Lecture 3:
Controllability results for other time-dependent PDEs

Abstract

This Lecture is devoted to present some controllability results for several time-dependent, mainly nonlinear, parabolic systems of PDEs. First, we will revisit the heat equation and some extensions. Then, we will consider several nonlinear systems from fluid mechanics: Burgers, Navier-Stokes, Boussinesq, micropolar, etc. Finally, some controllability results will be presented for systems governed by stochastic PDEs. Along this Lecture, several open questions will be stated.

1 Introduction. Recalling general ideas

Let us first give some general ideas, many of them already visited in the previous Lecture.

Suppose that we are considering an abstract *state equation* of the form

$$\left\{ \begin{array}{l} y_t - A(y) = Bv, \quad t \in (0,T), \\ y(0) = y^0, \end{array} \right. \tag{1}$$

which governs the behavior of a physical system. It is assumed that

- $y : [0,T] \mapsto H$ is the *state*, i. e. the variable that serves to identify the physical properties of the system,
- $v : [0,T] \mapsto U$ is the *control*, i.e. the variable we can choose (for simplicity, we assume that U and H are Hilbert spaces),
- $A : D(A) \subset H \mapsto H$ is a (generally nonlinear) operator with $A(0) = 0$, $B \in \mathcal{L}(U; H)$ and $y^0 \in H$.

Suppose that (22) is well-posed in the sense that, for each $y^0 \in H$ and each $v \in L^2(0,T;U)$, it possesses exactly one solution. Then the *null controllability* problem for (22) can be stated as follows:

For each $y^0 \in H$, find $v \in L^2(0,T;U)$ such that the corresponding solution of (22) *satisfies $y(T) = 0$.*

More generally, the *exact controllability to the trajectories* problem for (22) is the following:

For each free trajectory $\overline{y} : [0,T] \mapsto H$ and each $y^0 \in H$, find $v \in L^2(0,T;U)$ such that the corresponding solution of (22) *satisfies $y(T) = \overline{y}(T)$.*

Here, by a *free* or *uncontrolled* trajectory we mean any (sufficiently regular) function $\overline{y} : [0,T] \mapsto H$ satisfying $\overline{y}(t) \in D(A)$ for all t and

$$\overline{y}_t - A(\overline{y}) = 0, \quad t \in (0,T).$$

Notice that exact controllability to the trajectories is a very useful property from the view-point of applications: if we can find a control such that $y(T) = \overline{y}(T)$, then after time T we can switch off the control and let the system follow the "ideal" trajectory $\overline{y}$.

For each system of the form (22), these problems lead to several interesting questions. Among them, let us mention the following:

- First, are there controls v such that $y(T) = 0$ and/or $y(T) = \overline{y}(T)$?
- Then, if this is the case, which is the *cost* we have to pay to drive y to zero and/or $\overline{y}(T)$? In other words, which is the minimal norm of a control $v \in L^2(0,T;U)$ satisfying these properties?
- How can these controls be computed?

As indicated in Lecture 2, the controllability of differential systems is a very relevant area of research and has been the subject of a lot of work the last years. In particular, in the context of partial differential equations, the null controllability problem was first analyzed in [46, 47, 40, 41, 35, 38]. For semilinear systems of this kind, the first contributions have been given in [37, 49, 16, 29].

In this Lecture, I will consider several linear and nonlinear parabolic PDEs. First, we will recall the results satisfied by the classical heat equation in a bounded N-dimensional domain, complemented with appropriate initial and boundary-value conditions. Secondly, we will deal with similar stochastic PDEs. We will then consider the viscous Burgers equation. We will see that, for this PDE, the null controllability problem (with distributed and locally supported control) is well understood.[1] We will also consider the Navier-Stokes and Boussinesq equations and some other systems from mechanics. Several open problems will also be indicated.

[1] More precisely, if we denote by $T^*(r)$ the minimal time needed to drive any initial state with L^2 norm $\leq r$ to zero, we will show that $T^*(r) > 0$, with explicit sharp estimates from above and from below.

2 The classical heat equation. Observability and Carleman estimates

Let us consider the following control system for the heat equation:

$$
(2) \qquad \begin{cases} y_t - \Delta y = v 1_\omega, & (x,t) \in \Omega \times (0,T), \\ y(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ y(x,0) = y^0(x), & x \in \Omega. \end{cases}
$$

Here, we conserve the notation of Lecture 2. In particular, $\Omega \subset \mathbb{R}^N$ is a nonempty regular and bounded domain, $\omega \subset\subset \Omega$ is a (small) nonempty open subset (1_ω is the characteristic function of ω) and $y^0 \in L^2(\Omega)$.

It is well known that, for every $y^0 \in L^2(\Omega)$ and every $v \in L^2(\omega \times (0,T))$, there exists a unique solution y to (2), with $y \in L^2(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$.

In view of the results in Lecture 2, (2) is approximate, E-approximate and null controllable.

Also, if we introduce for each $\varphi^0 \in L^2(\Omega)$ the adjoint system

$$
(3) \qquad \begin{cases} -\varphi_t - \Delta \varphi = 0, & (x,t) \in \Omega \times (0,T), \\ \varphi(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ \varphi(x,T) = \varphi^0(x), & x \in \Omega, \end{cases}
$$

we know that the null controllability of (2) is equivalent to the *observability* of (3), that is, to the following estimate:

$$
(4) \qquad \|\varphi(\cdot,0)\|_{L^2}^2 \le C \iint_{\omega\times(0,T)} |\varphi|^2 \, dx\, dt \quad \forall \varphi^0 \in L^2(\Omega)
$$

(where C only depends on Ω, ω and T).

We have already seen that he estimate (4) is implied by the so called global Carleman inequalities. These have been introduced in the context of the controllability of PDEs by Fursikov and Imanuvilov; see [35, 29] When they are applied to the solutions of the adjoint system (3), they take the form

$$
(5) \qquad \iint_{\Omega\times(0,T)} \rho^{-2} |\varphi|^2 \, dx\, dt \le K \iint_{\omega\times(0,T)} \rho^{-2} |\varphi|^2 \, dx\, dt \quad \forall \varphi^0 \in L^2(\Omega),
$$

where $\rho = \rho(x,t)$ is an appropriate weight depending on Ω, ω and T and the constant K only depends on Ω and ω.[2] Combining (5) and the dissipativity of the backwards heat equation (3), it is not difficult to deduce (4) for some C only depending on Ω, ω and T.

Since (2) is linear, null controllability is equivalent in this case to *exact controllability to the trajectories.* This means that, for any uncontrolled solution $\overline{y}$ and any $y^0 \in L^2(\Omega)$, there exists $v \in L^2(\omega \times (0,T))$ such that the associated state y satisfies

$$
y(x,T) = \overline{y}(x,T) \quad \text{in} \quad \Omega.
$$

[2] In order to prove (5), we have to use a weight ρ that blows up as $t \to 0$ and also as $t \to T$, for instance exponentially.

REMARK 2.1 Notice that the null controllability of (2) holds for *any* ω and T. This is a consequence of the fact that, in a parabolic equation, the transmission of information is instantaneous. Recall that this was not the case for the wave equation. Again, this is not the case for the transport equation. Thus, let us consider the control system

$$\left\{ \begin{array}{ll} y_t + y_x = v1_\omega, & (x,t) \in (0,L) \times (0,T), \\ y(0,t) = 0, & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,L), \end{array} \right. \tag{6}$$

with $\omega = (a,b) \subset\subset (0,L)$. Then, if $0 < T < a$, null controllability does not hold, since the solution always satisfies

$$y(x,T) = y^0(x-T) \quad \forall x \in (T,a),$$

independently of the choice of v; see [12] for more details and similar results concerning other control systems for the wave, Schrödinger and Korteweg-De Vries equations. ■

There are many generalizations and variants of the previous argument that provide the null controllability of other similar linear (parabolic) state equations:

- Time-space dependent (and sufficiently regular) coefficients can appear in the equation, other boundary conditions can be used, boundary control (instead of distributed control) can be imposed, etc.; see [29]; see also [19] for a review of related results.

- The null controllability of Stokes-like systems of the form

  $$y_t - \Delta y + (a \cdot \nabla) y + (y \cdot \nabla) b + \nabla p = v1_\omega, \quad \nabla \cdot y = 0, \tag{7}$$

  where a and b are regular enough, can also be analyzed with these techniques. See for instance [22]; see also [15] for other controllability properties. We will come back in Section 5 to systems of this kind.

- Other linear parabolic (non-scalar) systems can also be considered, etc.

However, there are several interesting problems related to the controllability of linear parabolic systems that still remain open. Let us mention two of them.

First, let us consider the controlled system

$$\left\{ \begin{array}{ll} y_t - \nabla \cdot (a(x) \nabla y) = v1_\omega, & (x,t) \in \Omega \times (0,T), \\ y(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ y(x,0) = y^0(x), & x \in \Omega, \end{array} \right. \tag{8}$$

where y^0 and v are as before and the coefficient a is assumed to satisfy

$$a \in L^\infty(\Omega), \quad 0 < a_0 \le a(x) \le a_1 < +\infty \quad \text{a.e.} \tag{9}$$

It is natural to consider the null controllability problem for (8). Of course, this is equivalent to the observability of the associated adjoint system

$$\begin{cases} -\varphi_t - \nabla\cdot(a(x)\nabla\varphi) = 0, & (x,t)\in\Omega\times(0,T), \\ \varphi(x,t)=0, & (x,t)\in\partial\Omega\times(0,T), \\ y\varphi(x,T)=\varphi^1(x), & x\in\Omega, \end{cases} \tag{10}$$

that is to say, to the fact that an inequality like (4) holds for the solutions to (10).

To our knowledge, it is at present unknown whether (8) is null controllable. In fact, it is also unknown whether approximate controllability holds.

Recently, some partial results have been obtained in this context.

Thus, when $N=1$, the null controllability of (8) has been established in [1] for general a satisfying (9). The techniques in the proof rely on the theory of quasi-conformal complex mappings and can be applied only to the one-dimensional case, with a independent of t. Furthermore, they only serve to apply directly the Lebeau-Robbiano method (recall the proof of theorem 2.2 in Lecture 2), that is, they do not lead to a Carleman estimate of the form (5).

When $N\geq 2$, it is known that (8) is null controllable under the following assumption

$$\exists \textit{ smooth open set } \Omega_0 \subset\subset \Omega \textit{ such that } a \textit{ is } C^1 \textit{ in } \overline{\Omega_0} \textit{ and } \overline{\Omega\setminus\Omega_0}. \tag{11}$$

This has been proved in [39]. A slight improvement has been performed in [9], where Ω_0 is allowed to touch the boundary of Ω. Again, the proofs use that a is independent of t in an essential way and do not clarify whether (5) holds.

In fact, it is an open question whether a Carleman estimate like (5) holds for the solutions to (10) even if $N=1$ or (11) holds.

In order to have (5), we need more regularity for a; see [8] for a proof when $N=1$, a satisfies (9) and

$$a\in BV(\Omega); \tag{12}$$

see also [14] for a proof when $N\geq 2$, a is piecewise C^1 and satisfies (9) and some additional conditions.

At present, the following questions are open:

- Is (8) is null controllable when $N\geq 2$ and a satisfies (9) and (12)? Is (5) satisfied in this case?
- Is (5) satisfied when $N=1$ and a only satisfies (9)?

QUESTION 1: *Assume that $N=1$ and a is piecewise constant and satisfies* (9). *Is* (8) *approximaely controllable?*

A similar question can be asked when $N\geq 2$. Which is the rigorous question and which is the answer?

Our second open problem concerns the system

$$
(13) \qquad \begin{cases} y_t - D\Delta y = My + Bv1_\omega, & (x,t) \in \Omega \times (0,T), \\ y(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ y(x,0) = y^0(x), & x \in \Omega, \end{cases}
$$

where $y = (y_1, \dots, y_n)$ is the state, $v = (v_1, \dots, v_m)$ is the control and D, M and B are constant matrices, with $D, M \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$. It is assumed that D is definite positive, that is,

$$
(14) \qquad D\xi \cdot \xi \geq d_0 |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \quad d_0 > 0.
$$

When D is diagonal (or similar to a diagonal matrix), the null controllability problem for (13) is well understood. In view of the results in [2], (13) is null controllable if and only if

$$
(15) \qquad \text{rank } [(-\lambda_i D + M); B] = n \quad \forall i \geq 1,
$$

where the λ_i are the eigenvalues of the Dirichlet-Laplace operator and, for any matrix $H \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$, $[H; B]$ stands for the $n \times nm$ matrix

$$
[H; B] := [B | HB | \cdots | H^{n-1} B].
$$

Therefore, it is natural to search for (algebraic) conditions on D, M and B that ensure the null controllability of (13) in the general case. But, to our knowledge, this is unknown.

The results in [2] have been extended recently to the case of any D having no eigenvalue of geometric multiplicity > 4; see [18].

QUESTION 2: *Under which conditions the system*

$$
(16) \qquad \begin{cases} y_t - D\Delta y = M(x,t)y + Bv1_\omega, & (x,t) \in \Omega \times (0,T), \\ y(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ y(x,0) = y^0(x), & x \in \Omega, \end{cases}
$$

where D is a diagonal matrix satisfying (14)*, $M \in L^\infty(Q; \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n))$ and $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$, is null controllable?*

REMARK 2.2 As we have said, global Carleman estimates are the main tool we can use to establish the observability property (4). These open questions can be viewed, at least in part, as consequences of the limitations of Carleman estimates: first, they need regular coefficients; then, they are in fact a tool proper of *scalar* equations. ■

3 Some remarks on the controllability of stochastic PDEs

In this Section, we deal briefly with a system governed by a linear stochastic partial differential equation:

$$
(17) \qquad \left\{ \begin{array}{lll} y_t - \Delta y = v 1_\omega + B(t)\, \dot{w}_t & \text{in} & Q, \\ y = 0 & \text{on} & \Sigma, \\ y(x,0) = y^0(x) & \text{in} & \Omega. \end{array} \right.
$$

Here, v is again the control and $\dot{w}_t$ is a *Gaussian random field* (white noise in time). For instance, it can be regarded as the distributional time derivative of a Wiener process w_t. The equations are required to be satisfied $P-$a.e., i.e. P-almost surely, in a given probability space $\{\Lambda, \mathcal{F}, P\}$.

In the sequel, we are going to see that, for general y^0, y^1 and $B = B(t)$, one can obtain final states $y(\cdot, T)$ arbitrarily close to y^1 in quadratic mean by choosing v appropriately (an approximate controllability result). We will also see that, if B is not random and in some sense small, then one can also choose v such that $y(\cdot, T) = 0$ (a null controllability result).

3.1 Some basic results from probability calculus

In order to present the results without too much ambiguity, we will first recall some basic definitions and results.

Thus, assume that a *complete probability space* $\{\Lambda, \mathcal{F}, P\}$ is given. If X is a Banach space and $f \in L^1(\Lambda, \mathcal{F}; X)$, we will denote by Ef the expectation of f:

$$
Ef = \int_\Lambda f(\lambda)\, dP(\lambda).
$$

Assume that a separable Hilbert space K and a *Wiener process* w_t on $\{\Lambda, \mathcal{F}, P\}$ with values in K are given. This means that

$$
w_t = \sum_{k=1}^\infty \beta_t^k e_k \quad \forall t \geq 0,
$$

where $\{e_k\}$ is an orthonormal basis in K and the β_t^k are mutually independent *real Wiener processes* satisfying

$$
(18) \qquad E|\beta_t^k|^2 = \mu_k^2 t, \qquad \sum_{k=1}^\infty \mu_k^2 < +\infty.
$$

Roughly speaking, a normalized real Wiener process β_t is a measurable function $(\lambda, t) \mapsto \beta_t(\lambda)$ which is defined $P-$a.s. in Λ for all $t \in \mathbb{R}_+$ and satisfies the following:

(a) $\beta_0 = 0$,

(b) For each t, β_t is *normally distributed,* with mean 0 and variance t,

(c) $E(\beta_t \, \beta_s) = \sqrt{t}\sqrt{s}$ for all $t, s \geq 0$.

For other equivalent definitions and basic properties of real Wiener processes, see [4]. Recall that, in particular, the real processes β_t^k and the K-valued process w_t have Hölder-continuous sample paths $t \mapsto \beta_t^k(\lambda)$ and $t \mapsto w_t(\lambda)$.

In the sequel, we put

$$\mathcal{F}_t := \sigma(\, w_s,\ 0 \leq s \leq t \,)$$

($\mathcal{F}_t$ is the σ-algebra spanned by w_s for $0 \leq s \leq t$ and completed with the negligeable sets in $\mathcal{F}$). Obviously, $\{\mathcal{F}_t\}$ is an increasing family of sub σ-algebras of $\mathcal{F}$ and, among other things, one has:

$$\mathcal{F}_t = \sigma\left(\bigcup_{s<t} \mathcal{F}_s\right) \quad \forall t > 0. \tag{19}$$

Let H be a Hilbert space. For any $f \in L^1(\Lambda, \mathcal{F}; H)$, we denote by $E[f|\mathcal{F}_t]$ the conditional expectation of f with respect to $\mathcal{F}_t$, i.e. the unique element in $L^1(\Lambda, \mathcal{F}_t; H)$ such that

$$\int_A E[f|\mathcal{F}_t] \, dP = \int_A f \, dP \quad \forall A \in \mathcal{F}_t$$

The existence and uniqueness of $E[f|\mathcal{F}_t]$ is implied by the celebrated Radon-Nykodim theorem. For the main properties of the conditional expectation, see for instance [44]. In particular, recall that, if $f \in L^2(\Lambda, \mathcal{F}; H)$, then $E[f|\mathcal{F}_t] \in L^2(\Lambda, \mathcal{F}_t; H)$ and is in fact the orthogonal projection of f in $L^2(\Lambda, \mathcal{F}_t; H)$.

Let X be a Banach space. We denote by $I^2(0, T; X)$ the space formed by all stochastic processes $\Phi \in L^2(\Lambda \times (0, T), dP \otimes dt; X)$ which are $\mathcal{F}_t$-adapted a.e. in $(0, T)$, i.e. such that

$$\lambda \mapsto \Phi(\lambda, t) \text{ is } \mathcal{F}_t\text{-measurable for almost all } t \in (0, T)$$

(in the case $X = \mathcal{L}(K; H)$, measurability will be understood in the *strong* sense, i.e. the measurability of $\lambda \mapsto \Phi(\lambda, t)w$ for each $w \in K$). Then, $I^2(0, T; X)$ is a closed subspace of $L^2(\Lambda \times (0, T), dP \otimes dt; X)$.

Assume that a stochastic process B is given, with

$$B \in I^2(0, T; \mathcal{L}(K; H)) \tag{20}$$

(H is a Hilbert space). Then the stochastic integral of B with respect to w_t is defined by the formula

$$\int_0^t B(s) \, dw_s = \sum_{k=1}^{\infty} \int_0^t B(s) e_k \, d\beta_s^k \qquad \forall t \in [0, T].$$

Here, the convergence of the series is understood in the sense of $L^2(\Lambda, \mathcal{F}_t; H)$. The stochastic integrals in the right hand side are defined by the equalities

$$(\int_0^t B(s) e_k \, d\beta_s^k, h) = \int_0^t (B(s) e_k, h) \, d\beta_s^k \qquad \forall h \in H,$$

where the latter are usual *Ito stochastic integrals* with respect to the real-valued processes β_t^k; see [4].

Recall that, for any $b \in I^2(0,T;\mathbb{R})$, any real-valued Wiener process β_t and any fixed $t \in [0,T]$, we can introduce a random variable $I_t(f) : \Lambda \mapsto \mathbb{R}$ known as the Ito stochastic integral in $[0,t]$:

$$I_t(f) = \int_0^t f(s)\, d\beta_s\,.$$

The stochastic process $(\lambda, t) \mapsto I_t(f)(\lambda)$ again belongs to $I^2(0,T;\mathbb{R})$ and, among other properties, satisfies the following:

$$E \int_0^t f(s)\, d\beta_s = 0$$

and

$$E \left| \int_0^t f(s)\, d\beta_s \right|^2 = \int_0^t E|f(s)|^2\, ds$$

for all $t \in [0,T]$.

3.2 The controllability results

In the remainder of this Section, H and V will denote the Hilbert spaces $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively.

Assume we are given an arbitrary but fixed initial state

$$y^0 \in H, \tag{21}$$

a Wiener process w_t with values in the separable Hilbert space K and a stochastic process $B \in I^2(0,T;\mathcal{L}(K;H))$. Let $A = -\Delta$ be the usual Laplace-Dirichlet operator in Ω, with domain

$$D(A) = \{\, z \in H_0^1(\Omega) : -\Delta z \in L^2(\Omega) \,\}.$$

For each control $v \in I^2(0,T;H)$, there exists exactly one solution y to the state system

$$\begin{cases} y \in I^2(0,T;V) \cap L^2(\Lambda; C^0([0,T];H)), \\ y(\cdot,t) = y^0 + \displaystyle\int_0^t \{-Ay(\cdot,s) + 1_\omega v(\cdot,s)\}\, ds + \int_0^t B(s)\, dw_s \quad \forall t \in [0,T]. \end{cases} \tag{22}$$

In (22), the equalities have to be understood $P-$a.s. in V'.

Notice that we choose $\mathcal{F}_t$-adapted controls to govern the state system. This is a natural assumption from the stochastic viewpoint since, once w_t is given, only $\mathcal{F}_t$-adapted processes can be regarded as *statistically observable.*

Let $S(t)$ be the semigroup generated in H by A. Then, in accordance with the results in [13, 43], one has:

$$\begin{cases} y(\cdot,t) = S(t)y^0 + \displaystyle\int_0^t S(t-s)(1_\omega v(\cdot,s))\, ds + \int_0^t S(t-s)B(s)\, dw_s \\ \forall t \in [0,T] \end{cases} \tag{23}$$

Our first result deals with approximate controllability:

THEOREM 3.1 *The linear manifold* $Y_T = \{\, y(\cdot,T) : v \in I^2(0,T;H) \,\}$ *is dense in* $L^2(\Lambda, \mathcal{F}_T; H)$. *In other words: for any* $y^1 \in L^2(\Lambda, \mathcal{F}_T; H)$ *and any* $\varepsilon > 0$, *there exists a control* $v \in I^2(0,T;H)$ *such that the associated solution to* (22) *satisfies:*

$$E\|y(\cdot,T) - y^1\|_{L^2}^2 \leq \varepsilon.$$

Accordingly, it is said that (22) *is approximately controllable in quadratic mean.*

PROOF: We will argue as in the deterministic case. More precisely, in view of (23), it will suffice to check that, if $f \in L^2(\Lambda, \mathcal{F}_T; H)$ and

$$E(\int_0^T S(T-s)(1_\omega v(\cdot,s))\, ds,\, f)_{L^2} = 0 \quad \forall v \in I^2(0,T;H), \tag{24}$$

then necessarily $f = 0$.

Let f be a function in $L^2(\Lambda, \mathcal{F}_T; H)$ satisfying (24) and assume that $\phi \in I^2(0,T;H)$ is defined pathwise by

$$\begin{cases} -\phi_t + A\phi = 0 & \text{in} \quad Q, \\ \phi = 0 & \text{on} \quad \Sigma, \\ \phi(x,T) = f(x) & \text{in} \quad \Omega, \end{cases}$$

i.e. $\phi(\cdot,t) = S(T-t)f$ for all t. It will be sufficient to prove that

$$E[\phi(\cdot,t)|\mathcal{F}_t] = 0 \quad \forall t \in (0,T). \tag{25}$$

Indeed, this and the continuity property (19) of the family $\{\mathcal{F}_t\}$ clearly imply that $f = E[\phi(\cdot,T)|\mathcal{F}_T] = 0$.

QUESTION 3: *Why does* (25) *imply that* $f = 0$*?*

We know that

$$E\int_0^T (v(\cdot,s),\, 1_\omega \phi(\cdot,s))_{L^2}\, ds = 0 \qquad \forall v \in I^2(0,T;H).$$

Consequently, $1_\omega E[\phi(\cdot,t)|\mathcal{F}_t]$ is a stochastic process in $I^2(0,T;H)$ such that

$$E\int_0^T (v(\cdot,s),\, 1_\omega E[\phi(\cdot,s)|\mathcal{F}_s])\, ds = \int_0^T E(v(\cdot,s),\, 1_\omega \phi(\cdot,s))\, ds = 0$$

for all $v \in I^2(0,T;H)$ and, furthermore,

$$1_\omega E[\phi(\cdot,t)|\mathcal{F}_t] = 0. \tag{26}$$

For each $t \in (0,T)$, $E[\phi(\cdot,t)|\mathcal{F}_t] = S(T-t)E[f|\mathcal{F}_t]$ is real analytic in the variable $x \in \Omega$. Hence, one must necessarily have $E[\phi(\cdot,t)|\mathcal{F}_t] = 0$ for all $t \in (0,T)$ and the result is proved. $\Box$

A consequence of this theorem is that, for any $y^1 \in L^2(\Lambda, \mathcal{F}_T; H)$, $\varepsilon > 0$ and $\delta > 0$, a control v can be found such that

$$P\{\, \|y(\cdot,T) - y^1\|_{L^2} < \varepsilon \,\} \geq 1 - \delta.$$

However, the existence of a control $v \in I^2(0,T;H)$ such that $P\{\, \|y(\cdot,T) - y^1\|_{L^2} < \varepsilon \,\} = 1$ is an interesting open question.

The approximate controllability in quadratic mean remains true for systems governed by more general linear equations. More precisely, the following result is proved in [25]:

Theorem 3.2 *Assume that, in* (22)*, A is an operator of the form*

$$Ay = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial y}{\partial x_j}\right) + \sum_{j=1}^{N} b_j \frac{\partial y}{\partial x_j} + cy, \tag{27}$$

where the coefficients satisfy

$$a_{ij} \in C^1(\overline{\Omega}), \quad b_j, c \in L^\infty(\Omega)$$

and the usual ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_j\xi_i \geq \alpha|\xi|^2 \quad \forall\lambda \in \mathbb{R}^N, \quad \forall x \in \Omega, \quad \alpha > 0.$$

Then the corresponding $Y_T = \{\, y(\cdot,T) : v \in I^2(0,T;H) \,\}$ is dense in $L^2(\Lambda, \mathcal{F}_T; H)$.

We will now recall a null controllability result for (22) from [25]. Again, this is the analog of a deterministic result.

Theorem 3.3 *Let us set $\gamma(t) := t(T-t)$. Assume that B is not random, $B \in C^1([0,T]; \mathcal{L}(K;H))$ and, also, that the support of $B(t)w$ does not intersect ω for any t and $w \in K$. Then there exists a positive function $\rho = \rho(x)$ such that, if*

$$\iint_Q t\left(\gamma(t)^{-1}\|B\|^2_{\mathcal{L}(K;H)} + \gamma(t)^3\|B_t\|^2_{\mathcal{L}(K;H)}\right) e^{2\rho(x)/\gamma(t)}\,dx\,dt < +\infty, \tag{28}$$

for each $y^0 \in H$ there exists $v \in I^2(0,T;H)$ satisfying $y(x,T) \equiv 0$, i.e. (22) *is null controllable.*

As in the deterministic case, the proof relies on an observability estimate for the solution of the adjoint system.

The situation is more complicate in the case of a *multiplicative noise*, that is, for systems of the form

$$
(29) \qquad \begin{cases} y \in I^2(0,T;V) \cap L^2(\Omega; C^0([0,T],H)), \\ y(\cdot,t) = y^0 + \displaystyle\int_0^t \{-Ay(\cdot,s) + 1_\omega v(\cdot,s)\}\,ds + \int_0^t By(\cdot,s)\,dw_s \quad \forall t \in [0,T]. \end{cases}
$$

Here, B is given by $(By)(x) = b(x)y(x)$ for some $b \in W^{1,\infty}(\Omega)$ and (for simplicity) w_t is a real Wiener process.

From the theory of stochastic partial differential equations, it follows in particular that, for each $v \in I^2(0,T;H)$, there exists exactly one solution y to (29), see [43].

The approximate controllability in quadratic mean of (29) is equivalent to the unique continuation property for the backward (adjoint) stochastic system

$$
(30) \qquad \begin{cases} p \in I^2(0,T;V) \cap L^2(\Omega; C^0([0,T];H)), \quad q \in I^2(0,T;H), \\ p(\cdot,t) = f + \displaystyle\int_t^T \{A^*p(\cdot,s) + Bq(\cdot,s)\}\,ds - \int_t^T q(\cdot,s)\,dw_s \quad \forall t \in [0,T], \end{cases}
$$

In [6], a global Carleman estimate has been established for this system when $A = -\Delta$ and $b \in C^2(\overline{\Omega})$. Of course, this implies unique continuation for (30) and, consequently, approximate controllability in quadratic mean for (29) in this particular case.

On the other hand, an appropriate unique continuation property for (30) has been proved in [17] in the general case. As a consequence, one has approximate controllability in quadratic mean for (29). In fact, when b is a constant, and Γ is of class C^∞, we can also prove approximate controllability in all spaces $L^r(\Lambda, \mathcal{F}_T; L^q(\mathcal{O}))$ with $1 \le r, q < +\infty$.

The previous analysis can also be made for stochastic Stokes systems; see [24].

For more results concerning the approximate and null controllability of stochastic PDEs, see the recent review [48].

4 Positive and negative results for the one-dimensional Burgers equation

In this Section, we will be concerned with the null controllability of the following system for the viscous Burgers equation:

$$
(31) \qquad \begin{cases} y_t - y_{xx} + yy_x = v1_\omega, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = y(1,t) = 0, & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,1). \end{cases}
$$

Recall that some controllability properties of (31) have been studied in [29] (see Chapter 1, theorems 6.3 and 6.4). There, it is shown that, in general, a stationary solution of (31) with large L^2-norm cannot be reached (not even approximately) at any time T. In other words, with the help of one control, the solutions of the Burgers equation cannot go anywhere at any time.

For each $y^0 \in L^2(0,1)$, let us introduce

$$T(y^0) = \inf\{\, T > 0 : (31) \text{ is null controllable at time } T \,\}.$$

Then, for each $r > 0$, let us define the quantity

$$T^*(r) = \sup\{\, T(y^0) : \|y^0\|_{L^2} \le r \,\}.$$

Our main purpose is to show that $T^*(r) > 0$, with explicit sharp estimates from above and from below. In particular, this will imply that (global) null controllability at any positive time does not hold for (31).

More precisely, let us set $\phi(r) = (\log \frac{1}{r})^{-1}$. We have the following result from [21]:

THEOREM 4.1 *One has*

$$C_0\phi(r) \le T^*(r) \le C_1\phi(r) \quad \textit{as} \quad r \to 0, \tag{32}$$

for some positive constants C_0 *and* C_1 *not depending of* r.

REMARK 4.1 The same estimates hold when the control v acts on system (31) through the boundary *only* at $x = 1$ (or only at $x = 0$). Indeed, it is easy to transform the boundary controlled system

$$\begin{cases} y_t - y_{xx} + yy_x = 0, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = 0, \quad y(1,t) = w(t), & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,1) \end{cases} \tag{33}$$

into a system of the kind (31). The boundary controllability of the Burgers equation with *two* controls (at $x = 0$ and $x = 1$) has been analyzed in [32]. There, it is shown that even in this more favorable situation null controllability does not hold for small time. It is also proved in that paper that exact controllability does not hold for large time.[3] □

REMARK 4.2 It is proved in [10] that the Burgers equation is *globally* null controllable when we act on the system through two boundary controls and an additional right hand side only depending on t. In other words, for any $y^0 \in L^2(0,1)$, there exist w_1, w_2 and h in $L^2(0,T)$ such that the soution to

$$\begin{cases} y_t - y_{xx} + yy_x = h(t), & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = w_1(t), \quad y(1,t) = w_2(t), & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,1) \end{cases} \tag{34}$$

satisfies

$$y(x,T) = 0 \quad \text{in} \quad (0,1).$$

However, it is unknown whether this global property is conserved when one of the boundary controls w_1 or w_2 is eliminated. □

[3] Let us remark that the results in [32] do not allow to estimate $T(r)$; in fact, the proofs are based in contradiction arguments.

The proof of the estimate from above in (32) can be obtained by solving the null controllability problem for (31) via a (more or less) standard fixed point argument, using global Carleman inequalities to estimate the control and energy inequalities to estimate the state and being very careful with the role of T in these inequalities.

The proof of the estimate from below is inspired by the arguments in [3] and is implied by the following property: there exist positive constants C_0 and C_0' such that, for any sufficiently small $r > 0$, we can find initial data y^0 and associated states y satisfying $\|y^0\|_{L^2} \le r$ and

$$|y(x,t)| \ge C_0' r \quad \text{for some} \quad x \in (0,1) \quad \text{and any} \quad t : 0 < t < C_0 \phi(r).$$

For more details, see [21].

5 The Navier-Stokes and Boussinesq systems

There is a lot of more realistic nonlinear equations and systems from mechanics that can also be considered in this context. First, we have the well known Navier-Stokes equations:

$$(35) \qquad \begin{cases} y_t + (y \cdot \nabla) y - \Delta y + \nabla p = v 1_\omega, \quad \nabla \cdot y = 0, & (x,t) \in Q, \\ y = 0, & (x,t) \in \Sigma, \\ y(x,0) = y^0(x), & x \in \Omega. \end{cases}$$

Here and below, Q and Σ respectively stand for the sets $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^N$ is a nonempty regular and bounded domain, $N = 2$ or $N = 3$ and (again) $\omega \subset\subset \Omega$ is a nonempty open set.

In (35), (y,p) is the state (the velocity field and the pressure distribution) and v is the control (a field of external forces applied to the fluid particles located at ω). To our knowledge, the best results concerning the controllability of this system have been given in [22] and [23].[4] Essentially, these results establish the local exact controllability of the solutions of (35) to bounded uncontrolled trajectories.

In order to be more specific, let us recall the definition of some usual spaces in the context of Navier-Stokes equations:

$$V = \left\{ y \in H_0^1(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega \right\}$$

and

$$H = \{ y \in L^2(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega, \ y \cdot n = 0 \text{ on } \partial\Omega \}.$$

Of course, it will be said that (35) is *exactly controllable to the trajectories* if, for any trajectory $(\overline{y}, \overline{p})$, i. e. any solution of the uncontrolled Navier-Stokes system

$$(36) \qquad \begin{cases} \overline{y}_t + (\overline{y} \cdot \nabla) \overline{y} - \Delta \overline{y} + \nabla \overline{p} = 0, \quad \nabla \cdot \overline{y} = 0, & (x,t) \in Q, \\ \overline{y} = 0, & (x,t) \in \Sigma \end{cases}$$

[4] The main ideas come from [30, 36]; some additional results will appear soon in [20].

and any $y^0 \in H$, there exist controls $v \in L^2(\omega \times (0,T))^N$ and associated solutions (y,p) such that

$$y(x,T) = \overline{y}(x,T) \ \text{ in } \Omega. \tag{37}$$

At present, we do not know any global result concerning exact controllability to the trajectories for (35). However, the following local result holds:

THEOREM 5.1 *Let* $(\overline{y},\overline{p})$ *be a strong solution of* (36)*, with*

$$\overline{y} \in L^\infty(Q)^N, \quad \overline{y}(\cdot\,,0) \in V. \tag{38}$$

Then, there exists $\delta > 0$ *such that, for any* $y^0 \in H \cap L^{2N-2}(\Omega)^N$ *satisfying* $\|y^0 - \overline{y}^0\|_{L^{2N-2}} \le \delta$*, we can find a control* $v \in L^2(\omega \times (0,T))^N$ *and an associated solution* (y,p) *to* (35) *such that* (37) *holds.*

In other words, the local exact controllability to the trajectories holds for (35) in the space $X = L^{2N-2}(\Omega)^N \cap H$; see [22] for a slightly stronger result. Similar questions were addressed (and solved) in [28] and [27]. The fact that we consider here Dirichlet boundary conditions and locally supported distributed control increases a lot the mathematical difficulty of the control problem.

REMARK 5.1 It is clear that we cannot expect exact controllability for the Navier-Stokes equations with an arbitrary target function, because of the dissipative and non reversible properties of the system. On the other hand, approximate controllability is still an open question for this system. Some results in this direction have been obtained in [11] for different boundary conditions (Navier slip boundary conditions) and in [15] with a different nonlinearity. However, the notion of approximate controllability does not appear to be optimal from a practical viewpoint. Indeed, even if we could reach an arbitrary neighborhood of a given target y^1 at time T by the action of a control, the question of what to do afterwards to stay in the same neighbourhood would remain open. □

The proof of theorem 5.1 can be obtained as an application of *Liusternik's inverse mapping theorem* in an appropriate framework.

A key point in the proof is a related null controllability result for the linearized Navier-Stokes system at $(\overline{y},\overline{p})$, that is to say:

$$\begin{cases} y_t + (\overline{y}\cdot\nabla)y + (y\cdot\nabla)\overline{y} - \Delta y + \nabla p = v1_\omega, & (x,t) \in Q, \\ \nabla \cdot y = 0, & (x,t) \in Q, \\ y = 0, & (x,t) \in \Sigma, \\ y(x,0) = y^0(x), & x \in \Omega. \end{cases} \tag{39}$$

This control result is a consequence of a global Carleman inequality of the kind (5) that can be established for the solutions to the adjoint of (39), which is the following:

$$
\left\{
\begin{array}{ll}
-\varphi_t - (\nabla\varphi + \nabla\varphi^t)\,\overline{y} - \Delta\varphi + \nabla\pi = g, & (x,t) \in Q, \\
\nabla \cdot \varphi = 0, & (x,t) \in Q, \\
\varphi = 0, & (x,t) \in \Sigma, \\
\varphi(T) = \varphi^0, & x \in \Omega.
\end{array}
\right. \tag{40}
$$

The details can be found in [22].

Similar results have been given in [31] for the Boussinesq equations

$$
\left\{
\begin{array}{ll}
y_t + (y \cdot \nabla)y - \Delta y + \nabla p = v1_\omega + \theta\, e_N, \quad \nabla \cdot y = 0 & (x,t) \in Q, \\
\theta_t + y \cdot \nabla\theta - \Delta\theta = h1_\omega, & (x,t) \in Q, \\
y = 0, \quad \theta = 0, & (x,t) \in \Sigma, \\
y(x,0) = y^0(x), \quad \theta(x,0) = \theta^0(x), & x \in \Omega.
\end{array}
\right. \tag{41}
$$

Here, the state is the triplet (y,p,θ) (θ is interpreted as a temperature distribution) and the control is (v,h) (as before, v is a field of external forces; h is an external heat source).

QUESTION 4: *Can we deduce from theorem 5.1 a null controllability result for* (35) *for large* T*? What about* (41)*?*

QUESTION 5: *Does local null controllability imply local exact controllability to the trajectories in the context of* (35)*? What about* (41)*?*

An interesting question concerning both (35) and (41) is whether we can still get local exact controllability to the trajectories with a reduced number of scalar controls. This is partially answered in [23], where the following results are proved:

THEOREM 5.2 *Assume that the following property is satisfied:*

$$
\exists x^0 \in \partial\Omega,\ \exists \varepsilon > 0 \ \text{ such that } \overline{\omega} \cap \partial\Omega \supset B(x^0;\varepsilon) \cap \partial\Omega. \tag{42}
$$

Here, $B(x^0;\varepsilon)$ *is the ball centered at* x^0 *of radius* ε*. Then, for any* $T > 0$*,* (35) *is locally exactly controllable at time* T *to the trajectories satisfying* (38) *with controls* $v \in L^2(\omega \times (0,T))^N$ *having one component identically zero.*

THEOREM 5.3 *Assume that* ω *satisfies* (42) *with* $n_k(x^0) \neq 0$ *for some* $k < N$*. Then, for any* $T > 0$*,* (41) *is locally exactly controllable at time* T *to the trajectories* $(\overline{y}, \overline{p}, \overline{\theta})$ *satisfying* (38) *and*

$$
\overline{\theta} \in L^\infty(Q), \quad \overline{\theta}(\cdot\,,0) \in H_0^1(\Omega). \tag{43}
$$

with controls $v \in L^2(\omega \times (0,T))^N$ and $h \in L^2(\omega \times (0,T))$ such that $v_k \equiv v_N \equiv 0$. In particular, if $N = 2$, we have local exact controllability to these trajectories with controls $v \equiv 0$ and $h \in L^2(\omega \times (0,T))$.

The proofs of theorems 5.2 and 5.3 are similar to the proof of theorem 5.1. We have again to rewrite the controllability property as a nonlinear equation in a Hilbert space. Then, we have to check that the hypotheses of Liusternik's theorem are fulfilled.

Again, a crucial point is to prove the null controllability of certain linearized systems, this time with *reduced* controls. For instance, when dealing with (35), the task is reduced to prove that, for some ρ, ρ_0 and $K > 0$, the solutions to (39) satisfy the following Carleman-like estimates:

$$\iint_{\Omega\times(0,T)} \rho^{-2}|\varphi|^2 \, dx \, dt \le K \iint_{\omega\times(0,T)} \rho_0^{-2}(|\varphi_1|^2 + |\varphi_2|^2) \, dx \, dt \quad \forall \varphi^1 \in L^2(\Omega). \tag{44}$$

This inequality can be proved using the assumption (42) and the incompressibility identity $\nabla \cdot \varphi = 0$; see [23].

6 Some other nonlinear systems from mechanics

The previous arguments can be applied to other similar partial differential systems arising in mechanics. For instance, this is made in [**?**] in the context of micro-polar fluids.

To fix ideas, let us assume that $N = 3$. The behavior of a micro-polar three-dimensional fluid is governed by a system which has the form

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = \nabla \times w + v1_\omega, \quad \nabla \cdot y = 0, & (x,t) \in Q, \\ w_t + (y \cdot \nabla)w - \Delta w - \nabla(\nabla \cdot w) = \nabla \times y + u1_\omega, & (x,t) \in Q, \\ y = 0, \quad w = 0 & (x,t) \in \Sigma, \\ y(x,0) = y^0(x), \quad w(x,0) = w^0(x) & x \in \Omega. \end{cases} \tag{45}$$

Here, the state is (y,p,w) and the control is (v,u). As usual, y and p stand for the velocity field and pressure and w is the microscopic velocity of rotation of the fluid particles.

The following result holds:

THEOREM 6.1 *Let $(\overline{y}, \overline{p}, \overline{w})$ be such that*

$$\overline{y},\, \overline{w} \in L^\infty(Q) \cap L^2(0,T;H^2(\Omega)), \quad \overline{y}_t,\, \overline{w}_t \in L^2(Q) \tag{46}$$

and

$$\begin{cases} \overline{y}_t - \Delta\overline{y} + (\overline{y} \cdot \nabla)\overline{y} + \nabla\overline{p} = \nabla \times \overline{w}, \quad \nabla \cdot \overline{y} = 0, & (x,t) \in Q, \\ \overline{w}_t + (\overline{y} \cdot \nabla)\overline{w} - \Delta\overline{w} - \nabla(\nabla \cdot \overline{w}) = \nabla \times \overline{y}, & (x,t) \in Q, \\ \overline{y} = 0, \quad \overline{w} = 0 & (x,t) \in \Sigma. \end{cases} \tag{47}$$

Then, for each $T > 0$, (45) is locally exactly controllable to $(\overline{y}, \overline{p}, \overline{w})$ at time T. In other words, there exists $\delta > 0$ such that, for any initial data $(y^0, w^0) \in (H^2(\Omega) \cap V) \times H_0^1(\Omega)$ satisfying

$$\|(y^0, w^0) - (\overline{y}(\cdot\,,0), \overline{w}(\cdot\,,0))\|_{H^2 \times H_0^1} \leq \delta, \tag{48}$$

there exist L^2 controls u and v and associated solutions (y, p, w) satisfying

$$y(x,T) = \overline{y}(x,T), \quad w(x,T) = \overline{w}(x,T) \quad \text{in} \;\; \Omega. \tag{49}$$

Notice that this case involves a nontrivial difficulty. Indeed, w is a non-scalar variable and the equations satisfied by its components w_i are coupled through the second-order terms $\partial_i(\nabla \cdot w)$. This is a serious inconvenient. An appropriate strategy has to be applied in order to deduce the required Carleman estimates.

Let us also mention [5, 33, 34], where the controllability of the MHD and other related equations has been analyzed.

For all these systems, the proof of the controllability can be achieved arguing as in the first part of the proof of theorem 5.1. This is the general structure of the argument:

- First, rewrite the original controllability problem as a nonlinear equation in a space of admissible "state-control" pairs.

- Then, prove an appropriate global Carleman inequality and a regularity result and deduce that the linearized equation possesses at least one solution. This provides a controllability result for a related linear problem.

- Check that the hypotheses of a suitable implicit function theorem are satisfied and deduce a local result.

Remark 6.1 Recall that an alternative strategy was introduced in [49] in the context of the semilinear wave equation: first, consider a linearized similar problem and rewrite the original controllability problem in terms of a fixed point equation; then, prove a global Carleman inequality and deduce an observability estimate for the adjoint system and a controllability result for the linearized problem; finally, prove appropriate estimates for the control and the state (this usually needs some kind of *smallness* of the data), prove an appropriate compactness property of the state and deduce that there exists at least one fixed point. This method has been used in [16] and [26] in the context of semilinear heat equations and in [20] to prove a result similar to theorem 5.1. □

Remark 6.2 Observe that all these results are positive, in the sense that they provide local controllability properties. At present, no negative result is known to hold for these nonlinear systems (except for the already considered one-dimensional Burgers equation). □

To end this Section, let us mention two systems from fluid mechanics, apparently not much more complex than (35), for which local controllability is an open question.

The first system is the following:

$$
\left\{
\begin{array}{ll}
y_t + (y\cdot\nabla)y - \nabla\cdot(\nu(|Dy|)Dy) + \nabla p = v1_\omega\,, & (x,t)\in Q,\\
\nabla\cdot y = 0, & (x,t)\in Q,\\
y = 0, & (x,t)\in \Sigma,\\
y(x,0) = y^0(x), & x\in\Omega.
\end{array}
\right.
\tag{50}
$$

Here, $Dy = \frac{1}{2}(\nabla y + \nabla y^t)$ and $\nu : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a regular function (for example, we can take $\nu(s) \equiv a + bs^{r-1}$ for some $a, b, r > 0$). This models the behavior of a *quasi-Newtonian* fluid; for a mathematical analysis, see [7, 42].

In view of the new nonlinear diffusion term $\nabla\cdot(\nu(|Dy|)Dy)$, its control properties are much more difficult to analyze than for (35). In particular, it is unknown whether the local approximate and the local null controllability properties hold for (50).

For the second system, we suppose that $N = 2$. It reads:

$$
\left\{
\begin{array}{ll}
\theta_t + (y\cdot\nabla)\theta - \Delta\theta = v1_\omega\,, & (x,t)\in Q,\\
y = \nabla\times\left((-\Delta)^{-a}\theta\right), & (x,t)\in Q,\\
\theta = 0, & (x,t)\in \Sigma,\\
\theta(x,0) = \theta^0(x), & x\in\Omega,
\end{array}
\right.
\tag{51}
$$

where $a \in [1/2, 1]$. We are now modelling the behavior of a *quasi-geostrophic* fluid. The state variables θ and y may be viewed as a *generalized vorticity* and velocity field, respectively (notice that, for $a = 1$, we find again the Navier-Stokes system; see for instance [45]).

It is possible to prove a local null controllability result for (51). However, to our knowledge, the local approximate controllability and the local exact controllability to the trajectories are open problems.

References

[1] G. Alessandrini and L. Escauriaza, *Null-controllability of one-dimensional parabolic equations,* ESAIM Control Optim. Calc. Var. 14, no. 2, 2007, 284–293.

[2] F. Ammar-Khodja, A, Benabdallah, C. Dupaix and M. González-Burgos, *A Kalman rank condition for the localized distributed controllability of class of linear parabolic systems,* J. Evol. Equ. 9 (2009), no. 2, 267–291.

[3] S. Anita, D. Tataru, *Null controllability for the dissipative semilinear heat equation,* Appl. Math. Optim. 46, 2002, 97–105.

[4] L. Arnold, *Stochastic differential equations: theory and applications,* Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974.

[5] V. Barbu V, T. Havarneanu, C. Popa and S.S. Sritharan SS, *Exact controllability for the magnetohydrodynamic equations,* Comm. Pure Appl. Math. 56, 2003, 732–783.

[6] V. Barbu, A. Rascanu and G. Tesitore, *Carleman estimates and controllability of linear stochastic heat equations,* Appl. Math. Optim. 47 (2003), no. 2, 97–120.

[7] H. Bellout, F. Bloom, J. Nečas, *Young measure-valued solutions for non-Newtonian incompressible fluids,* Comm. PDE 19, no. 11–12, 1994, 1763–1803.

[8] A. Benabdallah, Y. Dermenjian, Yves and J. Le Rousseau, *Carleman estimates for the one-dimensional heat equation with a discontinuous coefficient and applications to controllability and an inverse problem,* J. Math. Anal. Appl. 336 (2007), no. 2, 865–887.

[9] A. Benabdallah, Y. Dermenjian, Yves and J. Le Rousseau, *to appear.*

[10] M. Chapouly, *Global controllability of nonviscous and viscous Burgers-type equations,* SIAM J. Control Optim. 48 (2009), no. 3, 1567–1599.

[11] J.-M. Coron, *On the controllability of the 2-D incompressible Navier-Stokes equations with the Navier slip boundary conditions,* ESAIM Control Optim. Calc. Var. 1, 1995/96, 35–75.

[12] J.-M. Coron, *Control and nonlinearity,* Mathematical Surveys and Monographs, 136. American Mathematical Society, Providence, RI, 2007.

[13] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge 1992.

[14] A. Doubova, A. Osses and J.-P. Puel, *Exact controllability to trajectories for semilinear heat equations with discontinuous diffusion coefficients,* ESAIM Control Optim. Calc. Var. 8 (2002), 621–661.

[15] C. Fabre, *Uniqueness results for Stokes equations and their consequences in linear and nonlinear control problems,* ESAIM Control Optim. Calc. Var. 1, 1995/96, 267–302.

[16] C. Fabre, J.-P. Puel JP, E. Zuazua, *Approximate controllability of the semilinear heat equation,* Proc. Royal Soc. Edinburgh, 125 A, 1995, 31–61.

[17] E. Fernández-Cara, M.J. Garrido-Atienza and J. Real, *On the approximate controllability of a stochastic parabolic equation with a multiplicative noise,* C.R. Acad. Sci. Paris, t. 328, Série I, p. 675–680, 1999.

[18] E. Fernández-Cara, M. González-Burgos and L. De Teresa, *in preparation.*

[19] E. Fernández-Cara and S.Guerrero, *Global Carleman inequalities for parabolic systems and applications to controllability,* SIAM J. Control Optim. 45, no. 4, 2006, 1399–1446.

[20] M. González-Burgos, S. Guerrero, J.-P. Puel, *Local exact controllability to the trajectories of the Boussinesq system via a fictitious control on the divergence equation,* Commun. Pure Appl. Anal. 8 (2009), no. 1, 311–333.

[21] E. Fernández-Cara, S. Guerrero, *Null controllability of the Burgers system with distributed controls,* Systems Control Lett. 56, no. 5, 2007, 366–372.

[22] E. Fernández-Cara, S. Guerrero, O.Yu. Imanuvilov and J.-P. Puel, *Local exact controllability to the trajectories of the Navier-Stokes equations,* J. Math. Pures Appl. 83 12, 2004, p. 1501–1542.

[23] E. Fernández-Cara, S. Guerrero, O.Yu. Imanuvilov and J.-P. Puel, *Some controllability results for the N-dimensional Navier-Stokes and Boussinesq systems with $N-1$ scalar controls,* SIAM J. Control and Optim. 45, no. 1, 2006, 146–173.

[24] E. Fernández-Cara. J.D. Martín and J. Real, *On the approximate controllability of stochastic Stokes systems,* Stoch. Anal. Appl., 17 (4), 563–577 (1999).

[25] E. Fernández-Cara and J. Real, *Remarks on the controllability of some stochastic partial differential equations,* in "Control and Estimation of Distributed Parameter Systems", p. 141–151, International Series of Numerical Mathematics, Birkhäuser Verlag, Basel 1998.

[26] E. Fernández-Cara and E. Zuazua, *Null and approximate controllability for weakly blowing up semilinear heat equations,* Ann. Inst. H. Poincar Anal. Non Linaire 17 (2000), no. 5, 583–616.

[27] A.V. Fursikov, *Exact boundary zero-controllability of three-dimensional Navier-Stokes equations,* J. Dynam. Control Systems 1, no. 3, 1995, 325–350.

[28] A.V. Fursikov an O. Yu. Imanuvilov O Yu, *On exact boundary zero-controllability of two-dimensional Navier-Stokes equations,* Acta Appl. Math. 37, no. 1–2, 1994, 67–76.

[29] A.V. Fursikov and O.Yu. Imanuvilov, *Controllability of evolution equations*, Lecture Notes Series # 34, Research Institute of Mathematics, Global Analysis Research Center, Seoul National University, 1996.

[30] A.V. Fursikov, O. Yu. Imanuvilov, *Exact controllability of the Navier-Stokes and Boussinesq equations (Russian),* Uspekhi Mat. Nauk 54 no. 3(327), 93–146; translation in Russian Math. Surveys 54, no. 3, 1999, 565–618.

[31] S. Guerrero, *Local exact controllability to the trajectories of the Boussinesq system,* Annales IHP, Anal. non linéaire, 23, 2006, 29–61.

[32] S. Guerrero, O. Yu. Imanuvilov, *Remarks on global controllability for the Burgers equation with two control forces,* Ann. Inst. H. Poincaré Anal. Non Linéaire 24, no. 6, 2007, 897–906.

[33] T. Havarneanu, C. Popa and S.S. Sritharan, *Exact internal controllability for the magnetohydrodynamic equations in multi-connected domains,* Adv. Differ. Equations 11, no. 8, 2006, 893–929.

[34] T. Havarneanu, C. Popa and S.S. Sritharan, *Exact internal controllability for the two-dimensional magnetohydrodynamic equations,* SIAM J. Control Optim. 46, no. 5, 2007, 1802–1830.

[35] O.Yu. Imanuvilov, *Boundary controllability of parabolic equations,* Russian Acad. Sci. Sb. Math., **186** (1995), 109–132 (in Russian).

[36] O. Yu. Imanuvilov O Yu, *Remarks on exact controllability for the Navier-Stokes equations,* ESAIM Control Optim. Calc. Var. 6, 2001, 39–72.

[37] I. Lasiecka and R. Triggiani, *Exact controllability of semilinear abstract systems with application to waves and plates boundary control problems,* Appl. Math. Optim. 23 (1991), no. 2, 109–154.

[38] G. Lebeau and L. Robbiano, *Contrôle exact de l'équation de la chaleur,* Comm. P.D.E., **20** (1995), 335–356.

[39] J. Le Rousseau, L. Robbiano, *Carleman estimate for elliptic operators with coefficients with jumps at an interface in arbitrary dimension and application to the null controllability of linear parabolic equations,* Arch. Ration. Mech. Anal. 195 (2010), no. 3, 953–990.

[40] J.L. Lions, Exact controllability, stabilizability and perturbations for distributed systems, SIAM Review, **30** (1988), 1–68.

[41] J.L. Lions, *Contrôlabilité Exacte, Stabilisation et Perturbations de Systèmes Distribués, Tomes 1 & 2,* Masson, RMA **8** & **9**, Paris 1988.

[42] J. Málek, J. Nečas, M. Rokyta, M. Ružička M, *Weak and measure-valued solutions to evolutionary PDEs,* Applied Mathematics and Mathematical Computation, 13, Chapman & Hall, London, 1996.

[43] E. Pardoux, *PhD Thesis,* Université Paris XI (France), 1975.

[44] E. Pardoux, *Intégrales stochastiques hilbertiennes,* Cahiers de Mathématiques de la Décision No. 7617, Université Paris IX,1976.

[45] J. Pedlosky, *Geophysical fluid dynamics, Second edition,* Springer, New York, 1987.

[46] D.L. Russell, *Controllability and stabilizability theory for linear partial differential equations. Recent progress and open questions,* SIAM Review **20** (1978), 639–739.

[47] D.L. Russell, *A unified boundary controllability theory for hyperbolic and parabolic partial differential equations,* Studies in Appl. Math., **52** (1973), 189–221.

[48] X. Zhang, *A unified controllability/observability theory for some stochastic and deterministic partial dfferential equations,* Proc. ICM, Hyderabad, India, 2010.

[49] E. Zuazua E, *Exact boundary controllability for the semilinear wave equation,* in “Nonlinear Partial Differential Equations and their Applications”, Vol. X (p. 357–391), H. Brezis and J.L. Lions eds., Pitman, New York, 1991.