Entire functions of exponential type and the Riemann zeta-function

Emanuel Carneiro

IMPA

VI ENAMA, Aracaju, Brazil

November, 2012

Contents

Preface

These are lectures notes for a mini-course at the VI ENAMA - Encontro Nacional de Análise Matemática e Aplicações - held in Aracaju, Brazil, in November 2012. I would like to thank the organizing and scientific committees of the VI ENAMA, and the colleagues from the Federal University of Sergipe for their hospitality. The purpose of these notes is to present briefly some recent advances in problems in approximation theory and its applications to the theory of the Riemann zeta-function.

The first chapter covers foundational material in harmonic analysis related to the theory of functions of exponential type. We start with Poisson summation formula and quickly move to two celebrated results: the Paley-Wiener theorem and the Plancherel-Polya theorem. These are then applied to obtain useful interpolation formulas and Bernstein's inequality.

The second chapter introduces the reader to a classical problem in the interface of approximation theory and harmonic analysis, the so called Beurling-Selberg extremal problem. In this setting the goal is to approximate (minimizing the $L^1(\mathbb{R})$-norm) a given real function by an entire function of prescribed exponential type. We have no ambition to cover in its full the vast material related to this topic. Our goal is to present some of the recent advances in this theory, in the form of a general method (which we call the Gaussian subordination method) to generate the solution of this problem for a wide class of even, odd and truncated real functions.

The third chapter describes three applications of these extremal functions to the theory of the Riemann zeta-function. Specifically, our goal is to provide the best (up to date) bounds, under the assumption of the Riemann hypothesis, for three objects related to $\zeta(s)$, namely, the size of $\zeta(s)$ in the critical line, the argument function $S(t)$, and its antiderivative $S_1(t)$.

I would like to thank in particular the outstanding group of mathematicians with whom I have collaborated in these research projects: Vorrapan Chandee (Univ. of Montreal), Friedrich Littmann (North Dakota State Univ.), Micah Milinovich (Univ. of Mississipi) and Jeffrey D. Vaaler (The Univ. of Texas at Austin). Chapters 2 and 3 basically summarize our recent joint projects.

Rio de Janeiro, October 2012,

Emanuel Carneiro.

Chapter 1

Harmonic analysis tools

1.1 The Poisson summation formula

If $f \in L^1(\mathbb{R})$ we define its Fourier transform by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)\,\mathrm{d}x.$$

In an analogous manner, if we identify periodic functions (of period 1) with functions defined on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and if $g \in L^1(\mathbb{T})$, we define its Fourier coefficients by

$$\widehat{g}(k) = \int_{\mathbb{T}} e^{-2\pi i x k} g(x)\,\mathrm{d}x,$$

where $k \in \mathbb{Z}$. Given a measurable function f on $\mathbb{R}$, we consider its periodization

$$P_f(x) = \sum_{m\in\mathbb{Z}} f(x+m). \tag{1.1}$$

Naturally, if we do not impose any decay on f, one can not infer any sort of convergence for the sum (1.1). However, if we assume $f \in L^1(\mathbb{R})$, we will have P_f with period 1 and

$$\begin{aligned}\|P_f\|_{L^1(\mathbb{T})} &= \int_0^1 \left|\sum_{m\in\mathbb{Z}} f(x+m)\right| \mathrm{d}x \le \int_0^1 \sum_{m\in\mathbb{Z}} |f(x+m)|\ \mathrm{d}x \\ &= \sum_{m\in\mathbb{Z}} \int_0^1 |f(x+m)|\ \mathrm{d}x = \sum_{m\in\mathbb{Z}} \int_m^{m+1} |f(y)|\ \mathrm{d}y = \|f\|_{L^1(\mathbb{R})}.\end{aligned}$$

Therefore $P_f \in L^1(\mathbb{T})$ and we can calculate its Fourier coefficients. An application of Fubini's theorem gives us

$$\begin{aligned}\widehat{P_f}(k) &= \int_0^1 e^{-2\pi i x k} \left(\sum_{m\in\mathbb{Z}} f(x+m) \right) \mathrm{d}x = \sum_{m\in\mathbb{Z}} \int_0^1 e^{-2\pi i x k}\, f(x+m)\, \mathrm{d}x \\ &= \sum_{m\in\mathbb{Z}} \int_m^{m+1} e^{-2\pi i y k}\, f(y)\, \mathrm{d}y = \int_{\mathbb{R}} e^{-2\pi i y k}\, f(y)\, \mathrm{d}y = \widehat{f}(k).\end{aligned} \tag{1.2}$$

If, for instance, we have the decay estimates

$$|f(x)| \leq \frac{C}{(1+|x|)^{1+\delta}} \quad \text{and} \quad |\widehat{f}(\xi)| \leq \frac{C}{(1+|\xi|)^{1+\delta}}, \tag{1.3}$$

for some $\delta > 0$ and some constant $C > 0$, the series (1.1) defining P_f will be absolutely convergent, thus defining a continuous function. Moreover, from (1.2) and (1.3), the Fourier inversion for the continuous periodic function P_f will hold pointwise, i.e.

$$P_f(x) = \sum_{k\in\mathbb{Z}} \widehat{P_f}(k)\, e^{2\pi i x k}\,,$$

and this is equivalent to

$$\sum_{m\in\mathbb{Z}} f(x+m) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)\, e^{2\pi i x k} \tag{1.4}$$

for all $x \in \mathbb{R}$. Expression (1.4) is known as the *Poisson summation formula.*

Our objective now is to weaken the decay conditions (1.3) in a way that we can still obtain (1.4) pointwise. Recall that a function $f : \mathbb{R} \to \mathbb{C}$ of bounded variation is said to be normalized if

$$f(x) = \frac{1}{2}\big\{ f(x^+) + f(x^-) \big\}$$

for all $x \in \mathbb{R}$, where $f(x^{\pm})$ are the lateral limits at the point x. From the basic theory of Fourier series (see for instance [13, Theorem 8.43]), we know that if $P : \mathbb{T} \to \mathbb{C}$ is a normalized function of bounded variation then the Fourier inversion holds pointwise , i.e.

$$P(x) = \lim_{N\to\infty} \sum_{k=-N}^{N} \widehat{P}(k)\, e^{2\pi i x k} \tag{1.5}$$

for all $x \in \mathbb{T}$.

Theorem 1.1 (Poisson summation formula)**.** *Let $f \in L^1(\mathbb{R})$ be a normalized function of bounded variation. Then we have*

$$\lim_{N\to\infty} \sum_{m=-N}^{N} f(x+m) = \lim_{N\to\infty} \sum_{k=-N}^{N} \widehat{f}(k)\, e^{2\pi i k x}$$

for all $x \in \mathbb{R}$.

Proof. As before, let $P_f(x) = \sum_{m\in\mathbb{Z}} f(x+m)$. We know that $P_f \in L^1(\mathbb{T})$, since $f \in L^1(\mathbb{R})$. Let $x_0 \in [-1/2, 1/2)$ be a point where the sum is absolutely convergent (in particular $|P_f(x_0)| < \infty$). For any other point $x \in [-1/2, 1/2)$, say $x > x_0$, the difference $|f(x+m) - f(x_0+m)|$ is less than or equal to the variation of f on the interval $[x_0+m, x+m]$, and therefore the sum of these increments is less than or equal to the total variation of f (let us call it Vf). Therefore

$$\begin{aligned}\sum_{m\in\mathbb{Z}} |f(x+m)| &\leq \sum_{m\in\mathbb{Z}} |f(x_0+m)| + \sum_{m\in\mathbb{Z}} |f(x+m) - f(x_0+m)| \\ &\leq \sum_{m\in\mathbb{Z}} |f(x_0+m)| + Vf < \infty\,,\end{aligned}$$

from which we conclude that the sum is absolutely convergent for each $x \in \mathbb{T}$. Now observe that for each partition $-1/2 = a_0 < a_1 < ... < a_k = 1/2$, we have

$$\begin{aligned}\sum_{i=1}^{k} |P_f(a_i) - P_f(a_{i-1})| &= \sum_{i=1}^{k} \left| \sum_{m\in\mathbb{Z}} \big\{ f(a_i+m) - f(a_{i-1}+m) \big\} \right| \\ &\leq \sum_{i=1}^{k} \sum_{m\in\mathbb{Z}} \big| f(a_i+m) - f(a_{i-1}+m) \big| \leq Vf,\end{aligned}$$

and therefore P_f has bounded variation. Finally, since f is normalized, we have for each point $x \in [-1/2, 1/2)$

$$\begin{aligned}\lim_{\epsilon\to 0} \frac{1}{2} \big\{ P_f(x+\epsilon) + P_f(x-\epsilon) \big\} &= \lim_{\epsilon\to 0} \frac{1}{2} \sum_{m\in\mathbb{Z}} \big\{ f(x+\epsilon+m) + f(x-\epsilon+m) \big\} \\ &= \sum_{m\in\mathbb{Z}} \frac{1}{2} \lim_{\epsilon\to 0} \big\{ f(x+\epsilon+m) + f(x-\epsilon+m) \big\} \\ &= \sum_{m\in\mathbb{Z}} f(x+m) = P_f(x),\end{aligned}$$

where we used dominated convergence to move the limit inside, since for $\epsilon < 1/2$ we have

$$|f(x+\epsilon+m)| \leq |f(x+m)| + Vf_{[x+m, x+m+1/2]}$$

and

$$|f(x-\epsilon+m)| \leq |f(x+m)| + Vf_{[x+m-1/2, x+m]}.$$

Therefore P_f is normalized and the result now follows from (1.5). □

1.2 The Paley-Wiener theorem

In this section we investigate the relation between the growth of a function and the size of the support of its Fourier transform. For $\delta > 0$, we shall say that an entire function $F : \mathbb{C} \to \mathbb{C}$ has *exponential type* $2\pi\delta$, if for each $\epsilon > 0$ there exists a constant C_ϵ such that

$$|F(z)| \leq C_\epsilon \, e^{(2\pi\delta+\epsilon)|z|},$$

for all $z \in \mathbb{C}$. In other words, for each $\epsilon > 0$, we have

$$|F(z)| = O\big(e^{(2\pi\delta+\epsilon)|z|}\big).$$

The class of entire functions with exponential type at most $2\pi\delta$ will be called $E^{2\pi\delta}$. We shall say that an entire function $F \in L^p(\mathbb{R})$ if the restriction of F to the real axis (call it $F|_{\mathbb{R}}$) belongs to L^p, i.e. if

$$\int_{-\infty}^{\infty} |F(x)|^p \, \mathrm{d}x < \infty,$$

when $1 \leq p < \infty$, and $\sup_{x\in\mathbb{R}} |F(x)| < \infty$, if $p = \infty$.

Theorem 1.2 (Paley-Wiener)**.** *For an entire function $F \in L^2(\mathbb{R})$ the two conditions below are equivalent:*

(i) F has exponential type $2\pi\delta$.

(ii) The Fourier transform of $F|_{\mathbb{R}}$ is supported on $[-\delta, \delta]$.

Proof. Here we shall essentially follow [44, Chapter XVI]. A different proof is presented in [40, Chapter III].

Step 1: (ii) $\Rightarrow$ (i). If $f = \widehat{F|_{\mathbb{R}}}$ is supported on $[-\delta, \delta]$, then $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and

$$F(x) = \int_{-\delta}^{\delta} f(t)\, e^{2\pi i t x} \, \mathrm{d}t. \tag{1.6}$$

The right-hand side of (1.6) extends to an entire function and thus

$$F(z) = \int_{-\delta}^{\delta} f(t)\, e^{2\pi i t z} \, \mathrm{d}t. \tag{1.7}$$

If $z = x + iy$, from (1.7) it is clear that

$$|F(z)| \leq e^{2\pi\delta|y|} \int_{-\delta}^{\delta} |f(t)| \, \mathrm{d}t \,,$$

which shows that F has exponential type $2\pi\delta$.

Step 2: (i) $\Rightarrow$ (ii). This is the deep part of the theorem. We must show that if $F \in E^{2\pi\delta} \cap L^2(\mathbb{R})$ then

$$f(t) = \int_{-\infty}^{\infty} F(\xi)\, e^{-2\pi i t \xi} \, \mathrm{d}\xi \tag{1.8}$$

is zero for almost all t outside $[-\delta,\delta]$, where the integral in (1.8) is meant as the L^2-limit of the truncated integrals $\int_{-R}^{R}$ as $R\to\infty$. Consider the following function

$$g(z;\theta)=\int_{\gamma_\theta} F(w)\,e^{-2\pi wz}\,\mathrm{d}w, \tag{1.9}$$

where the integral is taken over the ray $\gamma_\theta=\{\arg(w)=-\theta\}$, i.e. $w=\rho e^{-i\theta}$, with ρ varying from 0 to ∞.

Let $z=x+iy$. We first claim that the integral in (1.9) is absolutely and uniformly convergent on each closed half-plane contained in the open half-plane

$$x\cos\theta+y\sin\theta>\delta.$$

Note that this is the open half-plane (that does not contain the origin) delimited by the tangent to the circle $|w|=\delta$ at the point $\delta e^{i\theta}$. Let us call this open half-plane by H_θ. In fact, if z belongs to a closed half-plane Γ entirely contained in H_θ, we put $w=\rho\,e^{-i\theta}$ on the integrand of (1.9) to see that

$$\begin{aligned}\left|F(w)\,e^{-2\pi wz}\right| &= O\left\{e^{(2\pi\delta+\epsilon)\rho-\Re\left(2\pi(x+iy)\rho\,e^{-i\theta}\right)}\right\}\\ &= O\left\{e^{(2\pi\delta+\epsilon)\rho-2\pi(x\cos\theta+y\sin\theta)\rho}\right\},\end{aligned}$$

for all $\epsilon>0$. We can then choose ϵ sufficiently small such that the last expression decays exponentially fast with ρ, uniformly on Γ, thus proving our claim. In particular, by Morera's theorem, $z\mapsto g(z;\theta)$ is analytic in H_θ.

Secondly, we observe that if $0<|\theta'-\theta''|<\pi$, the functions $g(z;\theta')$ and $g(z;\theta'')$ coincide on the intersection of the half-planes $H_{\theta'}$ and $H_{\theta''}$. In fact, let us suppose without loss of generality that $\theta'<\theta''$, and let $z\in H_{\theta'}\cap H_{\theta''}$. It is easy to see geometrically that $z\in H_\theta$ for any $\theta'\le\theta\le\theta''$, and that the integrand $G(w)$ of (1.9), considered as a function of w alone, decays exponentially to 0 as $|w|\to\infty$ in the angle $[-\theta'',-\theta']$ (since this decay rate depends only on the distance from z to $H_{\theta'}$ and $H_{\theta''}$). By Cauchy's theorem we can change the ray of integration and thus $g(z;\theta')=g(z;\theta'')$.

Let $g_0(z)=g(z;0)$ and $g_1(z)=g(z;\pi)$. We now observe that g_0 and g_1 are analytic in the half-planes $x>0$ and $x<0$, respectively, and are analytic continuations of each other across the segments $y>\delta$ and $y<-\delta$ of the imaginary axis. To see this, let $x\ge\epsilon>0$ and note that

$$\begin{aligned}|g_0(z)| &= \left|\int_0^\infty F(w)\,e^{-2\pi w(x+iy)}\,\mathrm{d}w\right|\\ &\le\left(\int_0^\infty|F(w)|^2\,\mathrm{d}w\right)^{1/2}\left(\int_0^\infty e^{-4\pi\epsilon w}\,\mathrm{d}w\right)^{1/2}.\end{aligned}$$

Therefore, by Morera's theorem, g_0 is analytic for $x>0$. In an analogous manner, one shows that g_1 is analytic for $x<0$. Consider now $g_2=g(z,\pi/2)$.

We know that g_2 is analytic in $H_{\frac{\pi}{2}}$ and from the previous paragraph we also know it agrees with g_0 in $H_0 \cap H_{\frac{\pi}{2}}$. Since g_0 is analytic in the whole half-plane $x > 0$, it follows that g_2 is the analytic continuation of g_0 across the segment $y > \delta$ of the imaginary axis. Similarly, g_2 is the analytic continuation of g_1 across the same segment. A similar argument holds for $y < -\delta$.

We are now able to conclude the proof. Recall that

$$g_0(x+iy) = \int_0^\infty F(\xi)\, e^{-2\pi\xi(x+yi)}\, \mathrm{d}\xi.$$

Since

$$\int_0^\infty \left|F(\xi)\right|^2 \left|1 - e^{-2\pi\xi x}\right|^2 \mathrm{d}\xi \to 0$$

as $x \to 0^+$, we see by Plancherel's theorem that $g_0(x+iy)$ tends to

$$\int_0^\infty F(\xi)\, e^{-2\pi\xi yi}\, \mathrm{d}\xi$$

in L^2 as $x \to 0^+$. In an analogous manner

$$g_1(x+iy) = -\int_{-\infty}^0 F(\xi)\, e^{-2\pi\xi(x+yi)}\, \mathrm{d}\xi$$

tends to

$$-\int_{-\infty}^0 F(\xi)\, e^{-2\pi\xi yi}\, \mathrm{d}\xi$$

in L^2 as $x \to 0^-$. We conclude that $g_0(x+iy) - g_1(-x+iy)$ tends to

$$f(y) = \int_{-\infty}^\infty F(\xi)\, e^{-2\pi\xi yi}\, \mathrm{d}\xi$$

as $x \to 0^+$. However we know that $g_0(x+iy) - g_1(-x+iy) \to 0$ pointwise when $x \to 0^+$, for all $|y| > \delta$. Hence $f \equiv 0$ on $[-\delta, \delta]^c$ and the proof is complete.

$\square$

1.3 The Plancherel-Polya theorem

1.3.1 Statement and proof

The objective of this section is to prove the following result of Plancherel and Polya [33].

Theorem 1.3 (Plancherel-Polya)**.** *Let F be an entire function of exponential type $2\pi\delta$ such that its restriction to the real axis belongs to $L^p(\mathbb{R})$, for some p*

with $0 < p < \infty$. *Given* $\Delta > 0$ *let* $\{\lambda_m\}_{m\in\mathbb{Z}}$ *be a sequence of real numbers such that* $|\lambda_m - \lambda_n| \geq \Delta$, *for all* $m, n \in \mathbb{Z}$. *Then*

$$\sum_{m\in\mathbb{Z}} |F(\lambda_m)|^p \leq C \int_{-\infty}^{\infty} |F(x)|^p \,\mathrm{d}x,$$

where $C = C(p, \delta, \Delta)$.

Proof. We start by noticing that, since $z \mapsto |F(z)|^p$ is a subharmonic function, the mean value property gives us

$$\begin{aligned}|F(\lambda_m)|^p &\leq \frac{1}{m(B(\Delta/2))} \int_{(x,y)\in B(\Delta/2)} |F(\lambda_m + x + iy)|^p \,\mathrm{d}x\,\mathrm{d}y \\ &\leq \frac{1}{m(B(\Delta/2))} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} |F(\lambda_m + x + iy)|^p \,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

where $B(r)$ is the ball of radius r centered at the origin, and $m(B(r))$ is its volume. If we sum the last expression over $m \in \mathbb{Z}$, and use the fact that the λ_m's are at a distance Δ apart of each other, we find

$$\sum_{m\in\mathbb{Z}} |F(\lambda_m)|^p \leq \frac{1}{m(B(\Delta/2))} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \int_{-\infty}^{\infty} |F(x + iy)|^p \,\mathrm{d}x\,\mathrm{d}y. \tag{1.10}$$

We will show that, for any $y \in \mathbb{R}$, we have

$$\int_{-\infty}^{\infty} |F(x + iy)|^p \,\mathrm{d}x \leq e^{2\pi\delta p|y|} \int_{-\infty}^{\infty} |F(x)|^p \,\mathrm{d}x. \tag{1.11}$$

Clearly, the combination of (1.10) and (1.11) finish the proof of the theorem. The hard part is indeed the proof of (1.11) that shall make use of three lemmas as follows. □

1.3.2 Auxiliary lemmas

We keep denoting $z = x + iy$. Let $G(z)$ be an analytic function on the half-plane $y > 0$, that is continuous on $y \geq 0$. Let a be a positive number and define the function $\Psi(z)$ by

$$\Psi(z) = \int_{-a}^{a} |G(z + s)|^p \,\mathrm{d}s,$$

where the path of integration is the segment $[-a, a]$. Observe that $\Psi(z)$ is defined and continuous for $y \geq 0$.

Lemma 1.4. *Let* $\mathcal{D}$ *be a bounded and closed domain contained in the half-plane* $y \geq 0$. *Then the maximum of* $\Psi(z)$ *is attained in the boundary of* $\mathcal{D}$.

Proof. From the fact that $|G(z)|^p$ is subharmonic we have

$$|G(\zeta)|^p \leq \frac{1}{2\pi}\int_0^{2\pi} \big|G\big(\zeta + re^{i\varphi}\big)\big|^p \,\mathrm{d}\varphi,$$

where the circle $|z-\zeta| \leq r$ is contained in the half-plane $y \geq 0$. Therefore, for $z = x + iy$ and $r \leq y$ we have

$$\begin{aligned}\Psi(z) &= \int_{-a}^{a} |G(z+s)|^p \,\mathrm{d}s \\ &\leq \int_{-a}^{a} \left(\frac{1}{2\pi}\int_0^{2\pi} \big|G\big(z+s+re^{i\varphi}\big)\big|^p \,\mathrm{d}\varphi\right) \mathrm{d}s \\ &= \frac{1}{2\pi}\int_0^{2\pi} \Psi\big(z + re^{i\varphi}\big)\,\mathrm{d}\varphi.\end{aligned}$$

Thus Ψ is subharmonic on the half-plane $y \geq 0$ and the result now follows from the maximum principle. □

Lemma 1.5. *Let*

$$M = \sup_{x\in\mathbb{R}} \Psi(x)$$

and

$$N = \sup_{y\geq 0} \Psi(iy).$$

Suppose that M and N are finite and that $G(z)$ has exponential type on the half-plane $y \geq 0$. Then

$$\Psi(z) \leq \max\{M, N\}$$

for all z in the half-plane $y \geq 0$.

Proof. The hypothesis that $G(z)$ has exponential type on the half-plane $y \geq 0$ guarantees the existence of positive numbers B and b such that

$$|G(z)| \leq Be^{b|z|} \tag{1.12}$$

if $z = x + iy$, with $y \geq 0$. Let $\epsilon > 0$ and define

$$\begin{aligned}G_\epsilon(z) &= G(z)\, e^{-\epsilon\left[(z+a)\,e^{-\pi i/4}\right]^{3/2}}; \\ \Psi_\epsilon(z) &= \int_{-a}^{a} \big|G_\epsilon(z+s)\big|^p \,\mathrm{d}s.\end{aligned} \tag{1.13}$$

Above we choose the branch of $\left[(z+a)\,e^{-\pi i/4}\right]^{3/2}$ with positive real part when $x > -a$ and $y \geq 0$. From (1.12) and (1.13) we have

$$\big|G_\epsilon(z)\big| \leq B\, e^{b|z| - \epsilon\gamma|z+a|^{3/2}} \tag{1.14}$$

and

$$\big|G_\epsilon(z)\big| \le \big|G(z)\big|$$

for $x > -a$ and $y \ge 0$, where $\gamma = \cos\frac{3\pi}{8}$. Therefore we have

$$\big|\Psi_\epsilon(z)\big| \le \big|\Psi(z)\big|$$

when $x \ge 0$ and $y \ge 0$, and in particular

$$\big|\Psi_\epsilon(x)\big| \le M \tag{1.15}$$

for $x \ge 0$, and

$$\big|\Psi_\epsilon(iy)\big| \le N \tag{1.16}$$

for $y \ge 0$. Let z_0 be a point on the quadrant $x > 0$, $y > 0$. We now apply Lemma 1.4 to Ψ_ϵ and the domain $\mathcal{D} = \{z = x + iy;\, x \ge 0,\, y \ge 0,\, |z| \le R\}$. Assume that the radius R is sufficiently large so that $z_0 \in \mathcal{D}$ and that the maximum over the curved part of the boundary is at most $\max\{M, N\}$ (this can be done by (1.14)). By Lemma 1.4, (1.15) and (1.16) we arrive at

$$\big|\Psi_\epsilon(z_0)\big| \le \max\{M, N\}.$$

This reasoning holds for any $\epsilon > 0$. By considering $\epsilon \to 0^+$ we find that

$$\big|\Psi(z)\big| \le \max\{M, N\}$$

for any z in the first quadrant $x \ge 0$ and $y \ge 0$. The proof for the quadrant $x \le 0$ and $y \ge 0$ is analogous. □

Lemma 1.6. *In addition to the hypotheses of Lemma 1.5 assume that*

$$\lim_{y\to\infty} G(x + iy) = 0 \tag{1.17}$$

uniformly on the strip $-a \le x \le a$. *Then* $N \le M$ *and therefore, for* $y \ge 0$, *we have*

$$\int_{-a}^{a} |G(z+s)|^p \, \mathrm{d}s = \Psi(z) \le M.$$

Proof. Assume that $G(z)$ is not identically zero (otherwise the result is obviously true). Due to (1.17) we have $\Psi(iy) \to 0$ as $y \to \infty$, and thus the supremum N over the imaginary axis must be attained at a certain point iy_0. If $y_0 = 0$ we have

$$N = \Psi(iy_0) = \Psi(0) \le M.$$

If $y_0 > 0$ and $N > M$, we would have, by Lemma 1.5, the subharmonic function Ψ attaining its maximum over the half-plane $y \ge 0$ at an interior point, and thus Ψ would have to be constant, giving us a contradiction. Therefore we must have $N \le M$. □

Proof of the key inequality (1.11). After going through these three auxiliary lemmas, we are now in position to prove inequality (1.11) and complete the proof of the Plancherel-Polya theorem. It suffices to prove (1.11) in the case $y > 0$. Let $\epsilon > 0$ be given. We will apply Lemmas 1.5 and 1.6 to the function

$$G(z) = F(z)\, e^{i(2\pi\delta+\epsilon)z}.$$

Observe that $G(z)$ satisfies all the hypotheses of Lemmas 1.5 and 1.6. The number M defined in Lemma 1.5 is finite since

$$M \leq \int_{-\infty}^{\infty} |G(x)|^p \,\mathrm{d}x = \int_{-\infty}^{\infty} |F(x)|^p \,\mathrm{d}x.$$

Applying the conclusion of Lemma 1.6 to $z = iy$ we have

$$\begin{aligned} e^{-p(2\pi\delta+\epsilon)y} \int_{-a}^{a} |F(s+iy)|^p \,\mathrm{d}s &= \int_{-a}^{a} |G(s+iy)|^p \,\mathrm{d}s \\ &\leq \int_{-\infty}^{\infty} |F(x)|^p \,\mathrm{d}x. \end{aligned}$$

Since this result holds for any $a > 0$ and any $\epsilon > 0$ we obtain the desired result by making $a \to \infty$ and $\epsilon \to 0^+$. The proof is complete. □

1.4 Interpolation formulas

1.4.1 Basic results

We shall see here that entire functions of a prescribed exponential type (with an L^p condition on the real axis) are completely determined by their values on a set of well-spaced points. Recall that an entire function $F \in L^p(\mathbb{R})$, $1 \leq p < \infty$, if

$$\int_{-\infty}^{\infty} |F(x)|^p \,\mathrm{d}x < \infty.$$

In case $F \in L^2(\mathbb{R})$, its Fourier transform is defined as

$$\widehat{F}(t) = \lim_{R\to\infty} \int_{-R}^{R} F(x)\, e^{-2\pi i x t} \,\mathrm{d}t,$$

where the limit is taken in the L^2-sense. By the Paley-Wiener theorem, we know that F has exponential type $2\pi\delta$ and belongs to $L^2(\mathbb{R})$ if and only if $\widehat{F}$ is supported on $[-\delta, \delta]$, and so

$$F(z) = \int_{-\delta}^{\delta} \widehat{F}(t)\, e^{2\pi i t z} \,\mathrm{d}t \tag{1.18}$$

for all $z \in \mathbb{C}$.

Theorem 1.7. *Let F be an entire function of exponential type π such that $F \in L^p(\mathbb{R})$ for some p with $1 \le p < \infty$. Then*

$$F(z) = \frac{\sin \pi z}{\pi} \sum_{n \in \mathbb{Z}} (-1)^n \frac{F(n)}{(z-n)}, \tag{1.19}$$

where the expression on the right-hand side of (1.19) *converges uniformly on compact subsets of $\mathbb{C}$.*

If $p = 2$, the Fourier transform $\widehat{F}(t)$ occurring in (1.18) *has the following Fourier expansion (as functions in $L^2[-\frac{1}{2}, \frac{1}{2}]$)*

$$\widehat{F}(t) = \sum_{m \in \mathbb{Z}} F(m) e^{-2\pi i m t}. \tag{1.20}$$

If $p = 1$, the right-hand side of (1.20) *is absolutely convergent, and as $\widehat{F}$ is continuous, equality* (1.20) *holds pointwise. In particular*

$$\widehat{F}(0) = \sum_{m \in \mathbb{Z}} F(m)$$

and

$$0 = \widehat{F}\big(\tfrac{1}{2}\big) = \sum_{m \in \mathbb{Z}} (-1)^m F(m). \tag{1.21}$$

Proof. Let us start with the statements about the Fourier transforms. If $p = 2$, it is clear from (1.18) that $\{F(m)\}_{m \in \mathbb{Z}}$ are the Fourier coefficients of $\widehat{F}$ (thought as a periodic function of period 1). The Fourier expansion (1.20) is a consequence of this. If $p = 1$, the function $\widehat{F}$ will be a continuous function supported on $[-\frac{1}{2}, \frac{1}{2}]$ and again, by (1.18), its Fourier coefficients will be $\{F(m)\}_{m \in \mathbb{Z}}$. By the theorem of Plancherel and Polya, the sequence $\{|F(m)|\}_{m \in \mathbb{Z}}$ is summable and we conclude that (1.20) holds pointwise.

We now prove (1.19) when $p \le 2$. In this case, there is no loss of generality in assuming $p = 2$ since, if $p < 2$, we have F bounded on the real axis (by the theorem of Plancherel and Polya) and thus $F \in L^2(\mathbb{R})$. Define

$$v_N(t) = \sum_{m=-N}^{N} F(m)\, e^{-2\pi i m t}.$$

We already know that $v_N \to \widehat{F}$ in $L^2[-\frac{1}{2}, \frac{1}{2}]$ when $N \to \infty$ (therefore we have convergence in $L^1[-\frac{1}{2}, \frac{1}{2}]$ as well). Thus

$$\begin{aligned} F(z) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \widehat{F}(t)\, e^{2\pi i t z}\, \mathrm{d}t = \lim_{N \to \infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} v_N(t)\, e^{2\pi i t z}\, \mathrm{d}t \\ &= \lim_{N \to \infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{m=-N}^{N} F(m)\, e^{2\pi i t (z-m)}\, \mathrm{d}t \end{aligned}$$

$$= \lim_{N\to\infty} \sum_{m=-N}^{N} F(m) \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i t(z-m)}\,\mathrm{d}t$$

$$= \lim_{N\to\infty} \sum_{m=-N}^{N} F(m)\, \frac{\sin \pi(z-m)}{\pi(z-m)}. \tag{1.22}$$

Since $\{F(m)\}_{m\in\mathbb{Z}}$ is square summable, it is not hard to check that the sum on the right-hand side of (1.22) converges absolutely and uniformly on compact subsets of $\mathbb{C}$ (apply Hölder's inequality and use that $|\sin(\pi z)/\pi z| \ll e^{\pi|\Im(z)|}/(1+|z|)$), thus defining an entire function. We have then established (1.19) in these cases.

To treat the case where $F \in L^p(\mathbb{R})$ with $2 < p < \infty$, we consider the entire function

$$R(z) = \begin{cases} \dfrac{F(z)-F(0)}{z}, & \text{if } \ z \neq 0; \\[2ex] F'(0), & \text{if } \ z = 0. \end{cases} \tag{1.23}$$

It is clear that R is an entire function with the same exponential type as F. Moreover, since $F \in L^p(\mathbb{R})$, an application of Hölder's inequality will give $R \in L^1(\mathbb{R})$. We now use the interpolation formula for R (already established)

$$\frac{F(z)-F(0)}{z} = \frac{\sin \pi z}{\pi} \left\{ \frac{F'(0)}{z} + \sum_{n\neq 0} (-1)^n \frac{F(n)-F(0)}{(z-n)n} \right\},$$

together with (1.21)

$$0 = \widehat{R}\big(\tfrac{1}{2}\big) = \sum_{n\in\mathbb{Z}} (-1)^n\, R(n),$$

and the identity

$$\frac{\pi}{\sin \pi z} = \frac{1}{z} + \sum_{n\neq 0} (-1)^n \left(\frac{1}{(z-n)} + \frac{1}{n} \right).$$

We then get

$$\begin{aligned} F(z) = F(0) + \frac{\sin \pi z}{\pi} \Bigg\{ & F'(0) + \sum_{n\neq 0} (-1)^n F(n) \left(\frac{1}{(z-n)} + \frac{1}{n} \right) \\ & - F(0) \sum_{n\neq 0} (-1)^n \left(\frac{1}{(z-n)} + \frac{1}{n} \right) \Bigg\} \\ = \frac{\sin \pi z}{\pi} & \sum_{n\in\mathbb{Z}} (-1)^n \frac{F(n)}{(z-n)} \\ & + F(0) + \frac{\sin \pi z}{\pi} \left\{ \sum_{n\in\mathbb{Z}} (-1)^n R(n) - F(0) \frac{\pi}{\sin \pi z} \right\} \end{aligned}$$

$$= \frac{\sin \pi z}{\pi} \sum_{n\in\mathbb{Z}} (-1)^n \frac{F(n)}{(z-n)}.$$

Once more, it is clear that the last series converges absolutely and uniformly on compact subsets of $\mathbb{C}$ via Hölder's inequality, since $\sum_n |F(n)|^p$ converges and $|\sin(\pi z)/\pi z| \ll e^{\pi|\Im(z)|}/(1+|z|)$. □

A careful reading of the last details of the previous proof gives us the following corollary.

Corollary 1.8. *Let F be an entire function of exponential type π such that the entire function R defined in* (1.23) *belongs to $L^p(\mathbb{R})$ for some p with $1 \leq p < \infty$. Then*

$$F(z) = \frac{\sin \pi z}{\pi} \left\{ F'(0) + \frac{F(0)}{z} + \sum_{n\neq 0} (-1)^n F(n) \left(\frac{1}{(z-n)} + \frac{1}{n} \right) \right\}, \qquad (1.24)$$

where the expression on the right-hand side of (1.24) *converges uniformly on compact subsets of $\mathbb{C}$.*

Observe in particular that we can apply Corollary 1.8 when F is an entire function of type π bounded on $\mathbb{R}$. Both Theorem 1.7 and Corollary 1.8 assume $F \in E^{\pi}$. Similar interpolation formulas can be obtained when $F \in E^{2\pi\delta}$ by considering $G(z) = F(z/2\delta)$. In this case, the interpolation points will be $(1/2\delta)\mathbb{Z}$. One can also consider the translation $H(z) = F(z-\alpha)$ to interpolate H at $\mathbb{Z}+\alpha$, when $H \in E^{\pi}$.

1.4.2 Bernstein's inequality

We start here with the following proposition.

Proposition 1.9. *Let F be a function of exponential type $2\pi\delta$ such that the entire function R defined by*

$$R(z) = \begin{cases} \dfrac{F(z)-F(0)}{z}, & \text{if } \ z \neq 0; \\[2ex] F'(0), & \text{if } \ z = 0, \end{cases}$$

is such that $R \in L^2(\mathbb{R})$. Then F' has exponential type $2\pi\delta$ as well.

Proof. By the Paley-Wiener theorem we have $\widehat{R}$ with support in $[-\delta, \delta]$ and

$$R(z) = \int_{-\delta}^{\delta} \widehat{R}(t)\, e^{2\pi i t z}\, \mathrm{d}t,$$

for all $z \in \mathbb{C}$. Differentiating we have

$$\frac{F'(z)z - F(z) + F(0)}{z^2} = R'(z) = \int_{-\delta}^{\delta} 2\pi i t\, \widehat{R}(t)\, e^{2\pi i t z}\, \mathrm{d}t,$$

and from here we see that R' has exponential type $2\pi\delta$ and so does F'. □

We now want to show that if F has an L^p integrability when restricted to the real axis, so does F'. The case $p = \infty$ of this claim is known as Bernstein's inequality and is proved below.

Theorem 1.10 (Bernstein's inequality)**.** *Let F be an entire function of exponential type $2\pi\delta$ that is bounded on the real axis. Then*

$$\sup_{x\in\mathbb{R}} |F'(x)| \leq 2\pi\delta \sup_{x\in\mathbb{R}} |F(x)|.$$

Proof. We may suppose $\delta > 0$ since the case $\delta = 0$ follows by taking limits (hence for $\delta = 0$ the constants are the only admissible functions). We may also suppose $\delta = 1/2$, for otherwise we consider $G(z) = F(z/2\delta)$. From Corollary 1.8 we have

$$F(z) = \frac{\sin \pi z}{\pi} \left\{ F'(0) + \frac{F(0)}{z} + \sum_{n\neq 0} (-1)^n F(n) \left(\frac{1}{(z-n)} + \frac{1}{n} \right) \right\}. \tag{1.25}$$

The termwise differentiation of (1.25) leads to a series converging uniformly on compact subsets of $\mathbb{C}$. Therefore, denoting by $F_1(z)$ the function inside the curly brackets in (1.25) we have

$$F'(x) = \cos \pi x \, F_1(x) + \frac{\sin \pi x}{\pi} \sum_{n\in\mathbb{Z}} (-1)^{n+1} \frac{F(n)}{(x-n)^2}.$$

Taking $x = \frac{1}{2}$ in the last expression we have

$$F'\big(\tfrac{1}{2}\big) = \frac{4}{\pi} \sum_{n\in\mathbb{Z}} (-1)^{n+1} \frac{F(n)}{(2n-1)^2}, \tag{1.26}$$

and thus

$$\big|F'\big(\tfrac{1}{2}\big)\big| \leq \frac{4}{\pi} \left(\sum_{n\in\mathbb{Z}} \frac{1}{(2n-1)^2} \right) \sup_{x\in\mathbb{R}} |F(x)| = \pi \sup_{x\in\mathbb{R}} |F(x)|. \tag{1.27}$$

For the general case, we take $x_0 \in \mathbb{R}$ and consider $G(z) = F\big(x_0 + z - \frac{1}{2}\big)$. Applying (1.27) to G we have

$$\big|F'(x_0)\big| = \big|G'\big(\tfrac{1}{2}\big)\big| \leq \pi \sup_{x\in\mathbb{R}} |G(x)| = \pi \sup_{x\in\mathbb{R}} |F(x)|.$$

□

Note from the previous proof that, in fact, from (1.26) we have

$$F'(x) = \frac{4}{\pi} \sum_{n\in\mathbb{Z}} (-1)^{n+1} \frac{F\big(x + n - \frac{1}{2}\big)}{(2n-1)^2} \tag{1.28}$$

for any $x \in \mathbb{R}$. This formula is the source of many applications, in particular the next one.

Theorem 1.11. *Let F be an entire function of exponential type $2\pi\delta$ that is bounded on the real axis. Then for any $w : [0,\infty) \to [0,\infty)$ which is convex and non-decreasing we have*

$$\int_{-\infty}^{\infty} w\big((2\pi\delta)^{-1}|F'(x)|\big)\mathrm{d}x \leq \int_{-\infty}^{\infty} w(|F(x)|)\,\mathrm{d}x.$$

In particular, putting $w(u) = u^p$ for $p \geq 1$, we have

$$\left(\int_{-\infty}^{\infty} |F'(x)|^p \mathrm{d}x\right)^{1/p} \leq (2\pi\delta)\left(\int_{-\infty}^{\infty} |F(x)|^p\,\mathrm{d}x\right)^{1/p}. \tag{1.29}$$

Remark. Note that the limiting case of (1.29) (when $p = \infty$) is exactly Bernstein's inequality.

Proof. It suffices to consider the case $\delta = \frac{1}{2}$, since the general case follows by a simple change of variables. From formula (1.28) we know that

$$|F'(x_0)| \leq \frac{4}{\pi}\sum_{n\in\mathbb{Z}} \frac{\left|F\left(x_0 + n - \frac{1}{2}\right)\right|}{(2n-1)^2},$$

for any $x_0 \in \mathbb{R}$. We consider the probability measure μ_{x_0} given by

$$\mu_{x_0} = \frac{4}{\pi^2}\sum_{n\in\mathbb{Z}} \frac{1}{(2n-1)^2}\,\delta_{\left(x_0+n-\frac{1}{2}\right)},$$

where δ_a is the Dirac delta supported at the point $x = a$. From Jensen's inequality we have

$$w\left(\int_{-\infty}^{\infty} |F(x)|\,\mathrm{d}\mu_{x_0}(x)\right) \leq \int_{-\infty}^{\infty} w(|F(x)|)\,\mathrm{d}\mu_{x_0}(x).$$

We now integrate with respect to the variable x_0 and use the fact that w is non-decreasing to get

$$\begin{aligned}
\int_{-\infty}^{\infty} w\big(\pi^{-1}|F'(x_0)|\big)\,\mathrm{d}x_0 &\leq \int_{-\infty}^{\infty} w\left(\int_{-\infty}^{\infty} |F(x)|\,\mathrm{d}\mu_{x_0}(x)\right)\mathrm{d}x_0 \\
&\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(|F(x)|)\,\mathrm{d}\mu_{x_0}(x)\,\mathrm{d}x_0 \\
&= \int_{-\infty}^{\infty}\left(\frac{4}{\pi^2}\sum_{n\in\mathbb{Z}} \frac{w\left(\left|F\left(x_0+n-\frac{1}{2}\right)\right|\right)}{(2n-1)^2}\right)\mathrm{d}x_0 \\
&= \sum_{n\in\mathbb{Z}} \frac{4}{\pi^2(2n-1)^2}\int_{-\infty}^{\infty} w\left(\left|F\left(x_0+n-\tfrac{1}{2}\right)\right|\right)\mathrm{d}x_0 \\
&= \int_{-\infty}^{\infty} w(|F(x)|)\,\mathrm{d}x,
\end{aligned}$$

and this concludes the proof. □

1.4.3 Interpolation formulas involving derivatives

We saw in the previous sections that a function of exponential type π (with mild decay on the real axis) is completely determined by its values at the integers. We shall see in this section that, if we move to the bigger class of exponential type 2π, we will need more information (say, at the integers) to completely determine our function. It turns out the the values of the function and its derivative at the integers are sufficient to recover the original function, as the following theorem of Vaaler [43, Theorem 9] shows.

Theorem 1.12. *Let F be an entire function of exponential type 2π such that $F \in L^p(\mathbb{R})$ for some p with $1 \le p < \infty$. Then*

$$F(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{m\in\mathbb{Z}} \frac{F(m)}{(z-m)^2} + \sum_{n\in\mathbb{Z}} \frac{F'(n)}{(z-n)} \right\}, \tag{1.30}$$

where the expression on the right-hand side of (1.30) *converges uniformly on compact subsets of* $\mathbb{C}$.

If $p = 2$, the Fourier transform $\widehat{F}(t)$ occurring in (1.18) *has the form*

$$\widehat{F}(t) = (1 - |t|)\, u_F(t) + (2\pi i)^{-1} \operatorname{sgn}(t)\, v_F(t) \tag{1.31}$$

for almost all $t \in [-1, 1]$, where u_F and v_F are periodic functions in $L^2[0, 1]$ with period 1 and Fourier series expansions

$$u_F(t) = \sum_{m\in\mathbb{Z}} F(m)\, e^{-2\pi i m t}, \tag{1.32}$$

and

$$v_F(t) = \sum_{n\in\mathbb{Z}} F'(n)\, e^{-2\pi i n t}. \tag{1.33}$$

If $p = 1$, then (1.32) *and* (1.33) *are absolutely convergent, u_F and v_F are continuous, and* (1.31) *holds for all $t \in [-1, 1]$. In particular*

$$\widehat{F}(0) = u_F(0) = \sum_{m\in\mathbb{Z}} F(m), \tag{1.34}$$

and

$$0 = v_F(0) = \sum_{n\in\mathbb{Z}} F'(n). \tag{1.35}$$

Proof. We consider first the case $1 \le p \le 2$. In this case, by the theorem of Plancherel and Polya, we know that F is bounded, and thus $F \in L^2(\mathbb{R})$. From the Paley-Wiener theorem we have that $\widehat{F}$ is supported on $[-1, 1]$ and belongs to $L^2[-1, 1]$. For $0 \le t < 1$ we define

$$u_F(t) = \widehat{F}(t) + \widehat{F}(t-1) \tag{1.36}$$

and

$$v_F(t) = 2\pi i \left\{t\widehat{F}(t) + (t-1)\widehat{F}(t-1)\right\}. \tag{1.37}$$

We then extend the domain of u_F and v_F to $\mathbb{R}$ by requiring that both functions have period 1. Since $\widehat{F} \in L^2[-1,1]$, it is clear that both u_F and v_F are in $L^2[0,1]$. The identity (1.31) follows easily from (1.36), (1.37) and the periodicity of u_F and v_F. To obtain (1.32) and (1.33) we note that

$$F(n) = \int_0^1 \left\{\widehat{F}(t) + \widehat{F}(t-1)\right\} e^{2\pi i n t}\,\mathrm{d}t = \int_0^1 u_F(t)\, e^{2\pi i n t}\,\mathrm{d}t$$

and

$$\begin{aligned} F'(n) &= \int_{-1}^1 2\pi i t\, \widehat{F}(t)\, e^{2\pi i n t}\,\mathrm{d}t \\ &= \int_0^1 2\pi i \left\{t\widehat{F}(t) + (t-1)\widehat{F}(t)\right\} e^{2\pi i n t}\,\mathrm{d}t \\ &= \int_0^1 v_F(t)\, e^{2\pi i n t}\,\mathrm{d}t \end{aligned}$$

for each integer n. Thus $F(n)$ and $F'(n)$ are the Fourier coefficients of u_F and v_F, respectively. Observe now that

$$\left(\frac{\sin \pi z}{\pi z}\right)^2 = \int_{-1}^1 (1-|t|)\, e^{2\pi i t z}\,\mathrm{d}t$$

and

$$z\left(\frac{\sin \pi z}{\pi z}\right)^2 = \frac{1}{2\pi i}\int_{-1}^1 \operatorname{sgn}(t)\, e^{2\pi i t z}\,\mathrm{d}t.$$

From these two identities we have, for each positive integer N,

$$\begin{aligned} &\left(\frac{\sin \pi z}{\pi}\right)^2 \left\{\sum_{m=-N}^{N} \frac{F(m)}{(z-m)^2} + \sum_{n=-N}^{N} \frac{F'(n)}{(z-n)}\right\} \\ &\quad = \int_{-1}^1 \left\{(1-|t|)\, u_F(t,N) + (2\pi i)^{-1} \operatorname{sgn}(t)\, v_F(t,N)\right\} e^{2\pi i t z}\,\mathrm{d}t, \end{aligned} \tag{1.38}$$

where

$$u_F(t,N) = \sum_{m=-N}^{N} F(m)\, e^{-2\pi i m t}$$

and

$$v_F(t,N) = \sum_{n=-N}^{N} F'(n)\, e^{-2\pi i n t}.$$

Since the sequences $\{F(m)\}_{m\in\mathbb{Z}}$ and $\{F'(n)\}_{n\in\mathbb{Z}}$ are square summable (by the theorem of Plancherel-Polya and Theorem 1.11), the left-hand side of (1.38) converges uniformly on compact subsets of $\mathbb{C}$ as $N \to \infty$. On the right-hand side of (1.38) we have $u_F(\cdot, N) \to u_F$ and $v_F(\cdot, N) \to v_F$ in L^2, and from this we establish (1.30).

If $p = 1$, by the theorem of Plancherel-Polya and Theorem 1.11, we know that $\{F(m)\}_{m\in\mathbb{Z}}$ and $\{F'(n)\}_{n\in\mathbb{Z}}$ are summable, thus u_F and v_F have absolutely convergent Fourier series and we may take u_F and v_F to be continuous periodic functions. Since $\widehat{F}(t)$ is now continuous and supported on $[-1, 1]$, the identity (1.31) must hold pointwise for all $t \in [-1, 1]$. If we let $t = 0$ we easily derive (1.34) and (1.35).

When $2 < p < \infty$ we make use of the entire function

$$R(z) = \begin{cases} \dfrac{F(z) - F(0)}{z}, & \text{if } \ z \neq 0; \\ \\ F'(0), & \text{if } \ z = 0, \end{cases} \tag{1.39}$$

and its derivative

$$R'(z) = \begin{cases} \dfrac{zF'(z) - F(z) + F(0)}{z^2}, & \text{if } \ z \neq 0; \\ \\ \frac{1}{2}F''(0), & \text{if } \ z = 0. \end{cases}$$

Since R has the same exponential type of F, and $R \in L^2(\mathbb{R})$, we have already established that

$$R(z) = \lim_{N\to\infty} \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{m=-N}^{N} \frac{R(m)}{(z-m)^2} + \sum_{n=-N}^{N} \frac{R'(n)}{(z-n)} \right\} \tag{1.40}$$

uniformly on compact subsets of $\mathbb{C}$. We multiply both sides of the expression (1.40) by z and use the definitions of R and R' to rewrite

$$\begin{aligned} F(z) - F(0) = \lim_{N\to\infty} \left(\frac{\sin \pi z}{\pi}\right)^2 \Bigg\{ & \sum_{m=-N}^{N} \frac{F(m)}{(z-m)^2} + \sum_{n=-N}^{N} \frac{F'(n)}{(z-n)} \\ & + \sum_{k=-N}^{N} R'(k) - F(0) \sum_{l=-N}^{N} \frac{1}{(z-l)^2} \Bigg\}. \end{aligned} \tag{1.41}$$

As the identity

$$\sum_{l\in\mathbb{Z}} \frac{1}{(z-l)^2} = \left(\frac{\pi}{\sin \pi z}\right)^2$$

is well known, all we have to show is that

$$\lim_{N\to\infty} \sum_{k=-N}^{N} R'(k) = 0. \tag{1.42}$$

By Theorem 1.11 we know that F and F' are in $L^p(\mathbb{R})$ and an application of Hölder's inequality shows us that $R \in L^1(\mathbb{R})$. In this case, identity (1.42) follows from (1.35) and this finishes the proof. $\square$

Our next result extends this interpolation formula for the case when F has exponential type 2π, and $R \in L^p(\mathbb{R})$ for some finite p. This has appeared in [43, Theorem 10].

Corollary 1.13. *Let F be an entire function of exponential type 2π such that R defined by (1.39) belongs to $L^p(\mathbb{R})$ for some p with $1 \le p < \infty$. Then*

$$F(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{m\in\mathbb{Z}} \frac{F(m)}{(z-m)^2} + \frac{F'(0)}{z} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} F'(n)\left(\frac{1}{(z-n)} + \frac{1}{n}\right) + A_F \right\}, \tag{1.43}$$

where the expression on the right-hand side of (1.43) converges uniformly on compact subsets of $\mathbb{C}$ and A_F is a constant given by

$$A_F = \frac{1}{2}F''(0) + \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \frac{F(0)-F(k)}{k^2}. \tag{1.44}$$

Proof. Since $R \in L^p(\mathbb{R})$ we may apply Theorem 1.12. As in the previous proof we find that (1.40) and (1.41) hold. We now reorganize (1.41) using the expression for the derivative $R'(z)$ as follows

$$F(z) = \lim_{N\to\infty} \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{m=-N}^{N} \frac{F(m)}{(z-m)^2} + \frac{F'(0)}{z} + \sum_{\substack{n=-N\\ n\neq 0}}^{N} F'(n)\left(\frac{1}{(z-n)} + \frac{1}{n}\right) + \frac{1}{2}F''(0) + \sum_{\substack{k=-N\\ k\neq 0}}^{N} \frac{F(0)-F(k)}{k^2} \right\}. \tag{1.45}$$

For a function of exponential type 2π, we have already seen that the fact that $F \in L^p(\mathbb{R})$ implies that $F \in L^q(\mathbb{R})$ if $p < q$. We may therefore assume without loss of generality that $1 < p < \infty$. It follows that

$$\begin{aligned} \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} |F(m)m^{-1}|^p &= \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} |R(m) + F(0)m^{-1}|^p \\ &\le 2^p \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \left\{ |R(m)|^p + |F(0)m^{-1}|^p \right\} < \infty, \end{aligned} \tag{1.46}$$

and

$$\begin{aligned}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |F'(n)n^{-1}|^p &= \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |R'(n) + R(n)n^{-1}|^p \\ &\leq 2^p \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left\{|R'(n)|^p + |R(n)n^{-1}|^p\right\} < \infty.\end{aligned} \tag{1.47}$$

From the series defining A_F we have

$$\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \left|(F(0) - F(k))k^{-2}\right| = \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} |R(k)|\,|k|^{-1} < \infty. \tag{1.48}$$

Estimates (1.46), (1.47) and (1.48), together with (1.45), show that the right-hand side of (1.43) converges uniformly on compacts subsets of $\mathbb{C}$ (easy application of Hölder's inequality), with A_F given by the absolutely convergent series (1.44). This concludes the proof. □

Chapter 2

The Beurling-Selberg extremal problem

2.1 Introduction

After a brief review of some useful harmonic analysis tools in our first chapter, we now direct our interest to certain problems in approximation theory. Recall that an entire function $F : \mathbb{C} \to \mathbb{C}$ is of *exponential type* at most $2\pi\delta$ if for every $\epsilon > 0$ there exists a positive constant C_ϵ, such that the inequality

$$|F(z)| \leq C_\epsilon \, e^{(2\pi\delta+\epsilon)|z|}$$

holds for all complex numbers z.

Given a real function $f : \mathbb{R} \to \mathbb{R}$ and $\delta > 0$, we address here the problem of finding an entire function K of exponential type $2\pi\delta$ such that

$$\int_{-\infty}^{\infty} |K(x) - f(x)| \, \mathrm{d}x \tag{2.1}$$

is minimized. Such a function is called a *best approximation* of f. This is a classical problem in harmonic analysis and approximation theory, considered by Bernstein, Akhiezer, Krein, Nagy and others, since at least 1938. In particular, Krein [24] in 1938 and Nagy [41] in 1939 published seminal papers solving this problem for a wide class of functions $f(x)$.

For applications to analytic number theory, it is convenient to consider an additional restriction: we ask that $K(z)$ is real on $\mathbb{R}$ and that $K(x) \geq f(x)$ for all $x \in \mathbb{R}$. In this case, a minimizer of the integral (2.1) is called an extremal majorant of $f(x)$ (or extremal upper one-sided approximation). Extremal minorants are defined analogously. Beurling started working on this one-sided extremal problem, independently, in the late 1930's, and obtained the solution for $f(x) = \mathrm{sgn}(x)$ and an inequality for almost periodic functions in an unpublished manuscript. The one-sided extremals for the signum function were later used by Selberg [37] to obtain the solution of the extremal problem for characteristic functions of intervals (of integer size, the general case was settled later, by B. Logan and alternatively by F. Littmann [30]) and a sharp form of the large sieve inequality. In these notes we are mostly interested in the one-sided version of this problem and, therefore, we shall be referring to it as the *Beurling-Selberg extremal problem*. A further discussion of the early development of this theory with many of its applications is presented in the excellent survey [43] by J. D. Vaaler.

The problem (2.1) is hard in the sense that there is no general known way to produce a solution given any $f : \mathbb{R} \to \mathbb{R}$. Besides the original examples $f(x) = \operatorname{sgn}(x)$ of Beurling and $f(x) = \chi_{[a,b]}(x)$ of Selberg, the solution for the exponential family $f(x) = e^{-\lambda|x|}$, $\lambda > 0$, was discovered by Graham and Vaaler in [19], with a first glimpse of the technique of integration on the free parameter λ to produce solutions for a family of even and odd functions. Later, the problem for $f(x) = x^n \operatorname{sgn}(x)$ and $f(x) = (x^+)^n$, where n is a positive integer, was considered by Littmann in [27, 28, 29]. Using the exponential subordination, Carneiro and Vaaler in [8, 9] extended the construction of extremal approximations for a class of even functions that includes $f(x) = \log|x|$, $f(x) = \log\big((x^2+1)/x^2\big)$ and $f(x) = |x|^\alpha$, with $-1 < \alpha < 1$. The analogous exponential subordination framework for truncated and odd functions was treated in [6].

Other classical applications of the solutions of these problems to analytic number theory include Hilbert-type inequalities [8, 19, 28, 32, 43], Erdös-Turán discrepancy inequalities [8, 25, 43], optimal approximations of periodic functions by trigonometric polynomials [2, 8, 9, 43] and Tauberian theorems [19]. The extremal problem in higher dimensions, with applications, is considered in [1, 20]. Approximations in L^p-norms with $p \neq 1$ are treated, for instance, in [15].

Our focus in this section is to present the recent advances in this field, in the form of a general method to produce extremal majorants and minorants for classes of even, odd and truncated functions subject to a certain Gaussian subordination, as developed in the works [5] and [7]. This is the most general method up to date to produce such special functions.

2.2 The extremal problem for the Gaussian

2.2.1 Statements of the main theorems

We consider the problem of majorizing and minorizing the Gaussian function

$$x \mapsto G_\lambda(x) = e^{-\pi\lambda x^2} \tag{2.2}$$

on $\mathbb{R}$ by entire functions of exponential type. Here $\lambda > 0$ is a parameter. We make use of classical interpolation techniques (as developed in Chapter 1) and integral representations to achieve this goal.

For each positive value of λ we define two entire functions of the complex variable z as follows:

$$L_\lambda(z) = \Big(\frac{\cos \pi z}{\pi}\Big)^2 \bigg\{ \sum_{m=-\infty}^{\infty} \frac{G_\lambda\big(m+\frac{1}{2}\big)}{\big(z-m-\frac{1}{2}\big)^2} + \sum_{n=-\infty}^{\infty} \frac{G'_\lambda\big(n+\frac{1}{2}\big)}{\big(z-n-\frac{1}{2}\big)} \bigg\}, \tag{2.3}$$

$$M_\lambda(z) = \Big(\frac{\sin \pi z}{\pi}\Big)^2 \bigg\{ \sum_{m=-\infty}^{\infty} \frac{G_\lambda(m)}{(z-m)^2} + \sum_{n=-\infty}^{\infty} \frac{G'_\lambda(n)}{(z-n)} \bigg\}. \tag{2.4}$$

The function $L_\lambda(z)$ is a real entire function of exponential type 2π which interpolates both the values of $G_\lambda(z)$ and the values of its derivative $G'_\lambda(z)$ on the coset $\mathbb{Z}+\frac{1}{2}$. Similarly, the function $M_\lambda(z)$ is a real entire function of exponential type 2π which interpolates both the values of $G_\lambda(z)$ and the values of its derivative $G'_\lambda(z)$ on the integers $\mathbb{Z}$. We will show that these functions satisfy the basic inequality

$$L_\lambda(x) \le G_\lambda(x) \le M_\lambda(x) \tag{2.5}$$

for all real x. Moreover, we will show that the value of each of the two integrals

$$\int_{-\infty}^{\infty} \Big\{G_\lambda(x) - L_\lambda(x)\Big\}\, \mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty} \Big\{M_\lambda(x) - G_\lambda(x)\Big\}\, \mathrm{d}x,$$

is minimized.

In order to state a more precise form of our main results for the Gaussian function, we make use of the basic theta functions. Here v is a complex variable, τ is a complex variable with $\Im\{\tau\} > 0$, $q = e^{\pi i \tau}$, and $e(z) = e^{2\pi i z}$. Our notation for the theta functions is standard and follows that of Chandrasekharan [11]. Thus we define

$$\theta_1(v,\tau) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} e\big((n+\tfrac{1}{2})v\big), \tag{2.6}$$

$$\theta_2(v,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e(nv), \tag{2.7}$$

$$\theta_3(v,\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} e(nv). \tag{2.8}$$

We note that for a fixed value of τ with $\Im\{\tau\} > 0$, each of the functions $v \mapsto \theta_1(v,\tau)$, $v \mapsto \theta_2(v,\tau)$, and $v \mapsto \theta_3(v,\tau)$ is an *even* entire function of v. The function $v \mapsto \theta_1(v,\tau)$ is periodic with period 2, and satisfies the identity

$$\theta_1(v+1,\tau) = -\theta_1(v,\tau) \tag{2.9}$$

for all complex v. Both of the functions $v \mapsto \theta_2(v,\tau)$, and $v \mapsto \theta_3(v,\tau)$, are periodic with period 1. They are related by the identity

$$\theta_2(v+\tfrac{1}{2},\tau) = \theta_3(v,\tau). \tag{2.10}$$

The transformation formulas for the theta functions (see [11, Chapter V, Theorem 9, Corollary 1]) provide a connection with the Gaussian function $G_\lambda(z)$. In particular we have

$$\sum_{n=-\infty}^{\infty} (-1)^n G_\lambda(n-v) = \lambda^{-\frac{1}{2}}\theta_1\big(v, i\lambda^{-1}\big), \tag{2.11}$$

$$\sum_{n=-\infty}^{\infty} G_\lambda(n+\tfrac{1}{2}-v) = \lambda^{-\frac{1}{2}}\theta_2\big(v, i\lambda^{-1}\big), \tag{2.12}$$

$$\sum_{n=-\infty}^{\infty} G_\lambda(n-v) = \lambda^{-\frac{1}{2}}\theta_3\big(v, i\lambda^{-1}\big). \tag{2.13}$$

We consider now the problem of minorizing $G_\lambda(z)$ on $\mathbb{R}$ by a real entire function of exponential type at most 2π.

Theorem 2.1 (Extremal minorant for the Gaussian)**.** *Let $F(z)$ be a real entire function of exponential type at most 2π such that*

$$F(x) \le G_\lambda(x)$$

for all real x. Then

$$\int_{-\infty}^{\infty} F(x)\ \mathrm{d}x \le \lambda^{-\frac{1}{2}}\theta_2\big(0, i\lambda^{-1}\big), \tag{2.14}$$

and there is equality in (2.14) *if and only if $F(z) = L_\lambda(z)$.*

Here is the analogous result for the problem of majorizing $G_\lambda(z)$ on $\mathbb{R}$ by a real entire function of exponential type at most 2π.

Theorem 2.2 (Extremal majorant for the Gaussian)**.** *Let $F(z)$ be a real entire function of exponential type at most 2π such that*

$$G_\lambda(x) \le F(x)$$

for all real x. Then

$$\lambda^{-\frac{1}{2}}\theta_3\big(0, i\lambda^{-1}\big) \le \int_{-\infty}^{\infty} F(x)\ \mathrm{d}x, \tag{2.15}$$

and there is equality in (2.15) *if and only if $F(z) = M_\lambda(z)$.*

By a simple change of variables, using Theorem 2.1 and Theorem 2.2, one can check that the real entire functions $z \mapsto L_{\lambda\delta^{-2}}(\delta z)$ and $z \mapsto M_{\lambda\delta^{-2}}(\delta z)$ are the unique extremal minorant and majorant, respectively, of exponential type $2\pi\delta$ for the function $G_\lambda(x)$.

The entire functions $L_\lambda(z)$ and $M_\lambda(z)$ have exponential type 2π, and the restrictions of these functions to $\mathbb{R}$ are both integrable. Hence their Fourier transforms

$$\widehat{L}_\lambda(t) = \int_{-\infty}^{\infty} L_\lambda(x)\, e(-xt)\ \mathrm{d}x, \quad \text{and} \quad \widehat{M}_\lambda(t) = \int_{-\infty}^{\infty} M_\lambda(x)\, e(-xt)\ \mathrm{d}x,$$

are both continuous, and both Fourier transforms are supported on the compact interval $[-1,1]$. These Fourier transforms can be given explicitly in terms of the theta functions, as a simple application of Theorem 1.12.

Theorem 2.3. *If $-1 \le t \le 1$ then the Fourier transforms $t \mapsto \widehat{L}_\lambda(t)$ and $t \mapsto \widehat{M}_\lambda(t)$ are given by*

$$\widehat{L}_\lambda(t) = (1-|t|)\theta_1(t, i\lambda) - (2\pi)^{-1}\lambda \operatorname{sgn}(t)\frac{\partial\theta_1}{\partial t}(t, i\lambda), \tag{2.16}$$

and

$$\widehat{M}_\lambda(t) = (1-|t|)\theta_3(t, i\lambda) - (2\pi)^{-1}\lambda \operatorname{sgn}(t)\frac{\partial\theta_3}{\partial t}(t, i\lambda). \tag{2.17}$$

2.2.2 Integral representations

In this section we establish several representations for combinations of Gaussian functions that will be used in the proofs of Theorems 2.1 and 2.2.

Lemma 2.4. *Let z and w be distinct complex numbers. Then we have*

$$\begin{aligned}\frac{G_\lambda(z)-G_\lambda(w)}{z-w} &= 2\pi\lambda^{\frac{3}{2}}\int_{-\infty}^{0}\int_{-\infty}^{0} e^{-2\pi\lambda tu}G_\lambda(z-t)G_\lambda(w-u)\,\mathrm{d}u\,\mathrm{d}t \\ &\quad - 2\pi\lambda^{\frac{3}{2}}\int_{0}^{\infty}\int_{0}^{\infty} e^{-2\pi\lambda tu}G_\lambda(z-t)G_\lambda(w-u)\,\mathrm{d}u\,\mathrm{d}t.\end{aligned} \tag{2.18}$$

Proof. It suffices to prove the identity (2.18) for $\lambda = 1$, then the general case will follow from an elementary change of variables. Therefore we simplify our notation and write $G(z) = G_1(z)$. We note that $G(z)$ satisfies the identity

$$G(z)^{-1} = \int_{-\infty}^{\infty} e^{2\pi zt}G(t)\ \mathrm{d}t \tag{2.19}$$

for all complex numbers z, and the identity

$$G(z)G(w)e^{2\pi zw} = G(z-w) \tag{2.20}$$

for all pairs of complex numbers z and w. From (2.19) we get

$$\begin{aligned}\frac{G(z)-G(w)}{z-w} &= G(z)G(w)\left\{\frac{G(w)^{-1}-G(z)^{-1}}{z-w}\right\} \\ &= G(z)G(w)(z-w)^{-1}\int_{-\infty}^{\infty}\left\{e^{2\pi wt}-e^{2\pi zt}\right\}G(t)\ \mathrm{d}t.\end{aligned} \tag{2.21}$$

Then using Fubini's theorem we find that

$$
\begin{aligned}
(z-w)^{-1} & \int_{-\infty}^{\infty} \left\{ e^{2\pi wt} - e^{2\pi zt} \right\} G(t)\ \mathrm{d}t \\
&= 2\pi \int_{-\infty}^{0} \left\{ \int_{t}^{0} e^{2\pi(z-w)u}\ \mathrm{d}u \right\} e^{2\pi wt} G(t)\ \mathrm{d}t \\
&\quad - 2\pi \int_{0}^{\infty} \left\{ \int_{0}^{t} e^{2\pi(z-w)u}\ \mathrm{d}u \right\} e^{2\pi wt} G(t)\ \mathrm{d}t \\
&= 2\pi \int_{-\infty}^{0} \left\{ \int_{-\infty}^{u} e^{2\pi wt} G(t)\ \mathrm{d}t \right\} e^{2\pi(z-w)u}\ \mathrm{d}u \\
&\quad - 2\pi \int_{0}^{\infty} \left\{ \int_{u}^{\infty} e^{2\pi wt} G(t)\ \mathrm{d}t \right\} e^{2\pi(z-w)u}\ \mathrm{d}u \qquad (2.22) \\
&= 2\pi \int_{-\infty}^{0} \left\{ \int_{-\infty}^{0} e^{2\pi w(t+u)} G(t+u)\ \mathrm{d}t \right\} e^{2\pi(z-w)u}\ \mathrm{d}u \\
&\quad - 2\pi \int_{0}^{\infty} \left\{ \int_{0}^{\infty} e^{2\pi w(t+u)} G(t+u)\ \mathrm{d}t \right\} e^{2\pi(z-w)u}\ \mathrm{d}u \\
&= 2\pi \int_{-\infty}^{0} \int_{-\infty}^{0} e^{2\pi(wt+zu)} G(t+u)\ \mathrm{d}t\ \mathrm{d}u \\
&\quad - 2\pi \int_{0}^{\infty} \int_{0}^{\infty} e^{2\pi(wt+zu)} G(t+u)\ \mathrm{d}t\ \mathrm{d}u.
\end{aligned}
$$

Next we apply (2.20) twice and get

$$
\begin{aligned}
G(z)G(w)e^{2\pi(wt+zu)}G(t+u) &= G(z)G(w)G(u)G(t)e^{-2\pi tu+2\pi wt+2\pi zu} \\
&= G(z-u)G(w-t)e^{-2\pi tu}.
\end{aligned} \qquad (2.23)
$$

Then we combine (2.21), (2.22) and (2.23) to obtain the special case

$$
\begin{aligned}
\frac{G(z)-G(w)}{z-w} &= 2\pi \int_{-\infty}^{0} \int_{-\infty}^{0} e^{-2\pi tu} G(z-t)G(w-u)\ \mathrm{d}u\ \mathrm{d}t \\
&\quad - 2\pi \int_{0}^{\infty} \int_{0}^{\infty} e^{-2\pi tu} G(z-t)G(w-u)\ \mathrm{d}u\ \mathrm{d}t.
\end{aligned} \qquad (2.24)
$$

The more general identity (2.18) follows by replacing z with $\lambda^{\frac{1}{2}} z$, by replacing w with $\lambda^{\frac{1}{2}} w$, and by making a corresponding change of variables in each integral on the right of (2.24). □

Lemma 2.5. *Let z and w be distinct complex numbers. Then we have*

$$
\begin{aligned}
& \frac{G_\lambda(z)}{(z-w)^2} - \frac{G_\lambda(w)}{(z-w)^2} - \frac{G_\lambda'(w)}{z-w} \\
&= (2\pi)^2 \lambda^{\frac{5}{2}} \int_{-\infty}^{0} \int_{-\infty}^{0} t e^{-2\pi\lambda tu} G_\lambda(z-t) \left\{ G_\lambda(w) - G_\lambda(w-u) \right\}\ \mathrm{d}u\ \mathrm{d}t \qquad (2.25) \\
&\quad - (2\pi)^2 \lambda^{\frac{5}{2}} \int_{0}^{\infty} \int_{0}^{\infty} t e^{-2\pi\lambda tu} G_\lambda(z-t) \left\{ G_\lambda(w) - G_\lambda(w-u) \right\}\ \mathrm{d}u\ \mathrm{d}t.
\end{aligned}
$$

Proof. We differentiate both sides of (2.18) with respect to w and obtain the identity

$$\begin{aligned}
&\frac{G_\lambda(z)}{(z-w)^2} - \frac{G_\lambda(w)}{(z-w)^2} - \frac{G'_\lambda(w)}{z-w} \\
&= 2\pi\lambda^{\frac{3}{2}} \int_{-\infty}^{0}\int_{-\infty}^{0} e^{-2\pi\lambda tu} G_\lambda(z-t) G'_\lambda(w-u)\ \mathrm{d}u\ \mathrm{d}t \\
&\quad - 2\pi\lambda^{\frac{3}{2}} \int_{0}^{\infty}\int_{0}^{\infty} e^{-2\pi\lambda tu} G_\lambda(z-t) G'_\lambda(w-u)\ \mathrm{d}u\ \mathrm{d}t.
\end{aligned} \tag{2.26}$$

Using integration by parts we get

$$\begin{aligned}
&\int_{-\infty}^{0} e^{-2\pi\lambda tu} G'_\lambda(w-u)\ \mathrm{d}u \\
&\qquad = 2\pi\lambda \int_{-\infty}^{0} t e^{-2\pi\lambda tu} \big\{ G_\lambda(w) - G_\lambda(w-u) \big\}\ \mathrm{d}u,
\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}
&\int_{0}^{\infty} e^{-2\pi\lambda tu} G'_\lambda(w-u)\ \mathrm{d}u \\
&\qquad = 2\pi\lambda \int_{0}^{\infty} t e^{-2\pi\lambda tu} \big\{ G_\lambda(w) - G_\lambda(w-u) \big\}\ \mathrm{d}u.
\end{aligned} \tag{2.28}$$

The lemma now follows now by combining (2.26), (2.27) and (2.28). □

In order to apply the identities (2.11), (2.12) and (2.13), we require simple estimates for certain partial sums.

Lemma 2.6. *For all real u and positive integers N, we have*

$$\sum_{n=-N-1}^{N} (-1)^n G_\lambda(n + \tfrac{1}{2} - u) \ll_\lambda \min\{1, |u|\}, \tag{2.29}$$

$$\sum_{n=-N-1}^{N} \big\{ G_\lambda(n+\tfrac{1}{2}) - G_\lambda(n + \tfrac{1}{2} - u) \big\} \ll_\lambda \min\{1, |u|\}, \tag{2.30}$$

$$\sum_{n=-N}^{N} \big\{ G_\lambda(n) - G_\lambda(n-u) \big\} \ll_\lambda \min\{1, |u|\}, \tag{2.31}$$

where the constant implied by $\ll_\lambda$ depends on λ, but not on u or N. Moreover, if $S_{\lambda,N}(u)$ denotes the sum on the left of (2.29), then for $0 < t$ we have

$$\left| \int_0^\infty e^{-2\pi\lambda tu} S_{\lambda,N}(u)\ \mathrm{d}u \right| \le \lambda^{-\frac{1}{2}}, \tag{2.32}$$

and for $t < 0$ we have

$$\left| \int_{-\infty}^0 e^{-2\pi\lambda tu} S_{\lambda,N}(u)\ \mathrm{d}u \right| \le \lambda^{-\frac{1}{2}}. \tag{2.33}$$

Proof. For each positive integer N,

$$u \mapsto S_{\lambda,N}(u) = \sum_{n=-N-1}^{N} (-1)^n G_\lambda(n + \tfrac{1}{2} - u)$$

is an odd function of u. Hence its derivative is an even function of u. Therefore we get

$$\begin{aligned} \left|S_{\lambda,N}(u)\right| &= \left|\int_0^u S'_{\lambda,N}(v)\ \mathrm{d}v\right| \\ &\le \int_0^{|u|} \left\{ \sum_{n=-\infty}^{\infty} \left|G'_\lambda(n + \tfrac{1}{2} - v)\right| \right\} \mathrm{d}v \\ &\le C_\lambda |u|, \end{aligned}$$

where

$$C_\lambda = \sup_{v\in\mathbb{R}} \left\{ \sum_{n=-\infty}^{\infty} \left|G'_\lambda(n + \tfrac{1}{2} - v)\right| \right\}$$

is obviously finite. We also have

$$\left|S_{\lambda,N}(u)\right| \le \sup_{v\in\mathbb{R}} \left\{ \sum_{n=-\infty}^{\infty} \left|G_\lambda(n + \tfrac{1}{2} - v)\right| \right\} < \infty,$$

and the bound (2.29) follows. The proofs of (2.30) and (2.31) are very similar.

Let $0 < t$ and $0 < u$. For positive integers N we define

$$R_{\lambda,N}(u) = \int_0^u S_{\lambda,N}(v)\ \mathrm{d}v.$$

Then it follows, using integration by parts, that

$$\int_0^\infty e^{-2\pi\lambda tu} S_{\lambda,N}(u)\ \mathrm{d}u = 2\pi\lambda t \int_0^\infty e^{-2\pi\lambda tu} R_{\lambda,N}(u)\ \mathrm{d}u. \tag{2.34}$$

For $\alpha < \beta$, let

$$\chi_{\alpha,\beta}(x) = \tfrac{1}{2}\operatorname{sgn}(\beta - x) + \tfrac{1}{2}\operatorname{sgn}(x - \alpha)$$

denote the normalized characteristic function of the real interval with endpoints α and β. Using the inequality

$$\left| \sum_{n=-N-1}^{N} (-1)^n \chi_{n+\frac{1}{2}-u,n+\frac{1}{2}}(x) \right| \le 1,$$

we find that

$$\begin{aligned}
\left|R_{\lambda,N}(u)\right| &= \left|\sum_{n=-N-1}^{N} (-1)^n \int_0^u G_\lambda(n+\tfrac{1}{2}-v)\ \mathrm{d}v\right| \\
&= \left|\int_{-\infty}^{\infty} \Big\{ \sum_{n=-N-1}^{N} (-1)^n \chi_{n+\frac{1}{2}-u,n+\frac{1}{2}}(w)\Big\} G_\lambda(w)\ \mathrm{d}w\right| \\
&\le \int_{-\infty}^{\infty} G_\lambda(w)\ \mathrm{d}w \\
&= \lambda^{-\frac{1}{2}}.
\end{aligned}$$

Then using (2.34) we get

$$\begin{aligned}
\left|\int_0^\infty e^{-2\pi\lambda tu} S_{\lambda,N}(u)\ \mathrm{d}u\right| &\le 2\pi\lambda t \int_0^\infty e^{-2\pi\lambda tu} \left|R_{\lambda,N}(u)\right|\ \mathrm{d}u \\
&\le 2\pi\lambda^{\frac{1}{2}} t \int_0^\infty e^{-2\pi\lambda tu}\ \mathrm{d}u \\
&= \lambda^{-\frac{1}{2}}.
\end{aligned}$$

This verifies (2.32), and (2.33) follows from (2.32) because $u \mapsto S_{\lambda,N}(u)$ is an odd function. □

Because $z \mapsto L_\lambda(z)$ interpolates both the value of $G_\lambda(z)$ and the value of its derivative $G'_\lambda(z)$ at each point of the coset $\mathbb{Z} + \frac{1}{2}$, the entire function

$$z \mapsto G_\lambda(z) - L_\lambda(z)$$

has a zero of multiplicity at least 2 at each point of $\mathbb{Z} + \frac{1}{2}$. It follows that

$$z \mapsto \left(\frac{\pi}{\cos \pi z}\right)^2 \Big\{G_\lambda(z) - L_\lambda(z)\Big\}$$

is an entire function. In a similar manner, we find that

$$z \mapsto \left(\frac{\pi}{\sin \pi z}\right)^2 \Big\{M_\lambda(z) - G_\lambda(z)\Big\}$$

is an entire function.

Lemma 2.7. *For all complex z we have*

$$\begin{aligned}
&\left(\frac{\pi}{\cos \pi z}\right)^2 \Big\{G_\lambda(z) - L_\lambda(z)\Big\} \\
&= 2\pi^2\lambda^2 \int_{-\infty}^{\infty} \frac{tG_\lambda(z-t)}{\sinh \pi\lambda t} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi\lambda tu} \big\{\theta_3\big(u, i\lambda^{-1}\big) - \theta_3\big(\tfrac{1}{2}, i\lambda^{-1}\big)\big\}\, \mathrm{d}u\, \mathrm{d}t,
\end{aligned} \tag{2.35}$$

and

$$
\begin{aligned}
&\left(\frac{\pi}{\sin \pi z}\right)^2 \Big\{M_\lambda(z) - G_\lambda(z)\Big\} \\
&\quad = 2\pi^2\lambda^2 \int_{-\infty}^{\infty} \frac{tG_\lambda(z-t)}{\sinh \pi\lambda t} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi\lambda tu} \big\{\theta_2\big(\tfrac{1}{2}, i\lambda^{-1}\big) - \theta_2\big(u, i\lambda^{-1}\big)\big\}\, \mathrm{d}u\, \mathrm{d}t.
\end{aligned} \tag{2.36}
$$

Proof. In order to establish (2.35) we use the partial fraction expansion

$$
\lim_{N\to\infty} \sum_{n=-N-1}^{N} \frac{1}{\big(z-n-\frac{1}{2}\big)^2} = \left(\frac{\pi}{\cos \pi z}\right)^2, \tag{2.37}
$$

which converges uniformly on compact subsets of $\mathbb{C} \setminus \big\{\mathbb{Z} + \frac{1}{2}\big\}$. Then it follows from (2.3) and (2.37) that

$$
\begin{aligned}
&\left(\frac{\pi}{\cos \pi z}\right)^2 \Big\{G_\lambda(z) - L_\lambda(z)\Big\} \\
&\quad = \lim_{N\to\infty} \sum_{n=-N-1}^{N} \left\{\frac{G_\lambda(z)}{(z-n-\frac{1}{2})^2} - \frac{G_\lambda(n+\frac{1}{2})}{(z-n-\frac{1}{2})^2} - \frac{G'_\lambda(n+\frac{1}{2})}{z-n-\frac{1}{2}}\right\}.
\end{aligned} \tag{2.38}
$$

Note that the limit on the right of (2.38) converges uniformly on compact subsets of $\mathbb{C}$. For positive integers N and all real u let

$$
T_{\lambda,N}(u) = \sum_{n=-N-1}^{N} \big\{G_\lambda(n+\tfrac{1}{2}) - G_\lambda(n+\tfrac{1}{2}-u)\big\}.
$$

From (2.12) we conclude that

$$
\lim_{N\to\infty} T_{\lambda,N}(u) = \lambda^{-\frac{1}{2}} \big\{\theta_2\big(0, i\lambda^{-1}\big) - \theta_2\big(u, i\lambda^{-1}\big)\big\}. \tag{2.39}
$$

We apply the identity (2.25) with $w = n + \frac{1}{2}$ and sum over the integers n satisfying $-N-1 \le n \le N$. We get

$$
\begin{aligned}
&\sum_{n=-N-1}^{N} \left\{\frac{G_\lambda(z)}{(z-n-\frac{1}{2})^2} - \frac{G_\lambda(n+\frac{1}{2})}{(z-n-\frac{1}{2})^2} - \frac{G'_\lambda(n+\frac{1}{2})}{z-n-\frac{1}{2}}\right\} \\
&\qquad = (2\pi)^2\lambda^{\frac{5}{2}} \int_{-\infty}^{0}\int_{-\infty}^{0} te^{-2\pi\lambda tu} G_\lambda(z-t) T_{\lambda,N}(u)\ \mathrm{d}u\ \mathrm{d}t \\
&\qquad\quad - (2\pi)^2\lambda^{\frac{5}{2}} \int_{0}^{\infty}\int_{0}^{\infty} te^{-2\pi\lambda tu} G_\lambda(z-t) T_{\lambda,N}(u)\ \mathrm{d}u\ \mathrm{d}t.
\end{aligned} \tag{2.40}
$$

We now let $N \to \infty$ on both sides of (2.40). The limit on the left-hand side is determined by (2.38). On the right-hand side we use (2.30), the dominated convergence theorem and (2.39). In this way we obtain the identity

$$\begin{aligned}
&\left(\frac{\pi}{\cos\pi z}\right)^2\Big\{G_\lambda(z)-L_\lambda(z)\Big\}\\
&=(2\pi\lambda)^2\int_{-\infty}^0\int_{-\infty}^0 te^{-2\pi\lambda tu}G_\lambda(z-t)\big\{\theta_2\big(0,i\lambda^{-1}\big)-\theta_2\big(u,i\lambda^{-1}\big)\big\}\,\mathrm{d}u\,\mathrm{d}t \\
&-(2\pi\lambda)^2\int_0^\infty\int_0^\infty te^{-2\pi\lambda tu}G_\lambda(z-t)\big\{\theta_2\big(0,i\lambda^{-1}\big)-\theta_2\big(u,i\lambda^{-1}\big)\big\}\,\mathrm{d}u\,\mathrm{d}t.
\end{aligned} \tag{2.41}$$

If $0<t$, using that $v\mapsto\theta_2(v,\tau)$ has period 1 and (2.10), we get

$$\begin{aligned}
&\int_0^\infty e^{-2\pi\lambda tu}\big\{\theta_2\big(0,i\lambda^{-1}\big)-\theta_2\big(u,i\lambda^{-1}\big)\big\}\ \mathrm{d}u\\
&\qquad=\sum_{m=0}^\infty\int_0^1 e^{-2\pi\lambda t(u+m)}\big\{\theta_2\big(0,i\lambda^{-1}\big)-\theta_2\big(u+m,i\lambda^{-1}\big)\big\}\ \mathrm{d}u\\
&\qquad=\big\{1-e^{-2\pi\lambda t}\big\}^{-1}\int_0^1 e^{-2\pi\lambda tu}\big\{\theta_2\big(0,i\lambda^{-1}\big)-\theta_2\big(u,i\lambda^{-1}\big)\big\}\ \mathrm{d}u\\
&\qquad=\big\{2\sinh\pi\lambda t\big\}^{-1}\int_{-\frac12}^{\frac12} e^{-2\pi\lambda tu}\big\{\theta_3\big(\tfrac12,i\lambda^{-1}\big)-\theta_3\big(u,i\lambda^{-1}\big)\big\}\ \mathrm{d}u.
\end{aligned} \tag{2.42}$$

If $t<0$, in a similar manner, we find that

$$\begin{aligned}
&\int_{-\infty}^0 e^{-2\pi\lambda tu}\big\{\theta_2\big(0,i\lambda^{-1}\big)-\theta_2\big(u,i\lambda^{-1}\big)\big\}\ \mathrm{d}u\\
&\qquad=-\big\{2\sinh\pi\lambda t\big\}^{-1}\int_{-\frac12}^{\frac12} e^{-2\pi\lambda tu}\big\{\theta_3\big(\tfrac12,i\lambda^{-1}\big)-\theta_3\big(u,i\lambda^{-1}\big)\big\}\ \mathrm{d}u.
\end{aligned} \tag{2.43}$$

The identity (2.35) follows now by combining (2.41), (2.42) and (2.43).

The proof of (2.36) proceeds along the same lines using (2.13) and (2.31). We leave the details to the reader. □

Corollary 2.8. *For all real values of x we have*

$$0<\left(\frac{\pi}{\cos\pi x}\right)^2\Big\{G_\lambda(x)-L_\lambda(x)\Big\}, \tag{2.44}$$

and

$$0<\left(\frac{\pi}{\sin\pi x}\right)^2\Big\{M_\lambda(x)-G_\lambda(x)\Big\}. \tag{2.45}$$

In particular, the inequality (2.5) *holds for all real x.*

Proof. For real u the periodic function $u\mapsto\theta_3\big(u,i\lambda^{-1}\big)$ takes its maximum value at $u=0$ and its minimum values at $u=\frac12$. Therefore the function

$$t\mapsto\int_{-\frac12}^{\frac12} e^{-2\pi\lambda tu}\big\{\theta_3\big(u,i\lambda^{-1}\big)-\theta_3\big(\tfrac12,i\lambda^{-1}\big)\big\}\ \mathrm{d}u,$$

which appears in the integrand on the right of (2.35), is positive for all real values of t. This plainly verifies the inequality (2.44).

In a similar manner using (2.10), the periodic function $u \mapsto \theta_2\big(u, i\lambda^{-1}\big)$ takes its maximum value at $u = \frac{1}{2}$ and its minimum value at $u = 0$. Hence the function

$$t \mapsto \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi\lambda t u}\big\{\theta_2\big(\tfrac{1}{2}, i\lambda^{-1}\big) - \theta_2\big(u, i\lambda^{-1}\big)\big\}\ \mathrm{d}u,$$

which appears in the integrand on the right of (2.36), is positive for all real values of t. This establishes the inequality (2.45). □

2.2.3 Proofs of Theorems 2.1 and 2.2

Let $F(z)$ be an entire function of exponential type at most 2π such that

$$F(x) \le G_\lambda(x) \tag{2.46}$$

for all real x. Clearly we may assume that $x \mapsto F(x)$ is integrable on $\mathbb{R}$, for if not then (2.14) is trivial. By Theorem 1.11 we know that $F'(x)$ is also integrable and thus F has bounded variation. By the Poisson summation formula, (2.12) and (2.46), we find that

$$\begin{aligned} \int_{-\infty}^{\infty} F(x)\ \mathrm{d}x &= \lim_{N\to\infty} \sum_{n=-N}^{N} F(n+v) \\ &\le \lim_{N\to\infty} \sum_{n=-N}^{N} G_\lambda(n+v) \\ &= \lambda^{-\frac{1}{2}}\theta_2\big(\tfrac{1}{2} - v, i\lambda^{-1}\big) \end{aligned} \tag{2.47}$$

for all real v. We have already noted that $v \mapsto \theta_2\big(\frac{1}{2} - v, i\lambda^{-1}\big)$ takes its minimum value at $v = \frac{1}{2}$. Hence (2.47) implies that

$$\int_{-\infty}^{\infty} F(x)\ \mathrm{d}x \le \lambda^{-\frac{1}{2}}\theta_2\big(0, i\lambda^{-1}\big),$$

and this proves (2.14).

In Corollary 2.8 we proved that $F(z) = L_\lambda(z)$ satisfies the inequality (2.46) for all real x. In this special case there is equality in the inequality (2.47) when $v = \frac{1}{2}$. Thus we have

$$\int_{-\infty}^{\infty} L_\lambda(x)\ \mathrm{d}x = \lambda^{-\frac{1}{2}}\theta_2\big(0, i\lambda^{-1}\big). \tag{2.48}$$

Now *assume* that $F(z)$ is an entire function of exponential type at most 2π that satisfies (2.46) for all real x, and assume that there is equality in the inequality (2.47) when $v = \frac{1}{2}$. Then we must have

$$F(n + \tfrac{1}{2}) = G_\lambda(n + \tfrac{1}{2})$$

for all integers n. Then from (2.46) we also get

$$F'(n+\tfrac{1}{2}) = G'_\lambda(n+\tfrac{1}{2})$$

for all integers n. Of course this shows that the entire function

$$z \mapsto F(z) - L_\lambda(z) \tag{2.49}$$

is integrable on $\mathbb{R}$, has exponential type at most 2π, vanishes at each point of $\mathbb{Z}+\frac{1}{2}$, and its derivative also vanishes at each point of $\mathbb{Z}+\frac{1}{2}$. By an application of Theorem 1.12 (with an appropriate shift of $\frac{1}{2}$) we conclude that the entire function (2.49) is identically zero. This proves Theorem 2.1, and Theorem 2.2 can be proved by the same sort of argument.

2.3 Distribution framework for even functions

2.3.1 The Paley-Wiener theorem for distributions

Let $\mathcal{D}(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R}) \subseteq \mathcal{E}(\mathbb{R})$ be the usual spaces of C^∞ functions on $\mathbb{R}$ as defined in the work of L. Schwartz [36], and let $\mathcal{E}'(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$ be the corresponding dual spaces of distributions. Our notation and terminology for distributions follows that of [18], and precise definitions for these spaces are given in [18, Section 2.3]. We write $\varphi(x)$ for a generic element in the space $\mathcal{S}(\mathbb{R})$ of Schwartz functions. If T in $\mathcal{S}'(\mathbb{R})$ is a tempered distribution we write $T(\varphi)$ for the value of T at φ. Then the Fourier transform of T is the tempered distribution $\widehat{T}$ defined by

$$\widehat{T}(\varphi) = T(\widehat{\varphi}),$$

where

$$\widehat{\varphi}(y) = \int_{-\infty}^{\infty} \varphi(x) e(-yx)\, \mathrm{d}x$$

is the Fourier transform of the function φ. Functions $g : \mathbb{R} \to \mathbb{R}$ in any L^p class or with polynomial growth can be regarded as elements of $\mathcal{S}'(\mathbb{R})$ and we will usually make the identification

$$g(\varphi) = \int_{-\infty}^{\infty} g(x)\, \varphi(x)\, \mathrm{d}x$$

for all φ in $\mathcal{S}(\mathbb{R})$.

We recall the following form of the Paley-Wiener theorem for distributions, which is obtained by combining Theorem 1.7.5 and Theorem 1.7.7 in [21].

Theorem 2.9 (Paley-Wiener for distributions). *Let $\delta > 0$, and let U be a tempered distribution in $\mathcal{S}'(\mathbb{R})$ with Fourier transform $\widehat{U}$ supported in the compact interval $[-\delta, \delta]$. Then $\widehat{U}$ belongs to $\mathcal{E}'(\mathbb{R})$, and*

$$z \mapsto F(z) = \widehat{U}_\xi\big(e(\xi z)\big)$$

defines an entire function of the complex variable $z = x + iy$ *such that*

$$\big|F(z)\big| \ll_B \big(1+|z|\big)^B \exp\{2\pi\delta|y|\} \tag{2.50}$$

for some number $B \geq 0$ *and all* z *in* $\mathbb{C}$. *Moreover, the entire function* $F(z)$ *satisfies the identity*

$$U(\varphi) = \int_{-\infty}^{\infty} F(x)\,\varphi(x)\,\mathrm{d}x$$

for all φ *in* $\mathcal{S}(\mathbb{R})$.

Conversely, suppose that $F(z)$ *is an entire function of the complex variable* z *that satisfies the inequality* (2.50) *for some numbers* $B \geq 0$ *and* $\delta > 0$. *Then there exists a tempered distribution* V *in* $\mathcal{S}'(\mathbb{R})$ *such that* $\widehat{V}$ *belongs to* $\mathcal{E}'(\mathbb{R})$, $\widehat{V}$ *is supported on the compact interval* $[-\delta, \delta]$,

$$F(z) = \widehat{V}_\xi\big(e(\xi z)\big),$$

and

$$V(\varphi) = \int_{-\infty}^{\infty} F(x)\,\varphi(x)\,\mathrm{d}x$$

for all φ *in* $\mathcal{S}(\mathbb{R})$.

Here we write $\widehat{U}_\xi$ to indicate that the distribution $\widehat{U}$ is acting on the function $\xi \mapsto \big(e(\xi z)\big)$.

2.3.2 Integrating the free parameter

Our goal now is to be able to integrate the parameter λ with respect to a suitable non-negative Borel measure ν on $[0,\infty)$ and obtain the solution of the extremal problem for a different function. One might first guess that the class of suitable measures ν on $[0,\infty)$ would consist of those measures for which the function

$$g(x) = \int_0^{\infty} G_\lambda(x)\,\mathrm{d}\nu(\lambda)$$

is well defined, and that this would be the function to be approximated. Such a method was carried out in [8], [9] and [19] with the Gaussian replaced by exponential functions. It turns out that this condition is unnecessarily restrictive, and in order to find the very minimal conditions to be imposed on the measure ν one must look at things on the Fourier transform side.

We will illustrate what this condition should be in the minorant case. Define the difference function

$$D_\lambda(x) = G_\lambda(x) - L_\lambda(x) \geq 0.$$

The minimal integral corresponds to

$$\int_{-\infty}^{\infty} \{G_\lambda(x) - L_\lambda(x)\}\,\mathrm{d}x = \widehat{D}_\lambda(0).$$

If we succeed in our attempt to integrate the parameter λ, we will end up solving an extremal problem for which the value of the minimal integral is given by (and thus we want to impose this finiteness condition)

$$\int_0^\infty \int_{-\infty}^\infty \{G_\lambda(x) - L_\lambda(x)\}\,\mathrm{d}x\,\mathrm{d}\nu(\lambda) = \int_0^\infty \widehat{D}_\lambda(0)\,\mathrm{d}\nu(\lambda) < \infty. \qquad (2.51)$$

We will show that this is also a sufficient condition, provided we can define appropriately the real function to be minorized.

Suppose ν is a non-negative Borel measure on $[0,\infty)$ satisfying (2.51). Since

$$|\widehat{D}_\lambda(t)| \leq \widehat{D}_\lambda(0)$$

for all $t \in \mathbb{R}$, we observe that the function

$$t \mapsto \int_0^\infty \widehat{D}_\lambda(t)\,\mathrm{d}\nu(\lambda)$$

is well defined. In particular, from the classical Paley-Wiener theorem, the Fourier transform $t \mapsto \widehat{L}_\lambda(t)$ is supported on $[-1,1]$, and therefore

$$\int_0^\infty \widehat{D}_\lambda(t)\,\mathrm{d}\nu(\lambda) = \int_0^\infty \widehat{G}_\lambda(t)\,\mathrm{d}\nu(\lambda)$$

for $|t| \geq 1$. We are now in position to state the main results of this section. In the following theorems we write

$$[\alpha,\beta]^c = (-\infty,\alpha) \cup (\beta,\infty)$$

for the complement in $\mathbb{R}$ of a closed interval $[\alpha,\beta]$. Recall also that the Fourier transform of the Gaussian is given by

$$\widehat{G}_\lambda(t) = \lambda^{-\frac{1}{2}} e^{-\pi\lambda^{-1}t^2}.$$

Theorem 2.10 (Distribution Theorem - Minorant)**.** *Let ν be a non-negative Borel measure on $[0,\infty)$ satisfying*

$$\int_0^\infty \int_{-\infty}^\infty \{G_\lambda(x) - L_\lambda(x)\}\,\mathrm{d}x\,\mathrm{d}\nu(\lambda) < \infty.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a function of polynomial growth (thus an element of $\mathcal{S}'(\mathbb{R})$) that is continuous on $\mathbb{R}/\{0\}$, differentiable on $\mathbb{R}/\{0\}$, and such that

$$\widehat{g}(\varphi) = \int_{-\infty}^\infty \left\{\int_0^\infty \widehat{G}_\lambda(t)\,\mathrm{d}\nu(\lambda)\right\}\varphi(t)\,\mathrm{d}t$$

for all Schwartz functions φ supported on $[-1,1]^c$. Then there exists a unique extremal minorant $l(z)$ of exponential type 2π for $g(x)$. The function $l(x)$ interpolates the values of $g(x)$ at $\mathbb{Z}+\frac{1}{2}$ and satisfies

$$\int_{-\infty}^\infty \{g(x) - l(x)\}\,\mathrm{d}x = \int_0^\infty \int_{-\infty}^\infty \{G_\lambda(x) - L_\lambda(x)\}\,\mathrm{d}x\,\mathrm{d}\nu(\lambda).$$

Theorem 2.11 (Distribution Theorem - Majorant)**.** *Let ν be a non-negative Borel measure on $[0,\infty)$ satisfying*

$$\int_0^\infty \int_{-\infty}^\infty \{M_\lambda(x) - G_\lambda(x)\}\, \mathrm{d}x\, \mathrm{d}\nu(\lambda) < \infty.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a function of polynomial growth (thus an element of $\mathcal{S}'(\mathbb{R})$) that is continuous on $\mathbb{R}$, differentiable on $\mathbb{R}/\{0\}$, and such that

$$\widehat{g}(\varphi) = \int_{-\infty}^\infty \left\{ \int_0^\infty \widehat{G}_\lambda(t)\, \mathrm{d}\nu(\lambda) \right\} \varphi(t)\, \mathrm{d}t$$

for all Schwartz functions φ supported on $[-1,1]^c$. Then there exists a unique extremal majorant $m(z)$ of exponential type 2π for $g(x)$. The function $m(x)$ interpolates the values of $g(x)$ at $\mathbb{Z}$ and satisfies

$$\int_{-\infty}^\infty \{m(x) - g(x)\}\, \mathrm{d}x = \int_0^\infty \int_{-\infty}^\infty \{M_\lambda(x) - G_\lambda(x)\}\, \mathrm{d}x\, \mathrm{d}\nu(\lambda).$$

Similar results can be stated for the problem of majorizing or minorizing by functions of exponential type $2\pi\delta$. It is a matter of changing the interpolation points to $\delta\mathbb{Z}$ or $\delta(\mathbb{Z} + \frac{1}{2})$, and changing the support intervals to $[-\delta,\delta]^c$. For simplicity, we will proceed in our exposition only with type 2π.

The condition

$$\widehat{g}(\varphi) = \int_{-\infty}^\infty \left\{ \int_0^\infty \widehat{G}_\lambda(t)\, \mathrm{d}\nu(\lambda) \right\} \varphi(t)\, \mathrm{d}t$$

for all Schwartz functions φ supported on $[-\delta,\delta]^c$, that appears on the statements of the theorems, asserts that the Fourier transform $\widehat{g}$, which is a tempered distribution, is actually given by a function

$$t \mapsto \int_0^\infty \widehat{G}_\lambda(t)\, \mathrm{d}\nu(\lambda)$$

outside the interval $[-\delta,\delta]$. This is a typical behavior of functions with polynomial growth, that might have the Fourier transform given by a singular part supported on the origin plus an additional component given by a function outside the origin (e.g. the Fourier transform of $-\log|x|$ is given by $(2|t|)^{-1}$ away from the origin). It is clear in this context that the only information relevant for the Beurling-Selberg extremal problem is knowledge of the Fourier transform of the original function outside a compact interval.

Finally, we shall see that this method is quite powerful, producing most of the previously known examples in the literature, and a wide class of new ones. In particular, we will be able solve the extremal problem for functions such as

$$\log|x|, \quad |x|^\sigma, \quad -\log\left(\frac{x^2+\alpha^2}{x^2+\beta^2}\right) \quad \text{and} \quad 1 - x\arctan\left(\frac{1}{x}\right).$$

where $\sigma > -1$ and $0 \leq \alpha < \beta$. The last two of these functions will play an important role in the applications to the theory of the Riemann zeta-function developed in the next chapter.

2.3.3 Proofs of Theorems 2.10 and 2.11

Here we give a detailed proof of Theorem 2.10. The proof of Theorem 2.11 follows the same general method. First we construct the extreme minorant. Recall that

$$D_\lambda(x) = G_\lambda(x) - L_\lambda(x) \geq 0.$$

Then for each $x \in \mathbb{R}$ we define the non-negative valued function

$$d(x) = \int_0^\infty D_\lambda(x)\,\mathrm{d}\nu(\lambda). \tag{2.52}$$

It may happen that the value of $d(x)$ is $+\infty$ at some points x. However, the function $x \mapsto d(x)$ is integrable on $\mathbb{R}$, because

$$\int_{-\infty}^\infty d(x)\,\mathrm{d}x = \int_0^\infty \int_{-\infty}^\infty D_\lambda(x)\,\mathrm{d}x\,\mathrm{d}\nu(\lambda) = \int_0^\infty \widehat{D}_\lambda(0)\,\mathrm{d}\nu(\lambda) < \infty,$$

by the hypotheses of our theorem. Hence the Fourier transform $\widehat{d}(t)$ is a continuous function given by

$$\begin{aligned}\widehat{d}(t) = \int_{-\infty}^\infty d(x)\,e(-tx)\,\mathrm{d}x &= \int_{-\infty}^\infty \int_0^\infty D_\lambda(x)\,e(-tx)\,\mathrm{d}\nu(\lambda)\,\mathrm{d}x \\ &= \int_0^\infty \int_{-\infty}^\infty D_\lambda(x)\,e(-tx)\,\mathrm{d}x\,\mathrm{d}\nu(\lambda) = \int_0^\infty \widehat{D}_\lambda(t)\,\mathrm{d}\nu(\lambda),\end{aligned} \tag{2.53}$$

and for $|t| \geq 1$ we have

$$\widehat{d}(t) = \int_0^\infty \widehat{G}_\lambda(t)\,\mathrm{d}\nu(\lambda). \tag{2.54}$$

Let $U \in \mathcal{S}'(\mathbb{R})$ be the tempered distribution defined by

$$U(\varphi) = \int_{-\infty}^\infty \{g(x) - d(x)\}\,\varphi(x)\,\mathrm{d}x. \tag{2.55}$$

We shall prove that the Fourier transform $\widehat{U}$ is supported on $[-1,1]$. In fact, for any $\varphi \in \mathcal{S}(\mathbb{R})$ with support in $[-1,1]^c$ we have

$$\begin{aligned}\widehat{U}(\varphi) &= \widehat{g}(\varphi) - \widehat{d}(\varphi) \\ &= \int_{-\infty}^\infty \left\{\int_0^\infty \widehat{G}_\lambda(t)\,\mathrm{d}\nu(\lambda)\right\}\varphi(t)\,\mathrm{d}t - \int_{-\infty}^\infty \widehat{d}(t)\,\varphi(t)\,\mathrm{d}t = 0\,,\end{aligned}$$

by (2.54) and the hypotheses of the theorem. By the Paley-Wiener theorem for distributions we find that $\widehat{U} \in \mathcal{E}'(\mathbb{R})$, and therefore

$$z \mapsto l(z) = \widehat{U}_\xi\left(e(\xi z)\right)$$

defines an entire function of exponential type 2π such that

$$U(\varphi) = \int_{-\infty}^\infty l(x)\,\varphi(x)\,\mathrm{d}x \tag{2.56}$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$. From (2.55) and (2.56) we conclude that

$$d(x) = g(x) - l(x) \geq 0 \tag{2.57}$$

for almost all $x \in \mathbb{R}$. In particular, we get

$$\begin{aligned}\int_{-\infty}^{\infty} \{g(x) - l(x)\}\,\mathrm{d}x &= \int_{-\infty}^{\infty} d(x)\,\mathrm{d}x = \int_{0}^{\infty} \widehat{D}_\lambda(0)\,\mathrm{d}\nu(\lambda) \\ &= \int_{0}^{\infty}\int_{-\infty}^{\infty} \{G_\lambda(x) - L_\lambda(x)\}\,\mathrm{d}x\,\mathrm{d}\nu(\lambda) < \infty.\end{aligned}$$

Next we consider the interpolation points. The Poisson summation formula can be applied pointwise to D_λ, since it holds for the Gaussian G_λ and for the minorant L_λ, which is a continuous integrable function of bounded variation. This gives us

$$\lim_{N\to\infty} \sum_{n=-N}^{N} D_\lambda(x+n) = \lim_{N\to\infty} \sum_{k=-N}^{N} \widehat{D}_\lambda(k)\, e(xk). \tag{2.58}$$

Since the minorant L_λ interpolates the Gaussian G_λ at $\mathbb{Z} + \frac{1}{2}$, we have $D_\lambda\big(n + \frac{1}{2}\big) = 0$ for all $n \in \mathbb{Z}$. Therefore we apply (2.58) at $x = \frac{1}{2}$, and use the classical Paley-Wiener theorem. In this way we arrive at the identity

$$\widehat{D}_\lambda(0) = -\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} (-1)^k\, \widehat{G}_\lambda(k). \tag{2.59}$$

Now we define the function

$$d_1(x) = g(x) - l(x).$$

We note that $d_1(x)$ is a non-negative, continuous function on $\mathbb{R}/\{0\}$ that is equal almost everywhere to $d(x)$ defined in (2.52), and thus in $L^1(\mathbb{R})$. Define a periodic function $p : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^+ \cup \{\infty\}$ by

$$p(x) = \sum_{n\in\mathbb{Z}} d_1(n+x).$$

An application of Fubini's theorem provides

$$\int_{\mathbb{R}/\mathbb{Z}} p(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} d_1(x)\,\mathrm{d}x < \infty\,,$$

and therefore $p(x) \in L^1(\mathbb{R}/\mathbb{Z})$. Moreover, the Fourier coefficients of $p(x)$ satisfy

$$\widehat{p}(k) = \widehat{d}_1(k) = \widehat{d}(k)$$

for all $k \in \mathbb{Z}$. Convolution with the smoothing Féjer kernel

$$F_N(x) = \frac{1}{N+1} \left(\frac{\sin \pi (N+1) x}{\sin \pi x} \right)^2$$

produces the pointwise identity

$$\begin{aligned} p * F_N(x) &= \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N}\right) \widehat{p}(k)\, e(xk) \\ &= \widehat{d}(0) + \sum_{\substack{k=-N \\ k \neq 0}}^{N} \left(1 - \frac{|k|}{N}\right) \widehat{d}(k)\, e(xk) \\ &= \widehat{d}(0) + \sum_{\substack{k=-N \\ k \neq 0}}^{N} \left(1 - \frac{|k|}{N}\right) \int_0^\infty \widehat{G}_\lambda(k)\, \mathrm{d}\nu(\lambda)\, e(xk) \\ &= \widehat{d}(0) + \int_0^\infty \left\{ \sum_{\substack{k=-N \\ k \neq 0}}^{N} \left(1 - \frac{|k|}{N}\right) \widehat{G}_\lambda(k)\, e(xk) \right\} \mathrm{d}\nu(\lambda)\,, \end{aligned}$$

where we have used (2.54). In particular, at $x = \frac{1}{2}$ we obtain

$$\widehat{d}(0) = p * F_N\left(\tfrac{1}{2}\right) + \int_0^\infty \left\{ \sum_{\substack{k=-N \\ k \neq 0}}^{N} (-1)^{k+1} \left(1 - \frac{|k|}{N}\right) \widehat{G}_\lambda(k) \right\} \mathrm{d}\nu(\lambda). \qquad (2.60)$$

Note that the integrand in (2.60) in non-negative since $\widehat{G}_\lambda$ is radially decreasing and we can group the terms in consecutive pairs. Moreover, it converges absolutely to (2.59) as $N \to \infty$. Therefore, an application of Fatou's lemma together with (2.53) gives us

$$\begin{aligned} \widehat{d}(0) &\geq \liminf_{N \to \infty} p * F_N\left(\tfrac{1}{2}\right) \\ &\qquad + \liminf_{N \to \infty} \int_0^\infty \left\{ \sum_{\substack{k=-N \\ k \neq 0}}^{N} (-1)^{k+1} \left(1 - \frac{|k|}{N}\right) \widehat{G}_\lambda(k) \right\} \mathrm{d}\nu(\lambda) \\ &\geq \liminf_{N \to \infty} p * F_N\left(\tfrac{1}{2}\right) \\ &\qquad + \int_0^\infty \liminf_{N \to \infty} \left\{ \sum_{\substack{k=-N \\ k \neq 0}}^{N} (-1)^{k+1} \left(1 - \frac{|k|}{N}\right) \widehat{G}_\lambda(k) \right\} \mathrm{d}\nu(\lambda) \\ &= \liminf_{N \to \infty} p * F_N\left(\tfrac{1}{2}\right) + \int_0^\infty \widehat{D}_\lambda(0)\, \mathrm{d}\nu(\lambda) \end{aligned}$$

$$= \liminf_{N\to\infty} p * F_N\big(\tfrac{1}{2}\big) + \widehat{d}(0)\,,$$

and since $p * F_N(x)$ is non-negative we conclude that

$$\liminf_{N\to\infty} p * F_N\big(\tfrac{1}{2}\big) = 0.$$

We now use the definition of $p(x)$, Fubini's theorem and Fatou's lemma again to arrive at

$$\begin{aligned}
0 = \liminf_{N\to\infty} p * F_N\big(\tfrac{1}{2}\big) &= \liminf_{N\to\infty} \int_0^1 p(y)\, F_N\big(\tfrac{1}{2} - y\big)\,\mathrm{d}y \\
&= \liminf_{N\to\infty} \int_0^1 \Big\{ \sum_{n\in\mathbb{Z}} d_1(n+y) \Big\} F_N\big(\tfrac{1}{2} - y\big)\,\mathrm{d}y \\
&= \liminf_{N\to\infty} \sum_{n\in\mathbb{Z}} \Big\{ \int_0^1 d_1(n+y)\, F_N\big(\tfrac{1}{2} - y\big)\,\mathrm{d}y \Big\} \qquad (2.61) \\
&\geq \sum_{n\in\mathbb{Z}} \liminf_{N\to\infty} \int_0^1 d_1(n+y)\, F_N\big(\tfrac{1}{2} - y\big)\,\mathrm{d}y \\
&= \sum_{n\in\mathbb{Z}} d_1\big(n + \tfrac{1}{2}\big)\,,
\end{aligned}$$

where the last equality follows from the fact that $d_1(x)$ is continuous at the points $n + \frac{1}{2}$, $n \in \mathbb{Z}$. From (2.61) and the non-negativity of $d_1(x)$ we arrive at the implication

$$d_1\big(n + \tfrac{1}{2}\big) = 0 \Rightarrow g\big(n + \tfrac{1}{2}\big) = l\big(n + \tfrac{1}{2}\big) \qquad (2.62)$$

for all $n \in \mathbb{Z}$. From (2.57) and the fact that $g(x)$ is differentiable on $\mathbb{R}/\{0\}$ (by hypothesis) we also have

$$g'\big(n + \tfrac{1}{2}\big) = l'\big(n + \tfrac{1}{2}\big)$$

for all $n \in \mathbb{Z}$.

Finally, we show that the integral is minimal and we establish uniqueness. Assume that $\widetilde{l}(z)$ is a real entire function of exponential type 2π such that

$$\widetilde{l}(x) \leq g(x) \qquad (2.63)$$

for all $x \in \mathbb{R}$, and suppose that $\{g(x) - \widetilde{l}(x)\}$ is integrable. In this case the function

$$j(z) = l(z) - \widetilde{l}(z)$$

has exponential type 2π and is integrable on $\mathbb{R}$. Thus it has bounded variation and we can apply Poisson summation, together with (2.62) and (2.63), to get

$$\begin{aligned}\widehat{j}(0) &= \lim_{N\to\infty} \sum_{n=-N}^{N} j\big(n+\tfrac{1}{2}\big) \\ &= \lim_{N\to\infty} \sum_{n=-N}^{N} \Big(g\big(n+\tfrac{1}{2}\big) - \widetilde{l}\big(n+\tfrac{1}{2}\big)\Big) \geq 0\,.\end{aligned} \tag{2.64}$$

This plainly verifies that

$$\int_{-\infty}^{\infty} \{g(x) - \widetilde{l}(x)\}\,\mathrm{d}x \geq \int_{-\infty}^{\infty} \{g(x) - l(x)\}\,\mathrm{d}x\,,$$

and establishes the minimality of the integral. If equality occurs in (2.64) we must have

$$\widetilde{l}\big(n+\tfrac{1}{2}\big) = g\big(n+\tfrac{1}{2}\big) = l\big(n+\tfrac{1}{2}\big) \tag{2.65}$$

for all $n \in \mathbb{Z}$. From (2.63) we also have

$$\widetilde{l}'\big(n+\tfrac{1}{2}\big) = g'\big(n+\tfrac{1}{2}\big) = l'\big(n+\tfrac{1}{2}\big) \tag{2.66}$$

for all $n \in \mathbb{Z}$. The interpolation conditions (2.65) and (2.66) imply that

$$j\big(n+\tfrac{1}{2}\big) = j'\big(n+\tfrac{1}{2}\big) = 0$$

for all $n \in \mathbb{Z}$. By an application of Theorem 1.12 (with an appropriate shift of $\frac{1}{2}$), we conclude that the entire function $j(z)$ is identically zero. This proves the uniqueness of the extremal minorant $l(z)$, and completes the proof.

In the proof of uniqueness in the majorant case, we will obtain

$$j'(n) = 0$$

for all $n \neq 0$, since the original function $g(x)$ is not assumed to be differentiable at the origin. An application of Theorem 1.12 shows that $j'(0) = 0$ (since j must be integrable), and this leads to uniqueness.

2.3.4 Asymptotic analysis of the admissible measures

Recall that we are working with the family of Gaussian functions

$$G_\lambda(x) = e^{-\pi\lambda x^2}\,,$$

where $\lambda > 0$ is a parameter. The Fourier transform $t \mapsto \widehat{G}_\lambda(t)$ is given by

$$\widehat{G}_\lambda(t) = \lambda^{-\frac{1}{2}} e^{-\pi\lambda^{-1}t^2}.$$

In Theorems 2.1 and 2.2 we constructed, for each $\lambda > 0$, the extremal minorant $L_\lambda(z)$ and the extremal majorant $M_\lambda(z)$ for $G_\lambda(x)$. The values of the minimal integrals are given by

$$\begin{aligned}\int_{-\infty}^{\infty} &\{G_\lambda(x) - L_\lambda(x)\}\,\mathrm{d}x \\ &= \lambda^{-\frac{1}{2}}\Big(1 - \theta_2\big(0, i\lambda^{-1}\big)\Big) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^{n+1}\widehat{G}_\lambda(n)\,,\end{aligned} \tag{2.67}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} &\{M_\lambda(x) - G_\lambda(x)\}\,\mathrm{d}x \\ &= \lambda^{-\frac{1}{2}}\Big(\theta_3\big(0, i\lambda^{-1}\big) - 1\Big) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \widehat{G}_\lambda(n).\end{aligned} \tag{2.68}$$

From the two expressions above and the transformation formulas (2.12) and (2.13) we obtain the estimates

$$\int_{-\infty}^{\infty} \{G_\lambda(x) - L_\lambda(x)\}\,\mathrm{d}x = \begin{cases} O\big(\lambda^{-\frac{1}{2}}e^{-\pi\lambda^{-1}}\big) \text{ as } \lambda \to 0, \\ O\big(\lambda^{-\frac{1}{2}}\big) \text{ as } \lambda \to \infty, \end{cases} \tag{2.69}$$

and

$$\int_{-\infty}^{\infty} \{M_\lambda(x) - G_\lambda(x)\}\,\mathrm{d}x = \begin{cases} O\big(\lambda^{-\frac{1}{2}}e^{-\pi\lambda^{-1}}\big) \text{ as } \lambda \to 0, \\ O(1) \text{ as } \lambda \to \infty. \end{cases} \tag{2.70}$$

In order to apply Theorems 2.10 and 2.11 we require that the integrals with respect to ν of the functions of λ appearing in (2.67) and (2.68) are finite. The estimates (2.69) and (2.70) show that this is a wide class of measures because of the very fast decay when $\lambda \to 0$. One should compare this class of measures with the ones used in [8], [9] and [19], to fully notice the improvement and power of the Gaussian subordination method.

2.3.5 Examples

Positive definite functions

As a first application we present the following result.

Corollary 2.12. *Let ν be a finite non-negative Borel measure on $[0,\infty)$ and consider the function $g : \mathbb{R} \to \mathbb{R}$ given by*

$$g(x) = \int_0^{\infty} e^{-\pi\lambda x^2}\,\mathrm{d}\nu(\lambda)\,. \tag{2.71}$$

(i) There exists a unique extremal minorant $l(z)$ of exponential type 2π for $g(x)$. The function $l(x)$ interpolates the values of $g(x)$ at $\mathbb{Z}+\frac{1}{2}$ and satisfies

$$\int_{-\infty}^{\infty} \{g(x) - l(x)\}\,\mathrm{d}x = \int_0^{\infty} \left\{ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^{n+1} \widehat{G}_\lambda(n) \right\} \mathrm{d}\nu(\lambda).$$

(ii) There exists a unique extremal majorant $m(z)$ of exponential type 2π for $g(x)$. The function $m(x)$ interpolates the values of $g(x)$ at $\mathbb{Z}$ and satisfies

$$\int_{-\infty}^{\infty} \{m(x) - g(x)\}\,\mathrm{d}x = \int_0^{\infty} \left\{ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \widehat{G}_\lambda(n) \right\} \mathrm{d}\nu(\lambda).$$

Due to a classical result of Schoenberg [35, Theorems 2 and 3], a function $g : \mathbb{R} \to \mathbb{R}$ admits the representation (2.71) if and only if its radial extension to $\mathbb{R}^N$ is positive definite, for all $N \in \mathbb{N}$, or equivalently if the function $g\big(|x|^{1/2}\big)$ is completely monotone. Recall that a function $f(t)$ is *completely monotone* for $t \geq 0$ if

$$(-1)^n f^{(n)}(t) \geq 0 \quad \text{for} \quad 0 < t < \infty, \quad \text{and} \quad n = 1, 2, 3, \ldots,$$

and

$$f(0) = f(0+).$$

The last condition expresses the continuity of $f(t)$ at the origin. Using this characterization we arrive at the following interesting examples contemplated by our Corollary 2.12.

Example 1. $g(x) = e^{-\alpha|x|^{2r}}, \quad \alpha > 0, \quad \text{and} \quad 0 < r \leq 1.$

Example 2. $g(x) = \big(x^2 + \alpha^2\big)^{-\beta}, \quad \alpha > 0 \quad \text{and} \quad \beta > 0.$

The first example shows that we can recover all the theory for the exponential function $g(x) = e^{-\lambda|x|}$ developed in [8], [9] and [19], from the family of Gaussian functions and the distribution theorems. The second example includes the Poisson kernel $g(x) = 2\lambda/(\lambda^2 + 4\pi^2 x^2)$, $\lambda > 0$. Another application of Corollary 2.12 yields the following example.

Example 3. $g(x) = -\log\left(\dfrac{x^2+\alpha^2}{x^2+\beta^2}\right)$, for $0 \leq \alpha < \beta$.

Indeed, for $0 \leq \alpha < \beta$ consider the non-negative measure

$$\mathrm{d}\nu(\lambda) = \frac{\left\{e^{-\pi\lambda\alpha^2} - e^{-\pi\lambda\beta^2}\right\}}{\lambda}\,\mathrm{d}\lambda\,,$$

and observe that

$$-\log\left(\frac{x^2+\alpha^2}{x^2+\beta^2}\right) = \int_0^{\infty} e^{-\pi\lambda x^2}\,\frac{\left\{e^{-\pi\lambda\alpha^2} - e^{-\pi\lambda\beta^2}\right\}}{\lambda}\,\mathrm{d}\lambda\,. \tag{2.72}$$

When $0 < \alpha < \beta$ this is a finite measure and we fall under the scope of Corollary 2.12. When $\alpha = 0$, we still have $g(x)$ integrable and thus its Fourier transform (in the classical sense) is given by

$$\widehat{g}(t) = \int_0^\infty \widehat{G}_\lambda(t)\, \frac{\left\{1 - e^{-\pi\lambda\beta^2}\right\}}{\lambda}\, \mathrm{d}\lambda\,.$$

Thus we still fall under the hypothesis of Theorem 2.10 to obtain an extremal minorant in this case (note that an extremal majorant does not exist due to the singularity at the origin). In particular, the values of the minimal integrals in the one-sided approximations are given by

$$\int_{-\infty}^{\infty} \left\{-\log\left(\frac{x^2+\alpha^2}{x^2+\beta^2}\right) - l_{\alpha,\beta}(x)\right\} \mathrm{d}x = 2\log\left(\frac{1+e^{-2\pi\alpha}}{1+e^{-2\pi\beta}}\right)\,,$$

if $0 \leq \alpha$, and

$$\int_{-\infty}^{\infty} \left\{m_{\alpha,\beta}(x) + \log\left(\frac{x^2+\alpha^2}{x^2+\beta^2}\right)\right\} \mathrm{d}x = 2\log\left(\frac{1-e^{-2\pi\beta}}{1-e^{-2\pi\alpha}}\right)\,,$$

if $0 < \alpha$.

Example 4. $g(x) = 1 - x\arctan\left(\frac{1}{x}\right)$.

One can consider the non-negative and finite measure given by

$$\mathrm{d}\nu(\lambda) = \int_{1/2}^{3/2} \left\{\frac{e^{-\pi\lambda(\sigma-1/2)^2} - e^{-\pi\lambda}}{2\lambda}\right\} \mathrm{d}\sigma\, \mathrm{d}\lambda.$$

Using the fact that

$$g(x) = 1 - x\arctan\left(\frac{1}{x}\right) = \frac{1}{2}\int_{1/2}^{3/2} \log\left(\frac{x^2+1}{x^2+\left(\sigma-\frac{1}{2}\right)^2}\right) \mathrm{d}\sigma,$$

together with (2.72), we arrive at

$$g(x) = 1 - x\arctan\left(\frac{1}{x}\right) = \int_0^\infty e^{-\pi\lambda x^2}\, \mathrm{d}\nu(\lambda).$$

This was observed in [4] and shall be used when we consider bounds for $S_1(t)$ (the antiderivative of the argument function $S(t)$) under the Riemann hypothesis, in the next chapter.

Power functions

In this subsection we write $s = \sigma + it$ for a complex variable, and we define the meromorphic function $s \mapsto \gamma(s)$ by

$$\gamma(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right).$$

The function $\gamma(s)$ is analytic on $\mathbb{C}$ except for simple poles at the points $s = 0, -2, -4, \ldots$. It also occurs in the functional equation

$$\gamma(s)\zeta(s) = \gamma(1-s)\zeta(1-s), \tag{2.73}$$

where $\zeta(s)$ is the Riemann zeta-function.

Lemma 2.13. *Let* $0 < \delta$ *and let* $\varphi(t)$ *be a Schwartz function supported on* $[-\delta, \delta]^c$. *Then*

$$s \mapsto \int_{-\infty}^{\infty} |t|^{-s-1}\varphi(t)\ \mathrm{d}t \tag{2.74}$$

defines an entire function of s, *and the identity*

$$\gamma(s+1)\int_{-\infty}^{\infty} |t|^{-s-1}\varphi(t)\ \mathrm{d}t = \gamma(-s)\int_{-\infty}^{\infty} |x|^{s}\,\widehat{\varphi}(x)\ \mathrm{d}x \tag{2.75}$$

holds in the half-plane $\{s \in \mathbb{C} : -1 < \sigma\}$. *In particular, the function on the right of* (2.75) *is analytic at the points* $s = 0, 2, 4, \ldots$.

Proof. Because $\varphi(t)$ is supported in $[-\delta, \delta]^c$, the function $t \mapsto |t|^{-s-1}\varphi(t)$ is integrable on $\mathbb{R}$ for all complex values s. Hence by Morera's theorem the integral on the right of (2.74) defines an entire function. The identity (2.75) holds in the infinite strip $\{s \in \mathbb{C} : -1 < \sigma < 0\}$ by [39, Lemma 1, p. 117], and therefore it holds in the half-plane $\{s \in \mathbb{C} : -1 < \sigma\}$ by analytic continuation. The left-hand side of (2.75) is clearly analytic at each point of $\{s \in \mathbb{C} : -1 < \sigma\}$, hence the right-hand side of (2.75) is also analytic at each point of this half-plane. □

Lemma 2.13 asserts that, for $-1 < \sigma$ and $\sigma \neq 0, 2, 4, \ldots$, the Fourier transform of the function $x \mapsto \gamma(-\sigma)|x|^{\sigma}$ is given by the function

$$t \mapsto \gamma(\sigma+1)|t|^{-\sigma-1}$$

outside the interval $[-\delta, \delta]$. We intend to apply the distribution theorems, and towards this end, we consider the non-negative Borel measure ν_σ on $[0, \infty)$ given by

$$\mathrm{d}\nu_\sigma(\lambda) = \lambda^{-\frac{\sigma}{2}-1}\,\mathrm{d}\lambda\,,$$

and observe that we have

$$\int_0^{\infty} \widehat{G}_\lambda(t)\,\mathrm{d}\nu_\sigma(\lambda) = \gamma(\sigma+1)|t|^{-\sigma-1}. \tag{2.76}$$

For $-1 < \sigma$, the measure ν_σ is admissible for the minorant problem according to the asymptotics (2.69). For the majorant problem we shall require that $0 < \sigma$, according to the asymptotics (2.70).

Theorems 2.10 and 2.11 now apply. The values of the integrals in the following corollary can be obtained using Theorems 2.1 and 2.2, and then applying termwise integration to the series (2.7) and (2.8).

Corollary 2.14. *Let $-1 < \sigma$ with $\sigma \neq 0, 2, 4, \ldots$ and let*

$$g_\sigma(x) = \gamma(-\sigma)|x|^\sigma.$$

(i) There exists a unique extremal minorant $l_\sigma(z)$ of exponential type 2π for $g_\sigma(x)$. The function $l_\sigma(x)$ interpolates the values of $g_\sigma(x)$ at $\mathbb{Z} + \frac{1}{2}$ and satisfies

$$\int_{-\infty}^{\infty} \{g_\sigma(x) - l_\sigma(x)\}\, \mathrm{d}x = \big(2 - 2^{1-\sigma}\big)\gamma(1+\sigma)\,\zeta(1+\sigma). \tag{2.77}$$

(ii) If $0 < \sigma$, there exists a unique extremal majorant $m_\sigma(z)$ of exponential type 2π for $g_\sigma(x)$. The function $m_\sigma(x)$ interpolates the values of $g_\sigma(x)$ at $\mathbb{Z}$ and satisfies

$$\int_{-\infty}^{\infty} \{m_\sigma(x) - g_\sigma(x)\}\, \mathrm{d}x = 2\,\gamma(1+\sigma)\,\zeta(1+\sigma). \tag{2.78}$$

Corollary 2.14 provides a complete description of the extreme minorants and extreme majorants associated to $x \mapsto |x|^\sigma$. For $\sigma \leq -1$ these functions are not integrable at the origin, and therefore no extremals exist, and for $\sigma = 2k, k \in \mathbb{Z}^+$, these functions are entire, have only polynomial growth, and therefore the extremal problem is trivial. Previous results had been obtained in [8] and [9] for the functions $x \mapsto |x|^\sigma$, $-1 < \sigma < 1$, and in [28] for the functions $x \mapsto |x|^{2k+1}$, with $k \in \mathbb{Z}^+$.

Logarithm

We complete our list of applications (in the even case) with one additional example that follows from the distribution theorems.

Corollary 2.15. *Let $\alpha \geq 0$ and consider*

$$x \mapsto \tau_\alpha(x) = -\log(x^2 + \alpha^2).$$

(i) There exists a unique extremal minorant l_α of exponential type 2π for τ_α. The function l_α interpolates the values of τ_α at $\mathbb{Z} + \frac{1}{2}$, and satisfies

$$\int_{-\infty}^{\infty} \{\tau_\alpha(x) - l_\alpha(x)\}\, \mathrm{d}x = 2\log\big(1 + e^{-2\pi\alpha}\big).$$

(ii) If $0 < \alpha$, there exists a unique extremal majorant m_α of exponential type 2π for τ_α. The function m_α interpolates the values of τ_α at $\mathbb{Z}$, and satisfies

$$\int_{-\infty}^{\infty} \{m_\alpha(x) - \tau_\alpha(x)\}\, \mathrm{d}x = -2\log\big(1 - e^{-2\pi\alpha}\big).$$

Proof. For $0 \le \alpha$ we have the identity

$$-\log(x^2+\alpha^2) = \int_0^\infty \frac{\left\{e^{-\pi\lambda(x^2+\alpha^2)} - e^{-\pi\lambda}\right\}}{\lambda}\,\mathrm{d}\lambda\,. \tag{2.79}$$

Let φ be a Schwartz function supported in $[-\delta,\delta]^c$. An application of Fubini's theorem leads to the identity

$$\begin{aligned}
&\int_{-\infty}^{\infty} -\log(x^2+\alpha^2)\,\widehat{\varphi}(x)\,\mathrm{d}x \\
&\qquad = \int_{-\infty}^{\infty}\left\{\int_0^\infty \frac{\left\{e^{-\pi\lambda(x^2+\alpha^2)} - e^{-\pi\lambda}\right\}}{\lambda}\mathrm{d}\lambda\right\}\widehat{\varphi}(x)\,\mathrm{d}x \\
&\qquad = \int_0^\infty\int_{-\infty}^{\infty} \frac{\left\{e^{-\pi\lambda(x^2+\alpha^2)} - e^{-\pi\lambda}\right\}}{\lambda}\,\widehat{\varphi}(x)\,\mathrm{d}x\,\mathrm{d}\lambda \qquad (2.80)\\
&\qquad = \int_0^\infty\left\{\int_{-\infty}^{\infty}\widehat{G}_\lambda(t)\,\varphi(t)\,\mathrm{d}t\right\}\frac{e^{-\pi\lambda\alpha^2}}{\lambda}\,\mathrm{d}\lambda \\
&\qquad = \int_{-\infty}^{\infty}\left\{\int_0^\infty \widehat{G}_\lambda(t)\,\frac{e^{-\pi\lambda\alpha^2}}{\lambda}\,\mathrm{d}\lambda\right\}\varphi(t)\,\mathrm{d}t.
\end{aligned}$$

Equation (2.80) provides the Fourier transform of $-\log(x^2+\alpha^2)$ outside a compact interval $[-\delta,\delta]$. We can therefore apply the distribution theorems (Theorems 2.10 and 2.11) with measure ν on $[0,\infty)$ given by

$$\mathrm{d}\nu(\lambda) = \frac{e^{-\pi\lambda\alpha^2}}{\lambda}\,\mathrm{d}\lambda.$$

According to the asymptotics (2.69) and (2.70), if $\alpha > 0$ we can treat the two one-sided approximation problems, and if $\alpha = 0$ we can only treat the minorant problem (which is in agreement with the fact that $-\log|x|$ is unbounded from above). The special case of $-\log|x|$ (when $\alpha = 0$) was first obtained in [8] and [9]. □

2.4 The extremal problem for the truncated and odd Gaussians

We know move in the direction of developing the analogous extremal theory for the case of truncated and odd functions. This was carried out in the papers [5] and [6]. We present here briefly the developments of [5], that deal with the corresponding Gaussian subordination framework for the case of truncated and odd functions.

We keep the notation for the Gaussian

$$x \mapsto G_\lambda(x) = e^{-\pi\lambda x^2}$$

and the theta functions defined in (2.6), (2.7) and (2.8). We consider here the Beurling-Selberg extremal problem for the *truncated Gaussian* $x \mapsto G_\lambda^+(x)$ defined by

$$G_\lambda^+(x) = \begin{cases} G_\lambda(x) & \text{for } x > 0, \\ 1/2 & \text{for } x = 0, \\ 0 & \text{for } x < 0, \end{cases}$$

and the *odd Gaussian* $x \mapsto G_\lambda^o(x)$ defined by

$$G_\lambda^o(x) = \begin{cases} G_\lambda(x) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -G_\lambda(x) & \text{for } x < 0. \end{cases}$$

Recall that the Fourier transform of the Gaussian $G_\lambda(x) = e^{-\pi\lambda x^2}$ is given by

$$\widehat{G}_\lambda(t) = \int_{-\infty}^{\infty} e^{-2\pi i t x} G_\lambda(x)\,\mathrm{d}x = \lambda^{-1/2} e^{-\pi\lambda^{-1}t^2},$$

and, via contour integration, the Fourier transform of the truncated Gaussian $G_\lambda^+(x)$ is shown to be

$$\widehat{G}_\lambda^+(t) = \frac{1}{2}\,\lambda^{-1/2} e^{-\pi\lambda^{-1}t^2} + \frac{t}{i\lambda}\int_0^1 e^{-\pi\lambda^{-1}t^2(1-y^2)}\,\mathrm{d}y. \tag{2.81}$$

Define the following two entire functions of exponential type

$$L_\lambda^+(z) = \frac{\sin^2 \pi z}{\pi^2}\sum_{n=1}^{\infty}\left\{\frac{G_\lambda(n)}{(z-n)^2} + \frac{G_\lambda'(n)}{z-n} - \frac{G_\lambda'(n)}{z}\right\},$$

$$M_\lambda^+(z) = \frac{\sin^2 \pi z}{\pi^2}\sum_{n=1}^{\infty}\left\{\frac{G_\lambda(n)}{(z-n)^2} + \frac{G_\lambda'(n)}{z-n} - \frac{G_\lambda'(n)}{z}\right\} + \frac{\sin^2 \pi z}{\pi^2 z^2}.$$

Note that L_λ^+ and M_λ^+ are entire functions of exponential type 2π that interpolate the values of G_λ^+ and its derivative at $\mathbb{Z}\backslash\{0\}$. The following two theorems provide the solution of the extremal problem for the truncated Gaussian.

Theorem 2.16 (Extremal minorant for the truncated Gaussian)**.** *The inequality*

$$L_\lambda^+(x) \le G_\lambda^+(x)$$

holds for all real x. Let $z \mapsto L(z)$ be an entire function of exponential type at most 2π which satisfies the inequality $L(x) \le G_\lambda^+(x)$ for all real x. Then

$$\int_{-\infty}^{\infty}\left\{G_\lambda^+(x) - L(x)\right\}\mathrm{d}x \ge -\frac{\theta_3(0,i\lambda)}{2} + \frac{1}{2} + \frac{1}{2\sqrt{\lambda}}, \tag{2.82}$$

with equality if and only if $L = L_\lambda^+$.

Theorem 2.17 (Extremal majorant for the truncated Gaussian)**.** *The inequality*

$$G_\lambda^+(x) \leq M_\lambda^+(x)$$

holds for all real x. Let $z \mapsto M(z)$ be an entire function of exponential type at most 2π which satisfies the inequality $G_\lambda^+(x) \leq M(x)$ for all real x. Then

$$\int_{-\infty}^{\infty} \left\{M(x) - G_\lambda^+(x)\right\} \mathrm{d}x \geq \frac{\theta_3(0, i\lambda)}{2} + \frac{1}{2} - \frac{1}{2\sqrt{\lambda}}\,, \tag{2.83}$$

with equality if and only if $M = M_\lambda^+$.

The strategy for the proofs of the two theorems above is a decomposition of these functions into integral representations analogous to those developed in Section 2.2.2 for the Gaussian. The integrands will involve certain truncated theta functions that turn out to be solutions of the heat equation, and the maximum principle for the heat operator is used to obtain the necessary inequalities. The uniqueness part will follow from the interpolation properties at $\mathbb{Z}$ as done in the proofs of Theorems 2.1 and 2.2. A simple dilation argument provides the optimal approximations of exponential type $2\pi\delta$ for any $\delta > 0$. Since the proofs of these results are rather lengthy and technical, we decided not to include them here, and instead refer the interested reader to the original source [5].

Once we have established the solution of the extremal problem for the truncated Gaussian as described in Theorems 2.16 and 2.17, we can easily derive the solution of this problem for the odd Gaussian $x \mapsto G_\lambda^o(x)$. Observe that

$$G_\lambda^o(x) = G_\lambda^+(x) - G_\lambda^+(-x)$$

and define the entire functions

$$\begin{aligned} L_\lambda^o(z) &= L_\lambda^+(z) - M_\lambda^+(-z), \\ M_\lambda^o(z) &= M_\lambda^+(z) - L_\lambda^+(-z). \end{aligned} \tag{2.84}$$

Theorems 2.16 and 2.17 imply that

$$L_\lambda^o(x) \leq G_\lambda^o(x) \leq M_\lambda^o(x).$$

These functions preserve the interpolation properties at $\mathbb{Z}$ and are the extremal minorant and majorant for the odd Gaussian, respectively. This follows by arguments analogous to the proofs of Theorems 2.1 and 2.2, and plainly guarantees the odd counterparts of all the results we present here for truncated functions.

2.5 Framework for truncated and odd functions

2.5.1 Integrating the free parameter

Having solved the Beurling-Selberg extremal problem for a family of functions with a free parameter $\lambda > 0$, we are now interested in integrating this parameter against a set of admissible non-negative Borel measures ν on $[0,\infty)$ to generate a new class of truncated (and odd) functions for which the extremal problem has a solution.

We now determine the set of admissible measures ν. For the minorant problem, the minimal condition we must impose on the measure ν is that the function on the right-hand side of (2.82) should be ν-integrable. The well-known asymptotics for the theta functions (given by the transformation formulas) lead us to consider non-negative Borel measures ν on $[0,\infty)$ satisfying

$$\int_0^\infty \frac{1}{1+\sqrt{\lambda}}\,\mathrm{d}\nu(\lambda) < \infty. \tag{2.85}$$

On the other hand, for the majorant problem, the minimal condition we must impose on the measure ν is that the function on the right-hand side of (2.83) should be ν-integrable. This is equivalent to the measure being finite, i.e.

$$\int_0^\infty \mathrm{d}\nu(\lambda) < \infty\,. \tag{2.86}$$

Define the truncation x_+^0 by

$$x_+^0 = \tfrac{1}{2}(1+\operatorname{sgn}(x))$$

and consider the truncated function $g:\mathbb{R}\to\mathbb{R}$ given by

$$g(x) = x_+^0 \int_0^\infty e^{-\pi\lambda x^2}\,\mathrm{d}\nu(\lambda)\,.$$

We are now able to state the two main results of this section.

Theorem 2.18 (Extremal minorant - general truncated case)**.** *Let ν satisfy (2.85). Then there exists a unique extremal minorant $z\mapsto l(z)$ of exponential type 2π for $x\mapsto g(x)$. The function l interpolates the values of g and its derivative at $\mathbb{Z}\backslash\{0\}$ and satisfies*

$$\int_{-\infty}^\infty \{g(x)-l(x)\}\,\mathrm{d}x = \int_0^\infty \left\{-\frac{\theta_3(0,i\lambda)}{2}+\frac{1}{2}+\frac{1}{2\sqrt{\lambda}}\right\}\mathrm{d}\nu(\lambda).$$

Theorem 2.19 (Extremal majorant - general truncated case)**.** *Let ν satisfy (2.86). Then there exists a unique extremal majorant $z\mapsto m(z)$ of exponential type 2π for $x\mapsto g(x)$. The function m interpolates the values of g and its derivative at $\mathbb{Z}\backslash\{0\}$ and satisfies*

$$\int_{-\infty}^\infty \{m(x)-g(x)\}\,\mathrm{d}x = \int_0^\infty \left\{\frac{\theta_3(0,i\lambda)}{2}+\frac{1}{2}-\frac{1}{2\sqrt{\lambda}}\right\}\mathrm{d}\nu(\lambda).$$

Observe that the class of measures allowed by (2.85) and (2.86) is more restrictive than the class we worked in the even Gaussian problem, thus making the method less powerful in this truncated/odd case (one might also see this from the fact that we did not have to appeal to the Fourier space for the definition of $g(x)$). When adapting Theorems 2.18 and 2.19 to the context of odd functions, we must ask for the more restrictive condition (2.86), due to the construction (2.84).

2.5.2 Proofs of Theorems 2.18 and 2.19

We start with the minorant case, where we have seen by Theorem 2.16 that

$$L_\lambda^+(z) = \frac{\sin^2 \pi z}{\pi^2} \sum_{n=1}^{\infty} \left\{ \frac{G_\lambda(n)}{(z-n)^2} + \frac{G'_\lambda(n)}{z-n} \right\} - \frac{\sin^2 \pi z}{\pi^2 z} \sum_{n=1}^{\infty} G'_\lambda(n)$$

satisfies

$$L_\lambda^+(x) \le G_\lambda^+(x) \tag{2.87}$$

for all $x \in \mathbb{R}$, with

$$L_\lambda^+(n) = G_\lambda^+(n) \tag{2.88}$$

if $n \in \mathbb{Z}/\{0\}$, and

$$L_\lambda^+(0) = \lim_{x \to 0^-} G_\lambda^+(x) = 0. \tag{2.89}$$

We consider a non-negative Borel measure ν satisfying (2.85) and we need to show that

$$l(z) = \int_0^\infty L_\lambda^+(z)\, \mathrm{d}\nu(\lambda)$$

is a well defined entire function of exponential type at most 2π. If this is the case, by integrating expressions (2.87), (2.88) and (2.89) against ν, these properties will be carried on to $l(x)$ and $g(x) = \int_0^\infty G_\lambda^+(x)\, \mathrm{d}\nu(\lambda)$ making $l(x)$ the unique extremal minorant of exponential type at most 2π for $g(x)$ via the same arguments used in the proof of Theorem 2.1.

For this purpose we need to collect some estimates. For $n \in \mathbb{N}$ using (2.85) we have

$$\int_0^\infty G_\lambda(n)\, \mathrm{d}\nu(\lambda) = \int_0^1 G_\lambda(n)\, \mathrm{d}\nu(\lambda) + \int_1^\infty \sqrt{\lambda}\, G_\lambda(n)\, \frac{\mathrm{d}\nu(\lambda)}{\sqrt{\lambda}} \le C_1 + \frac{C_2}{n}, \tag{2.90}$$

and

$$\begin{aligned} \int_0^\infty \left|G'_\lambda(n)\right| \mathrm{d}\nu(\lambda) &= 2\pi \int_0^1 \lambda\, n\, G_\lambda(n)\, \mathrm{d}\nu(\lambda) + 2\pi \int_1^\infty \lambda^{3/2}\, n\, G_\lambda(n)\, \frac{\mathrm{d}\nu(\lambda)}{\sqrt{\lambda}} \\ &\le \frac{C_3}{n} + \frac{C_4}{n^2}, \end{aligned} \tag{2.91}$$

where C_1, C_2, C_3 and C_4 are positive constants depending exclusively on ν.

To analyze the remaining term observe that

$$\lambda^{1/2} \sum_{n=1}^{\infty} \left|G'_\lambda(n)\right| = \sum_{n=1}^{\infty} \frac{2\pi}{n^2}\, \lambda^{3/2}\, n^3\, G_\lambda(n) \le C_5 \sum_{n=1}^{\infty} \frac{2\pi}{n^2},$$

which proves that $\sum_{n=1}^{\infty} \left|G'_\lambda(n)\right|$ is $\mathcal{O}\big(\lambda^{-1/2}\big)$ as $\lambda \to \infty$. On the other hand, using the arithmetic-geometric mean inequality and the fact that

$$\sum_{n=0}^{\infty} e^{-tn^2}\big(1 - 2tn^2\big) \ge \frac{1}{2},$$

obtained by differentiating both sides of the transformation formula (2.13), we arrive at

$$\begin{aligned}\sum_{n=1}^{\infty} \left|G'_\lambda(n)\right| = \sum_{n=1}^{\infty} 2\pi\lambda\, n\, G_\lambda(n) &\le \sum_{n=1}^{\infty} \pi \left\{\lambda^{3/2}\, n^2 + \lambda^{1/2}\right\} G_\lambda(n) \\ &\le \frac{\lambda^{1/2}}{4} + \left(\tfrac{1}{2}+\pi\right) \lambda^{1/2} \sum_{n=1}^{\infty} G_\lambda(n) \\ &= \frac{\lambda^{1/2}}{4} + \left(\tfrac{1}{2}+\pi\right) \lambda^{1/2} \left(\frac{\theta_3(0,i\lambda)-1}{2}\right).\end{aligned}$$

We know $\theta_3(0,i\lambda) \to \lambda^{-1/2}$ as $\lambda \to 0$, by the transformation formula (2.13). Therefore we may conclude that $\sum_{n=1}^{\infty} \left|G'_\lambda(n)\right|$ is $\mathcal{O}(1)$ as $\lambda \to 0$.

This shows that $\sum_{n=1}^{\infty} \left|G'_\lambda(n)\right|$ is ν-integrable, and together with (2.90) and (2.91) we can can move the integration inside the summation series since it converges absolutely to obtain

$$\begin{aligned}l(z) &= \int_0^\infty L^+_\lambda(z)\,\mathrm{d}\nu(\lambda) \\ &= \frac{\sin^2 \pi z}{\pi^2} \sum_{n=1}^{\infty} \left\{ \frac{\int_0^\infty G_\lambda(n)\,\mathrm{d}\nu(\lambda)}{(z-n)^2} + \frac{\int_0^\infty G'_\lambda(n)\,\mathrm{d}\nu(\lambda)}{z-n} \right\} \\ &\qquad - \frac{\sin^2 \pi z}{\pi^2 z} \int_0^\infty \sum_{n=1}^{\infty} G'_\lambda(n)\,\mathrm{d}\nu(\lambda).\end{aligned}$$

An application of Morera's theorem shows that this is an entire function and the exponential type 2π is given by the main term $\sin^2 \pi z$. The proof of the majorizing case is analogous.

2.5.3 Examples

We highlight some interesting choices of non-negative Borel measures ν that can be applied in Theorems 2.18 and 2.19 . We will mainly present the truncated functions here. Similar examples can be given for the odd functions. The first of these examples considers $\nu = \delta$ (the Dirac delta). In this case we obtain the following.

Example 1. $g(x) = x^0_+$.

This reproves the classical extremal functions to the signum function contained in [43, Theorems 4 and 8]. In our setting the values of the minimal integrals can be found via the asymptotics of $\theta_3(0,i\lambda)$.

More generally, as in the case of even functions, one can consider any *finite* non-negative Borel measure ν on $[0,\infty)$. With the complete monotone characterization of the positive definite functions (see Section 2.3.5) we arrive at the following truncated and odd counterparts.

Example 2. $g(x) = x_+^0\, e^{-\alpha|x|^{2r}}$, $\alpha > 0$ and $0 \le r \le 1$.

Example 3. $g(x) = x_+^0\, (x^2+\alpha^2)^{-\beta}$, $\alpha > 0$ and $\beta > 0$.

The family in Example 2 includes the truncated exponential $g(x) = x_+^0 e^{-\lambda|x|}$ treated in [19], while the family in Example 3 includes the truncated Poisson kernel $g(x) = x_+^0[2\lambda/(\lambda^2+4\pi^2x^2)]$. Despite not knowing the exact expression of the measures ν that produce these families, one can arrive at the value of the minimal integral with the knowledge of the Fourier transforms of these functions via Poisson summation.

Observe that the non-negative measure

$$\mathrm{d}\nu(\lambda) = \frac{\left\{e^{-\pi\lambda\alpha^2} - e^{-\pi\lambda\beta^2}\right\}}{\lambda}\,\mathrm{d}\lambda\,,$$

for $0 \le \alpha < \beta$ is a finite measure if $0 < \alpha$. If $\alpha = 0$, then ν still satisfies (2.85), and we can solve the minorant problem. This generates the following family.

Example 4. $g(x) = -x_+^0\, \log(x^2+\alpha^2)/(x^2+\beta^2)\,,\quad 0 \le \alpha < \beta$.

Finally, recall the definition of the meromorphic function $s \mapsto \gamma(s)$ by

$$\gamma(s) = \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right),$$

which is analytic on $\mathbb{C}$ except for simple poles at the points $s = 0, -2-4, \ldots$ The family of measures

$$\mathrm{d}\nu_\sigma(\lambda) = \lambda^{-\frac{\sigma}{2}-1}\,\mathrm{d}\lambda$$

satisfies (2.85) when $-1 < \sigma < 0$ and thus we can solve the minorant problem for the truncated power functions they produce.

Example 5. $g(x) = \gamma(-\sigma)\, x_+^0\, |x|^\sigma$, $\quad -1 < \sigma < 0$.

We close this section with a particular example of an odd function that will be relevant when we study the argument of the Riemann zeta-function in the next chapter.

Example 6. $g(x) = \arctan\left(\frac{1}{x}\right) - \frac{x}{1+x^2}$.

In fact, it was observed in [4] that the measure

$$\mathrm{d}\nu(\lambda) = \left\{\int_0^\infty \frac{t}{2\sqrt{\pi\lambda^3}}\, e^{-\frac{t^2}{4\lambda}} \left(\frac{1}{t}\sin\left(\sqrt{\pi}t\right) - \sqrt{\pi}\cos\left(\sqrt{\pi}t\right)\right)\mathrm{d}t\right\}\mathrm{d}\lambda \tag{2.92}$$

is non-negative, finite and verifies

$$g(x) = \arctan\left(\frac{1}{x}\right) - \frac{x}{1+x^2} = \mathrm{sgn}(x)\int_0^\infty e^{-\pi\lambda x^2}\,\mathrm{d}\nu(\lambda). \tag{2.93}$$

Let us verify these facts. First we prove identity (2.93), for all $x > 0$. By making the change of variables $y = \sqrt{\pi}t$ and using Fubini's theorem, we see that the right-hand side of (2.93) is equal to

$$\int_0^\infty \left\{ \int_0^\infty \frac{e^{-\pi\lambda x^2 - \frac{y^2}{4\pi\lambda}}}{2\pi\lambda^{3/2}}\, y \,\mathrm{d}\lambda \right\} \left(\frac{\sin y}{y} - \cos y \right) \mathrm{d}y. \tag{2.94}$$

Call $W(x, y)$ the quantity inside the brackets in (2.94). To prove (2.93), it suffices to show that $W(x, y) = e^{-xy}$. For this, consider the change of variables $k = \frac{\sqrt{2\pi x\lambda}}{\sqrt{y}}$ which implies that

$$W(x, y) = \frac{\sqrt{2xy}}{\sqrt{\pi}}\, e^{-xy} \int_0^\infty \frac{e^{-\frac{xy}{2}\left(k - \frac{1}{k}\right)^2}}{k^2}\, \mathrm{d}k.$$

Now from the symmetry $k \to \frac{1}{k}$, we can rewrite the last expression as

$$W(x, y) = \frac{1}{2}\frac{\sqrt{2xy}}{\sqrt{\pi}}\, e^{-xy} \int_0^\infty e^{-\frac{xy}{2}\left(k - \frac{1}{k}\right)^2} \left(1 + \frac{1}{k^2}\right) \mathrm{d}k.$$

Finally, from the change of variables $w = k - \frac{1}{k}$, we arrive at

$$W(x, y) = \frac{1}{2}\frac{\sqrt{2xy}}{\sqrt{\pi}}\, e^{-xy} \int_{-\infty}^\infty e^{-\frac{xy}{2}w^2}\, \mathrm{d}w = e^{-xy}.$$

This proves (2.93).

We now prove that the measure ν given by (2.92) is non-negative. We do so by establishing that the density function

$$D(\lambda) = \int_0^\infty \frac{t}{2\sqrt{\pi\lambda^3}}\, e^{-\frac{t^2}{4\lambda}} \left(\frac{1}{t}\sin(\sqrt{\pi}t) - \sqrt{\pi}\cos(\sqrt{\pi}t) \right) \mathrm{d}t$$

is non-negative for all $\lambda > 0$. Again, we make the variable change $y = \sqrt{\pi}t$ and obtain that

$$D(\lambda) = \frac{1}{2\pi\lambda^{3/2}} \int_0^\infty e^{-\frac{y^2}{4\pi\lambda}} (\sin y - y\cos y)\, \mathrm{d}y.$$

Setting $\pi\lambda = a^2$, it suffices to prove that

$$\int_0^\infty e^{-\frac{y^2}{4a^2}} (\sin y - y\cos y)\, \mathrm{d}y \geq 0$$

for all $a > 0$. Using integration by parts and the Fourier transform of the odd Gaussian, we obtain that

$$\begin{aligned}
\int_0^\infty e^{-\frac{y^2}{4a^2}} (\sin y - y\cos y)\, \mathrm{d}y &= \left\{ (1 + 2a^2) \int_0^\infty e^{-\frac{y^2}{4a^2}} \sin y \,\mathrm{d}y \right\} - 2a^2 \\
&= \left\{ (1 + 2a^2)\, 2a\, e^{-a^2} \int_0^a e^{w^2}\, \mathrm{d}w \right\} - 2a^2.
\end{aligned}$$

We are left to prove that

$$h(a) = \int_0^a e^{w^2} \mathrm{d}w - \frac{a\, e^{a^2}}{1+2a^2} \geq 0$$

for all $a \geq 0$. This follows from observing that $h(0) = 0$ and

$$h'(a) = e^{a^2} \left(\frac{4a^2}{(1+2a^2)^2} \right) \geq 0$$

for all $a \geq 0$. This concludes the proof of the non-negativity of the measure.

Finally, we verify that ν is indeed a finite measure on $(0, \infty)$. In fact, note that (2.93) and the monotone convergence theorem imply

$$\int_0^\infty \mathrm{d}\nu(\lambda) = \lim_{x \to 0^+} \int_0^\infty e^{-\pi\lambda x^2}\, \mathrm{d}\nu(\lambda) = \lim_{x \to 0^+} \left\{ \arctan\left(\frac{1}{x}\right) - \frac{x}{x^2+1} \right\} = \frac{\pi}{2},$$

and this concludes the verification of the original claims.

Chapter 3

Applications to the theory of the Riemann zeta-function

3.1 Bounds under the Riemann hypothesis

After a brief introduction to the Beurling-Selberg extremal problem and some of its recent advances in the previous chapter, our objective in this chapter is to provide some applications of these extremal tools. Here we will focus our attention on the interesting connection between some special functions and the theory of the Riemann zeta-function. These special functions are the following:

$$f(x) = \log\left(\frac{x^2+1}{x^2}\right), \tag{3.1}$$

$$g(x) = \arctan\left(\frac{1}{x}\right) - \frac{x}{1+x^2}, \tag{3.2}$$

and

$$h(x) = 1 - x\arctan\left(\frac{1}{x}\right). \tag{3.3}$$

Throughout this chapter we fix this notation for $f(x)$, $g(x)$ and $h(x)$, and recall that we obtained in the previous chapter the solution of the Beurling-Selberg extremal problem for these three functions.

Bernhard Riemann published his paper "*Über die Anzahl der Primzahlen unter einer gegebenen Grösse*" in the *Monatsberichte der Berliner Akademie* in November, 1859. There we find the statement that the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

initially defined for $\mathrm{Re}(s) > 1$, and then suitably extended meromorphically to the complex plane, "probably" has its complex zeros all aligned over the line $\Re(s) = 1/2$.

His hypothesis has not been proved until this day (despite the fact that modern computers can verify that the first 10^{12} zeros are on the critical line), but considerable effort has been put in order to understand the different objects in the theory of the Riemann zeta-function assuming its validity.

For instance, J. E. Littlewood in 1924 [26] showed that under the Riemann hypothesis (RH) we have the following estimate:

$$\log\left|\zeta\left(\tfrac{1}{2}+it\right)\right| \leq \big(C+o(1)\big)\frac{\log t}{\log\log t},$$

for sufficiently large t. This estimate was never improved in its order of magnitude, and the advances have rather focused on diminishing the value of the admissible constant C. In [34] Ramachandra and Sankaranarayanan obtained $C = 0.466$, while in [38] Soundararajan improved this bound, obtaining $C = 0.373$. Recently, Chandee and Soundararajan in [10, Theorem 1] obtained another improvement, currently the best bound, as shown below.

Theorem 3.1 (Upper bound for $\zeta(s)$ in the critical line)**.** *Assume RH. For large real numbers t, we have*

$$\log\left|\zeta\left(\tfrac{1}{2}+it\right)\right| \leq \frac{\log 2}{2}\frac{\log t}{\log\log t} + O\left(\frac{\log t\log\log\log t}{(\log\log t)^2}\right).$$

A generalization of this result to the critical strip $0 < \Re(s) < 1$ was later obtained by Carneiro and Chandee in [3, Theorem 1].

Another object of interest is the argument function defined by (here $t > 0$)

$$S(t) = \tfrac{1}{\pi}\arg\zeta\left(\tfrac{1}{2}+it\right),$$

where the argument is defined by a continuous variation along the line segments joining the points $2, 2+it$ and $\frac{1}{2}+it$, taking $\arg\zeta(2) = 0$, if t is not an ordinate of a zero of $\zeta(s)$. If t is an ordinate of a zero we set

$$S(t) = \tfrac{1}{2}\lim_{\epsilon\to 0}\big\{S(t+\epsilon)+S(t-\epsilon)\big\}.$$

This function appears for instance when counting the number of zeros $N(t)$ of $\zeta(s)$ with imaginary ordinate in the interval $[0,t]$

$$N(t) = \frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O\left(\frac{1}{t}\right).$$

In the work [26] Littlewood also showed that under RH we have

$$|S(t)| \leq \big(C+o(1)\big)\frac{\log t}{\log\log t},$$

and, as in the case of the size of $\zeta\left(\frac{1}{2}+it\right)$, this estimate has not been improved in its order of magnitude over the years. Efforts to bring down the value of the admissible constant C were carried out by Ramachandra and Sankaranarayanan [34] who proved that $C = 1.119$ is admissible, and later by Fujii [13] who obtained the result for $C = 0.67$.

The application of certain extremal functions of exponential type, that majorize and/or minorize the characteristic functions of intervals, to problems related to the theory of the Riemann zeta-function dates back to the works of Montgomery [31] and Gallagher [14], on the pair correlation of zeros of $\zeta(s)$. In [16] Goldston and Gonek were the first to realize a distinct connection between the Riemann hypothesis and these extremal functions, via the so called Guinand-Weil explicit formula (the method we shall be presenting here). Using this connection they obtained the following bound [16, Theorem 2] for the argument function

$$|S(t)| \leq \left(\frac{1}{2} + o(1)\right)\frac{\log t}{\log\log t}.$$

We shall present here a sharper version of this bound, recently obtained by Carneiro, Chandee and Milinovich in [4, Theorem 2].

Theorem 3.2 (Bound for $S(t)$)**.** *Assume RH. For t sufficiently large we have*

$$|S(t)| \leq \frac{1}{4}\frac{\log t}{\log\log t} + O\left(\frac{\log t \log\log\log t}{(\log\log t)^2}\right).$$

Finally, another important function in the theory of the Riemann zeta-function is the antiderivative of $S(t)$ defined by

$$S_1(t) = \int_0^t S(u)\,\mathrm{d}u.$$

There has been earlier work on establishing explicit bounds for $S_1(t)$. Littlewood [26] was the first to prove that $S_1(t) \ll \log t/(\log\log t)^2$ under the assumption of the Riemann hypothesis. More recently, Karatsuba and Korolëv [23] obtained that

$$\big|S_1(t)\big| \leq (40+o(1))\,\frac{\log t}{(\log\log t)^2},$$

and Fujii [13] obtained that

$$-\,(0.51+o(1))\,\frac{\log t}{(\log\log t)^2} \leq S_1(t) \leq (0.32+o(1))\,\frac{\log t}{(\log\log t)^2}.$$

We present here the following improvement obtained in [4, Theorem 1].

Theorem 3.3 (Bounds for $S_1(t)$)**.** *Assume RH. For t sufficiently large we have*

$$-\left(\frac{\pi}{24}+o(1)\right)\frac{\log t}{(\log\log t)^2} \leq S_1(t) \leq \left(\frac{\pi}{48}+o(1)\right)\frac{\log t}{(\log\log t)^2},$$

where the terms $o(1)$ in the above inequalities are $O(\log\log\log t/\log\log t)$.

The objective of this chapter is to prove the three theorems above. The general strategy for their proofs is essentially the same. It consists of three main steps: (i) expressing the considered object as a certain sum over the zeros of $\zeta(s)$; (ii) making use of suitable extremal majorants/minorants of exponential type; (iii) applying an appropriate explicit formula to evaluate the sums by taking advantage of the compactly supported Fourier transforms. We shall see in the next section how the functions $f(x)$, $g(x)$ and $h(x)$ are naturally related to Theorems 3.1, 3.2 and 3.3, respectively.

3.2 Representation lemmas and the explicit formula

In this section we let

$$\xi(s) = \frac{1}{2}s(1-s)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

be Riemann's ξ-function. This function is an entire function of order 1 and satisfies the functional equation

$$\xi(s) = \xi(1-s).$$

Hadamard's factorization formula (cf. [12, Chapter 12]) gives us

$$\xi(s) = e^{A+Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho},$$

where $\rho = \frac{1}{2} + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$. We have $B = -\sum_{\rho}\mathrm{Re}(1/\rho)$, with this sum being absolutely convergent. Under RH, γ is real. For $\frac{1}{2} \le \alpha \le \frac{3}{2}$ define

$$f_{\alpha}(x) = \log\left(\frac{x^2+1}{x^2+\left(\alpha-\frac{1}{2}\right)^2}\right). \tag{3.4}$$

Note that our $f(x)$ initially defined in (3.1) is the same $f_{1/2}(x)$ defined above.

Lemma 3.4 (Representation for $\log|\zeta(\alpha+it)|$)**.** *Assume RH and let $f_{\alpha}(x)$ be defined by* (3.4)*, where $\frac{1}{2} \le \alpha \le \frac{3}{2}$. For large t we have*

$$\log|\zeta(\alpha+it)| = \left(\frac{3}{4}-\frac{\alpha}{2}\right)\log t - \frac{1}{2}\sum_{\gamma} f_{\alpha}(t-\gamma) + O(1), \tag{3.5}$$

uniformly on α, where the sum runs over the non-trivial zeros $\rho = \frac{1}{2}+i\gamma$ of $\zeta(s)$.

Proof. We apply Hadamard's factorization formula at the points $s = \alpha + it$ and $s = \frac{3}{2} + it$ and divide. The absolute convergence of the product allows us to divide term by term to find

$$\left| \frac{\xi(\alpha + it)}{\xi\left(\frac{3}{2} + it\right)} \right| = \prod_{\rho = 1/2 + i\gamma} \left(\frac{(\alpha - \frac{1}{2})^2 + (t - \gamma)^2}{1 + (t - \gamma)^2} \right)^{1/2},$$

and therefore

$$\log|\xi(\alpha + it)| = \log\left|\xi\left(\tfrac{3}{2} + it\right)\right| + \frac{1}{2}\sum_{\gamma} \log\left(\frac{(\alpha - \frac{1}{2})^2 + (t - \gamma)^2}{1 + (t - \gamma)^2} \right). \tag{3.6}$$

Recall Stirling's formula for the Gamma function [12, Chapter 10]

$$\log\Gamma(z) = \tfrac{1}{2}\log 2\pi - z + \left(z - \tfrac{1}{2}\right)\log z + O\big(|z|^{-1}\big),$$

for large $|z|$. Using Stirling's formula and the fact that $\left|\zeta\left(\frac{3}{2} + it\right)\right| \asymp 1$ in (3.6), we obtain (3.5). □

Similar representations hold for the argument function $S(t)$ and for the function $S_1(t)$, as reported in [4].

Lemma 3.5 (Representation for $S(t)$)**.** *Assume RH and let $g(x)$ be defined by (3.2). Then, for large t not coinciding with an ordinate of a zero of $\zeta(s)$, we have*

$$S(t) = \frac{1}{\pi}\sum_{\gamma} g(t - \gamma) + O(1), \tag{3.7}$$

where the sum runs over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$.

Proof. For t not coinciding with an ordinate of a zero of $\zeta(s)$, we have

$$S(t) = -\frac{1}{\pi}\int_{\frac{1}{2}}^{\infty} \Im\frac{\zeta'}{\zeta}(\sigma + it)\,\mathrm{d}\sigma = \frac{1}{\pi}\int_{\frac{3}{2}}^{\frac{1}{2}} \Im\frac{\zeta'}{\zeta}(\sigma + it)\,\mathrm{d}\sigma + O(1).$$

We now replace the integrand on the right-hand side of the above expression by a sum over the non-trivial zeros of $\zeta(s)$. Let $s = \sigma + it$. If s is not a zero of $\zeta(s)$, then the partial fraction decomposition for $\zeta'(s)/\zeta(s)$ (cf. [12, Chapter 12]) and Stirling's formula

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z + O\big(|z|^{-1}\big), \tag{3.8}$$

valid for large $|z|$ with $\Re(z) > 0$, imply that

$$\begin{aligned} \frac{\zeta'}{\zeta}(s) &= \sum_{\rho}\left(\frac{1}{s - \rho} + \frac{1}{\rho}\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2} + 1\right) + O(1) \\ &= \sum_{\rho}\left(\frac{1}{s - \rho} + \frac{1}{\rho}\right) - \frac{1}{2}\log t + O(1) \end{aligned} \tag{3.9}$$

uniformly for $\frac{1}{2} \leq \sigma \leq \frac{3}{2}$ and $t \geq 2$, where the sum runs over the non-trivial zeros ρ of $\zeta(s)$. From (3.9) and the Riemann hypothesis, it follows that

$$\begin{aligned} S(t) &= \frac{1}{\pi} \int_{\frac{3}{2}}^{\frac{1}{2}} \Im \frac{\zeta'}{\zeta}(\sigma + it)\,\mathrm{d}\sigma + O(1) \\ &= \frac{1}{\pi} \int_{\frac{3}{2}}^{\frac{1}{2}} \Im \left(\frac{\zeta'}{\zeta}(\sigma + it) - \frac{\zeta'}{\zeta}(\tfrac{3}{2} + it) \right) \mathrm{d}\sigma + O(1) \\ &= \frac{1}{\pi} \int_{\frac{1}{2}}^{\frac{3}{2}} \sum_{\gamma} \left\{ \frac{(t-\gamma)}{(\sigma - \frac{1}{2})^2 + (t-\gamma)^2} - \frac{(t-\gamma)}{1 + (t-\gamma)^2} \right\} \mathrm{d}\sigma + O(1) \\ &= \frac{1}{\pi} \sum_{\gamma} \int_{\frac{1}{2}}^{\frac{3}{2}} \left\{ \frac{(t-\gamma)}{(\sigma - \frac{1}{2})^2 + (t-\gamma)^2} - \frac{(t-\gamma)}{1 + (t-\gamma)^2} \right\} \mathrm{d}\sigma + O(1) \\ &= \frac{1}{\pi} \sum_{\gamma} \left\{ \arctan\left(\frac{1}{(t-\gamma)} \right) - \frac{(t-\gamma)}{1 + (t-\gamma)^2} \right\} + O(1) \\ &= \frac{1}{\pi} \sum_{\gamma} g(t-\gamma) + O(1), \end{aligned}$$

where the interchange of the integral and the sum is justified by dominated convergence since $g(x) = O(x^{-3})$. This proves the lemma. □

Lemma 3.6 (Representation for $S_1(t)$)**.** *Assume RH and let $h(x)$ be defined by* (3.3)*. For large t we have*

$$S_1(t) = \frac{1}{4\pi} \log t - \frac{1}{\pi} \sum_{\gamma} h(t-\gamma) + O(1),$$

where the sum runs over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$.

Proof. From [42, Theorem 9.9] we have

$$S_1(t) = \frac{1}{\pi} \int_{1/2}^{3/2} \log \left| \zeta(\alpha + it) \right| \mathrm{d}\alpha + O(1).$$

We replace the integrand by the absolutely convergent sum over the zeros of $\zeta(s)$ given by Lemma 3.4 and integrate term-by-term to obtain

$$S_1(t) = \frac{1}{4\pi} \log t - \frac{1}{\pi} \sum_{\rho} h(t-\gamma) + O(1),$$

where the interchange between integration and sum is justified since all terms are non-negative. Notice that we have used the fact that

$$h(x) = 1 - x \arctan\left(\frac{1}{x} \right) = \frac{1}{2} \int_{1/2}^{3/2} \log \left(\frac{x^2 + 1}{x^2 + \left(\alpha - \frac{1}{2}\right)^2} \right) \mathrm{d}\alpha. \tag{3.10}$$

This completes the proof of the lemma. □

We note the similarity of the representations obtained on Lemmas 3.4, 3.5 and 3.6. We were able to write each of our objects (initially a function of t) as a simple function of t plus a sum over the zeros of $\zeta(s)$ plus a small error term. Naturally the hard part to be analyzed is the sum over the zeros of $\zeta(s)$, but fortunately for this matter we can invoke the following version of the Guinand-Weil explicit formula [22, Theorem 5.12] which connects sums over the zeros of $\zeta(s)$ to sums of the Fourier transforms evaluated at the prime powers.

Lemma 3.7 (Guinand-Weil explicit formula)**.** *Let $\Phi(s)$ be analytic in the strip $|\Im(s)| \leq 1/2+\varepsilon$ for some $\varepsilon > 0$, and assume that $|\Phi(s)| \ll (1+|s|)^{-(1+\delta)}$ for some $\delta > 0$ when $|\Re(s)| \to \infty$. Let $\Phi(x)$ be real-valued for real x, and set $\widehat{\Phi}(\xi) = \int_{-\infty}^{\infty} \Phi(x)\, e^{-2\pi i x\xi}\, \mathrm{d}x$. Then*

$$\sum_{\rho} \Phi(\gamma) = \Phi\left(\frac{1}{2i}\right) + \Phi\left(-\frac{1}{2i}\right) - \frac{1}{2\pi}\widehat{\Phi}(0)\log\pi + \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\mathrm{d}u - \frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left(\widehat{\Phi}\left(\frac{\log n}{2\pi}\right) + \widehat{\Phi}\left(\frac{-\log n}{2\pi}\right)\right).$$

where Γ'/Γ is the logarithmic derivative of the Gamma function, and $\Lambda(n)$ is the von Mangoldt function defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m,\ p \text{ prime},\ m \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Observe however that we cannot apply the explicit formula to evaluate the sum of our particular functions f, g and h over the zeros of ζ, since f has singularities on the strip $|\Im(s)| \leq 1/2$, g is not continuous and h is not differentiable at the origin. To overcome this difficulty we adopt the following strategy:

(i) We want to replace each of our functions f, g and h by an appropriate majorant or minorant (to create an inequality), that satisfies the hypothesis of the explicit formula (a real entire function, integrable on $\mathbb{R}$).

Now that we believe we will be able to use the explicit formula, we might want to choose which of its expressions we would like to “keep" or “simplify". For this we will focus on two of its terms.

(ii) We will ask that the term $\widehat{\Phi}(0)$ for these majorants be as close as possible to the original $\widehat{f}(0)$, which is the same as saying that $\int_{\mathbb{R}}\{\Phi - f\}\,\mathrm{d}x$ should be minimal.

(iii) Finally, in order to simplify the sum of the Fourier transforms of over prime powers, we will consider the instances in which this sum is finite, i.e. $\widehat{\Phi}$ has compact support.

With this framework we are essentially asking for the solution of the Beurling-Selberg problem for each of the functions f, g and h.

3.3 Extremal functions revisited

In this section we revisit the extremal function theory to state and prove the main facts concerning the majorants/minorants of the functions $f(x)$, $g(x)$ and $h(x)$ in the precise format we need. Let us start with the minorants for the function $f(x)$, contemplated in the Gaussian subordination framework for even functions in Chapter 2. Observe that there can be no discussion about real entire majorants for this function because of its singularity at the origin.

Lemma 3.8 (Extremal minorants for f)**.** *Let $1 \leq \Delta$ and f be defined by* (3.1)*. Then there is a unique real entire function $m_\Delta^- : \mathbb{C} \to \mathbb{C}$ satisfying the following properties:*

(i) For all real x we have

$$\frac{-C}{1+x^2} \leq m_\Delta^-(x) \leq f(x)$$

for some positive constant C. For any complex number $x+iy$ we have

$$\big|m_\Delta^-(x+iy)\big| \ll \frac{\Delta^2}{1+\Delta|x+iy|}e^{2\pi\Delta|y|}.$$

(ii) The Fourier transform of m_Δ^-, namely

$$\widehat{m}_\Delta^-(\xi) = \int_{-\infty}^{\infty} m_\Delta^-(x)\, e^{-2\pi i x\xi}\,\mathrm{d}x,$$

is a continuous real-valued function supported on the interval $[-\Delta, \Delta]$ and satisfies

$$\big|\widehat{m}_\Delta^-(\xi)\big| \ll 1$$

for each $\xi \in [-\Delta, \Delta]$.

(iii) The L^1-distance to f is given by

$$\int_{-\infty}^{\infty} \big\{f(x) - m_\Delta^-(x)\big\}\,\mathrm{d}x = \frac{2}{\Delta}\Big\{\log 2 - \log\big(1+e^{-2\pi\Delta}\big)\Big\}.$$

Proof. (i) Observe that the desired function $m_\Delta^-(x)$ we seek is the extremal minorant of exponential type $2\pi\Delta$ of $f(x)$. To adjust to our work in Chapter 2 (done for exponential type 2π) we simply consider the function

$$F_\Delta(x) := f\left(\frac{x}{\Delta}\right) = \log\left(\frac{x^2+\Delta^2}{x^2}\right).$$

We do know (by Section 2.3.5 - Example 3) that such F_Δ admits an minorant of type 2π, that we shall call $L_\Delta(z)$, given by

$$L_\Delta(z) = \Big(\frac{\cos \pi z}{\pi}\Big)^2 \sum_{n=-\infty}^{\infty} \left\{ \frac{F_\Delta\big(n-\frac{1}{2}\big)}{\big(z-n+\frac{1}{2}\big)^2} + \frac{F'_\Delta\big(n-\frac{1}{2}\big)}{\big(z-n+\frac{1}{2}\big)} \right\},$$
$$= \sum_{n=-\infty}^{\infty} \left(\frac{\sin \pi\big(z-n+\frac{1}{2}\big)}{\pi\big(z-n+\frac{1}{2}\big)}\right)^2 \left\{ f\left(\frac{n-\frac{1}{2}}{\Delta}\right) + \frac{\big(z-n+\frac{1}{2}\big)}{\Delta} f'\left(\frac{n-\frac{1}{2}}{\Delta}\right)\right\}. \tag{3.11}$$

For any complex number ξ we have

$$\left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2 \ll \frac{e^{2\pi|\Im(\xi)|}}{1+|\xi|^2},$$

and we also have $f(x) \leq 1/x^2$ and $|f'(x)| \leq 2/(|x|(x^2+1))$ for real x. From (3.11) we conclude that

$$\big|L_\Delta(x+iy)\big| \leq \frac{\Delta^2}{1+|x+iy|}\, e^{2\pi|y|}. \tag{3.12}$$

Since $f(x) \geq 0$ and $f'(-x) = -f'(x)$, by pairing the terms $n \geq 1$ with the terms $1-n \leq 0$ in the sum (3.11) we find, for $x \in \mathbb{R}$, that

$$L_\Delta(x) \geq \sum_{n=1}^{\infty} \left(\frac{\sin \pi\big(x-n+\frac{1}{2}\big)}{\pi\big(x^2-\big(n-\frac{1}{2}\big)\big)^2}\right)^2 \frac{2\,\big(n-\frac{1}{2}\big)}{\Delta} f'\left(\frac{n-\frac{1}{2}}{\Delta}\right), \tag{3.13}$$

and from this we can deduce that there is a constant C such that

$$-\,C\frac{\Delta^2}{\Delta^2+x^2} \leq L_\Delta(x) \leq F_\Delta(x). \tag{3.14}$$

We now consider $m_\Delta^-(z) = L_\Delta(\Delta z)$. Part (i) of the lemma plainly follows from (3.12) and (3.14).

(ii) We know that m_Δ^- is an (even) entire function of exponential type $2\pi\Delta$ that is uniformly integrable on $\mathbb{R}$ (with integral independent of the parameter $\Delta \geq 1$, by part (i)). From the Paley-Wiener theorem we have that $\widehat{m}_\Delta^-$ is a continuous real-valued function supported on the interval $[-\Delta, \Delta]$ and satisfies

$$\big|\widehat{m}_\Delta^-(\xi)\big| \ll 1$$

for each $\xi \in [-\Delta, \Delta]$.

(iii) Recall from Section 2.3.5 - Example 3 that

$$\int_{-\infty}^{\infty} \{F_\Delta(x) - L_\Delta(x)\}\, \mathrm{d}x = 2\Big\{\log 2 - \log\big(1+e^{-2\pi\Delta}\big)\Big\}.$$

A simple change of variables $x \mapsto \Delta x$ gives the desired result. □

The proofs of next two lemmas are similar to the previous one and are omitted here. The interested reader can check the details in [4, Lemmas 6 and 8].

Lemma 3.9 (Extremal functions for g)**.** *Let $1 \leq \Delta$ and g be defined by* (3.2)*. Then there are unique real entire functions $m_\Delta^+ : \mathbb{C} \to \mathbb{C}$ and $m_\Delta^- : \mathbb{C} \to \mathbb{C}$ satisfying the following properties:*

(i) For all real x we have

$$\frac{-C}{1+x^2} \leq m_\Delta^-(x) \leq g(x) \leq m_\Delta^+(x) \leq \frac{C}{1+x^2},$$

for some positive constant C. For any complex number $x + iy$ we have

$$\left|m_\Delta^\pm(x+iy)\right| \ll \frac{\Delta^2}{1+\Delta|x+iy|} e^{2\pi\Delta|y|}.$$

(ii) The Fourier transforms of $m_\Delta^\pm$ are continuous functions supported on the interval $[-\Delta, \Delta]$ and satisfy

$$\left|\widehat{m}_\Delta^\pm(\xi)\right| \ll 1$$

for each $\xi \in [-\Delta, \Delta]$.

(iii) The L^1-distances to g are given by

$$\int_{-\infty}^{\infty} \left\{m_\Delta^+(x) - g(x)\right\} \,\mathrm{d}x = \int_{-\infty}^{\infty} \left\{g(x) - m_\Delta^-(x)\right\} \,\mathrm{d}x = \frac{\pi}{2\Delta}.$$

Lemma 3.10 (Extremal functions for h)**.** *Let $1 \leq \Delta$ and h be defined by* (3.3)*. Then there are unique real entire functions $m_\Delta^+ : \mathbb{C} \to \mathbb{C}$ and $m_\Delta^- : \mathbb{C} \to \mathbb{C}$ satisfying the following properties:*

(i) For all real x we have

$$\frac{-C}{1+x^2} \leq m_\Delta^-(x) \leq h(x) \leq m_\Delta^+(x) \leq \frac{C}{1+x^2},$$

for some positive constant C. For any complex number $x + iy$ we have

$$\left|m_\Delta^\pm(x+iy)\right| \ll \frac{\Delta^2}{1+\Delta|x+iy|} e^{2\pi\Delta|y|}.$$

(ii) The Fourier transforms of $m_\Delta^\pm$ are continuous real-valued functions supported on the interval $[-\Delta, \Delta]$ and satisfy

$$\left|\widehat{m}_\Delta^\pm(\xi)\right| \ll 1$$

for each $\xi \in [-\Delta, \Delta]$.

(iii) The L^1-distances to h are given by

$$\int_{-\infty}^{\infty} \big\{h(x) - m_\Delta^-(x)\big\}\,\mathrm{d}x = \int_{1/2}^{3/2} \frac{1}{\Delta}\Big\{\log\big(1+e^{-(2\sigma-1)\pi\Delta}\big) - \log\big(1+e^{-2\pi\Delta}\big)\Big\}\,\mathrm{d}\sigma,$$

and

$$\int_{-\infty}^{\infty} \big\{m_\Delta^+(x) - h(x)\big\}\,\mathrm{d}x = \int_{1/2}^{3/2} \frac{1}{\Delta}\Big\{\log\big(1-e^{-2\pi\Delta}\big) - \log\big(1-e^{-(2\sigma-1)\pi\Delta}\big)\Big\}\,\mathrm{d}\sigma.$$

3.4 Proofs of the main theorems

We now make use of the extremal functions described on the last section, together with the representation formulas to provide the proofs of the main theorems.

3.4.1 Proof of Theorem 3.1

With f defined by (3.1) and m_Δ^- defined as in Lemma 3.8, we can use Lemma 3.4 to obtain

$$\begin{aligned}\log\big|\zeta\big(\tfrac{1}{2}+it\big)\big| &= \frac{1}{2}\log t - \frac{1}{2}\sum_\gamma f(t-\gamma) + O(1)\\ &\leq \frac{1}{2}\log t - \frac{1}{2}\sum_\gamma m_\Delta^-(t-\gamma) + O(1).\end{aligned} \tag{3.15}$$

We now apply the explicit formula (Lemma 3.7) with $\Phi(z) = m_\Delta^-(t-z)$. In this context we have $\widehat{\Phi}(\xi) = \widehat{m}_\Delta^-(-\xi)e^{-2\pi i\xi t}$ and therefore

$$\begin{aligned}\sum_\rho m_\Delta^-(t-\gamma) &= \big\{m_\Delta^-\big(t-\tfrac{1}{2i}\big) + m_\Delta^-\big(t+\tfrac{1}{2i}\big)\big\} - \frac{1}{2\pi}\widehat{m}_\Delta^-(0)\log\pi\\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^-(t-u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\mathrm{d}u\\ &\quad - \frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left\{n^{-it}\,\widehat{m}_\Delta^-\left(-\frac{\log n}{2\pi}\right) + n^{it}\,\widehat{m}_\Delta^-\left(\frac{\log n}{2\pi}\right)\right\}.\end{aligned} \tag{3.16}$$

Let us split this sum into four terms and quote each of these separately.

First term. From Lemma 3.8 (i) we see that

$$\left|m_\Delta^-\left(t - \tfrac{1}{2i}\right) + m_\Delta^-\left(t + \tfrac{1}{2i}\right)\right| \ll \frac{\Delta^2}{1+\Delta t} e^{\pi\Delta}. \tag{3.17}$$

Second term. From Lemma 3.8 (ii) we have

$$\left|\widehat{m}_\Delta^-(0)\right| \ll 1. \tag{3.18}$$

Third term. Using Stirling's formula (3.8), Lemma 3.8 (i) and (iii), and the fact that $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi$ we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^-(t-u)\,\Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\right]\,\mathrm{d}u &= \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^-(u)\left(\log t + O(\log(2+|u|))\right)\mathrm{d}u \qquad (3.19)\\ &= \log t - \frac{\log t}{\pi\Delta}\log\left(\frac{2}{1+e^{-2\pi\Delta}}\right) + O(1).\end{aligned}$$

Fourth term. Finally, we use the fact that the Fourier transform of m_Δ^- is compactly supported on the interval $[-\Delta,\Delta]$, as given in Lemma 3.8 (ii), to bound the sum over the prime powers

$$\begin{aligned}&\left|\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left\{n^{-it}\widehat{m}_\Delta^-\left(-\frac{\log n}{2\pi}\right) + n^{it}\widehat{m}_\Delta^-\left(\frac{\log n}{2\pi}\right)\right\}\right| \\ &\le \frac{1}{2\pi}\sum_{n=2}^{e^{2\pi\Delta}}\frac{\Lambda(n)}{\sqrt{n}}\left\{\left|\widehat{m}_\Delta^-\left(-\frac{\log n}{2\pi}\right)\right| + \left|\widehat{m}_\Delta^-\left(\frac{\log n}{2\pi}\right)\right|\right\} \qquad (3.20)\\ &\ll \sum_{n=2}^{e^{2\pi\Delta}}\frac{\Lambda(n)}{\sqrt{n}} \ll e^{\pi\Delta},\end{aligned}$$

where the last expression was evaluated via summation by parts.

Conclusion. Combining expressions (3.15)-(3.20) we arrive at

$$\log\left|\zeta\left(\tfrac{1}{2}+it\right)\right| \le \frac{\log t}{2\pi\Delta}\log\left(\frac{2}{1+e^{-2\pi\Delta}}\right) + O\left(\frac{\Delta^2 e^{\pi\Delta}}{(1+\Delta t)} + e^{\pi\Delta} + 1\right). \tag{3.21}$$

Until now we did all of our estimates without prescribing any particular value for Δ. It turns out that the choice

$$\pi\Delta = \log\log t - 3\log\log\log t$$

in (3.21) concludes the proof of Theorem 3.1.

3.4.2 Proof of Theorem 3.2

This follows by a very similar argument. With g defined as in (3.2), and $m_\Delta^\pm$ defined as in Lemma 3.9, we can use Lemma 3.5 to obtain

$$\begin{aligned}&\frac{1}{\pi}\sum_\gamma m_\Delta^-(t-\gamma)+O(1)\\&\qquad\leq S(t)=\frac{1}{\pi}\sum_\gamma g(t-\gamma)+O(1)\leq\frac{1}{\pi}\sum_\gamma m_\Delta^+(t-\gamma)+O(1).\end{aligned}$$

We then use the explicit formula with $m_\Delta^\pm$ and bound the first, second and fourth terms as done in the proof of Theorem 3.1, now using Lemma 3.9. For the third term we use Stirling's formula (3.8), Lemma 3.9 (i) and (iii), and the fact that $\int_{-\infty}^{\infty} g(x)\,\mathrm{d}x=0$ to get

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^\pm(t-u)\,\Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\right]\,\mathrm{d}u=\\&\qquad=\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^\pm(u)\left(\log t+O(\log(2+|u|)\right)\mathrm{d}u\\&\qquad=\pm\frac{\log t}{4\Delta}+O(1).\end{aligned}$$

We thus arrive at

$$|S(t)|\leq\frac{\log t}{4\pi\Delta}+O\left(\frac{\Delta^2e^{\pi\Delta}}{(1+\Delta t)}+e^{\pi\Delta}+1\right).$$

and again it is just a matter of choosing $\pi\Delta=\log\log t-3\log\log\log t$ to conclude the proof of Theorem 3.2.

3.4.3 Proof of Theorem 3.3

Let h be defined as in (3.3), and $m_\Delta^\pm$ defined as in Lemma 3.10. From Lemma 3.6 we have

$$\begin{aligned}&\frac{1}{4\pi}\log t-\frac{1}{\pi}\sum_\gamma m_\Delta^+(t-\gamma)+O(1)\\&\qquad\leq S_1(t)=\frac{1}{4\pi}\log t-\frac{1}{\pi}\sum_\gamma h(t-\gamma)+O(1)\\&\qquad\qquad\leq\frac{1}{4\pi}\log t-\frac{1}{\pi}\sum_\gamma m_\Delta^-(t-\gamma)+O(1).\end{aligned}$$

Once more we apply the explicit formula with $m_\Delta^\pm$ and bound the first, second and fourth terms as done in the proof of Theorem 3.1, now using Lemma 3.10.

For the third term we use Stirling's formula (3.8), Lemma 3.10 (i) and (iii), and the fact that

$$\int_{-\infty}^{\infty} h(x)\,\mathrm{d}x = \frac{\pi}{2}$$

to get

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^-(t-u)\,\Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\right]\,\mathrm{d}u \\
&\quad= \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^-(u)\Big(\log t + O\big(\log(2{+}|u|)\big)\Big)\,\mathrm{d}u \\
&\quad= \frac{1}{4}\log t - \frac{\log t}{2\pi\Delta}\int_{1/2}^{3/2}\left(\log\left(1+e^{-(2\sigma-1)\pi\Delta}\right) - \log\left(1+e^{-2\pi\Delta}\right)\right)\mathrm{d}\sigma + O(1) \\
&\quad\geq \frac{1}{4}\log t - \frac{\log t}{2\pi\Delta}\int_{1/2}^{\infty}\log\left(1+e^{-(2\sigma-1)\pi\Delta}\right)\mathrm{d}\sigma + O(1) \\
&\quad= \frac{1}{4}\log t - \frac{\log t}{2\pi^2\Delta^2}\int_{0}^{\infty}\log\left(1+e^{-2\alpha}\right)\mathrm{d}\alpha + O(1).
\end{aligned}$$

Now observe that (cf. [17, §4.291])

$$\int_0^{\infty}\log\left(1{+}e^{-2\alpha}\right)\mathrm{d}\alpha = \frac{1}{2}\int_0^1\frac{\log(1{+}u)}{u}\,\mathrm{d}u = \frac{\pi^2}{24}.$$

Therefore, by combining these estimates, we arrive at

$$S_1(t) \leq \frac{\log t}{48\pi\Delta^2} + O\left(\frac{\Delta^2 e^{\pi\Delta}}{(1+\Delta t)} + e^{\pi\Delta} + 1\right).$$

Choosing $\pi\Delta = \log\log t - 3\log\log\log t$ in the inequality above gives us

$$S_1(t) \leq \frac{\pi}{48}\frac{\log t}{(\log\log t)^2} + O\left(\frac{\log t\log\log\log t}{(\log\log t)^3}\right),$$

which is the upper bound for $S_1(t)$ stated in Theorem 3.3. To prove the lower bound we proceed similarly by observing that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^+(t-u)\,\Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\right]\,\mathrm{d}u \\
&\qquad= \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^+(u)\Big(\log t + O\big(\log(2{+}|u|)\big)\Big)\,\mathrm{d}u \\
&\qquad= \frac{1}{4}\log t - \frac{\log t}{2\pi\Delta}\int_{1/2}^{3/2}\left(\log\left(1-e^{-(2\sigma-1)\pi\Delta}\right) - \log\left(1-e^{-2\pi\Delta}\right)\right)\mathrm{d}\sigma + O(1) \\
&\qquad\leq \frac{1}{4}\log t - \frac{\log t}{2\pi\Delta}\int_{1/2}^{\infty}\log\left(1-e^{-(2\sigma-1)\pi\Delta}\right)\mathrm{d}\sigma + O(1) \\
&\qquad= \frac{1}{4}\log t - \frac{\log t}{2\pi^2\Delta^2}\int_{0}^{\infty}\log\left(1-e^{-2\alpha}\right)\mathrm{d}\alpha + O(1).
\end{aligned}$$

We now invoke the identity (cf. [17, §4.291])

$$\int_0^\infty \log\left(1-e^{-2\alpha}\right)\, \mathrm{d}\alpha = \frac{1}{2}\int_0^1 \frac{\log(1-u)}{u}\,\mathrm{d}u = -\frac{\pi^2}{12}$$

to arrive at

$$S_1(t) \geq -\frac{\log t}{24\pi\Delta^2} + O\left(\frac{\Delta^2 e^{\pi\Delta}}{(1+\Delta t)} + e^{\pi\Delta} + 1\right).$$

Finally, choosing $\pi\Delta = \log\log t - 3\log\log\log t$ in the inequality above gives us

$$S_1(t) \geq -\frac{\pi}{24}\frac{\log t}{(\log\log t)^2} + O\left(\frac{\log t \log\log\log t}{(\log\log t)^3}\right),$$

and this completes the proof of Theorem 3.3.

Bibliography

[1] J. T. Barton, H. L. Montgomery, and J. D. Vaaler, Note on a Diophantine inequality in several variables, Proc. Amer. Math. Soc. 129 (2001), 337–345.

[2] E. Carneiro, Sharp approximations to the Bernoulli periodic functions by trigonometric polynomials, J. Approx. Theory 154 (2008), 90–104.

[3] E. Carneiro and V. Chandee, Bounding $\zeta(s)$ in the critical strip, J. Number Theory 131 (2011), 363–384.

[4] E. Carneiro, V. Chandee and M. Milinovich, Bounding $S(t)$ and $S_1(t)$ under the Riemann hypothesis, to appear in Math. Ann.

[5] E. Carneiro and F. Littmann, Bandlimited approximations to the truncated Gaussian and applications, to appear in Const. Approx.

[6] E. Carneiro and F. Littmann, Entire approximations for a class of truncated and odd functions, preprint.

[7] E. Carneiro, F. Littmann, and J. D. Vaaler, Gaussian subordination for the Beurling-Selberg extremal problem, to appear in Trans. Amer. Math. Soc.

[8] E. Carneiro and J. D. Vaaler, Some extremal functions in Fourier analysis, II, Trans. Amer. Math. Soc. 362 (2010), 5803–5843.

[9] E. Carneiro and J. D. Vaaler, Some extremal functions in Fourier analysis, III, Constr. Approx. 31, No. 2 (2010), 259–288.

[10] V. Chandee and K. Soundararajan, Bounding $|\zeta(\frac{1}{2} + it)|$ on the Riemann Hypothesis, Bull. London Math. Soc. 43(2) (2011), 243–250.

[11] K. Chandrasekharan, *Elliptic Functions*, Springer-Verlag, Berlin, 1985.

[12] H. Davenport, *Multiplicative number theory*, Third edition, Graduate Texts in Mathematics 74, Springer-Verlag, New York, 2000.

[13] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, John Wiley and Sons Inc., 1999.

[14] P. X. Gallagher, Pair correlation of zeros of the zeta-function, J. Reine Angew. Math. 362 (1985), 72–86.

[15] M. I. Ganzburg and D. S. Lubinsky, Best approximating entire functions to $|x|^\alpha$ in L^2, Complex analysis and dynamical systems III, 93–107, Contemp. Math. 455, Amer. Math. Soc., Providence, RI, 2008.

[16] D. A. Goldston and S. M. Gonek, A note on $S(t)$ and the zeros of the Riemann zeta-function, Bull. Lond. Math. Soc. 39 (2007), no. 3, 482–486.

[17] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, Translated from Russian. Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger. Seventh edition. Elsevier/Academic Press, Amsterdam, 2007.

[18] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson, Upper Saddle River, New Jersey, 2004.

[19] S. W. Graham and J. D. Vaaler, A class of Extremal Functions for the Fourier Transform, Tran. Amer. Math. Soc. 265, (1981), 283–302.

[20] J. Holt and J. D. Vaaler, The Beurling-Selberg extremal functions for a ball in the Euclidean space, Duke Math. Journal, 83, (1996), 203–247.

[21] L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, 1964.

[22] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium Publications, vol. 53, 2004.

[23] A. A. Karatsuba and M. A. Korolëv, The argument of the Riemann zeta-function, Russian Math. Surveys 60 (2005) no. 3, 433–488 (in English), Uspekhi Mat. Nauk 60 (2005), no. 3, 41–96 (in Russian).

[24] M. G. Krein, On the best approximation of continuous differentiable functions on the whole real axis, Dokl. Akad. Nauk SSSR 18 (1938), 615-624. (Russian).

[25] X. J. Li and J. D. Vaaler, Some trigonometric extremal functions and the Erdös-Turán type inequalities, Indiana Univ. Math. J. 48 (1999), no. 1, 183–236.

[26] J. E. Littlewood, On the zeros of the Riemann zeta-function, Proc. Camb. Philos. Soc. 22 (1924), 295–318.

[27] F. Littmann, One-sided approximation by entire functions, J. Approx. Theory 141 (2006), no. 1, 1–7.

[28] F. Littmann, Entire majorants via Euler-Maclaurin summation, Trans. Amer. Math. Soc. 358 (2006), no. 7, 2821–2836.

[29] F. Littmann, Entire approximations to the truncated powers, Constr. Approx. 22 (2005), no. 2, 273–295.

[30] F. Littmann, Quadrature and extremal bandlimited functions, preprint.

[31] H. L. Montgomery, The pair correlation of zeros of the zeta-function, Proc. Symp. Pure Math. 24, Providence (1973), 181–193.

[32] H. L. Montgomery and R. C. Vaughan, Hilbert's Inequality, J. London Math. Soc. (2), 8, (1974), 73–81.

[33] M. Plancherel and G. Polya, Fonctions entiéres et intégrales de Fourier multiples, (Seconde partie) Comment. Math. Helv. 10, (1938), 110–163.

[34] K. Ramachandra and A. Sankaranarayanan, On some theorems of Littlewood and Selberg. I., J. Number Theory 44 (1993), 281–291.

[35] I. J. Schoenberg, Metric spaces and completely monotone functions, Ann. of Math. (2) 39 (1938), 811–841.

[36] L. Schwartz, *Théorie des Distributions*, Vol. I et II, Hermann 1950, 1951.

[37] A. Selberg, Lectures on Sieves, *Atle Selberg: Collected Papers*, Vol. II, Springer-Verlag, Berlin, 1991, pp. 65–247.

[38] K. Soundararajan, Moments of the Riemann zeta-function, Ann. of Math. (2) 170 (2009), no. 2, 981–993.

[39] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[40] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean spaces*, Princeton University Press, 1973.

[41] B. Sz.-Nagy, Über gewisse Extremalfragen bei transformierten trigonometrischen Entwicklungen II, Ber. Math.-Phys. Kl. Sächs. Akad. Wiss. Leipzig 91, 1939.

[42] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd edition (The Clarendon Press/Oxford University Press, New York, 1986).

[43] J. D. Vaaler, Some extremal functions in Fourier analysis, Bull. Amer. Math. Soc. 12 (1985), 183–215.

[44] A. Zygmund, *Trigonometric Series*, Vol II, Cambridge University Press, 1959.