Lecture 1:
Optimal control of systems governed by PDEs

Abstract

In this Lecture, I will review part of the existing theory for the optimal control of partial differential systems. This is a very broad subject and there have been so many contributions in this field over the last years that we will have to limit considerably the scope. In fact, I will only analyze a few questions concerning some very particular PDEs. We shall focus on the Laplace, the stationary Navier-Stokes and the heat equations. Of course, the existing theory allows to handle much more complex situations. The optimal control of (elliptic, parabolic and hyperbolic) partial differential systems was addressed in [17]. A lot of work has also been made in this field and many details can be found for instance in [9, 11, 15, 16] and the references therein.

1 Some examples

It will be assumed that $\Omega \subset \mathbb{R}^N$ is a bounded, regular and connected open set, with boundary $\Gamma = \partial\Omega$.

The first example concerns the optimal control of a *capacitor*.

Let $\omega \subset\subset \Omega$ be a non-empty open set. For each $u \in L^2(\omega)$, we consider the state system

$$\left\{ \begin{array}{lll} -\Delta y = 1_\omega u & \text{in} & \Omega, \\ y = 0 & \text{on} & \Gamma, \end{array} \right. \tag{1}$$

where 1_ω is the characteristic function of ω.

The solution $y = y(x)$ to (1) can be interpreted as the *electric potential* of a capacitor to which a *density of charge* $1_\omega v$ is applied; $E = -\nabla y$ is the associate electric field.

In practice, it might be important to know how to choose v in a subset $\mathcal{U}_{\text{ad}} \subset L^2(\omega)$ in order to obtain a potential y as close as possible to a prescribed function y_d without too much effort. For instance, $\mathcal{U}_{\text{ad}}$ can be a ball in $L^2(\omega)$. It can also be a set of the form

$$\mathcal{U}_{\text{ad}} = \{\, u \in L^2(\omega) : \underline{u} \le u(x) \le \overline{u} \ \text{ a.e.} \,\}, \tag{2}$$

where $\underline{u}, \overline{u} \in \mathbb{R}$.

Thus, let us fix $y_d \in L^2(\Omega)$ and let us introduce the *cost functional* J, with

$$J(u) = \frac{a}{2} \int_\Omega |y - y_d|^2 \, dx + \frac{b}{2} \int_\omega |u|^2 \, dx \tag{3}$$

where $a, b > 0$. The optimal control problem we want to solve is then:

> PROBLEM P1: *To find $\hat{u} \in \mathcal{U}_{\rm ad}$ such that $J(\hat{u}) \le J(u)$ for all $u \in \mathcal{U}_{\rm ad}$, where J is given by* (3)*.*

We will see below that this problem can be solved. We will also see the way the solution (the optimal control) can be characterized by an appropriate *optimality system.* Additionally, we will present some generalizations and variants.

In our second problem, the control is performed through the coefficients of the system.

Assume that Ω is composed of two *dielectric materials* whose properties and prices are different. We want to build a *nonhomogeneous plate* with these two materials in such an optimal way. Here, the word optimal means that, under an applied density of charge (fixed and known), the associate potential is close to a prescribed state y_d.

Let α and β be the *permeability coefficients* of the first and the second material, respectively. We assume that $0 < \alpha < \beta$. Let $\{G_1, G_2\}$ be a *partition* of Ω (G_1 and G_2 are measurable sets) and set

$$a(x) = \begin{cases} \alpha & \text{if } x \in G_1\,, \\ \beta & \text{if } x \in G_2\,. \end{cases} \tag{4}$$

Then the *electrostatic potential* $y = y(x)$ corresponding to this distribution of the materials is the solution of the system

$$\begin{cases} -\nabla \cdot (a(x)\nabla y) = f(x) & \text{in} \quad \Omega, \\ y = 0 & \text{on} \quad \Gamma, \end{cases} \tag{5}$$

where $f \in H^{-1}(\Omega)$ (for instance) is given. In this example, the coefficient $a = a(x)$ is the control and y is the state.

Let us put

$$j(a) = \frac{1}{2} \int_\Omega |y - y_d|^2 \, dx \quad \forall a \in \mathcal{U}_{\rm ad}\,, \tag{6}$$

where $y_d \in L^2(\Omega)$ and, by definition, we have

$$\mathcal{A}_{\rm ad} = \{\, a \in L^\infty(\Omega) : \; a(x) = \alpha \text{ or } a(x) = \beta \text{ a.e.} \,\} \tag{7}$$

The second problem we want to consider in this Section is then:

PROBLEM P2: *To find* $\hat{a} \in \mathcal{A}_{\rm ad}$ *such that* $j(\hat{a}) \le j(a)$ *for all* $a \in \mathcal{A}_{\rm ad}$*, where* j *is given by* (6).

It is well known that, in general, this problem has no solution and that a "generalized" or "relaxed" version has to be introduced in order to describe the limitting behavior of the minimizing sequences. This is in fact typical in control problems where the control enters in the system through its coefficients and, specially, in the principal part of the operator. Phenomena of this kind have led to a very rich development of the theory. We will see later what can be done and which is the physical interpretation of the "generalized" or "relaxed solution".

The third example is an *optimal design* problem.

We will assume that Ω is filled with a viscous incompressible fluid and we will try to search for the optimal shape of a body travelling at constant velocity in Ω. Thus, assume that $B \subset \Omega$ is a non-empty closed subset whose shape is in principle unknown. We will assume that B is the closure of a connected open set and ∂B is piecewise Lipschitz-continuous. Let us choose a reference system fixed with respect to B. We will consider the following Navier-Stokes system in $\Omega \setminus B$:

$$
(8) \qquad \begin{cases} -\nu \Delta y + (y \cdot \nabla) y + \nabla \pi = 0, \quad \nabla \cdot y = 0 & \text{in} \quad \Omega \setminus B, \\ y = y_\infty & \text{on} \quad \Gamma, \\ y = 0 & \text{on} \quad \partial B. \end{cases}
$$

Here, (y, π) is the state (the velocity field and the pressure of the fluid). The positive coefficient ν is the viscosity of the fluid. We have assumed that the velocity of the fluid particles is $-y_\infty$ (a constant vector) on the *exterior* boundary Γ, that is, far from the body B. We have also imposed the usual *no-slip condition* on ∂B. The boundary conditions in (8) mean that the body travels with velocity $-y_\infty$ and the fluid particles on ∂B adhere to the body.

For each B in a family $\mathcal{B}_{\rm ad}$ of *admissible bodies*, the state system (8) possesses at least one *weak solution* (y, π), with $y \in H^1(\Omega; \mathbb{R}^2)$ and $\pi \in L^2(\Omega)$. Now, we can associate to each solution the quantity

$$
(9) \qquad T(B, y) = 2\nu \int_\Omega |Dy|^2 \, dx,
$$

where

$$
Dy = \frac{1}{2}(\nabla y + \nabla y^t)
$$

is the symmetric part of the gradient ∇y. It can be seen that $T(B, y)$ is in fact the *hydrodynamical drag* of the fluid, that is

$$
T(B, y) = -y_\infty \cdot \int_{\partial B} \big(-\pi \, I + \nu D(y) \big) \cdot n \, ds
$$

(the projection in the direction of the velocity of the body of the force exerted by the fluid).

Our third problem is the following:

PROBLEM P3: *To find $\hat{B} \in \mathcal{B}_{\rm ad}$ such that the corresponding system* (8) *possesses a solution* $(\hat{y}, \hat{\pi})$ *satisfying* $T(\hat{B}, \hat{y}) \le T(B, y)$ *whenever* (y, π) *is a solution to* (8) *and* $B \in \mathcal{B}_{\rm ad}$.

We will see below that, unless the family $\mathcal{B}_{\rm ad}$ satisfies particular and in some sense artificial conditions, it is not possible to prove an existence result for Problem P3.

Besides existence, another interesting question is to analyze the way $T(B, y)$ depends on B. In fact, we will show that, at least when y_∞ is small, the mapping $B \mapsto T(B, y)$ is well-defined and in some sense of class C^∞. We will also indicate how to compute its "derivative".

We will now consider an optimal control problem for a parabolic system with origin in medicine. As shown below, the control is oriented to the determination of therapy strategies.

The state system is nonlinear and reads:

$$(10)\qquad \begin{cases} c_t - \nabla\cdot(D(x)\nabla c) = f(c) - F(c,\beta) & \text{in} \quad Q = \Omega\times(0,T), \\ \beta_t - \mu\Delta\beta = -h(\beta) - H(c,\beta) + v1_\omega & \text{in} \quad Q = \Omega\times(0,T), \\ c = 0 & \text{on} \quad \Sigma = \partial\Omega\times(0,T), \\ \beta = 0 & \text{on} \quad \Sigma = \partial\Omega\times(0,T), \\ c(0) = c_0 & \text{in} \quad \Omega, \\ \beta(0) = \beta_0 & \text{in} \quad \Omega. \end{cases}$$

We assume that Ω is an organ, where we find a population of cancer cells with density $c = c(x,t)$ and a distribution of inhibitors (or antibodies), of density $\beta = \beta(x,t)$. The antibodies are generated through a therapy process, determined by the control v and localized in a small open set $\omega \subset \Omega$. This can be used to model the evolution of a glioblastoma, i.e. a brain tumor, after radiotherapy, see [30, 31].

The functions f and h define the proliferation and death rates of c and β, respectively. On the other hand, F and H determine the way c and β interact. In the simplest cases we simply take

$$(11)\qquad f(c) = \rho c, \quad h(\beta) = -m\beta, \quad F(c,\beta) = Rc\beta, \quad H(c,\beta) = Mc\beta,$$

for some positive constants ρ, m, R and M.

For a large family of functions f, h, F and H, for any $v \in L^2(\omega\times(0,T))$ there exists at least one solution (c, β) to (10).

Obviously, in order to make the problem realistic, we have to impose constraints on v. Thus, we will assume that $v \in \mathcal{V}_{\rm ad}$, where $\mathcal{V}_{\rm ad}$ a bounded, closed and convex set of $L^2(\omega\times(0,T))$. A natural choice is the following:

$$\mathcal{V}_{\rm ad} = \{\, v \in L^2(\omega\times(0,T)) : 0 \le v \le A, \quad \int_0^T v\,dt \le B, \quad v = 0 \ \text{ for } \ t \notin \mathcal{I}\,\},$$

where $\mathcal{I}$ is a (small) closed set of times where the therapy is applied.

There are different possible choices for the cost function. A reasonable choice is the following:

$$K(v, c, \beta) = \frac{a}{2} \int_{\Omega} |c(T)|^2 \, dx + \frac{b}{2} \int_{\omega \times (0,T)} |v|^2 \, dx \, dt. \tag{12}$$

The fourth considered problem is thus:

PROBLEM P4: *To find* $\hat{v} \in \mathcal{V}_{\rm ad}$ *such that the corresponding system* (10) *possesses a solution* $8(\hat{c}, \hat{\beta})$ *satisfying* $K(\hat{c}, \hat{\beta}, \hat{v}) \leq K(c, \beta, v)$ *whenever* (c, β) *is a solution to* (10) *and* $v \in \mathcal{V}_{\rm ad}$.

Under very general conditions, we will give below an existence result for Problem P4. We will also find the *optimality system* for this problem.

2 Existence, uniqueness and optimality results for optimal control problems governed by elliptic PDEs

Our first result is the following:

THEOREM 2.1 *Assume that* $\mathcal{U}_{\rm ad}$ *is a non-empty closed convex set of* $L^2(\omega)$. *Then, Problem* P1 *possesses exactly one solution.*

PROOF: For the proof we only have to check that $u \mapsto J(u)$ is a strictly convex, coercive and weakly lower semicontinuous function on $L^2(\omega)$.

But this is very easy to verify. In fact, $u \mapsto J(u)$ can be written in the form

$$J(u) = \frac{1}{2} a_0(u, u) + a_1(u) + a_2 \quad \forall u \in \mathcal{U}_{\rm ad} \,, \tag{13}$$

where $a_0(\cdot\,,\cdot)$ is a continuous and coercive bilinear form on $L^2(\omega)$, $a_1(\cdot)$ is a continuous linear form on $L^2(\omega)$ and $a_2 \in \mathbb{R}$.

These are given as follows:

$$a_0(u, v) = a \int_{\Omega} yz \, dx + b \int_{\omega} uv \, dx$$

and

$$a_1(u) = -a \int_{\Omega} y_d y \, dx,$$

where y (resp. z) is the solution to (1) (resp. (1) with u replaced by v). On the other hand,

$$a_3 = \frac{a}{2} \int_{\Omega} |y_d|^2 \, dx.$$

Hence, the usual arguments of the *direct method of the Calculus of Variations* lead to the existence and uniqueness of solution, as assserted. □

QUESTION 1: *What can be said if, in* (3), *we assume that* $b = 0$? *Which interpretation can be given to the corresponding optimal control problem?*

We will now be concerned with the computation of $J'(u)$ and the obtention of an optimality system. Our result is the following:

THEOREM 2.2 *Assume that* $\mathcal{U}_{\mathrm{ad}} \subset L^2(\omega)$ *is a non-empty closed convex set and let* $\hat{u}$ *be the solution to Problem* P1. *Then there exists* $\hat{y}$ *and* $\hat{p}$ *such that the following optimality system is satisfied:*

$$\left\{ \begin{array}{lll} -\Delta\hat{y} = \hat{u}1_\omega & \textit{in} & \Omega, \\ \hat{y} = 0 & \textit{on} & \Gamma, \end{array} \right. \tag{14}$$

$$\left\{ \begin{array}{lll} -\Delta\hat{p} = \hat{y} - y_d & \textit{in} & \Omega, \\ \hat{p} = 0 & \textit{on} & \Gamma, \end{array} \right. \tag{15}$$

$$\int_\omega (a\hat{p} + b\hat{u})(u - \hat{u})\,dx \geq 0 \quad \forall u \in \mathcal{U}_{\mathrm{ad}}. \tag{16}$$

PROOF: For the proof, we argue as follows. Since $\hat{u}$ is the solution to Problem P1, we must have

$$\langle J'(\hat{u}), u - \hat{u}\rangle \geq 0 \quad \forall u \in \mathcal{U}_{\mathrm{ad}}, \quad \hat{u} \in \mathcal{U}_{\mathrm{ad}}\,. \tag{17}$$

Here, $\langle \cdot\,, \cdot\rangle$ denotes the scalar product in $L^2(\omega)$. Taking into account (13), this can be written as follows:

$$a_0(\hat{u}, u - \hat{u}) + a_1(u - \hat{u}) \geq 0$$

that is to say,

$$a\int_\Omega (\hat{y} - y_d)(y - \hat{y})\,dx + b\int_\omega \hat{u}\,(u - \hat{u})\,dx \geq 0 \tag{18}$$

for all $u \in \mathcal{U}_{\mathrm{ad}}$. Of course, in (18) y is the solution to (1) and $\hat{y}$ is the solution to (1) with u replaced by $\hat{u}$.

Let $\hat{p}$ be the solution to (15), the *adjoint system*. It is then clear that

$$\int_\Omega (\hat{y} - y_d)(y - \hat{y})\,dx = \int_\Omega \nabla\hat{p}\cdot\nabla(y - \hat{y})\,dx = \int_\omega \hat{p}\,(u - \hat{u})\,dx.$$

Consequently, (18) is equivalent to (16). This proves that the optimality system (14) – (16) must hold. $\square$

It is usual to say that $\hat{p}$ is the *adjoint state* associate to the optimal control $\hat{u}$. In fact, in view of the previous argument, for each $u \in \mathcal{U}_{\rm ad}$, we have

$$\langle J'(u), v \rangle = \int_\omega (ap + bu)\, v\, dx \quad \forall v \in \mathcal{U}_{\rm ad}\,, \tag{19}$$

where p is the adjoint state associate to u, i.e. the solution to

$$\begin{cases} -\Delta p = y - y_d & \text{in} \quad \Omega, \\ p = 0 & \text{on} \quad \Gamma. \end{cases} \tag{20}$$

This provides a very useful technique to compute the derivative $J'(u)$ for a given u. From the practical viewpoint this is very important, since a method to compute $J'(u)$ permits the use of *descent methods* in order to determine the optimal control $\hat{u}$.

QUESTION 2: *The optimality system in theorem 2.2 suggests the following iterative method for the computation of* $\hat{u}$*:*

$$\begin{cases} -\Delta y^n = u^{n-1} 1_\omega & \textit{in} \quad \Omega, \\ y^n = 0 & \textit{on} \quad \Gamma, \end{cases} \tag{21}$$

$$\begin{cases} -\Delta p^n = y^n - y_d & \textit{in} \quad \Omega, \\ p^n = 0 & \textit{on} \quad \Gamma, \end{cases} \tag{22}$$

$$\int_\omega (ap^n + bu^n)(u - u^n)\, dx \geq 0 \quad \forall u \in \mathcal{U}_{\rm ad}. \tag{23}$$

What can be said on the convergence of these iterates?

QUESTION 3: *In view of* (19) – (20)*, how can we apply (for instance) the fixed-step gradient method to produce a sequence* $\{u^n\}$ *of controls converging to the optimal control* $\hat{u}$*? What about the optimal-step gradient method? What about the fixed-step and optimal-step conjugate gradient methods?*

The previous ideas can be generalized in several directions. We will present a generalization involving nonlinear elliptic state systems and nonquadratic cost functionals.

Thus, let us introduce the system

$$\begin{cases} Ay + f(y) = 1_\omega u & \text{in} \quad \Omega, \\ y = 0 & \text{on} \quad \Gamma, \end{cases} \tag{24}$$

where A is the linear second order operator given by

$$Ay = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial y}{\partial x_j} \right) + \sum_{j=1}^{2} b_j(x) \frac{\partial y}{\partial x_j} + c(x) y \tag{25}$$

and $f : \mathbb{R} \mapsto \mathbb{R}$ is (for instance) a nondecreasing C^1 function satisfying

$$|f(s)| \leq C(1 + |s|) \quad \forall s \in \mathbb{R}. \tag{26}$$

We will assume that the coefficients a_{ij} , b_i and c satisfy:

$$\begin{gathered} a_{ij}\,, b_i\,, c \in L^\infty(\Omega), \quad c \geq 0, \\ \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \quad \forall \xi \in \mathbb{R}^2 \quad \text{a.e. in } \Omega, \quad \alpha > 0. \end{gathered} \tag{27}$$

For each $u \in L^2(\omega)$, the corresponding system (24) possesses exactly one solution $y \in H_0^1(\Omega)$. Let $\mathcal{U}_{\rm ad} \subset L^2(\omega)$ be a family of admissible controls. We will now set

$$J(u) = \int_\Omega F(x, y(x), u(x))\, dx \quad \forall u \in \mathcal{U}_{\rm ad}\,, \tag{28}$$

where $F = F(x, s, v)$ is assumed to be a *Carathéodory function*, defined for $(x, s, v) \in \Omega \times \mathbb{R} \times \mathbb{R}$. We consider the following generalization of Problem P1:

> PROBLEM P1$'$: *To find $\hat{u} \in \mathcal{U}_{\rm ad}$ such that $J(\hat{u}) \leq J(u)$ for all $u \in \mathcal{U}_{\rm ad}$, where J is given by* (24),(28).

Among all possible results that can be established in this context, let us indicate the following:

THEOREM 2.3 *Assume that $\mathcal{U}_{\rm ad}$ is a closed convex subset of $L^2(\omega)$. Also, assume that F is of the form*

$$F(x, s, v) = F_0(x, s) + F_1(x, v)\, 1_\omega(x),$$

where F_0 and F_1 are Carathéodory functions satisfying:

$$\left\{ \begin{array}{l} |F_0(x, s)| \leq C(1 + |s|^2) \quad \forall (x, s) \in \Omega \times \mathbb{R}, \\ a|v|^2 \leq F_1(x, v) \leq C(1 + |v|^2) \quad \forall (x, v) \in \omega \times \mathbb{R}, \quad a > 0, \\ F_1(x, \cdot) \textit{ is convex for each } x \in \omega. \end{array} \right. \tag{29}$$

Then Problem P1$'$ *possesses at least one solution $\hat{u}$.*

The proof relies on arguments similar to those above but technically more involved. It will not be given here. For instance, see [4] for the details.

QUESTION 4: *What can be said if, in* (29)*, we have $a = 0$?*

Notice that, in the previous result, the convexity hypothesis on $F_1(x, \cdot)$ is essential. Indeed, let us consider the particular case in which the state system is

$$\left\{ \begin{array}{lll} -\Delta y = u & \text{in} & \Omega, \\ y = 0 & \text{on} & \Gamma, \end{array} \right. \tag{30}$$

the set $\mathcal{U}_{\text{ad}}$ is

$$\mathcal{U}_{\text{ad}} = \{\, u \in L^2(\Omega) : |u| \leq 1 \quad \text{a.e. in } \Omega \,\} \tag{31}$$

and the cost functional is given by

$$J(u) = \int_\Omega \left(|u|^2 - 1\right)^2 \, dx + \frac{1}{2} \int_\Omega |y|^2 \, dx \quad \forall u \in \mathcal{U}_{\text{ad}} \, . \tag{32}$$

Then, it can be shown that

$$\inf_{u \in \mathcal{U}_{\text{ad}}} J(u) = 0$$

and, however,

$$J(u) > 0 \quad \forall u \in \mathcal{U}_{\text{ad}} \, ,$$

whence the optimal control problem associate to (30), (31) and (32) has no solution.

To end this Subsection, let us state a result concerning the optimality system for Problem P1$'$. We will need the adjoint operator A^*, which is given as follows:

$$A^* p = - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial p}{\partial x_i} + b_j(x) p \right) + c(x) p. \tag{33}$$

Then, one has:

Theorem 2.4 *Assume that F is as above, that F_0 and F_1 possess bounded partial derivatives and, also, that (29) is satisfied. Let $\hat{u}$ be a solution to Problem* P1$'$. *Then there exists $\hat{y}$ and $\hat{p}$ such that the following optimality system is satisfied:*

$$\begin{cases} A\hat{y} + f(\hat{y}) = \hat{u} 1_\omega & \textit{in} \quad \Omega, \\ \hat{y} = 0 & \textit{on} \quad \Gamma, \end{cases} \tag{34}$$

$$\begin{cases} A^* \hat{p} + f'(\hat{y}) \hat{p} = \dfrac{\partial F_0}{\partial s}(x, \hat{y}) & \textit{in} \quad \Omega, \\ \hat{p} = 0 & \textit{on} \quad \Gamma, \end{cases} \tag{35}$$

$$\int_\omega \left(\hat{p} + \frac{\partial F_1}{\partial v}(x, \hat{u}) \right) (u - \hat{u}) \, dx \geq 0 \quad \forall u \in \mathcal{U}_{\text{ad}} \, . \tag{36}$$

As before, the method of proof of this result provides an expression for the derivative $J'(u)$ of J at each u. More precisely, one finds that

$$\langle J'(u), v \rangle = \int_\omega \left(p + \frac{\partial F_1}{\partial v}(x, u) \right) v \, dx \quad \forall v \in \mathcal{U}_{\text{ad}} \, , \tag{37}$$

where p is the adjoint state associate to u, i.e. the solution to

$$(38) \qquad \begin{cases} A^*p + f'(y)p = \dfrac{\partial F_0}{\partial s}(x,y) & \text{in} \quad \Omega, \\ p = 0 & \text{on} \quad \Gamma \end{cases}$$

and y is the state, i.e. the solution to (24).

For other similar results, see for instance [3] and [5].

QUESTION 5: *Is there a way to use the optimality system in theorem 2.4 to prove a uniqueness result?*

QUESTION 6: *The optimality system in theorem 2.4 also suggests a "natural" iterative method for the computation of $\hat{u}$. Which one? What can be said on the convergence of the iterates?*

QUESTION 7: *In view of* $(37)-(38)$*, how can we apply gradient and conjugate gradient method to produce a sequence of controls that converge to an optimal control?*

3 Control on the coefficients, nonexistence and relaxation

In this Section we assume that $N = 2$ and we consider Problem P2.

We will try to show the complexity of the problems in which the control is applied through coefficients in the principal part of the operator. We will first see that, in general, there exists no solution to this problem.

The following notation is needed. For given α and β with $\alpha, \beta > 0$, let us denote by $\mathcal{A}(\alpha,\beta)$ the family of 2×2 matrices A with components $A_{ij} \in L^\infty(\Omega)$ such that

$$(39) \qquad A(x)\xi\cdot\xi \geq \alpha|\xi|^2, \quad (A(x))^{-1}\xi\cdot\xi \geq \frac{1}{\beta}|\xi|^2 \quad \forall \xi \in \mathbb{R}^2, \quad x \text{ a.e. in } \Omega.$$

It will be useful to recall the concept of *H-convergence*, which was introduced by F. Murat in 1978 (see [19],[20] and [23]):

Definition 3.1 *Assume that $A^n \in \mathcal{A}(\alpha,\beta)$ for each $n \geq 1$ and that $A^0 \in \mathcal{A}(\alpha,\beta)$. It will be said that A^n H-converges to A^0 in Ω if, for any non-empty open set $\mathcal{O} \in \Omega$ and any $g \in H^{-1}(\mathcal{O})$, the solution y^n of the elliptic problem*

$$(40) \qquad \begin{cases} -\nabla\cdot(A^n(x)\nabla y) = g(x) & \text{in} \quad \mathcal{O}, \\ y = 0 & \text{on} \quad \partial\mathcal{O}, \end{cases}$$

satisfies

$$y^n \to y^0 \quad \text{weakly in } H^1_0(\mathcal{O})$$

and

$$A^n\nabla y^n \to A^0\nabla y^0 \quad \text{weakly in } L^2(\mathcal{O}),$$

where y^0 is the unique solution of the problem

$$\left\{ \begin{array}{lll} -\nabla \cdot (A^0(x)\nabla y) = g(x) & in & \mathcal{O}, \\ y = 0 & on & \partial\mathcal{O}. \end{array} \right. \tag{41}$$

It can be seen that the family $\mathcal{A}(\alpha,\beta)$ is *closed* for the H-convergence. The following is also true:

THEOREM 3.1 *The family $\mathcal{A}(\alpha,\beta)$ is compact for the H-convergence. In other words, any sequence in $\mathcal{A}(\alpha,\beta)$ possesses subsequences that H-converge in $\mathcal{A}(\alpha,\beta)$.*

A key point is that we can have all A^n of the form

$$A^n = a^n I \quad \forall n \geq 1,$$

while the H-limit A^0 can have extra-diagonal terms. In fact, explicit examples can be constructed and, in particular, we can find $A^0 \in \mathcal{A}(\alpha,\beta)$ and $f^0 \in H^{-1}(\Omega)$ with the following two properties:

(a) A^0 is the H-limit of a sequence of the form $a^n I$, with $a^n(x) = \alpha$ or $a^n(x) = \beta$ a.e.

(b) Let y^0 be the solution to (41) with g replaced by f^0. Then there is no function a with $a(x) = \alpha$ or $a(x) = \beta$ a.e. such that y^0 solves (5) with f replaced by f^0.

We are now ready to prove that Problem P2 has no solution in general. Let us take $f = f^0$ and $y_d = y^0$, where y^0 is the solution of (41) with g replaced by f^0. In view of the properties of A^0, it is clear that

$$\inf_{a \in \mathcal{A}_{\text{ad}}} j(a) = 0$$

(recall that $\mathcal{A}_{\text{ad}}$ is given by (7)). However, in view of the properties of f^0, we also have

$$j(a) > 0 \quad \forall a \in \mathcal{A}_{\text{ad}}\,.$$

As a consequence, we must modify the definition of *optimal material.* Note that minimizing sequences do exist and that, in fact, they "describe" the optimal behavior. Consequently, it seems natural to adopt a new formulation in which the limits of minimizing sequences are distinguished material configurations. A satisfactory strategy consists of introducing a *relaxed problem.*

Relaxation is a useful tool in Optimization. Roughly speaking, to *relax* an extremal problem, say (P), is to introduce a second one, denoted by (Q), satisfying the following three conditions:

(a) (Q) possesses at least one solution.

(b) Any solution to (Q) can be written as the limit (in some sense) of a minimizing sequence for (P).

(c) Conversely, any minimizing sequence for (P) contains a subsequence that converges (in the same sense) to a solution of (Q).

For an overview on the role of the notion of relaxation in control problems, see [13] and [24]. We will only present here an intuitive and very simple argument which leads to a relaxed problem for P2.

The main point is to determine the "closure" in $\mathcal{A}(\alpha,\beta)$ of the family formed by the matrices of the form aI, with $a \in \mathcal{A}_{\rm ad}$. The answer is given by the following result:

THEOREM 3.2 *Let $\tilde{\mathcal{A}}_{\rm ad}$ be the family of all $A \in \mathcal{A}(\alpha,\beta)$ with the following two properties:*

(a) $A(x)$ is symmetric for x a.e. in Ω.

(b) For almost all x, the eigenvalues $\lambda_1(x)$ and $\lambda_2(x)$ of the matrix $A(x)$ satisfy:

$$\alpha \le \lambda_1(x) \le \lambda_2(x) \le \beta, \quad \frac{\alpha\beta}{\alpha+\beta-\lambda_2(x)} \le \lambda_1(x). \tag{42}$$

Then, if A is given in $\mathcal{A}(\alpha,\beta)$, one has $A \in \tilde{\mathcal{A}}_{\rm ad}$ if and only if A can be written as the H-limit of a sequence of matrices of the form $a^n I$, with $a^n \in \mathcal{A}_{\rm ad}$ for all n.

This is proved in [32] (see also [23]). At this respect, it is worth mentioning that, in a similar N-dimensional situation with $N \ge 3$, the determination of the set of H-limits of the matrices of the form aI with $a \in \mathcal{A}_{\rm ad}$ is an open problem.

The previous result permits to introduce a new control problem which is nothing but the relaxation of Problem P2.

Namely, for each $A \in \tilde{\mathcal{A}}_{\rm ad}$, let us consider the (relaxed) state system

$$\begin{cases} -\nabla\cdot(A(x)\nabla Y) = f(x) & \text{in} \quad \Omega, \\ Y = 0 & \text{on} \quad \Gamma \end{cases} \tag{43}$$

and let us set

$$k(A) = \frac{1}{2}\int_\Omega |Y-y_d|^2\,dx \quad \forall A \in \tilde{\mathcal{A}}_{\rm ad}\,. \tag{44}$$

The relaxed problem is then:

PROBLEM P2′: *To find $\hat{A} \in \tilde{\mathcal{A}}_{\rm ad}$ such that $k(\hat{A}) \le k(A)$ for all $A \in \tilde{\mathcal{A}}_{\rm ad}$, where $\tilde{j}$ is given by* (44).

Indeed, the following can be proved:

THEOREM 3.3 *Assume that $f \in H^{-1}(\Omega)$ and $y_d \in L^2(\Omega)$ are given. Then, there exists at least one solution $\hat{A}$ to Problem* P2′. *This can be written as the H-limit of a minimizing sequence for Problem* P2. *Furthermore, any minimizing sequence for Problem* P2 *contains a subsequence that H-converges to a solution of Problem* P2′.

The proof of this result is not difficult, taking into account the definition of H-convergence and the fact that $\tilde{\mathcal{A}}_{\rm ad}$ is the H-closure of $\mathcal{A}_{\rm ad}$.

From a physical viewpoint, we see that the "generalized" solution to the original problem is a *composite material*. In general, it is anisotropic, i.e. $\hat{A}_{ij}(x)$ may be $\neq 0$ for $i \neq j$.

QUESTION 8: *Is it possible to deduce an optimality system for the solutions to Problem* P2′*? Which one? Does this optimality system lead to convergent iterates?*

QUESTION 9: *Is it possible to compute* $k'(A)$ *easily and use this computation to apply gradient and/or conjugate gradient methods in the context of Problem* P2′*?*

The reader is referred to [18] and the references therein for more details on the control of coefficients, the generation of composite materials and other related topics.

4 Optimal design and domain variations

We will now consider Problem P3.

This is an *optimal design* problem. The feature is that, now, the control is a geometric data in (8) (the set B). Accordingly, we have to minimize a function over a set $\mathcal{B}_{\rm ad}$ where there is no vector structure at our disposal. It is thus reasonable to expect a higher level of difficulty than for other optimal control problems.

As mentioned above, the existence of a solution to Problem P3 is not clear at all. To simplify our arguments, let us introduce two non-empty open sets D_0 and D_1 , with

$$D_0 \subset\subset D_1 \subset\subset \Omega$$

and let us first assume that $\mathcal{B}_{\rm ad}$ is the family of the non-empty closed sets B with piecewise Lipschitz-continuous boundary that satisfy

$$\overline{D_0} \subset B \subset \overline{D_1}\,. \tag{45}$$

Also, assume that $|y_\infty|$ is small enough (depending on ν and Ω). Then, for each $B \in \mathcal{B}_{\rm ad}$, the state system (8) possesses exactly one solution (y, π) (the pressure π is unique up to an additive constant). Consequently, we can assign to B a *drag* $D(B) = T(B, y)$, given by (9).

In other words, in this case the function $B \mapsto D(B)$ is well-defined and Problem P3 reads:

To find $\hat{B} \in \mathcal{B}_{\rm ad}$ *such that* $D(\hat{B}) \leq D(B)$ *for all* $B \in \mathcal{B}_{\rm ad}$.

Let $\{B^n\}$ be a minimizing sequence. For each $n \geq 1$, let us denote by y^n the velocity field associated to B^n by (8). Then, it is clear that y^n is uniformly bounded in the H^1-norm. More precisely, the extensions-by-zero of y^n to the whole domain Ω, that we denote by $\widetilde{y}^n$, are uniformly bounded in $H^1(\Omega; \mathbb{R}^2)$. We can thus assume that $\widetilde{y}^n$ converges weakly in $H^1(\Omega; \mathbb{R}^2)$, strongly in $L^2(\Omega;\mathbb{R}^2)$ and a.e. to a function $\widetilde{y}^0$. This is a consequence of the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, see for instance [1].

On the other hand, since $\{B^n\}$ is a sequence of closed sets of Ω, we can also assume that B^n converges in the sense of the *Haussdorf distance* d_H to a closed set B^0. This is a consequence of the fact that the family of closed subsets of Ω is compact for d_H , see [7].

At this respect, recall that, when B and B' closed sets in $\mathbb{R}^2$, the Haussdorf distance $d_H(B,B')$ is given by

$$d_H(B,B') = \max\{\rho(B,B'), \rho(B',B)\},$$

where

$$\rho(B,B') = \sup_{x\in B} d(x,B') \quad \text{and} \quad d(x,B') = \inf_{x'\in B'} |x-x'| \quad \text{for all } B'$$

and a similar definition holds for $\rho(B',B)$.

The set B^0 satisfies (45). However, the uniform bound in the H^1 norm does not give enough regularity for B^0 and it is not clear whether the restriction of $\widetilde{y}^0$ to the limit set $\Omega \setminus B^0$ is, together with some π^0, the solution of (8) with B replaced by B^0.

We can overcome this difficulty by introducing a more restrictive family $\mathcal{B}_{\rm ad}$.

For instance, let us now assume that $\mathcal{B}_{\rm ad}$ is the family of the non-empty closed sets B satisfying (45) whose boundaries are *uniformly Lipschitz-continuous* with constant $L > 0$. By this we mean that the boundary ∂B of any $B \in \mathcal{B}_{\rm ad}$ can be written in the form

$$\partial B = \{\, x(\theta) : \theta \in [0,1] \,\}, \tag{46}$$

where the function $\theta \mapsto x(\theta)$ satisfies $x(0) = x(1)$ and is Lipschitz-continuous on $[0,1]$ with Lipschitz constant L. Obviously, $\mathcal{B}_{\rm ad}$ is non-empty if L is large enough.

It is clear that we can argue as before and find a limit set B^0 and a vector field $\widetilde{y}^0$, defined in Ω. In this particular case, the set B^0 belongs to $\mathcal{B}_{\rm ad}$, that is, its boundary is also of the form (46), see [6]. In view of this regularity property for B^0, it can also be proved that the restriction y^0 to $\Omega \setminus B^0$ is, together with an appropriate π^0, the solution of (8) with $B = B^0$.

QUESTION 10: *Why is this true?*

Unfortunately, this new definition of the admissible set $\mathcal{B}_{\rm ad}$ can be too restrictive.

Actually, this is a common fact for optimal design problems: either we choose the apparently natural definition of $\mathcal{B}_{\rm ad}$ (and then existence is not known) or we make it more restrictive (and then the problem can become unrealistic). For more details on these and other similar results, see [26, 27, 12].

We will now study the behavior of the function $B \mapsto D(B)$. Let $\hat{B}$ be a reference shape for the body (arbitrary in $\mathcal{B}_{\rm ad}$ but fixed). The body variations are described by a field $u = u(x)$ and we search for a formula of the kind

$$D(\hat{B}+u) = D(\hat{B}) + D'(\hat{B};u) + o(u), \tag{47}$$

where the modified fluid domain is

$$(\Omega \setminus \hat{B}) + u = \Omega \setminus (\hat{B} + u) = \{\, x \in \mathbb{R}^2 : x = (I + u)(\xi),\ \xi \in \Omega \setminus \hat{B} \,\}$$

and

$$o(u)\|u\|^{-1}_{W^{1,\infty}} \to 0 \quad \text{as} \quad \|u\|_{W^{1,\infty}} \to 0.$$

We are thus led to an analysis of the differentiability of the function $u \mapsto D(\hat{B} + u)$.

A lot of work has been made for the definition and computation of the variations with respect to a domain of functionals defined through the solutions to boundary value problems. The reader is referred to [29] and the references therein.

We will recall briefly a variant of a general method introduced by F. Murat and J. Simon in [21] and [22][1]. This is taken from [2]. Notice that some formal computations of the derivative were previously carried out by O. Pironneau in [25] (see also [27]), using "normal" variations.

We will choose fields $u \in W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$ such that $u = 0$ on Γ. This includes many interesting situations in which $\partial(\Omega \setminus (\hat{B} + u))$ possesses "corner" points. Furthermore, the equality $u = 0$ on Γ expresses the fact that the outer boundary limiting the fluid is fixed.

We will also assume that $\|u\|_{W^{1,\infty}} \le \eta$, with η being small enough to ensure that the boundary of $\Omega \setminus (\hat{B} + u)$ is Lipschitz-continuous and also that $\hat{B} + u$ is included in a fixed open set D_2 satisfying

$$\hat{B} \subset\subset D_2 \subset\subset \Omega$$

(such a constant $\eta > 0$ exists, see [2] for a proof).

For the sequel, we introduce

$$\mathcal{W} = \{\, u \in W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2) : \ \|u\|_{W^{1,\infty}} \le \eta, \quad u = 0 \text{ on } \partial\Omega \,\}.$$

Now, we choose g satisfying

$$\nabla \cdot g = 0, \quad g = y_\infty \text{ in a neighborhood of } \partial\Omega, \quad g = 0 \text{ in a neighborhood of } D_2$$

(such a function g always exists; see for instance [10]). If $u \in \mathcal{W}$, one has $g = 0$ in a neighborhood of $\partial\hat{B} + u$. After normalization of the pressure, the Navier-Stokes problem in $\Omega \setminus (\hat{B} + u)$ can be written as follows:

$$\begin{cases} -\nu\Delta y(u) + (y(u) \cdot \nabla)y(u) + \pi(u) = 0, \quad \nabla \cdot y(u) = 0 \quad \text{in } \Omega \setminus (\hat{B} + u), \\ y(u) - g \in H^1_0(\Omega \setminus (\hat{B} + u); \mathbb{R}^2), \\ \pi(u) \in L^2(\Omega \setminus (\hat{B} + u)), \quad \displaystyle\int_{\Omega\setminus\hat{B}} \pi(u) \circ (I + u)\, dx = 0. \end{cases} \tag{48}$$

The drag associated to $\hat{B} + u$ can be defined and is given by

$$D(\hat{B} + u) = 2\nu \int_{\Omega\setminus(\hat{B}+u)} |Dy(u)|^2\, dx, \tag{49}$$

[1] The general method in [21] and [22] cannot be directly applied to the Stokes and Navier-Stokes cases. This is due to the incompressibility condition.

where $Dy(u) = \frac{1}{2}(\nabla y(u) + \nabla y(u)^t)$.

Under these conditions, it is proved in [2] that the equality (47) is satisfied, with the first order term $D'(\hat{B}; u)$ given by

$$D'(\hat{B}; u) = 4\nu \int_{\Omega\setminus\hat{B}} Dy \cdot \left(D\dot{y}(u) - E(u, y) + \frac{1}{2}(\nabla \cdot u) Dy \right) dx.$$

Here, we have introduced the following notation:

(a) $(\dot{y}(u), \dot{\pi}(u))$ is the unique solution to the linear problem

$$\begin{cases} -\nu\Delta\dot{y}(u) + (y\cdot\nabla)\dot{y}(u) + (\dot{y}(u)\cdot\nabla)y + \dot{\pi}(u) = G(u, y, \pi), \quad \nabla\cdot\dot{y}(u) = 0 \quad \text{in } \Omega\setminus\hat{B}, \\ \dot{y}(u) \in H_0^1(\Omega\setminus\hat{B}; \mathbb{R}^2), \\ \dot{\pi}(u) \in L^2(\Omega\setminus\hat{B}), \quad \displaystyle\int_{\Omega\setminus\hat{B}} \dot{\pi}(u)\, dx = 0, \end{cases}$$

where

$$G(u, y, \pi) = -\nu\Delta((u\cdot\nabla)y) + \big(((u\cdot\nabla)y)\cdot\nabla\big)y + (y\cdot\nabla)((u\cdot\nabla)y) + \nabla(u\cdot\nabla\pi).$$

(b) $E(u, y)$ is the 2×2 tensor whose (i, j)-th component is given by

$$E_{ij}(u, y) = \frac{1}{2}\sum_k \left(\frac{\partial u_k}{\partial x_i}\frac{\partial y_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\frac{\partial y_i}{\partial x_k} \right).$$

(c) $y = y(0)$ and $\pi = \pi(0)$, i.e. (y, π) is the solution to (48) for $u = 0$.

It can also be proved that, if B and Ω are $W^{2,\infty}$ domains and $u \in W^{2,\infty}(\mathbb{R}^2; \mathbb{R}^2)$, then $y \in H^2(\Omega; \mathbb{R}^2)$, $\pi \in H^1(\Omega)$ and

$$D'(\hat{B}; u) = \int_{\partial\hat{B}} \left(\frac{\partial w}{\partial n} - \frac{\partial y}{\partial n} \right) \cdot \frac{\partial y}{\partial n} (u\cdot n)\, d\sigma, \tag{50}$$

with (w, q) being the unique solution to the "adjoint" problem

$$\begin{cases} -\nu\Delta w_i + \sum_j \partial_i y_j\, w_j - \sum_j y_j\, \partial_j w_i + \partial_i q = -2\nu\,\Delta y_i\,, \quad 1 \le i \le 2, \quad \nabla\cdot w = 0, \\ w \in H_0^1(\Omega\setminus\hat{B}; \mathbb{R}^2) \cap H^2(\Omega\setminus\hat{B}; \mathbb{R}^2), \\ q \in H^1(\Omega\setminus\hat{B}), \quad \displaystyle\int_{\Omega\setminus\hat{B}} q\, dx = 0, \end{cases} \tag{51}$$

Notice that, in order to compute the derivative of the drag in several directions u, it is interesting to use the identity (50). Indeed, it suffices to solve (8) and (51) only once. Then, to determine $D'(\hat{B}; u)$ for a given u, we will only have to compute one integral on $\partial\hat{B}$.

QUESTION 11: *Assume that $\mathcal{B}_{\rm ad}$ is the family of the non-empty closed sets B satisfying* (45) *whose boundaries are uniformly Lipschitz-continuous with Lipschitz constant L. How can* (50) *be used to produce a sequence $\{B^n\}$ "converging" to a solution to Problem* P3*?*

To end this Section, let us state another result from [2]:

THEOREM 4.1 *There exists $\alpha > 0$ such that, if $|y_\infty| \leq \alpha\nu$, then $u \mapsto D(\hat{B}+u)$ is a C^∞ mapping in the set $\mathcal{W}$.*

One can also obtain expressions for the derivatives of higher orders. This must be made with caution; indeed, $D''(\hat{B};\cdot,\cdot)$ (i.e. the second derivative at 0 of $u \mapsto D(\hat{B}+u)$) does not coincide with $(D'(\hat{B};\cdot)';\cdot)$ (i.e. the derivative at 0 of the mapping $u \mapsto D'(\hat{B}+u;\cdot)$), see [28].

5 Optimal control for a system modelling tumor growth

This Section deals with Problem P4. For simplicity, we will assume that the functions f, h, F and H are given by (11), where ρ, m, R and M are positive constants. We will also assume that the initial data in (10) satisfy:

$$c_0\,, \beta_0 \in L^\infty(\Omega) \cap H^1_0(\Omega), \quad c_0\,, \beta_0 \geq 0.$$

For each $v \in L^2(\omega \times (0,T))$ with $v \geq 0$, there exists at least one solution (c, β) to (10), with

$$c \in L^\infty(Q), \quad c_t\,, \frac{\partial c}{\partial x_i}\,, \frac{\partial^2 c}{\partial x_i \partial x_j} \in L^2(Q)$$

and the same properties for β.

QUESTION 12: *Why is this true? What about uniqueness?*

Then the following results can be proved:

THEOREM 5.1 *Assume that $\mathcal{V}_{\rm ad}$ is a non-empty closed convex set of $L^2(\omega)$ and all $v \in \mathcal{V}_{\rm ad}$ satisfy $v \geq 0$. Then Problem* P4 *possesses at least one solution.*

THEOREM 5.2 *Let the assumptions of theorem 5.1 be satisfied and let $\hat{u}$ be a solution to Problem* P4*. Then there exists $(\hat{c}, \hat{\beta})$ and $(\hat{p}, \hat{\eta})$ such that*

$$(52) \qquad \begin{cases} \hat{c}_t - \nabla\cdot(D(x)\nabla\hat{c}) = \rho\hat{c} - R\hat{c}\hat{\beta} & in \quad Q = \Omega\times(0,T), \\ \hat{\beta}_t - \mu\Delta\hat{\beta} = -m\hat{\beta} - M\hat{c}\hat{\beta} + v1_\omega & in \quad Q = \Omega\times(0,T), \\ \hat{c} = 0 & on \quad \Sigma = \partial\Omega\times(0,T), \\ \hat{\beta} = 0 & on \quad \Sigma = \partial\Omega\times(0,T), \\ \hat{c}(0) = c_0 & in \quad \Omega, \\ \hat{\beta}(0) = \beta_0 & in \quad \Omega, \end{cases}$$

$$
\left\{
\begin{array}{lll}
-\hat{p}_t - \nabla \cdot (D(x)\nabla \hat{p}) = \rho \hat{p} - R\hat{\beta}\hat{p} - M\hat{\beta}\hat{\eta} & \mathit{in} & Q = \Omega \times (0,T), \\
-\hat{\eta}_t - \mu\Delta\hat{\eta} = -m\hat{\eta} - R\hat{c}\hat{p} - M\hat{c}\hat{\eta} & \mathit{in} & Q = \Omega \times (0,T), \\
\hat{p} = 0 & \mathit{on} & \Sigma = \partial\Omega \times (0,T), \\
\hat{\eta} = 0 & \mathit{on} & \Sigma = \partial\Omega \times (0,T), \\
\hat{p}(T) = \hat{c}(T) & \mathit{in} & \Omega, \\
\hat{\eta}(0) = 0 & \mathit{in} & \Omega,
\end{array}
\right.
\tag{53}
$$

$$
\iint_{\omega\times(0,T)} (a\hat{p} + b\hat{u})(u - \hat{u})\, dx\, dt \geq 0 \quad \forall u \in \mathcal{V}_{\mathrm{ad}}.
\tag{54}
$$

For the proofs, the arguments are not too different from those in Section 2.

Again, it is common to say that $(\hat{p}, \hat{\eta})$ is the *adjoint state* associate to the optimal control $\hat{u}$. Also,

$$
\langle J'(u), v \rangle = \iint_{\omega\times(0,T)} (ap + bu)\, v\, dx\, dt \quad \forall v \in \mathcal{V}_{\mathrm{ad}}\,,
\tag{55}
$$

where (p, η) is the adjoint state associate to u, i.e. the solution to

$$
\left\{
\begin{array}{lll}
-p_t - \nabla \cdot (D(x)\nabla p) = \rho p - R\beta p - M\beta\eta & \text{in} & Q = \Omega \times (0,T), \\
-\eta_t - \mu\Delta\eta = -m\eta - Rcp - Mc\eta & \text{in} & Q = \Omega \times (0,T), \\
p = 0 & \text{on} & \Sigma = \partial\Omega \times (0,T), \\
\eta = 0 & \text{on} & \Sigma = \partial\Omega \times (0,T), \\
p(T) = c(T) & \text{in} & \Omega, \\
\eta(0) = 0 & \text{in} & \Omega.
\end{array}
\right.
$$

Once more, this provides very useful techniques to compute, for any control u, the associate $J'(u)$.

QUESTION 13: *Can the optimality system in theorem 5.2 be used to prove a uniqueness result for Problem* P4*?*

QUESTION 14: *Again, a "natural" iterative method for the computation of $\hat{u}$ is suggested by the optimality system in theorem 5.2. Which is this method? What can be said on the convergence of the iterates?*

QUESTION 15: *How can we apply gradient and conjugate gradient method to produce a sequence of controls that converge to an optimal control in he context of Problem* P4*?*

This optimal control problem has been solved numerically in [8]; more results will be given in a forthcoming paper.

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