TUG-OF-WAR GAMES AND PDES.
GAMES THAT PDE PEOPLE LIKE TO PLAY.

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Abstract. In these notes we briefly review some recent results concerning Tug-of-War games and their relation to some well known PDEs. In particular, we will show that solutions to certain PDEs can be obtained as limits of values of Tug-of-War games when the parameter that controls the length of the possible movements goes to zero. Since the equations under study are nonlinear and not in divergence form we will make extensive use of the concept of viscosity solutions.

1. Introduction

The goal of these notes is to introduce the reader (expert or not) to some important techniques and results in the theory of second order elliptic PDEs and their connections with game theory.

The fundamental works of Doob, Hunt, Kakutani, Kolmogorov and many others have shown the profound and powerful connection between the classical linear potential theory and the corresponding probability theory. The idea behind the classical interplay is that harmonic functions and martingales share a common origin in mean value properties. This approach turns out to be useful in the nonlinear theory as well.

First, our aim is to explain through elementary examples a way in which elliptic PDEs arise in Probability. For instance, first we show how simple is the relation between probabilistic issues on random walks and the Laplace operator and also other elliptic operators, as well as the heat equation.

Next, we will enter in what is the core of these notes, the approximation by means of values of games of solutions to nonlinear problems like $p-$harmonic functions, that is, solutions to the PDE, $\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$ (including the nowadays popular case $p = \infty$). Here we will focus on the main ideas and statements, without making details of the proofs.

We will assume that the reader is familiar with basic tools from probability theory (like conditional expectations) and with the (not so basic) concept of viscosity solutions for second order elliptic and parabolic PDEs (we refer to the book [10] for this last issue).

The Bibliography of these notes does not escape the usual rule of being incomplete. In general, we have listed those papers which are closer to the topics discussed here (some of them are not explicitly cited in the text). But, even for those papers, the list is far from being exhaustive and we apologize for omissions.

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2. Linear PDEs and probability

2.1. **The probability of hitting the exit and harmonic functions.** Let us begin by considering a bounded and smooth two-dimensional domain $\Omega \subset \mathbb{R}^2$ and assume that the boundary, $\partial\Omega$ is decomposed in two parts, Γ_1 and Γ_2 (that is, $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$). We begin with a position $(x, y) \in \Omega$ and ask the following question: assume that you move completely at random beginning at (x, y) (we assume that we are in an homogeneous environment and that we do not privilege any direction. In addition, we assume that every time the particle moves independently of its past history) what is the probability $u(x, y)$ of hitting the first part of the boundary Γ_1 the first time that the particle hits the boundary ?.

We will call Γ_1 the "open part" of the boundary and think that when we hit this part we can "exit" the domain, while we will call Γ_2 the "closed part" of the boundary, when we hit it we are dead.

This problem of describing the random movement in a precise mathematical way is the central subject of Brownian Motion. It originated in 1827, when the botanist Robert Brown observed this type of random movement in pollen particles suspended in water.

A clever and simple way to get some insight to solve the question runs as follows: First, we simplify the problem and approximate the movement by random increments of step h in each of the axes directions, with $h > 0$ small. From (x, y) the particle can move to $(x + h, y)$, $(x - h, y)$, $(x, y + h)$, or $(x, y - h)$, each movement being chosen at random with probability $1/4$.

Starting at (x, y), let $u_h(x, y)$ be the probability of hitting the exit part $\Gamma_1 + B_\delta(0)$ the first time that $\partial\Omega + B_\delta(0)$ is hitt when we move on the lattice of side h. Observe that we need to enlarge a little the boundary to capture points on the lattice of size h (that do not necessarily lie on $\partial\Omega$).

Applying conditional expectations we get

$$u_h(x, y) = \frac{1}{4}u_h(x + h, y) + \frac{1}{4}u_h(x - h, y) + \frac{1}{4}u_h(x, y + h) + \frac{1}{4}u_h(x, y - h). \qquad (1)$$

That is,

$$0 = \{u_h(x + h, y) - 2u_h(x, y) + u_h(x - h, y)\} + \{u_h(x, y + h) - 2u_h(x, y) + u_h(x, y - h)\}. \qquad (2)$$

Now, assume that u_h converges as $h \to 0$ to a function u uniformly in $\overline{\Omega}$. Note that this convergence can be proved rigorously.

Let ϕ be a smooth function such that $u - \phi$ has a strict minimum at $(x_0, y_0) \in \Omega$. By the uniform convergence of u_h to u there are points (x_h, y_h) such that

$$(u_h - \phi)(x_h, y_h) \le (u_h - \phi)(x, y) + o(h^2) \qquad (x, y) \in \Omega$$

and

$$(x_h, y_h) \to (x_0, y_0) \qquad h \to 0.$$

Note that u_h is not necessarily continuous.

Hence, from (2) at the point (x_h, y_h) and using that

$$u_h(x, y) - u_h(x_h, y_h) \ge \phi(x, y) - \phi(x_h, y_h) + o(h^2) \qquad (x, y) \in \Omega,$$

we get

$$
(3)\qquad \begin{aligned} 0 \geq\ & \{\phi(x_h+h,y_h) - 2\phi(x_h,y_h) + \phi(x_h-h,y_h)\} \\ & + \{\phi(x_h,y_h+h) - 2\phi(x_h,y_h) + \phi(x_h,y_h-h)\} + o(h^2). \end{aligned}
$$

Now, we just observe that

$$
\{\phi(x_h+h,y_h) - 2\phi(x_h,y_h) + \phi(x_h-h,y_h)\} = h^2 \frac{\partial^2\phi}{\partial x^2}(x_h,y_h) + o(h^2)
$$

$$
\{\phi(x_h,y_h+h) - 2\phi(x_h,y_h) + \phi(x_h,y_h-h)\} = h^2 \frac{\partial^2\phi}{\partial y^2}(x_h,y_h) + o(h^2).
$$

Hence, substituting in (3), dividing by h^2 and taking limit as $h \to 0$ we get

$$
0 \geq \frac{\partial^2\phi}{\partial x^2}(x_0,y_0) + \frac{\partial^2\phi}{\partial y^2}(x_0,y_0).
$$

Therefore, a uniform limit of the approximate values u_h, u, has the following property: *each time that a smooth function ϕ touches u from below at a point (x_0,y_0) the derivatives of ϕ must satisfy,*

$$
0 \geq \frac{\partial^2\phi}{\partial x^2}(x_0,y_0) + \frac{\partial^2\phi}{\partial y^2}(x_0,y_0).
$$

An analogous argument considering ψ a smooth function such that $u-\psi$ has a strict maximum at $(x_0,y_0) \in \Omega$ shows a reverse inequality. Therefore, *each time that a smooth function ψ touches u from above at a point (x_0,y_0) the derivatives of ψ must verify*

$$
0 \leq \frac{\partial^2\psi}{\partial x^2}(x_0,y_0) + \frac{\partial^2\psi}{\partial y^2}(x_0,y_0).
$$

But at this point we realize that this is exactly the definition of being u a **viscosity solution to the Laplace equation**

$$
\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.
$$

Hence, we obtain that the uniform limit of the sequence of solutions to the approximated problems u_h, u is the unique viscosity solution (that is also a classical solution in this case) to the following boundary value problem

$$
(4)\qquad \begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = 1 & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2. \end{cases}
$$

The boundary conditions can be easily obtained from the fact that $u_h \equiv 1$ in a neighborhood (of width h) of Γ_1 and $u_h \equiv 0$ in a neighborhood of Γ_2.

Note that we have only required *uniform* convergence to get the result, and hence no requirement is made on derivatives of the approximating sequence u_h. Moreover, we do not assume that u_h is continuous.

Now, we just notice that in higher dimensions $\Omega \subset \mathbb{R}^N$, the discretization method described above leads in the same simple way to viscosity solutions to the Laplace operator in higher dimensions and then to the fact that exiting probabilities are harmonic functions.

Another way to understand this strong relation between probabilities and the Laplacian is through the *mean value property of harmonic functions*. In the same context of the problem solved above, assume that a closed ball $B_r(x_0, y_0)$ of radius r and centered at a point (x_0, y_0) is contained in Ω. Starting at (x_0, y_0), the probability density of hitting first a given point on the sphere $\partial B_r(x_0, y_0)$ is constant on the sphere, that is, it is uniformly distributed on the sphere. Therefore, the probability $u(x_0, y_0)$ of exiting through Γ_1 starting at (x_0, y_0) is the average of the exit probabilities u on the sphere, here we are using again the formula of conditional probabilities. That is, u satisfies the mean value property on spheres:

$$u(x_0, y_0) = \frac{1}{|\partial B_r(x_0, y_0)|} \int_{\partial B_r(x_0,y_0)} u(x, y)\, dS(x, y)$$

with r small enough. It is well known that this property leads to u being harmonic.

We can also say that, if the movement is completely random and equidistributed in the ball $B_h(x_0, y_0)$, then, by the same conditional expectation argument used before, we have

$$u_h(x_0, y_0) = \frac{1}{|B_h(x_0, y_0)|} \int_{B_h(x_0,y_0)} u_h(x, y)\, dx\, dy.$$

Again one can take the limit as $h \to 0$ and obtain that a uniform the limit of the u_h, u, is harmonic (in the viscosity sense).

2.2. **Counting the number of steps needed to reach the exit.** Another motivating problem is the following: with the same setting as before (Ω a bounded smooth domain in $\mathbb{R}^2$) we would like to compute the expected time, that we call T, that we have to spend starting at (x, y) before hitting the boundary $\partial\Omega$ for the first time.

We can proceed exactly as before computing the time of the random walk at the discrete level, that is, in the lattice of size h. This amounts to adding a constant (the unit of time that we spend in each movement), which depends on the step h, to the right hand side of (1). We have

$$T_h(x, y) = \frac{1}{4} T_h(x + h, y) + \frac{1}{4} T_h(x - h, y) + \frac{1}{4} T_h(x, y + h) + \frac{1}{4} T_h(x, y - h) + t(h).$$

That is,

$$0 = \{T_h(x + h, y) - 2T_h(x, y) + T_h(x - h, y)\} + \{T_h(x, y + h) - 2T_h(x, y) + T_h(x, y - h)\} + t(h).$$

Proceeding as we did before, and since we need to divide by h^2, a natural choice is to set that $t(h)$ is of order h^2. Choosing

$$t(h) = Kh^2$$

and letting $h \to 0$, we conclude that a uniform limit of the approximate solutions T_h, T is the unique solution to

$$\begin{cases} -\Delta T = 4K & \text{in } \Omega, \\ T = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

The boundary condition is natural since if we begin on the boundary the expected time needed to reach it is zero.

From the previous probabilistic interpretations for the solutions of problems (4) and (5) as limits of random walks, one can imagine a probabilistic model for which the solution of the limit process is a solution to the general Poisson problem

$$\begin{cases} -\Delta u(x) = g(x) & \text{in } \Omega, \\ u(x) = F(x) & \text{on } \partial\Omega. \end{cases} \tag{6}$$

In this general model the functions g and F can be thought as costs that one pays, respectively, along the random movement and at the stopping time on the boundary.

2.3. **Anisotropic media.** Suppose now that the medium in which we perform our random movements is neither isotropic (that is, it is directionally dependent) nor homogeneous (that is, it differs from one point to another).

We can imagine a random discrete movement as follows. We move from a point (x, y) to four possible points at distance h located at two orthogonal axis forming a given angle α with the horizontal, and with different probabilities $q/2$ for the two points on the first axis and $(1-q)/2$ for the two points on the second axis. The angle α (that measures the orientation of the axis) and the probability q depend on the point (x, y). After the same analysis as above, we encounter now the general elliptic equation

$$\sum_{ij} a_{ij}(x, y) u_{x_i x_j}(x, y) = 0.$$

2.4. **The heat equation.** Now assume that we are on the real line $\mathbb{R}$ and consider that we move in a two dimensional lattice as follows: when we are at the point (x_0, t_0), the time increases by $\delta t := h^2$ and the spacial position moves with an increment of size $\delta x = h$ and goes to $x_0 - h$ or to $x_0 + h$ with the same probability. In this way the new points in the lattice that can be reached starting from (x_0, t_0) are $(x_0 - h, t_0 + h^2)$ or to $(x_0 + h, t_0 + h^2)$, each one with probability $1/2$.

As we will see, the choice $\delta t = (\delta x)^2$ is made to ensure that a certain limit as $h \to 0$ exists.

Let us start at $x = 0$, $t = 0$ and let $u_h(x, t)$ be the probability that we are at x at time t (here $x = kh$ and $t = lh^2$ is a point in the two dimensional lattice).

As in the previous subsection, conditional probabilities give the identity

$$u_h(x, t) = \frac{1}{2} u_h(x - h, t - h^2) + \frac{1}{2} u_h(x + h, t - h^2).$$

That is,

$$\frac{u_h(x, t) - u_h(x, t - h^2)}{h^2} = \frac{1}{h^2} \Big\{ \frac{1}{2} u_h(x - h, t - h^2) + \frac{1}{2} u_h(x + h, t - h^2) - u_h(x, t - h^2) \Big\}.$$

Now, as before, we let $h \to 0$ and, assuming uniform convergence (that can be proved !), we arrive to the fact that the limit should be a viscosity solution to

$$u_t(x, t) = \frac{1}{2} u_{xx}(x, t).$$

It is at this point where the relation $\delta t = (\delta x)^2$ is needed.

2.5. **Comments.**

(1) We want to remark that two facts are crucial in the previous analysis. The first one is the formula for the discrete version of the problem obtained using conditional expectations, (1), and the second one is the use of the theory of viscosity solutions to perform the passage to the limit without asking for more than uniform convergence.

These ideas are going to be used again in the next section.

(2) One can show the required uniform convergence from the following two facts: first, the values u_h are uniformly bounded (they all lie between 0 and 1 in the case of the problem of exiting the domain through Γ_1 and by the maximum and the minimum of the datum F in the general case assuming $g \equiv 0$) and second the family u_h is equicontinuous for "not too close points" (see the following sections), this can be proved using coupling methods. In fact, one can mimic a path of the game starting at x but starting at y and when one of the two paths hits the boundary the other is at a position that is close (at a distance smaller than $|x-y|$) of the boundary. It remains to prove a uniform in h estimate that says that "given $\epsilon > 0$ there exists $\delta > 0$ such that if one begins close to the boundary Γ_1 (with $\text{dist}(x_0, \Gamma_1) < \delta$) then the probability of hitting this part of the boundary is bounded below by $1-\epsilon$" (and of course an analogous statement for positions that start close to Γ_2). For this argument to work in the general case one can impose that F is uniformly continuous.

(3) The initial condition for the heat equation in subsection 2.4 is $u(x,0) = \delta_0$ and the solution can be explicitly obtained as the Gaussian

$$u(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

For an initial distribution of particles given by $u(x,0) = u_0(x)$ we get

$$u(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} * u_0(x).$$

(4) Also note that the identities obtained using conditional expectations are the same that correspond to the discretization of the equations using finite differences. This gives a well-known second order numerical scheme for the Laplacian.

Hence, we remark that probability theory can be used to obtain numerical schemes for PDEs.

(5) Finally, note that when a general Poisson problem is considered in (6) the functions g and F that appear can be thought as costs that one pays. The first one is a *running cost* that is payed at each movement, while the second one is a *final cost* that is payed when the game ends reaching the boundary.

We will use this terminology in the next section.

3. Tug-of-War games and the $\infty-$Laplacian

In this section we will look for a probabilistic approach to approximate solutions to the $\infty-$Laplacian. This is the nonlinear degenerate elliptic operator, usually denoted by Δ_∞, given by,

$$\Delta_\infty u := \left(D^2 u \, \nabla u\right) \cdot \nabla u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i x_j},$$

and arises from taking limit as $p \to \infty$ in the p-Laplacian operator in the viscosity sense, see [4] and [9]. In fact, let us present a formal derivation. First, expand (formally) the $p-$laplacian:

$$\begin{aligned} \Delta_p u &= \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \\ &= |\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4}\sum_{i,j} u_{x_i}u_{x_j}u_{x_i,x_j} = \\ &= (p-2)|\nabla u|^{p-4}\left\{\frac{1}{p-2}|\nabla u|^2\Delta u + \sum_{i,j} u_{x_i}u_{x_j}u_{x_i,x_j}\right\} \end{aligned}$$

and next, using this formal expansion, pass to the limit in the equation $\Delta_p u = 0$, to obtain

$$\Delta_\infty u = \sum_{i,j} u_{x_i}u_{x_j}u_{x_i,x_j} = Du \cdot D^2u \cdot (Du)^t = 0.$$

Note that this calculation can be made rigorous in the viscosity sense.

The ∞-laplacian operator appears naturally when one considers absolutely minimizing Lipschitz extensions of a boundary function F; see [26] and also the survey [4]. A fundamental result of Jensen [26] establishes that the Dirichlet problem for Δ_∞ is well posed in the viscosity sense. Solutions to $-\Delta_\infty u = 0$ (that are called infinity harmonic functions) are also used in several applications, for instance, in optimal transportation and image processing (see, e.g., [17], [20] and the references therein). Also the eigenvalue problem related to the ∞-laplacian has been exhaustively studied, see [12], [29], [30], [31].

Let us recall the definition of an absolutely minimizing Lipschitz extension. Let $F : \partial\Omega \to \mathbb{R}$. We denote by $L(F, \partial\Omega)$ the smallest Lipschitz constant of F in $\partial\Omega$, i.e.,

$$L(F, \partial\Omega) := \sup_{x,y\in\partial\Omega} \frac{|F(x) - fF(y)|}{|x-y|}.$$

If we are given a Lipschitz function $F : \partial\Omega \to \mathbb{R}$, i.e., $L(F, \partial\Omega) < +\infty$, then it is well-known that there exists a *minimal Lipschitz extension* (MLE for short) of F to Ω, that is, a function $h : \overline{\Omega} \to \mathbb{R}$ such that $h_{|\partial\Omega} = F$ and $L(F, \partial\Omega) = L(h, \overline{\Omega})$. Extremal extensions were explicitly constructed by McShane [36] and Whitney [50],

$$\Psi(F)(x) := \inf_{y\in\partial\Omega} \left(F(y) + L(F, \partial\Omega)|x-y|\right), \quad x \in \overline{\Omega},$$

and

$$\Lambda(F)(x) := \sup_{y\in\partial\Omega} \left(F(y) - L(F, \partial\Omega)|x-y|\right), \quad x \in \overline{\Omega},$$

are MLE of F to $\overline{\Omega}$ and if u is any other MLE of F to $\overline{\Omega}$ then

$$\Lambda(F) \le u \le \Psi(F).$$

The notion of a minimal Lipschitz extension is not completely satisfactory since it involves only the global Lipschitz constant of the extension and ignore what may happen locally. To solve this problem, in the particular case of the euclidean space $\mathbb{R}^N$, Arosson [2] introduce the concept of *absolutely minimizing Lipschitz extension* (AMLE for short) and proved the existence of AMLE by means of a variant of the Perron's method. The AMLE is given by the following definition. Here we consider the general case of extensions of Lipschitz functions defined on a subset $A \subset \overline{\Omega}$, but the reader may consider $A = \partial\Omega$.

Definition 3.1. Let A be any nonempty subset of $\overline{\Omega}$ and let $F : A \subset \overline{\Omega} \to \mathbb{R}$ be a Lipschitz function. A function $u : \overline{\Omega} \to \mathbb{R}$ is an *absolutely minimizing Lipschitz extension* of F to $\overline{\Omega}$ if

(i) u is an MLE of F to $\overline{\Omega}$,
(ii) whenever $B \subset \overline{\Omega}$ and $g : \overline{\Omega} \to \mathbb{R}$ is and MLE of F to $\overline{\Omega}$ such that $g = u$ in $\overline{\Omega} \setminus B$, then
$$L(u, B) \le L(g, B).$$

Remark 3.2. *The definition of AMLE can be extended to any metric space (X, d), and existence of such an extension can be proved when (X, d) is a separable length space,* [28].

It turns out (see [4]) that *the unique AMLE of F (defined on $\partial\Omega$) to $\overline{\Omega}$ is the unique solution to*
$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \Omega, \\ u(x) = F(x) & \text{on } \partial\Omega. \end{cases}$$

Our main aim in this section is to describe a game that approximates this problem in the same way as problems involving the random walk described in the previous section approximate harmonic functions.

3.1. **Description of the game.** We follow [47] and [11], but we restrict ourselves to the case of a game in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ (the results presented in [47] are valid in general length spaces).

A Tug-of-War is a two-person, zero-sum game, that is, two players are in contest and the total earnings of one are the losses of the other. Hence, one of them, say Player I, plays trying to maximize his expected outcome, while the other, say Player II is trying to minimize Player I's outcome (or, since the game is zero-sum, to maximize his own outcome).

Let us describe briefly the game introduced in [47] by Y. Peres, O. Schramm, S. Sheffield and D. Wilson. Consider a bounded domain $\Omega \subset \mathbb{R}^N$, and take $\Gamma_D \subset \partial\Omega$ and $\Gamma_N \equiv \partial\Omega \setminus \Gamma_D$. Let $F : \Gamma_D \to \mathbb{R}$ be a Lipschitz continuous function. At an initial time, a token is placed at a point $x_0 \in \overline{\Omega} \setminus \Gamma_D$. Then, a (fair) coin is tossed and the winner of the toss is allowed to move the game position to any $x_1 \in \overline{B_\epsilon(x_0)} \cap \overline{\Omega}$. At each turn, the coin is tossed again, and the winner chooses a new game state $x_k \in \overline{B_\epsilon(x_{k-1})} \cap \overline{\Omega}$. Once the token has reached some $x_\tau \in \Gamma_D$, the game ends and Player I earns $F(x_\tau)$ (while Player II earns $-F(x_\tau)$). This is the reason why we will refer to F as the *final payoff function.* In more general models, it is considered also a *running payoff* $g(x)$ defined in Ω, which represents the reward (respectively, the cost) at each intermediate state x, and gives rise to nonhomogeneous problems. We will assume here that $g \equiv 0$. This procedure yields a sequence of game states $x_0, x_1, x_2, \ldots, x_\tau$, where every x_k except x_0 are random variables, depending on the coin tosses and the strategies adopted by the players.

Now we want to give a precise definition of the *value of the game.* To this end we have to introduce some notation and put the game into its normal or strategic form (see [48] and [43]). The initial state $x_0 \in \overline{\Omega} \setminus \Gamma_D$ is known to both players (public knowledge). Each player i chooses an *action* $a_0^i \in \overline{B_\epsilon(0)}$ which is announced to the other player; this defines an action profile $a_0 = \{a_0^1, a_0^2\} \in \overline{B_\epsilon(0)} \times \overline{B_\epsilon(0)}$. Then, the new state $x_1 \in \overline{B_\epsilon(x_0)}$ (namely, the current state plus the action) is selected according to a probability distribution $p(\cdot|x_0, a_0)$ in $\overline{\Omega}$ which, in our case, is given by the fair coin toss. At stage k, knowing the history $h_k = (x_0, a_0, x_1, a_1, \ldots, a_{k-1}, x_k)$, (the sequence of states and actions up to that stage), each player i chooses an action a_k^i. If the game ends at time $j < k$, we set $x_m = x_j$ and $a_m = 0$ for $j \le m \le k$. The current state x_k and the profile $a_k = \{a_k^1, a_k^2\}$ determine the distribution $p(\cdot|x_k, a_k)$ (again given by the fair coin toss) of the new state x_{k+1}.

Denote $H_k = (\overline{\Omega} \setminus \Gamma_D) \times \left(\overline{B_\epsilon(0)} \times \overline{B_\epsilon(0)} \times \overline{\Omega}\right)^k$, the set of *histories up to stage* k, and by $H = \bigcup_{k\geq 1} H_k$ the set of all histories. Notice that H_k, as a product space, has a measurable structure. The *complete history space* H_∞ is the set of plays defined as infinite sequences $(x_0, a_0, \ldots, a_{k-1}, x_k, \ldots)$ endowed with the product topology. Then, the final payoff for Player I, i.e. F, induces a Borel-measurable function on H_∞. A *pure strategy* $S_i = \{S_i^k\}_k$ for Player i, is a sequence of mappings from histories to actions, namely, a mapping from H to $\overline{B_\epsilon(0)}$ such that S_i^k is a Borel-measurable mapping from H_k to $\overline{B_\epsilon(0)}$ that maps histories ending with x_k to elements of $\overline{B_\epsilon(0)}$ (roughly speaking, at every stage the strategy gives the next movement for the player, provided he win the coin toss, as a function of the current state and the past history). The initial state x_0 and a profile of strategies $\{S_I, S_{II}\}$ define (by Kolmogorov's extension theorem) a unique probability $\mathbb{P}^{x_0}_{S_I,S_{II}}$ on the space of plays H_∞. We denote by $\mathbb{E}^{x_0}_{S_I,S_{II}}$ the corresponding expectation.

Then, if S_I and S_{II} denote the strategies adopted by Player I and II respectively, we define the expected payoff for player I as

$$V_{x_0,I}(S_I, S_{II}) = \begin{cases} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)], & \text{if the game terminates a.s.} \\ -\infty, & \text{otherwise.} \end{cases}$$

Analogously, we define the expected payoff for player II as

$$V_{x_0,II}(S_I, S_{II}) = \begin{cases} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)], & \text{if the game terminates a.s.} \\ +\infty, & \text{otherwise.} \end{cases}$$

Finally, we can define the ϵ*-value of the game for Player I* as

$$u_I^\epsilon(x_0) = \sup_{S_I} \inf_{S_{II}} V_{x_0,I}(S_I, S_{II}),$$

while the ϵ-value of the game for Player II is defined as

$$u_{II}^\epsilon(x_0) = \inf_{S_{II}} \sup_{S_I} V_{x_0,II}(S_I, S_{II}).$$

In some sense, $u_I^\epsilon(x_0), u_{II}^\epsilon(x_0)$ are the least possible outcomes that each player expects to get when the ϵ-game starts at x_0. Notice that, as in [47], we penalize severely the games that never end.

If $u_I^\epsilon = u_{II}^\epsilon := u_\epsilon$, we say that *the game has a value*. In [47] it is shown that, under very general hypotheses, that are fulfilled in the present setting, the ϵ-Tug-of-War game has a value.

All these $\epsilon-$values are Lipschitz functions with respect to the discrete distance d^ϵ given by

$$d_\varepsilon(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \varepsilon\left(\left[\!\left[\frac{|x-y|}{\varepsilon}\right]\!\right] + 1\right) & \text{if } x \neq y. \end{cases} \tag{7}$$

where $|.|$ is the Euclidean norm and $[\![r]\!]$ is defined for $r > 0$ by $[\![r]\!] := n$, if $n < r \leq n+1,\ n = 0, 1, 2, \ldots$, that is,

$$d_\varepsilon(x,y) = \begin{cases} 0 & \text{if } \ x = y, \\ \varepsilon & \text{if } \ 0 < |x-y| \leq \varepsilon, \\ 2\varepsilon & \text{if } \ \varepsilon < |x-y| \leq 2\varepsilon \\ \vdots & \end{cases}$$

see [47] (but in general they are not continuous). Let us present a simple example where we can compute the value of the game.

3.2. **The $1-d$ game.** Let us analyze in detail the one-dimensional game and its limit as $\epsilon \to 0$.

We set $\Omega = (0,1)$ and play the ϵ−game. To simplify we assume that $\epsilon = 1/2^n$ and that the running payoff is zero. Concerning the final payoff, we end the game at $x = 0$ (with zero final payoff) and at $x = 1$ (with final payoff equals to one). Note that, the general result from [47] applies and hence we can assert the existence of a value for this game. Nevertheless, in this simple $1-d$ case we can obtain the existence of such value by direct computations. For the moment, let us assume that there exists a value that we call u_ϵ and proceed, in several steps, with the analysis of this sequence of functions u_ϵ for ϵ small. All the calculations below hold both for u_I^ϵ and for u_{II}^ϵ.

Step 1. $u_\epsilon(0) = 0$ and $u_\epsilon(1) = 1$. Moreover, $0 \leq u_\epsilon(x) \leq 1$ (the value functions are uniformly bounded).

Step 2. u_ϵ is increasing in x and strictly positive in $(0,1]$.

Indeed, if $x < y$ then for every pair of strategies S_I, S_{II} for Player I and II beginning at x we can construct strategies beginning at y in such a way that

$$x_{i,x} \leq x_{i,y}$$

(here $x_{i,x}$ and $x_{i,y}$ are the positions of the game after i movements beginning at x and y respectively). In fact, just reproduce the movements shifting points by $y-x$ when possible (if not, that is, if the jump is too large and ends outside the interval, just remain at the larger interior position $x = 1$). In this way we see that the probability of reaching $x = 1$ beginning at y is bigger than the probability of reaching $x = 0$ and hence, taking expectations, infimum and supremum, it follows that

$$u_\epsilon(x) \leq u_\epsilon(y).$$

Now, we just observe that there is a positive probability of obtaining a sequence of $1/\epsilon$ consecutive heads (exactly $2^{-1/\epsilon}$), hence the probability of reaching $x = 1$ when the first player uses the strategy that points ϵ to the right is strictly positive. Therefore,

$$u_\epsilon(x) > 0,$$

for every $x \neq 0$.

Step 3. In this one dimensional case it is easy to identify the optimal strategies for players I and II: to jump ϵ to the right for Player I and to jump ϵ to the left for Player II. That is, if we are at x, the optimal strategies lead to

$$x \to \min\{x+\epsilon, 1\}$$

for Player I, and to

$$x \to \max\{x-\epsilon, 0\}$$

for Player II.

This follows from step 2, where we have proved that the function u_ϵ is increasing in x. As a consequence, the optimal strategies follow: for instance, Player I will choose the point where the expected payoff is maximized and this is given by $\min\{x+\epsilon, 1\}$,

$$\sup_{z\in[x-\epsilon,x+\epsilon]\cap[0,1]} u_\epsilon(z) = \max_{z\in[x-\epsilon,x+\epsilon]\cap[0,1]} u_\epsilon(z) = u_\epsilon(\min\{x+\epsilon, 1\}),$$

since u_ϵ is increasing.

This is also clear from the following intuitive fact: player I wants to maximize the payoff (reaching $x = 1$) and player II wants to minimize the payoff (hence pointing to 0).

Step 4. u_ϵ is constant in every interval of the form $(k\epsilon, (k+1)\epsilon)$ for $k = 1, ..., N$ (we denote by N the total number of such intervals in $(0, 1]$).

Indeed, from step 3 we know what are the optimal strategies for both players, and hence the result follows noticing that the number of steps that one has to advance to reach $x = 0$ (or $x = 1$) is the same for every point in $(k\epsilon, (k+1)\epsilon)$.

Remark 3.3. *Note that u_ϵ is necessarily discontinuos at every point of the form $y_k = k\epsilon \in (0, 1)$.*

Step 5. Let us call $a_k := u_\epsilon |_{(k\epsilon,(k+1)\epsilon)}$. Then we have

$$a_0 = 0, \qquad a_k = \frac{1}{2}(a_{k-1} + a_{k+1}),$$

for every $i = 2, ..., n-1$, and

$$a_n = 1.$$

Notice that these identities follow from the Dynamic Programming Principle, using that from step 3 we know the optimal strategies, that from step 4 u_ϵ is constant in every subinterval of the form $(k\epsilon, (k+1)\epsilon)$, we immediately get the conclusion.

Remark 3.4. *Note the similarity with a finite difference scheme used to solve $u_{xx} = 0$ in $(0, 1)$ with boundary conditions $u(0) = 0$ and $u(1) = 1$. In fact, a discretization of this problem in a uniform mesh of size ϵ leads to the same formulas obtained in step 5.*

Step 6. We have

$$u_\epsilon(x) = \epsilon k, \qquad x \in (k\epsilon, (k+1)\epsilon). \tag{8}$$

Indeed, the constants

$$a_k = \epsilon k$$

are the unique solution to the formulas obtained in step 5.

Remark 3.5. *Since formula (8) is in fact valid for u_I^ϵ and u_{II}^ϵ, this proves that the game has a value.*

Remark 3.6. *Note that u_ϵ verifies that*

$$0 \le u_\epsilon(x) - u_\epsilon(y) \le 2(x - y)$$

for every $x > y$ with $x - y > \epsilon$. This is a sort of equicontinuity valid for "far apart points".

In this one dimensional case, we can pass to the limit directly, by using the explicit formula for u_ϵ (see Step 7 below). However, in the N-dimensional case there is no explicit formula, and then we will need a compactness result (a sort of Arzela-Ascoli lemma).

Step 7.

$$\lim_{\epsilon \to 0} u_\epsilon(x) = x,$$

uniformly in $[0, 1]$.

Indeed, this follows from the explicit formula for u_ϵ in every interval of the form $(k\epsilon, (k+1)\epsilon)$ found in step 6 and from the monotonicity stated in step 2 (to take care of the values of u_ϵ at points of the form $k\epsilon$, we have $a_{k-1} \le u_\epsilon(k\epsilon) \le a_k$).

Remark 3.7. *Note that the limit function*

$$u(x) = x$$

is the unique viscosity (and classical) solution to

$$\Delta_\infty u(x) = (u_{xx}(u_x)^2)(x) = 0 \qquad x \in (0,1),$$

with boundary conditions

$$u(0) = 0, \qquad u(1) = 1.$$

Remark 3.8. *Notice that an alternative approach to the previous analysis can be done by using the theory of Markov chains.*

3.3. **Mixed boundary conditions for Δ_∞.** Now we continue the analysis of the Tug-of-War game described previously. As before we assume that we are in the general case of a bounded domain Ω in $\mathbb{R}^N$. The game ends when the position reaches one part of the boundary Γ_D (where there is a specified final payoff F) and look for the condition that the limit must verify on the rest of it, $\partial\Omega \setminus \Gamma_D$.

All these $\epsilon-$values are Lipschitz functions with respect to the discrete distance d^ϵ defined in (7), see [47] (but in general they are not continuous as the one-dimensional example shows), which converge uniformly when $\epsilon \to 0$. The uniform limit as $\epsilon \to 0$ of the game values u_ϵ is called *the continuous value* of the game that we will denote by u and it can be seen (see below) that u is a viscosity solution to the problem

$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \Omega, \\ u(x) = F(x) & \text{on } \Gamma_D, \end{cases} \tag{9}$$

When $\Gamma_D \equiv \partial\Omega$ it is known that this problem has a unique viscosity solution, (as proved in [26]; see also [6], [14], and in a more general framework, [47]).

However, when $\Gamma_D \not\equiv \partial\Omega$ the PDE problem (9) is incomplete, since there is a missing boundary condition on $\Gamma_N = \partial\Omega \setminus \Gamma_D$. Our concern now is to find the boundary condition that completes the problem. Assuming that Γ_N is regular, in the sense that the normal vector field $\vec{n}(x)$ is well defined and continuous for all $x \in \Gamma_N$, it is proved in [11] that it is in fact the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial n}(x) = 0\,, \quad x \in \Gamma_N.$$

On the other hand, instead of using the beautiful and involved proof based on game theory arguments, written in [47], we give here (based on [11]) an alternative proof of the property $-\Delta_\infty u = 0$ in Ω, by using direct viscosity techniques, perhaps more natural in this context. The key point in our proof is the *Dynamic Programming Principle*, that in our case reads as follows: the value of the game u_ϵ verifies

$$2u_\epsilon(x) = \sup_{y \in \overline{B_\epsilon(x)} \cap \bar\Omega} u_\epsilon(y) + \inf_{y \in \overline{B_\epsilon(x)} \cap \bar\Omega} u_\epsilon(y) \qquad \forall x \in \bar\Omega \setminus \Gamma_D,$$

where $B_\epsilon(x)$ denotes the open ball of radius ϵ centered at x.

This Dynamic Programming Principle, in some sense, plays the role of the mean property for harmonic functions in the infinity-harmonic case. This principle turns out to be an important qualitative property of the approximations of infinity-harmonic functions, and is the main tool to construct convergent numerical methods in this kind of problems; see [44].

We have the following result, for the proof we refer to [11].

Theorem 3.9. *Let $u(x)$ be the continuous value of the Tug-of-War game described above (as introduced in* [47]*). Assume that $\partial\Omega = \Gamma_N \cup \Gamma_D$, where Γ_N is of class C^1, and F is a Lipschitz function defined on Γ_D. Then,*

i) *$u(x)$ is a viscosity solution to the mixed boundary value problem*

$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}(x) = 0 & \text{on } \Gamma_N, \\ u(x) = F(x) & \text{on } \Gamma_D. \end{cases} \tag{10}$$

ii) *Reciprocally, assume that Ω verifies that for every $z \in \overline{\Omega}$ and every $x^* \in \Gamma_N$ $z \neq x^*$ that*

$$\left\langle \frac{x^* - z}{|x^* - z|}; n(x^*) \right\rangle > 0.$$

Then, if $u(x)$ is a viscosity solution to (10)*, it coincides with the unique continuous value of the game.*

The hypothesis imposed on Ω in part ii) holds whenever Ω is strictly convex. The first part of the theorem comes as a consequence of the Dynamic Programming Principle read in the viscosity sense.

The proof of the second part is not included in this work. We refer to [11] for details and remark that the proof uses that the continuous value of the game is determined by the fact that it enjoys comparison with quadratic functions in the sense described in [47].

We have found a PDE problem, (10), which allows to find both the continuous value of the game and the AMLE of the Dirichlet data F (which is given only on a subset of the boundary) to $\overline{\Omega}$. To summarize, we point out that a complete equivalence holds, in the following sense (see [11] for details):

Theorem 3.10. *It holds*

$$u \text{ is AMLE of } F|_{\Gamma_D} \text{ in } \overline{\Omega} \Leftrightarrow u \text{ is the limit of the values of the game} \Leftrightarrow u \text{ solves } (10).$$

The first equivalence was proved in [47] and the second one is just Theorem 3.9.

Another consequence of Theorem 3.9 is the following:

Corollary 3.11. *There exists a unique viscosity solution to* (10).

The existence of a solution is a consequence of the existence of a continuous value for the game together with part i) in the previous theorem, while the uniqueness follows by uniqueness of the value of the game and part ii).

Note that to obtain uniqueness we have to invoke the uniqueness of the game value. It should be desirable to obtain a direct proof (using only PDE methods) of existence and uniqueness for (10) but it is not clear how to find the appropriate perturbations near Γ_N to obtain uniqueness (existence follows easily by taking the limit as $p \to \infty$ in the mixed boundary value problem problem for the $p-$laplacian).

3.4. **Comments.**

(1) When we add a running cost of the form $\epsilon^2 g(x)$ we obtain a solution to

$$\begin{cases} -\Delta_\infty u(x) = g(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}(x) = 0 & \text{on } \Gamma_N, \\ u(x) = F(x) & \text{on } \Gamma_D. \end{cases}$$

(2) Many of the results presented here are valid in general length spaces (see [47]) and on graphs. In case of a graph the next positions that can be selected by both players are the points that are connected with the actual position of the game. This leads to consider the infinity laplacian on a graph.

(3) One can consider games in which the coin is biased, that is the probability of getting a head is p (with $p \neq 1/2$). In this case the limit as $\epsilon \to 0$ (with $p = p(\epsilon) \to 1/2$) was analyzed in [46] and the PDE that appears reads as

$$\Delta_\infty u + \beta |\nabla u| = 0.$$

4. $p-$harmonious functions

The aim of this section is to describe games that approximate the $p-$Laplacian that is give by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

We assume that $2 \le p < \infty$. The case $p = \infty$ was considered in the previous section.

4.1. **Description of the game.** Consider a two-player zero-sum-game in a domain Ω described as follows: starting from a point $x_0 \in \Omega$, Players I and II play the original tug-of-war game described in [47] (see the previous section for details) with probability α, and with probability β (recall that $\alpha + \beta = 1$), a random point in $B_\varepsilon(x_0)$ is chosen. Once the game position reaches a strip near the boundary of width ϵ, Player II pays Player I the amount given by a pay-off function. Naturally, Player I tries to maximize and Player II to minimize the payoff. Hence, the equation

$$u_\varepsilon(x) = \frac{\alpha}{2} \left\{ \sup_{\overline{B}_\varepsilon(x)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x)} u_\varepsilon \right\} + \beta \fint_{B_\varepsilon(x)} u_\varepsilon \, dy,$$

describes the expected payoff of the above game. Intuitively, the expected payoff at the point can be calculated by summing up all the three cases: Player I moves, Player II moves, or a random point is chosen, with their corresponding probabilities.

In this variant of tug-of-war with noise the noise is distributed uniformly on $B_\varepsilon(x)$. This approach allows us to use the dynamic programming principle in the form

$$u_\varepsilon(x) = \frac{\alpha}{2} \left\{ \sup_{\overline{B}_\varepsilon(x)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x)} u_\varepsilon \right\} + \beta \fint_{B_\varepsilon(x)} u_\varepsilon \, dy, \tag{11}$$

to conclude that our game has a value and that the value is p-harmonious. There are indications, see Barles-Souganidis [7] and Oberman [44], that our results based on the mean value approach are likely to be useful in applications for example to numerical methods as well as in problems of analysis, cf. Armstrong-Smart [3]. For further results on games see [8] and [35].

4.2. $p-$**harmonious functions.** The goal of this section is to study functions that satisfy (11) with fixed $\varepsilon > 0$ and suitable nonnegative α and β, with $\alpha + \beta = 1$ (we will call such functions $p-$harmonious functions).

Consider a bounded domain $\Omega \subset \mathbb{R}^N$ and fix $\varepsilon > 0$. To prescribe boundary values for $p-$harmonious functions, let us denote the compact boundary strip of width ε by

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^N \setminus \Omega \,:\, \operatorname{dist}(x, \partial\Omega) \le \varepsilon\}.$$

Definition 4.1. *The function u_ε is p-harmonious in Ω with boundary values a bounded Borel function $F : \Gamma_\varepsilon \to \mathbb{R}$ if*

$$u_\varepsilon(x) = \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x)} u_\varepsilon \right\} + \beta \fint_{B_\varepsilon(x)} u_\varepsilon \, dy \qquad \text{for every} \quad x \in \Omega,$$

where α, β are defined in (18), *and*

$$u_\varepsilon(x) = F(x), \qquad \text{for every} \quad x \in \Gamma_\varepsilon.$$

The reason for using the boundary strip Γ_ε instead of simply using the boundary $\partial\Omega$ is the fact that for $x \in \Omega$ the ball $\overline{B}_\varepsilon(x)$ is not necessarily contained in Ω.

Let us first explain the name p-harmonious. When u is harmonic, then it satisfies the well known mean value property

$$u(x) = \fint_{B_\varepsilon(x)} u \, dy, \tag{12}$$

that is (11) with $\alpha = 0$ and $\beta = 1$. On the other hand, functions satisfying (11) with $\alpha = 1$ and $\beta = 0$

$$u_\varepsilon(x) = \frac{1}{2}\left\{ \sup_{\overline{B}_\varepsilon(x)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x)} u_\varepsilon \right\} \tag{13}$$

are called *harmonious* functions in [24] and [25] and are values of Tug-of-War games like the ones described in the previous section. As we have seen, as ε goes to zero, they approximate solutions to the infinity Laplacian.

Now, recall that the p-Laplacian is given by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2}\left\{(p-2)\Delta_\infty u + \Delta u\right\}. \tag{14}$$

Since the p-Laplace operator can be written as a combination of the Laplace operator and the infinity Laplacian, it seems reasonable to expect that the combination (11) of the averages in (12) and (13) give an approximation to a solution to the p-Laplacian. We will show that this is indeed the case. To be more precise, we prove that p-harmonious functions are uniquely determined by their boundary values and that they converge uniformly to the p-harmonic function with the given boundary data. Furthermore, we show that p-harmonious functions satisfy the strong maximum and comparison principles. Observe that the validity of the strong comparison principle is an open problem for the solutions of the p-Laplace equation in $\mathbb{R}^N$, $N \ge 3$.

4.3. **A heuristic argument.** It follows from expansion (14) that u is a solution to $\Delta_p u = 0$ if and only if

$$(p-2)\Delta_\infty u + \Delta u = 0, \tag{15}$$

because this equivalence can be justified in the viscosity sense even when $\nabla u = 0$ as shown in [32]. Averaging the classical Taylor expansion

$$u(y) = u(x) + \nabla u(x) \cdot (y - x) + \frac{1}{2}\langle D^2 u(x)(y - x), (y - x)\rangle + O(|y - x|^3),$$

over $B_\varepsilon(x)$, we obtain

$$u(x) - \fint_{B_\varepsilon(x)} u\,dy = -\frac{\varepsilon^2}{2(n+2)}\Delta u(x) + O(\varepsilon^3), \tag{16}$$

when u is smooth. Here we used the shorthand notation

$$\fint_{B_\varepsilon(x)} u\,dy = \frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} u\,dy.$$

Then observe that gradient direction is almost the maximizing direction. Thus, summing up the two Taylor expansions roughly gives us

$$\begin{aligned} & u(x) - \frac{1}{2}\left\{ \sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u \right\} \\ & \approx u(x) - \frac{1}{2}\left\{ u\left(x + \varepsilon\frac{\nabla u(x)}{|\nabla u(x)|}\right) + u\left(x - \varepsilon\frac{\nabla u(x)}{|\nabla u(x)|}\right) \right\} \\ & = -\frac{\varepsilon^2}{2}\Delta_\infty u(x) + O(\varepsilon^3). \end{aligned} \tag{17}$$

Next we multiply (16) and (17) by suitable constants α and β, $\alpha+\beta = 1$, and add up the formulas to obtain

$$u(x) - \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x)} u - \inf_{\overline{B}_\varepsilon(x)} u \right\} + \beta\fint_{B_\varepsilon(x)} u\,dy = -\alpha\frac{\varepsilon^2}{2}\Delta_\infty u(x) - \beta\frac{\varepsilon^2}{2(n+2)}\Delta u(x) + O(\varepsilon^3)$$

Next, we choose α and β so that we have the operator in (15) on the right hand side. This process gives us the choices of the constants

$$\alpha = \frac{p-2}{p+N}, \quad \text{and} \quad \beta = \frac{2+N}{p+N}. \tag{18}$$

and we deduce

$$u(x) = \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u \right\} + \beta\fint_{B_\varepsilon(x)} u\,dy + O(\varepsilon^3)$$

as $\varepsilon \to 0$.

4.4. ***p*-harmonious functions and Tug-of-War games.** In this section, we describe the connection between p-harmonious functions and tug-of-war games. Fix $\varepsilon > 0$ and consider the two-player zero-sum-game described before. At the beginning, a token is placed at a point $x_0 \in \Omega$ and the players toss a biased coin with probabilities α and β, $\alpha+\beta = 1$. If they get heads (probability α), they play a tug-of-war, that is, a fair coin is tossed and the winner of the toss is allowed to move the game position to any $x_1 \in \overline{B}_\varepsilon(x_0)$. On the other hand, if they get tails (probability β), the game state moves according to the uniform probability to a random point in the ball $B_\varepsilon(x_0)$. Then they continue playing the same game from x_1.

This procedure yields a possibly infinite sequence of game states $x_0, x_1, \ldots$ where every x_k is a random variable. We denote by $x_\tau \in \Gamma_\varepsilon$ the first point in Γ_ε in the sequence, where τ refers to the first time we hit Γ_ε. The payoff is $F(x_\tau)$, where $F : \Gamma_\varepsilon \to \mathbb{R}$ is a given *payoff function.* Player I earns $F(x_\tau)$ while Player II earns $-F(x_\tau)$.

Note that, due to the fact that $\beta > 0$, or equivalently $p < \infty$, the game ends almost surely

$$\mathbb{P}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}(\{\omega \in H^\infty : \tau(\omega) < \infty\}) = 1$$

for any choice of strategies.

The *value of the game for Player I* is given by

$$u^\varepsilon_{\mathrm{I}}(x_0) = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}[F(x_\tau)]$$

while the *value of the game for Player II* is given by

$$u^\varepsilon_{\mathrm{II}}(x_0) = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}[F(x_\tau)].$$

The values $u^\varepsilon_{\mathrm{I}}(x_0)$ and $u^\varepsilon_{\mathrm{II}}(x_0)$ are the best expected outcomes each player can guarantee when the game starts at x_0.

We start by the statement of the *Dynamic Programming Principle* (DPP) applied to our game.

Lemma 4.2 (DPP)**.** *The value function for Player I satisfies*

$$\begin{aligned} u^\varepsilon_I(x_0) &= \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x_0)} u^\varepsilon_I + \inf_{\overline{B}_\varepsilon(x_0)} u^\varepsilon_I\right\} + \beta \fint_{B_\varepsilon(x_0)} u^\varepsilon_I\,dy, \qquad x_0 \in \Omega, \\ u^\varepsilon_I(x_0) &= F(x_0), \qquad x_0 \in \Gamma_\varepsilon. \end{aligned} \tag{19}$$

The value function for Player II, u^ε_{II}, satisfies the same equation.

Formulas similar to (19) can be found in Chapter 7 of [37]. A detailed proof adapted to our case can also be found in [40].

Let us explain intuitively why the DPP holds by considering the expectation of the payoff at x_0. Whenever the players get heads (probability α) in the first coin toss, they toss a fair coin and play the tug-of-war. If Player I wins the fair coin toss in the tug-of-war (probability 1/2), she steps to a point maximizing the expectation and if Player II wins, he steps to a point minimizing the expectation. Whenever they get tails (probability β) in the first coin toss, the game state moves to a random point according to a uniform probability on $B_\varepsilon(x_0)$. The expectation at x_0 can be obtained by summing up these different alternatives.

We warn the reader that, as happens for the tug-of-war game without noise described previously, the value functions are discontinuous in general as the next example shows.

Example 4.3. Consider the case $\Omega = (0, 1)$ and

$$F(x) = u^\varepsilon_{\mathrm{I}}(x) = \begin{cases} 1, & x \geq 1 \\ 0, & x \leq 0. \end{cases}$$

In this case the optimal strategies for both players are clear: Player I moves ε to the right and Player II moves ε to the left. Now, there is a positive probability of reaching $x \geq 1$ that can be uniformly bounded from below in $(0, 1)$ by $C = (2/\alpha)^{-(1/\varepsilon+1)}$. This can be seen by considering the probability of Player I winning all the time until the game ends with $x \geq 1$. Therefore $u^\varepsilon_{\mathrm{I}} > C > 0$

in the whole $(0,1)$. This implies a discontinuity at $x=0$ and hence a discontinuity at $x=\varepsilon$. Indeed, first, note that u_ε is nondecreasing and hence

$$u_{\mathrm{I}}^{\varepsilon}(\varepsilon-) = \lim_{x\nearrow\varepsilon} \frac{\alpha}{2} \sup_{|x-y|\le\varepsilon} u_{\mathrm{I}}^{\varepsilon}(y) + \frac{\beta}{2\varepsilon}\int_0^{2\varepsilon} u_{\mathrm{I}}^{\varepsilon}\,dy = \frac{\alpha}{2} u_{\mathrm{I}}^{\varepsilon}(2\varepsilon-) + \frac{\beta}{2\varepsilon}\int_0^{2\varepsilon} u_{\mathrm{I}}^{\varepsilon}\,dy,$$

because $\sup_{|x-y|\le\varepsilon} u_{\mathrm{I}}^{\varepsilon}(y) = u_{\mathrm{I}}^{\varepsilon}(x+\varepsilon)$ and $\inf_{|x-\varepsilon|\le\varepsilon} u_{\mathrm{I}}^{\varepsilon}$ is zero for $x\in(0,\varepsilon)$. However,

$$u_{\mathrm{I}}^{\varepsilon}(\varepsilon+) \ge \frac{\alpha}{2}C + \lim_{x\searrow\varepsilon} \frac{\alpha}{2} \sup_{|x-y|\le\varepsilon} u_{\mathrm{I}}^{\varepsilon}(y) + \frac{\beta}{2\varepsilon}\int_0^{2\varepsilon} u_{\mathrm{I}}^{\varepsilon}\,dy \ge \frac{\alpha}{2}C + u_{\mathrm{I}}^{\varepsilon}(\varepsilon-)$$

because $\sup_{|x-y|\le\varepsilon} u_{\mathrm{I}}^{\varepsilon}(y) = u_{\mathrm{I}}^{\varepsilon}(x+\varepsilon) \ge u_{\mathrm{I}}^{\varepsilon}(2\varepsilon-)$ and $\inf_{|x-\varepsilon|\le\varepsilon} u_{\mathrm{I}}^{\varepsilon} \ge C$ for $x>\varepsilon$.

By adapting the martingale methods used in [47], we prove a comparison principle. This also implies that $u_{\mathrm{I}}^{\varepsilon}$ and $u_{\mathrm{II}}^{\varepsilon}$ are respectively the smallest and the largest p-harmonious function, see [41] for the details of the proof.

Theorem 4.4. *Let $\Omega\subset\mathbb{R}^N$ be a bounded open set. If v_ε is a p-harmonious function with boundary values F_v in Γ_ε such that $F_v \ge F_{u_I^\varepsilon}$, then $v \ge u_I^\varepsilon$.*

Similarly, we can prove that $u_{\mathrm{II}}^{\varepsilon}$ is the largest p-harmonious function. Next we state that the game has a value. This together with the previous comparison principle proves the uniqueness of p-harmonious functions with given boundary values. We refer again to [41] for details.

Theorem 4.5. *Let $\Omega\subset\mathbb{R}^N$ be a bounded open set, and F a given boundary data in Γ_ε. Then $u_I^\varepsilon = u_{II}^\varepsilon$, that is, the game has a value.*

Theorems 4.4 and 4.5 imply that with a fixed boundary data there exists a unique p-harmonious function.

Theorem 4.6. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set. Then there exists a unique p-harmonious function in Ω with given boundary values F.*

Corollary 4.7. *The value of the game with pay-off function F coincides with the p-harmonious function with boundary values F.*

4.5. **Maximum principles for p-harmonious functions.** In this section, we show that the strong maximum and strong comparison principles hold for p-harmonious functions.

First let us state that p-harmonious functions satisfy the *strong maximum principle.*

Theorem 4.8. *Let $\Omega\subset\mathbb{R}^N$ be a bounded, open, and connected set. If u_ε is p-harmonious in Ω with boundary values F, then*

$$\sup_{\Gamma_\varepsilon} F \ge \sup_{\Omega} u_\varepsilon.$$

Moreover, if there is a point $x_0\in\Omega$ such that $u_\varepsilon(x_0) = \sup_{\Gamma_\varepsilon} F$, then u_ε is constant in Ω.

Proof. See [41] □

In addition, p-harmonious functions with *continuous* boundary values satisfy the *strong comparison principle.* Note that the validity of the strong comparison principle is not known for the p-harmonic functions in $\mathbb{R}^N$, $N\ge 3$.

Theorem 4.9. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open and connected set, and let u_ε and v_ε be p-harmonious functions with continuous boundary values $F_u \geq F_v$ in Γ_ε. Then if there exists a point $x_0 \in \Omega$ such that $u_\varepsilon(x_0) = v_\varepsilon(x_0)$, it follows that*

$$u_\varepsilon = v_\varepsilon \quad in \quad \Omega,$$

and, moreover, the boundary values satisfy

$$F_u = F_v \quad in \quad \Gamma_\varepsilon.$$

4.6. **Convergence as $\varepsilon \to 0$.** We have that p-harmonious functions with a fixed boundary datum converge to the unique p-harmonic function.

Theorem 4.10. *Let Ω be a bounded domain satisfying Condition* **??** *and F be a continuous function. Consider the unique viscosity solution u to*

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u)(x) = 0, & x \in \Omega \\ u(x) = F(x), & x \in \partial\Omega, \end{cases} \tag{20}$$

and let u_ε be the unique p-harmonious function with boundary values F. Then

$$u_\varepsilon \to u \qquad uniformly\ in \quad \overline{\Omega}$$

as $\varepsilon \to 0$.

4.7. **Comments.**

(1) When we add a running cost of the form $\epsilon^2 g(x)$ we obtain a solution to the following inhomogeneous problem that involves the 1−homogeneous p−Laplacian,

$$\begin{cases} -|\nabla u|^{2-p}\Delta_p u(x) = kg(x) & \text{in } \Omega, \\ u(x) = F(x) & \text{on } \partial\Omega. \end{cases}$$

Here k is a constant that depends only on p and N.

Note that the 1−homogeneous p−Laplacian is not variational (it is not in divergence form).

(2) There is also a "parabolic" analogous, see [42]. In this case the equation that is obtained in the limit is also given by the 1−homogeneous p−Laplacian, that is,

$$\begin{cases} u_t(x,t) - |\nabla u|^{2-p}\Delta_p u(x,t) = 0 & \text{in } \Omega \times (0,T), \\ u(x,t) = F(x,t) & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = F(x,0) & \text{in } \Omega. \end{cases}$$

5. A mean value property that characterizes p−harmonic functions

Inspired by the analysis performed in the previous section we can guess a mean value formula for p−harmonic functions.

A well known fact that one can find in any elementary PDE textbook states that u is harmonic in a domain $\Omega \subset \mathbb{R}^N$ (that is u satisfies $\Delta u = 0$ in Ω) if and only if it satisfies the mean value property

$$u(x) = \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u(y)\, dy,$$

whenever $B_\varepsilon(x) \subset \Omega$. In fact, we can relax this condition by requiring that it holds asymptotically

$$u(x) = \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u(y)\, dy + o(\varepsilon^2),$$

as $\varepsilon \to 0$. This follows easily for C^2 functions by using the Taylor expansion and for continuous functions by using the theory of viscosity solutions. Interestingly, a weak asymptotic mean value formula holds in some nonlinear cases as well. Our goal in this paper is to characterize p-harmonic functions, $1 < p \le \infty$, by means of this type of asymptotic mean value properties.

We begin by stating what we mean by weak asymptotic expansions and why is it reasonable to say that our asymptotic expansions hold in "a viscosity sense". As is the case in the theory of viscosity solutions, we test the expansions of a function u against test functions ϕ that touch u from below or above at a particular point.

Select α and β determined by the conditions $\alpha + \beta = 1$ and $\alpha/\beta = (p-2)/(N+2)$. That is, we have

$$\alpha = \frac{p-2}{p+N}, \quad \text{and} \quad \beta = \frac{2+N}{p+N}. \tag{21}$$

Observe that if $p = 2$ above, then $\alpha = 0$ and $\beta = 1$, and if $p = \infty$, then $\alpha = 1$ and $\beta = 0$.

As before we follow the usual convention to denote the mean value of a function

$$⨍_B f(y)\, dy = \frac{1}{|B|} \int_B f(y)\, dy.$$

Definition 5.1. *A continuous function u satisfies*

$$u(x) = \frac{\alpha}{2} \left\{ \max_{B_\varepsilon(x)} u + \min_{B_\varepsilon(x)} u \right\} + \beta ⨍_{B_\varepsilon(x)} u(y)\, dy + o(\varepsilon^2), \quad \textit{as } \varepsilon \to 0, \tag{22}$$

in the viscosity sense if

(1) *for every $\phi \in C^2$ such that $u - \phi$ has a strict minimum at the point $x \in \overline{\Omega}$ with $u(x) = \phi(x)$, we have*

$$0 \ge -\phi(x) + \frac{\alpha}{2} \left\{ \max_{B_\varepsilon(x)} \phi + \min_{B_\varepsilon(x)} \phi \right\} + \beta ⨍_{B_\varepsilon(x)} \phi(y)\, dy + o(\varepsilon^2).$$

(2) *for every $\phi \in C^2$ such that $u - \phi$ has a strict maximum at the point $x \in \overline{\Omega}$ with $u(x) = \phi(x)$, we have*

$$0 \le -\phi(x) + \frac{\alpha}{2} \left\{ \max_{B_\varepsilon(x)} \phi + \min_{B_\varepsilon(x)} \phi \right\} + \beta ⨍_{B_\varepsilon(x)} \phi(y)\, dy + o(\varepsilon^2).$$

Observe that a C^2-function u satisfies (22) in the classical sense if and only if it satisfies it in the viscosity sense. However, the viscosity sense is actually weaker than the classical sense for non C^2-functions as the following example shows.

Example: Set $p = \infty$ and consider Aronsson's function

$$u(x, y) = |x|^{4/3} - |y|^{4/3}$$

near the point $(x, y) = (1, 0)$. Aronsson's function is ∞-harmonic in the viscosity sense but it is not of class C^2, see [4]. It will follow from Theorem 5.2 below that u satisfies

$$u(x) = \frac{1}{2}\left\{\max_{B_\varepsilon(x)} u + \min_{B_\varepsilon(x)} u\right\} + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0,$$

in the viscosity sense of Definition 5.1. However, let us verify that the expansion does not hold in the classical sense.

Clearly, we have

$$\max_{B_\varepsilon(1,0)} u = u(1+\varepsilon, 0) = (1+\varepsilon)^{4/3}.$$

To find the minimum, we set $x = \varepsilon\cos(\theta)$, $y = \varepsilon\sin(\theta)$ and solve the equation

$$\frac{d}{d\theta}u(1+\varepsilon\cos(\theta), \varepsilon\sin(\theta)) = -\frac{4}{3}(1+\varepsilon\cos(\theta))^{1/3}\varepsilon\sin(\theta) - \frac{4}{3}(\varepsilon\sin(\theta))^{1/3}\varepsilon\cos(\theta) = 0.$$

By symmetry, we can focus our attention on the solution

$$\theta_\varepsilon = \arccos\left(\frac{\varepsilon - \sqrt{4+\varepsilon^2}}{2}\right).$$

Hence, we obtain

$$\begin{aligned}\min_{B_\varepsilon(1,0)} u &= u(1+\varepsilon\cos(\theta_\varepsilon), \varepsilon\sin(\theta_\varepsilon)) \\ &= \left(1 + \frac{1}{2}\varepsilon\left(\varepsilon - \sqrt{4+\varepsilon^2}\right)\right)^{4/3} - \left(\varepsilon\sqrt{1 - \frac{1}{4}\left(\varepsilon - \sqrt{4+\varepsilon^2}\right)^2}\right)^{4/3}.\end{aligned}$$

We are ready to compute

$$\lim_{\varepsilon\to 0+} \frac{\frac{1}{2}\left\{\max_{B_\varepsilon(1,0)} u + \min_{B_\varepsilon(1,0)} u\right\} - u(1,0)}{\varepsilon^2} = \frac{1}{18}.$$

But if an asymptotic expansion held in the classical sense, this limit would have to be zero.

The following theorem states our main result and provides a characterization to the p-harmonic functions.

Theorem 5.2. *Let $1 < p \le \infty$ and let u be a continuous function in a domain $\Omega \subset \mathbb{R}^N$. The asymptotic expansion*

$$u(x) = \frac{\alpha}{2}\left\{\max_{B_\varepsilon(x)} u + \min_{B_\varepsilon(x)} u\right\} + \beta \fint_{B_\varepsilon(x)} u(y)\,dy + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0,$$

holds for all $x \in \Omega$ in the viscosity sense if and only if

$$\Delta_p u(x) = 0$$

in the viscosity sense. Here α and β are determined by (21).

We use the notation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \Delta_p u$$

for the regular p-Laplacian and

$$\Delta_\infty u = |\nabla u|^{-2}\left\langle D^2 u\,\nabla u, \nabla u\right\rangle$$

for the 1-homogeneous infinity Laplacian.

We observe that the notions of a viscosity solution and a Sobolev weak solution for the p-Laplace equation agree for $1 < p < \infty$, see Juutinen-Lindqvist-Manfredi [32]. Therefore, Theorem 5.2 characterizes weak solutions when $1 < p < \infty$.

Also, we note that Wang [50] has also used Taylor series to give sufficient conditions for p-subharmonicity in terms of asymptotic mean values of $(u(x) - u(0))^p$.

5.1. **Proof of Theorem 5.2.** As we did before, to gain some intuition on why such asymptotic mean value formula might be true, let us formally expand the p-Laplacian as follows

$$\Delta_p u = (p-2)|\nabla u|^{p-4} \langle D^2 u \, \nabla u, \nabla u \rangle + |\nabla u|^{p-2} \Delta u. \tag{23}$$

This formal expansion was used by Peres and Sheffield in [48] (see also Peres et. al. [47]) to find $p-$harmonic functions as limits of values of Tug-of-War games.

Suppose that u is a smooth function with $\nabla u \neq 0$. We see from (23), that u is a solution to $\Delta_p u = 0$ if and only if

$$(p-2)\Delta_\infty u + \Delta u = 0. \tag{24}$$

It follows from the classical Taylor expansion that

$$u(x) - \fint_{B_\varepsilon(x)} u \, dy = -\varepsilon^2 \Delta u(x) \frac{1}{2N} \fint_{B(0,1)} |z|^2 \, dz + o(\varepsilon^2) \tag{25}$$

and

$$\begin{aligned} & u(x) - \frac{1}{2} \left\{ \max_{\overline{B_\varepsilon(x)}} u + \min_{\overline{B_\varepsilon(x)}} u \right\} \\ & \approx u(x) - \frac{1}{2} \left\{ u \left(x + \varepsilon \frac{\nabla u(x)}{|\nabla u(x)|} \right) + u \left(x - \varepsilon \frac{\nabla u(x)}{|\nabla u(x)|} \right) \right\} \\ & = -\frac{\varepsilon^2}{2} \Delta_\infty u(x) + o(\varepsilon^2). \end{aligned} \tag{26}$$

The volume of the unit ball in $\mathbb{R}^N$ will be denoted by ω_N and the $N-1$ dimensional area of the unit sphere will be denoted by σ_{N-1}. Observe that since $\sigma_{N-1}/\omega_N = N$ we have

$$\frac{1}{N} \fint_{B(0,1)} |z|^2 \, dz = \frac{1}{N+2}.$$

Multiply (25) and (26) by suitable constants and add up the formulas so that we have the operator from (24) on the right hand side. This process gives us the choices of the constants α and β in (21) needed to obtain the asymptotic expansion of Theorem 5.2.

The main idea of the proof of Theorem 5.2 is just to work in the viscosity setting and use the expansions (25) and (26). The derivation of (26) also needs some care. We start by recalling the viscosity characterization of p-harmonic functions for $p < \infty$, see [32].

Definition 5.3. *For $1 < p < \infty$ consider the equation*

$$-div\left(|\nabla u|^{p-2} \nabla u\right) = 0.$$

(1) *A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in C^2$ such that $u - \phi$ has a strict minimum at the point $x \in \Omega$ with $\nabla \phi(x) \neq 0$ we have*

$$-(p-2)\Delta_\infty \phi(x) - \Delta \phi(x) \geq 0.$$

(2) *An upper semi-continuous function u is a subsolution if for every $\phi \in C^2$ such that $u - \phi$ has a strict maximum at the point $x \in \Omega$ with $\nabla\phi(x) \neq 0$ we have*

$$-(p-2)\Delta_\infty \phi(x) - \Delta\phi(x) \leq 0.$$

(3) *Finally, u is a viscosity solution if it is both a supersolution and a subsolution.*

For the case $p = \infty$ we must restrict the class of test functions as in [47]. Let $S(x)$ denote the class of C^2 functions ϕ such that either $\nabla\phi(x) \neq 0$ or $\nabla\phi(x) = 0$ and the limit

$$\lim_{y\to x} \frac{2(\phi(y) - \phi(x))}{|y-x|^2} = \Delta_\infty \phi(x)$$

exists.

Definition 5.4. *Consider the equation $-\Delta_\infty u = 0$.*

(1) *A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in S(x)$ such that $u - \phi$ has a strict minimum at the point $x \in \Omega$ we have*

$$-\Delta_\infty \phi(x) \geq 0.$$

(2) *An upper semi-continuous function u is a subsolution if for every $\phi \in S(x)$ such that $u-\phi$ has a strict maximum at the point $x \in \Omega$ we have*

$$-\Delta_\infty \phi(x) \leq 0.$$

(3) *Finally, u is a viscosity solution if it is both a supersolution and a subsolution.*

Proof of Theorem 5.2. We first consider asymptotic expansions for smooth functions that involve the infinity Laplacian ($p = \infty$) and the regular Laplacian ($p = 2$).

Choose a point $x \in \Omega$ and a C^2-function ϕ defined in a neighborhood of x. Let x_1^ε and x_2^ε be the point at which ϕ attains its minimum and maximum in $\overline{B_\varepsilon(x)}$ respectively; that is,

$$\phi(x_1^\varepsilon) = \min_{y\in\overline{B_\varepsilon(x)}} \phi(y) \quad \text{and} \quad \phi(x_2^\varepsilon) = \max_{y\in\overline{B_\varepsilon(x)}} \phi(y).$$

Next, we use some ideas from [11]. Consider the Taylor expansion of the second order of ϕ

$$\phi(y) = \phi(x) + \nabla\phi(x)\cdot(y-x) + \frac{1}{2}\langle D^2\phi(x)(y-x),(y-x)\rangle + o(|y-x|^2)$$

as $|y-x| \to 0$. Evaluating this Taylor expansion of ϕ at the point x with $y = x_1^\varepsilon$ and $y = 2x - x_1^\varepsilon = \tilde{x}_1^\varepsilon$, we get

$$\phi(x_1^\varepsilon) = \phi(x) + \nabla\phi(x)(x_1^\varepsilon - x) + \frac{1}{2}\langle D^2\phi(x)(x_1^\varepsilon - x),(x_1^\varepsilon - x)\rangle + o(\varepsilon^2)$$

and

$$\phi(\tilde{x}_1^\varepsilon) = \phi(x) - \nabla\phi(x)(x_1^\varepsilon - x) + \frac{1}{2}\langle D^2\phi(x)(x_1^\varepsilon - x),(x_1^\varepsilon - x)\rangle + o(\varepsilon^2)$$

as $\varepsilon \to 0$. Adding the expressions, we obtain

$$\phi(\tilde{x}_1^\varepsilon) + \phi(x_1^\varepsilon) - 2\phi(x) = \langle D^2\phi(x)(x_1^\varepsilon - x),(x_1^\varepsilon - x)\rangle + o(\varepsilon^2).$$

Since x_1^ε is the point where the minimum of ϕ is attained, it follows that

$$\phi(\tilde{x}_1^\varepsilon) + \phi(x_1^\varepsilon) - 2\phi(x) \leq \max_{y\in\overline{B_\varepsilon(x)}} \phi(y) + \min_{y\in\overline{B_\varepsilon(x)}} \phi(y) - 2\phi(x),$$

and thus

$$\frac{1}{2}\left\{\max_{y\in\overline{B_\varepsilon(x)}}\phi(y)+\min_{y\in\overline{B_\varepsilon(x)}}\phi(y)\right\}-\phi(x)\geq\frac{1}{2}\langle D^2\phi(x)(x_1^\varepsilon-x),(x_1^\varepsilon-x)\rangle+o(\varepsilon^2). \tag{27}$$

Repeating the same process at the point x_2^ε we get instead

$$\frac{1}{2}\left\{\max_{y\in\overline{B_\varepsilon(x)}}\phi(y)+\min_{y\in\overline{B_\varepsilon(x)}}\phi(y)\right\}-\phi(x)\leq\frac{1}{2}\langle D^2\phi(x)(x_2^\varepsilon-x),(x_2^\varepsilon-x)\rangle+o(\varepsilon^2). \tag{28}$$

Next we derive a counterpart for the expansion with the usual Laplacian ($p=2$). Averaging both sides of the classical Taylor expansion of ϕ at x we get

$$⨍_{B_\varepsilon(x)}\phi(y)\,dy=\phi(x)+\sum_{i,j=1}^N\frac{\partial^2\phi}{\partial x_i^2}(x)⨍_{B_\varepsilon(0)}\frac{1}{2}z_iz_j\,dz+o(\varepsilon^2).$$

The values of the integrals in the sum above are zero when $i\neq j$. Using symmetry, we compute

$$⨍_{B_\varepsilon(0)}z_i^2\,dz=\frac{1}{N}⨍_{B_\varepsilon(0)}|z|^2\,dz=\frac{1}{N\omega_N\varepsilon^N}\int_0^\varepsilon\int_{\partial B_\rho}\rho^2\,dS\,d\rho=\frac{\sigma_{N-1}\varepsilon^2}{N(N+2)\omega_N}=\frac{\varepsilon^2}{(N+2)},$$

with the notation introduced after (26). We end up with

$$⨍_{B_\varepsilon(x)}\phi(y)\,dy-\phi(x)=\frac{\varepsilon^2}{2(N+2)}\Delta\phi(x)+o(\varepsilon^2). \tag{29}$$

Assume for the moment that $p\geq 2$ so that $\alpha\geq 0$. Multiply (27) by α and (29) by β and add. We arrive at the expansion valid for any smooth function ϕ:

$$\begin{aligned}&\frac{\alpha}{2}\left\{\max_{y\in\overline{B_\varepsilon(x)}}\phi(y)+\min_{y\in\overline{B_\varepsilon(x)}}\phi(y)\right\}+\beta⨍_{B_\varepsilon(x)}\phi(y)\,dy-\phi(x)\\&\geq\frac{\beta\varepsilon^2}{2(N+2)}\left((p-2)\left\langle D^2\phi(x)\left(\frac{x_1^\varepsilon-x}{\varepsilon}\right),\left(\frac{x_1^\varepsilon-x}{\varepsilon}\right)\right\rangle+\Delta\phi(x)\right)\\&\quad+o(\varepsilon^2).\end{aligned} \tag{30}$$

We remark that $x_1^\varepsilon\in\partial B_\varepsilon(x)$ for $\varepsilon>0$ small enough whenever $\nabla\phi(x)\neq 0$. In fact, suppose, on the contrary, that there exists a subsequence $x_1^{\varepsilon_j}\in B_{\varepsilon_j}(x)$ of minimum points of ϕ. Then, $\nabla\phi(x_1^{\varepsilon_j})=0$ and, since $x_1^{\varepsilon_j}\to x$ as $\varepsilon_j\to 0$, we have by continuity that $\nabla\phi(x)=0$. A simple argument based on Lagrange multipliers then shows that

$$\lim_{\varepsilon\to 0}\frac{x_1^\varepsilon-x}{\varepsilon}=-\frac{\nabla\phi}{|\nabla\phi|}(x). \tag{31}$$

We are ready to prove that if the asymptotic mean value formula holds for u, then u is a viscosity solution. Suppose that function u satisfies the asymptotic expansion in the viscosity sense according to Definition 5.1. Consider a smooth ϕ such that $u-\phi$ has a strict minimum at x and $\phi\in S(x)$ if $p=\infty$. We obtain

$$0\geq-\phi(x)+\frac{\alpha}{2}\left\{\max_{\overline{B_\varepsilon(x)}}\phi+\min_{\overline{B_\varepsilon(x)}}\phi\right\}+\beta⨍_{B_\varepsilon(x)}\phi(y)\,dy+o(\varepsilon^2),$$

and thus, by (30),

$$0 \geq \frac{\beta\varepsilon^2}{2(N+2)} \left((p-2) \left\langle D^2\phi(x) \left(\frac{x_1^\varepsilon - x}{\varepsilon} \right), \left(\frac{x_1^\varepsilon - x}{\varepsilon} \right) \right\rangle + \Delta\phi(x) \right) + o(\varepsilon^2).$$

If $\nabla\phi(x) \neq 0$ we take limits as $\varepsilon \to 0$. Taking into consideration (31) we get

$$0 \geq \frac{\beta}{2(N+2)} \left((p-2)\Delta_\infty\phi(x) + \Delta\phi(x) \right).$$

Suppose now that $p = \infty$ and that the limit

$$\lim_{y \to x} \frac{\phi(y) - \phi(x)}{|y-x|^2} = L$$

exists. We need to deduce that $L \leq 0$ from

$$0 \geq \frac{1}{2} \left\{ \max_{B_\varepsilon(x)} \phi + \min_{B_\varepsilon(x)} \phi \right\} - \phi(x).$$

Let us argue by contradiction. Suppose that $L > 0$ and choose $\eta > 0$ small enough so that $L - \eta > 0$. Use the limit condition to obtain the inequalities

$$(L-\eta)|x-y|^2 \leq \phi(x) - \phi(y) \leq (L+\eta)|x-y|^2,$$

for small $|x-y|$. Therefore, we get

$$\begin{aligned} 0 \geq\ & \frac{1}{2} \max_{B_\varepsilon(x)} (\phi - \phi(x)) + \frac{1}{2} \min_{B_\varepsilon(x)} (\phi - \phi(x)) \\ \geq\ & \frac{1}{2} \max_{B_\varepsilon(x)} (\phi - \phi(x)) \geq (\frac{L-\eta}{2})\varepsilon^2, \end{aligned}$$

which is a contradiction. Thus, we have proved that $L \geq 0$.

To prove that u is a viscosity subsolution, we first derive a reverse inequality to (30) by considering the maximum point of the test function, that is, using (28) and (29), and then choose a function ϕ that touches u from above. We omit the details.

To prove the converse implication, assume that u is a viscosity solution. In particular u is a subsolution. Let ϕ be a smooth test function such that $u - \phi$ has a strict local maximum at $x \in \Omega$. If $p = \infty$, we also assume $\phi \in S(x)$. If $\nabla\phi(x) \neq 0$, we get

$$-(p-2)\Delta_\infty\phi(x) - \Delta\phi(x) \leq 0. \tag{32}$$

The statement to be proven is

$$\liminf_{\varepsilon \to 0+} \frac{1}{\varepsilon^2} \left(-\phi(x) + \frac{\alpha}{2} \left\{ \max_{B_\varepsilon(x)} \phi + \min_{B_\varepsilon(x)} \phi \right\} + \beta \fint_{B_\varepsilon(x)} \phi(y)\, dy \right) \geq 0.$$

This again follows from (30). Indeed, divide (30) by ε^2, use (31), and deduce from (32) that the limit on the right hand side is bounded from below by zero.

For the case $p = \infty$ with $\nabla\phi(x) = 0$ we assume the existence of the limit

$$\lim_{y \to x} \frac{\phi(y) - \phi(x)}{|y-x|^2} = L \geq 0$$

and observe that

$$\liminf_{\varepsilon\to 0+} \frac{1}{\varepsilon^2}\left(-\phi(x)+\frac{1}{2}\left\{\max_{B_\varepsilon(x)}\phi+\min_{B_\varepsilon(x)}\phi\right\}\right)\geq 0.$$

The argument for the case of supersolutions is analogous.

Finally, we need to address the case $1 < p < 2$. Since $\alpha \le 0$ we use (28) instead of (27) to get a version of (30) with x_2^ε in place of x_1^ε. The argument then continues in the same way as before. □

5.2. **Comments.**

(1) There is also a "parabolic" mean value characterization of solutions to the evolution governed by the 1−homogeneous p−Laplacian, that is,

$$\begin{cases} u_t(x,t) - |\nabla u|^{2-p}\Delta_p u(x,t) = 0 & \text{in } \Omega\times(0,T),\\ u(x,t) = F(x,t) & \text{on } \partial\Omega\times(0,T),\\ u(x,0) = F(x,0) & \text{in } \Omega, \end{cases}$$

see [42].

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