO eq. for which there are

$$t_1, \ t_2 \in \mathbb{R}, \ t_1 \neq t_2$$

such that

$$u(\cdot, t_j) \in \dot{Z}_{5,4} \subset \dot{Z}_{7/2,7/2}, \qquad j = 1, 2.$$

More precisely:

Theorem 7 (Fonseca-L-Ponce (2011)) *For any* $u_0 \in \dot{Z}_{5,4}$ *such that*

$$\int_{-\infty}^{\infty} x\, u_0(x)dx \neq 0,$$

the corresponding solution $u \in C(\mathbb{R} : \dot{Z}_{5,7/2-})$ *of the BO eq. satisfies that* $\exists! t^*$ *such that*

$$u(\cdot, t^*) \in \dot{Z}_{5,4}$$

where

$$t^* = -\frac{4}{\|u_0\|_2^2} \int_{-\infty}^{\infty} x\, u_0(x)dx.$$

Idea proof of Theorem 6

We take two solutions $u_1,\ u_2$ of (1) whose data $u_{1,0},\ u_{2,0}$ satisfy

$$u_1(x,0) = u_{1,0}(x),\ \ u_2(x,0) = u_{2,0}(x) \in Z_{4,4}, \tag{5}$$

with

$$\begin{cases} \displaystyle\int_{-\infty}^{\infty} u_1(x,0)\,dx = \int_{-\infty}^{\infty} u_2(x,0)\,dx, \\ \\ \displaystyle\int_{-\infty}^{\infty} x\,u_1(x,0)\,dx = \int_{-\infty}^{\infty} x\,u_2(x,0)\,dx, \\ \|u_{1,0}\|_2 = \|u_{2,0}\|_2, \qquad u_{1,0} \neq u_{2,0}. \end{cases} \tag{6}$$

Thus, from the result of Fonseca and Ponce it follows that

$$u_1,\ u_2 \in C(\mathbb{R} : Z_{4,5/2^-}).$$

Defining

$$v(x,t) = u_1(x,t) - u_2(x,t),$$

one sees that v verifies the linear equation

$$\partial_t v + \mathcal{H}\partial_x^2 v + u_1 \partial_x v + \partial_x u_2 v = 0, \tag{7}$$

with

$$v \in C(\mathbb{R} : Z_{4,5/2^-}), \tag{8}$$

and

$$\int_{-\infty}^{\infty} v(x,0)\,dx = \int_{-\infty}^{\infty} x\,v(x,0)\,dx = 0, \qquad \forall t \in \mathbb{R}. \tag{9}$$

The identities in (9) follow by combining our hypothesis (6), the first conservation law

$$\int_{-\infty}^{\infty} u_j(x,t)\,dx = \int_{-\infty}^{\infty} u_{j,0}(x)\,dx, \qquad \forall\, t \in \mathbb{R}, \quad j = 1,2,$$

and the identity

$$\frac{d}{dt}\int_{-\infty}^{\infty} x\,u_j(x,t)\,dx = \frac{1}{2}\|u_j(t)\|_2^2 = \frac{1}{2}\|u_{j,0}\|_2^2, \qquad \forall\, t \in \mathbb{R}, \quad j = 1,2.$$

Idea proof Theorem 7

Let $W(t)u_0(x)$ be the solution of

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u = 0, \\ u(x,0) = u_0 \in \dot{Z}_{5,4}. \end{cases}$$

If $\int x\, u_0(x)dx \neq 0$, then

$$W(t)u_0(x) = c(e^{-it|\xi|\xi}\widehat{u_0}(\xi))^\vee \in L^2(|x|^{7-}) - L^2(|x|^7), \qquad \forall\, t \neq 0.$$

If $\int x\, u_0(x)dx = 0$, then

$$W(t)u_0(x) = c(e^{-it|\xi|\xi}\widehat{u_0}(\xi))^\vee \in L^2(|x|^8), \qquad \forall\, t.$$

Motivation for t^*

Look for t such that

$$\int_0^t \int_{-\infty}^{\infty} x\, u(x,t) dx dt = \int_0^t (\int_{-\infty}^{\infty} x\, u_0(x) dx + \frac{t'}{2} \|u_0\|_2^2) dt' = 0,$$

since

$$\frac{d}{dt} \int_{-\infty}^{\infty} x u(x,t) dx = \frac{1}{2} \|u(\cdot, t)\|_2^2 = \frac{1}{2} \|u_0\|_2^2.$$

Extensions

Now we consider the initial value problem (IVP) for the dispersion generalized Benjamin-Ono (DGBO) equation

$$\begin{cases} \partial_t u + D^{1+a}\partial_x u + u\partial_x u = 0, & t, x \in \mathbb{R}, \quad 0 < a < 1, \\ u(x,0) = u_0(x), & \end{cases} \tag{10}$$

where D^s denotes the homogeneous derivative of order $s \in \mathbb{R}$,

$D^s = (-\Delta)^{s/2}$ so $D^s f = c_s\big(|\xi|^s \widehat{f}\,\big)^{\vee}$, with $D^s = (\mathcal{H}\,\partial_x)^s$ if $n = 1$.

These equations model vorticity waves in the coastal zone. When $a = 1$ the equation in (10) becomes the Korteweg-de Vries (KdV) equation

$$\partial_t u - \partial_x^3 u + u\partial_x u = 0, \qquad t, x \in \mathbb{R}, \tag{11}$$

and when $a = 0$ the equation in (10) agrees with the Benjamin-Ono (BO) equation

$$\partial_t u + \mathcal{H}\partial_x^2 u + u\partial_x u = 0, \qquad t, x \in \mathbb{R}. \tag{12}$$

Remarks.

- We notice that for $a \in (0,1)$ the dispersive effect is stronger than the one for the BO equation but still too weak compared to that of the KdV equation.

Indeed it was shown by Molinet, Saut and Tzetkov (2001) that for the IVP associated to the DGBO equation (10) the flow map data-solution from $H^s(\mathbb{R})$ to $C([0,T] : H^s(\mathbb{R}))$ fails to be locally C^2 at the origin for any $T > 0$ and any $s \in \mathbb{R}$ as in the case of the BO equation. Therefore, so far local well-posedness in classical Sobolev spaces $H^s(\mathbb{R})$ for (10) cannot be obtained by an argument based only on the contraction principle.

- Local well-posedness in classical Sobolev spaces for (10) has been studied by Kenig, Ponce and Vega (1991), Molinet and Ribaud (2004), Herr (2007), Guo (2008), and Herr, Ionescu, Kenig and H. Koch (2010) where local well-posedness was established for $s \geq 0$.

- Real solutions of the IVP (10) satisfy at least three conserved quantities:

$$I_1(u) = \int_{-\infty}^{\infty} u(x,t)dx, \qquad I_2(u) = \int_{-\infty}^{\infty} u^2(x,t)dx,$$
$$I_3(u) = \int_{-\infty}^{\infty} (|D_x^{\frac{1+a}{2}} u|^2 + \frac{u^3}{6})(x,t)dx.$$

To state our results we recall the weighted Sobolev spaces

$$Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx), \quad s, r \in \mathbb{R},$$

and

$$\dot{Z}_{s,r} = \{f \in H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx) \; : \; \widehat{f}(0) = 0\}, \qquad s,\, r \in \mathbb{R}.$$

The persistence property of the solution $u = u(x,t)$ of the IVP (10) in the weighted Sobolev spaces $Z_{s,\,r}$ can only hold if $s \geq (1+a)r$. This can be seen from the fact that the linear part of the equation (10)

$$L = \partial_t + D^{1+a}\partial_x \quad \text{commutes with} \qquad \Gamma = x - (a+2)tD^{1+a}.$$

Hence, it is natural to consider well-posedness in the weighted Sobolev spaces $Z_{s,\,r},\; s \geq (1+a)r$.

Theorem 8 (a) *Let* $a \in (0,1)$. *If* $u_0 \in Z_{s,r}$, *then the solution* u *of the IVP* (10) *satisfies* $u \in C([-T,T] : Z_{s,r})$ *if either*

(i) $s \geq (1+a)$ *and* $r \in (0,1]$, *or*

(ii) $s \geq 2(1+a)$ *and* $r \in (1,2]$, *or*

(iii) $s \geq [(r+1)^-](1+a)$ *and* $2 < r < 5/2+a$, *with* $[\cdot]$ *denoting the integer part function.*

(b) *If* $u_0 \in \dot{Z}_{s,r}$, *then the solution* u *of the IVP* (10) *satisfies*

$$u \in C([-T,T] : \dot{Z}_{s,r}),$$

whenever

(iv) $s \geq [(r+1)^-](1+a)$ *and* $5/2+a < r < 7/2+a$.

Theorem 9

Let $a \in (0,1)$. If $u_0 \in Z_{s,r}$, $r \geq 2$ and $s \geq r(1+a) + (1-a)/2$. Then there exists a unique solution u of the IVP (10) such that

$$u \in C([-T,T] \,:\, Z_{s,r})$$

with

$$D_x^l u \in L^\infty(\mathbb{R} : L^2(-T,T)), \qquad l \in [(1+a)/2, s + (1+a)/2].$$

Theorem 10 *Let* $u \in C([-T,T] : Z_{s,(5/2+a)^-})$ *with*

$$T > 0 \qquad \textit{and} \qquad s \geq (1+a)(5/2+a) + (1-a)/2$$

be a solution of the IVP (10). If there exist two times $t_1, t_2 \in [-T,T]$, $t_1 \neq t_2$, *such that*

$$u(\cdot, t_j) \in Z_{s,5/2+a}, \quad j = 1,2.$$

Then

$$\widehat{u}(0,t) = \int u(x,t)dx = \int u_0(x)\,dx = \widehat{u}_0(0) = 0 \ \textit{ for all } \ t \in [-T,T].$$

Remark.

Theorem 10 shows that persistence in $Z_{s,r}$ with $r = (5/2 + a)^-$ is the best possible for general initial data. In fact, it shows that for data $u_0 \in Z_{s,r}$, $s \geq (1+a)r + (1-a)/2$, $r \geq 5/2 + a$ with $\widehat{u}_0(0) \neq 0$ the corresponding solution $u = u(x,t)$ verifies that

$$|x|^{(5/2+a)^-} u \in L^\infty([0,T] : L^2(\mathbb{R})), \qquad T > 0,$$

but there does not exist a non-trivial solution u corresponding to data u_0 with $\widehat{u}_0(0) \neq 0$ such that

$$|x|^{5/2+a} u \in L^\infty([0,T'] : L^2(\mathbb{R})), \qquad \text{for some} \quad T' > 0.$$

Theorem 11 *Let* $u \in C([-T,T] : Z_{s,(7/2+a)^-})$ *with*

$$T > 0 \quad \textit{and} \quad s \geq (1+a)(7/2+a) + \frac{1-a}{2},$$

be a solution of the IVP (10). If there exist three different times $t_1, t_2, t_3 \in [-T,T]$ *such that*

$$u(\cdot, t_j) \in \dot{Z}_{s,7/2+a}, \quad j = 1,2,3. \tag{13}$$

Then

$$u \equiv 0.$$

Remark

Theorem 11 shows that the decay $r = (7/2 + a)^-$ is the largest possible. More precisely, Theorem 8 part (b) tells us that there are non trivial solutions $u = u(x, t)$ verifying

$$|x|^{(7/2+a)^-} u \in L^\infty([0, T] : L^2(\mathbb{R})), \qquad T > 0,$$

and Theorem 11 guarantees that there does not exist a non-trivial solution such that

$$|x|^{7/2+a} u \in L^\infty([0, T'] : L^2(\mathbb{R})), \qquad \text{for some} \quad T' > 0.$$

Theorem 12 *Let* $u \in C([-T,T] : Z_{s,(7/2+a)^-})$ *with*

$$T > 0 \quad \textit{and} \quad s \geq (1+a)(7/2+a) + (1-a)/2,$$

be a solution of the IVP (10). *If there exist* $t_1, t_2 \in [-T,T]$, $t_1 \neq t_2$ *such that*

$$u(\cdot, t_j) \in \dot{Z}_{s,7/2+a}, \quad j = 1, 2,$$

and

$$\int x u(x,t_1)dx = 0 \quad \textit{or} \quad \int x u(x,t_2)dx = 0, \tag{14}$$

then

$$u \equiv 0.$$

Theorem 12 tells us that the conditions of Theorem 11 can be reduced to two times provided the first momentum of the solution u vanishes at one of them.

Theorem 13 *Let* $u \in C([-T,T] : Z_{s,(7/2+a)^-})$ *with*

$$T > 0 \qquad \textit{and} \qquad s \geq (1+a)(7/2+[1+2a]/2)+(1-a)/2$$

be a non-trivial solution of the IVP (10) *such that*

$$u_0 \in \dot{Z}_{s,\frac{7}{2}+\tilde{a}},\;\; \tilde{a} = [1+2a]/2, \qquad \textit{and} \qquad \int_{-\infty}^{\infty} x\, u_0(x)\, dx \neq 0. \quad (15)$$

Then there exists $t^* \neq 0$ *with*

$$t^* = -\frac{4}{\|u_0\|_2^2} \int_{-\infty}^{\infty} x\, u_0(x)\, dx, \quad (16)$$

such that $u(t^*) \in \dot{Z}_{s,\frac{7}{2}+\tilde{a}}$.

Remarks.

Notice that $\tilde{a} > a$, so Theorem 13 shows that the condition of Theorem 11 at two times is in general not sufficient to guarantee that $u \equiv 0$. So, in this regard Theorem 12 is optimal.

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