Lecture 2: Controllability of the linear heat and wave PDEs

Abstract

This Lecture is devoted to the controllability of some systems governed by linear time-dependent PDEs. I will consider the heat and the wave equations. I will try to explain which is the meaning of controllability and which kind of controllability properties can be expected to be satisfied by each of these PDEs. The main related results, together with the main ideas in their proofs, will be recalled.

1 Introduction

Let us first make some very general considerations on the following abstract problem:

$$\begin{cases} y_t - Ay = Bv, \quad t \in (0,T), \\ y(0) = y_0, \end{cases} \tag{1}$$

where A and B are linear operators, $v = v(t)$ is the control and $y = y(t)$ is the state.

For fixed $T > 0$, we choose y^0 and y^1 in the space of states (the space where y "lives") and we try to answer the following question:

> *Can one find a control v such that the solution y associated to v and y^0 takes the value y^1 at $t = T$?*

This is an *exact controllability* problem. The control requirement $y(T) = y^1$ can be relaxed in various ways, leading to other notions of controllability.

Of course, the solvability of problems of this kind depends very much on the nature of the system under consideration; in particular, the following features may play a crucial role: time reversibility, regularity of the state, structure of the set of admissible controls, etc.

The controllability of partial differential equations has been the object of intensive research since more than 30 years. However, the subject is older than that. In 1978, D.L. Russell [36] made a rather complete survey of the most relevant results that were available in the literature at that time. In that paper, the author described a number of different tools that were developed to address controllability problems, often inspired and related to other subjects concerning partial differential equations: multipliers, moment problems, nonharmonic Fourier series, etc. More recently, J.-L. Lions introduced the so called *Hilbert Uniqueness Method* (H.U.M.; see [28, 29]). That was the starting point of a fruitful period for this subject.

It would be impossible to present here all the relevant results that have been proved in this area. I will thus only consider some model examples where the most relevant intrinsic difficulties of controllability analysis are found.

Several important related topics, like numerical computation and simulation in controllability problems, stabilizability, connections with finite dimensional controllability theory, etc. have been left out. However, some useful references for these issues have been included; see [6, 7, 19, 20, 21, 41].

2 Basic results for the linear heat equation

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain ($N \geq 1$), with boundary Γ of class C^2. Let ω be an open and non-empty subset of Ω. Let $T > 0$ and consider the linear controlled heat equation in the cylinder $Q = \Omega \times (0, T)$:

$$\left\{ \begin{array}{lll} y_t - \Delta y = v 1_\omega & \text{in} & Q, \\ y = 0 & \text{on} & \Sigma, \\ y(x,0) = y^0(x) & \text{in} & \Omega. \end{array} \right. \tag{2}$$

In (2), $\Sigma = \Gamma \times (0,T)$ is the lateral boundary of Q, 1_ω is the characteristic function of the set ω, $y = y(x,t)$ is the state and $v = v(x,t)$ is the control. Since v is multiplied by 1_ω, the action of the control is constrained to $\omega \times (0,T)$.

We assume that $y^0 \in L^2(\Omega)$ and $v \in L^2(\omega \times (0,T))$, so that (2) admits a unique solution

$$y \in C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)).$$

We will set $R(T; y^0) = \{ \, y(\cdot, T) : v \in L^2(Q) \, \}$. Then,

(a) System (2) is said to be *approximately controllable* (at time T) if $R(T; y^0)$ is dense in $L^2(\Omega)$ for all $y^0 \in L^2(\Omega)$.

(b) It is said to be *exactly controllable* if $R(T; y^0) = L^2(\Omega)$ for all $y^0 \in L^2(\Omega)$.

(c) Finally, it is said to be *null controllable* if $0 \in R(T; y^0)$ for all $y^0 \in L^2(\Omega)$.

It will be seen below that approximate and null controllability hold for every non-empty open non-empty set $\omega \subset \Omega$ and every $T > 0$.

On the other hand, it is clear that exact controllability cannot hold, except possibly in the case in which $\omega = \Omega$. Indeed, due to the regularizing effect of the heat equation, the solutions of (2) at time $t = T$ are smooth in $\Omega \setminus \overline{\omega}$. Therefore, if $\omega \neq \Omega$, $R(T; y^0)$ is strictly contained in $L^2(\Omega)$ for all $y^0 \in L^2(\Omega)$.

Our first remark is that null controllability implies that the whole range of the semigroup generated by the heat equation is reachable too. Let us make this statement more precise.

Let us denote by $S(t)$ the semigroup generated by the heat equation (2) without control, i.e. with $v = 0$. Then, if null controllability holds, it follows that for any $y^0 \in L^2(\Omega)$ and any $y^1 \in S(T)(L^2(\Omega))$ there exists $v \in L^2(\omega \times (0,T))$ such that the solution of (2) satisfies $y(x,T) \equiv y^1(x)$. In other words,

$$S(T)(L^2(\Omega)) \subset R(T; y^0) \quad \forall y^0 \in L^2(\Omega).$$

QUESTION 1: *Why is this true?*

The space $S(T)(L^2(\Omega))$ is dense in $L^2(\Omega)$. Therefore, null controllability implies approximate controllability. Observe however that the reachable states we obtain by this argument are smooth, due to the regularizing effect of the heat equation.

Observe that proving that null controllability implies approximate controllability requires the use of the density of $S(T)(L^2(\Omega))$ in $L^2(\Omega)$. In the case of the linear heat equation this is easy to check developing solutions in Fourier series. However, if the equation contains time or space-time dependent coefficients, this is true but not so immediate. In those cases, the density of the range of the "semigroup", can be reduced by duality to a backward uniqueness property in the spirit of J.-L. Lions and B. Malgrange [32].

Our first main result is the following:

THEOREM 2.1 *System* (2) *is approximately controllable for any non-empty open set* $\omega \subset \Omega$ *and any* $T > 0$.

PROOF: This is an easy consequence of Hahn-Banach theorem. For completeness, we will reproduce the argument here.

Let us fix ω and $T > 0$. Then, it is clear that (2) is approximately controllable if and only if $R(T; 0)$ is dense in $L^2(\Omega)$. But this is true if and only if any φ^0 in the orthogonal complement $R(T; 0)^\perp$ is necessarily zero.

Let $\varphi^0 \in L^2(\Omega)$ be given and assume that it belongs to $R(T; 0)^\perp$. Let us introduce the following system:

$$(3) \qquad \begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in} \quad Q, \\ \varphi = 0 & \text{on} \quad \Sigma, \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega. \end{cases}$$

Then, if $v \in L^2(\omega \times (0,T))$ is given and $y = y(x,t)$ is the associate state (the solution to (2) with $y^0 = 0$), we must have

$$\iint_{\omega \times (0,T)} \varphi v \, dx \, dt = \int_\Omega \varphi^0(x) y(x,T) \, dx = 0.$$

Consequently, approximate controllability holds if and only if the following uniqueness property is true:

If φ solves (3) *and $\varphi = 0$ in $\omega \times (0,T)$, then necessarily $\varphi \equiv 0$, i.e. $\varphi^0 = 0$.*

But this is a well known uniqueness property for the heat equation, a consequence of the fact that the solutions to (3) are analytic in space.

This proves that approximate controllability holds for (2). □

Following the variational approach in [31], we can also determine the way the "good" control can be constructed. First of all, observe that it is sufficient to consider the particular case $y^0 = 0$. Then, let us fix $y^1 \in L^2(\Omega)$ and $\varepsilon > 0$ and let us introduce the following functional on $L^2(\Omega)$:

$$J_\varepsilon(\varphi^0) = \frac{1}{2} \iint_{\omega \times (0,T)} |\varphi|^2 \, dx \, dt + \varepsilon \|\varphi^0\|_{L^2} - \int_\Omega \varphi^0 y^1 \, dx, \tag{4}$$

where, for each φ^0, we have denoted by φ the solution to the corresponding problem (3).

The functional J_ε is continuous and strictly convex in $L^2(\Omega)$. On the other hand, in view of the unique continuation property above, it can be proved that

$$\lim_{\|\varphi^0\|_{L^2} \to \infty} \frac{J_\varepsilon(\varphi^0)}{\|\varphi^0\|_{L^2}} \geq \varepsilon. \tag{5}$$

Hence, J_ε admits a unique minimizer $\widehat{\varphi}^0$ in $L^2(\Omega)$. The control $u = \widehat{\varphi}$, where $\widehat{\varphi}$ solves (3) with $\widehat{\varphi}^0$ as final data is such that the solution of (2) (with $y^0 = 0$) satisfies

$$\|y(\cdot, T) - y^1\|_{L^2} \leq \varepsilon. \tag{6}$$

QUESTION 2: *Why is* (5) *true? How can we prove* (6) *for this control?*

With a slight change in the definition of J_ε, we are also able to build *bang-bang* controls. Indeed, it suffices to consider the new functional

$$\tilde{J}_\varepsilon(\varphi^0) = \frac{1}{2} \left(\iint_{\omega \times (0,T)} |\varphi| \, dx \, dt \right)^2 + \varepsilon \|\varphi^0\|_{L^2} - \int_\Omega \varphi^0 y^1 \, dx. \tag{7}$$

Then $\tilde{J}_\varepsilon$ is continuous and convex in $L^2(\Omega)$ and satisfies the coercivity property (5) too.

Let $\widehat{\varphi}^0$ be a minimizer of $\tilde{J}_\varepsilon$ in $L^2(\Omega)$ and let $\widehat{\varphi}$ be the corresponding solution of (3). Let us set

$$u = \iint_{\omega \times (0,T)} |\widehat{\varphi}| \, dx \, dt \;\; \mathrm{sgn}(\widehat{\varphi}), \tag{8}$$

where sgn is the multivalued sign function: $\operatorname{sgn}(s) = 1$ if $s > 0$, $\operatorname{sgn}(0) = [-1, 1]$ and $\operatorname{sgn}(s) = -1$ when $s < 0$. Again, the control u given by (8) is such that the solution to (2) with zero initial data satisfies (6).

Due to the regularizing effect of the heat equation, the zero set of nontrivial solutions of (3) is of zero $(n+1)-$dimensional Lebesgue measure. Thus, the control u in (8) belongs to $L^\infty(Q)$ and is of *bang-bang* form, i.e. $u = \pm\lambda$ a.e. in Q, where

$$\lambda = \iint_{\omega\times(0,T)} |\widehat{\varphi}| \, dx \, dt.$$

In fact, it can be proved that u minimizes the L^∞-norm in the set of all controls such that (6) is satisfied (we refer to [9] for a proof of this assertion).

Following [40], we can improve the previous argument and show that, for any ω, any $T > 0$ and any finite-dimensional subspace $E \subset L^2(\Omega)$, (2) is E-approximate controllable. This means that, for arbitrary $y^0, y^1 \in L^2(\Omega)$ and any $\varepsilon > 0$, there exists a control $v \in L^2(Q)$ such that the corresponding solution to (2) satisfies:

$$\|y(\cdot,T) - y^1\|_{L^2} \leq \varepsilon, \quad \pi_E(y(\cdot,T)) = \pi_E(y^1). \tag{9}$$

Here, $\pi_E : L^2(\Omega) \mapsto E$ stands for the usual orthogonal projector on E.

Indeed, it suffices to modify J_ε (or $\tilde{J}_\varepsilon$) and use instead the functional J_ε^E , where

$$J_\varepsilon^E(\varphi^0) = \frac{1}{2}\iint_{\omega\times(0,T)} |\varphi|^2 \, dx \, dt + \varepsilon\|(I - \pi_E)\varphi^0\|_{L^2} - \int_\Omega \varphi^0 y^1 \, dx. \tag{10}$$

As before, J_ε^E is continuous, strictly convex and coercive in $L^2(\Omega)$. Once again, let us denote by $\widehat{\varphi}^0$ its unique minimizer and let us set $u = \widehat{\varphi}$. Then the associate state y satisfies (9).

QUESTION 3: *Which is in this case the argument leading to* (9)*? Is the hypothesis "E is finite-dimensional" essential?*

Let us now analyze the null controllability of (2).

The null controllability property for system (2), together with a L^2- estimate of the control, is equivalent to the following observability inequality for the adjoint system (3):

$$\|\varphi(\cdot,0)\|_{L^2}^2 \leq C\iint_{\omega\times(0,T)} |\varphi|^2 \, dx \, dt \quad \forall\varphi^0 \in L^2(\Omega). \tag{11}$$

QUESTION 4: *Which is the proof of this assertion?*

Due to the regularizing effect of the heat equation, the norm in the left hand side of (11) is very weak. However, the irreversibility of the system makes (11) difficult to prove. For instance, multiplier methods do not apply in this context.

Thus, we see that the approximate (resp. null) controllability of (2) is related to the unique continuation property (resp. the observability) of (3).

Historically, it seems that the first null controllability results established for the heat equation involved boundary controls. They were given in [36] in the one-dimensional case, using moment problems and classical results on the linear independence in $L^2(0,T)$ of families of real exponentials. Later, in [37], a deep general result was proved. Roughly speaking, the following was shown:

> *If the wave equation is controllable for some $T > 0$ with controls supported in ω, then the heat equation* (2) *is null controllable for every $T > 0$ with controls supported in ω.*

In view of the controllability results in Section 3, according to this principle, it follows that the heat equation (2) is null controllable for all $T > 0$ provided ω satisfies a specific geometric control condition. However, this geometric condition does not seem to be natural in the context of the heat equation and, therefore, this result is not completely satisfactory.

More recently, the following was shown by G. Lebeau and L. Robbiano [26]:

THEOREM 2.2 *System* (2) *is null controllable for any non-empty open set $\omega \subset \Omega$ and any $T > 0$.*

SKETCH OF THE PROOF: A slightly simplified proof of this result was given in [27]. The main ingredient is an observability estimate for the eigenfunctions of the Dirichlet-Laplace operator:

$$\left\{ \begin{array}{lll} -\Delta w_j = \lambda_j w_j & \text{in} & \Omega, \\ w_j = 0 & \text{on} & \partial\Omega. \end{array} \right. \tag{12}$$

Recall that the eigenvalues $\{\lambda_j\}$ form a nondecreasing sequence of positive numbers such that $\lambda_j \to \infty$ as $j \to \infty$ and the associated eigenfunctions $\{w_j\}$ form an orthonormal basis in $L^2(\Omega)$.

The following holds:

> *Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. For any open set $\omega \subset \Omega$, there exist positive constants $C_1, C_2 > 0$ such that*
>
> $$\sum_{\lambda_j \le \mu} |a_j|^2 \le C_1 e^{C_2\sqrt{\mu}} \int_\omega \Big| \sum_{\lambda_j \le \mu} a_j w_j(x) \Big|^2 dx \tag{13}$$
>
> *whenever $\{a_j\} \in \ell^2$ and $\mu > 0$.*

This result was implicitly used in [26] and is proved in [27]. A consequence is that the observability inequality (11) holds for the solutions of (3) with initial data in

$$E_\mu = \operatorname{span}\{ \varphi_j : \lambda_j \le \mu \},$$

the constant being of the order of $\exp\left(C\sqrt{\mu}\right)$.

This shows that the projection on E_μ of the solution of (3) can be controlled to zero with a control of size $\exp\left(C\sqrt{\mu}\right)$. Thus, when controlling the frequencies $\lambda_j \le \mu$, one increases the L^2-norm of the high frequencies $\lambda_j > \mu$ by a multiplicative factor of the order of $\exp\left(C\sqrt{\mu}\right)$.

However, it was observed in [26] that any solution of the heat equation (2) with $v = 0$ such that the projection on E_μ of $y(\cdot, 0)$ vanishes decays in $L^2(\Omega)$ at a rate of the order of $\exp(-\mu t)$.

Consequently, if we divide the time interval $[0, T]$ in two parts $[0, T/2]$ and $[T/2, T]$, we control to zero the frequencies $\lambda_j \le \mu$ in the interval $[0, T/2]$ and then allow the equation to evolve without control in the interval $[T/2, T]$, it follows that, at time $t = T$, the projection of the solution y over E_μ vanishes and the norm of the high frequencies does not exceed the norm of the initial data.

This argument allows to control to zero the projection over E_μ for any $\mu > 0$, but not the whole solution. To do that, an iterative argument is needed. Thus, we decompose the interval $[0, T)$ in disjoint subintervals of the form $[T_j, T_{j+1})$ for $j \in \mathbb{N}$, with a suitable choice of the sequence $\{T_j\}$. In each interval $[T_j, T_{j+1}]$, we control to zero the frequencies $\lambda_k \le 2j$. By letting $j \to \infty$, we obtain a control $v \in L^2(Q)$ such that the solution of (2) satisfies

$$y(x, T) \equiv 0. \tag{14}$$

Once (11) is known to hold, one can obtain the control with minimal L^2-norm satisfying (14). It suffices to minimize the functional

$$J(\varphi^0) = \frac{1}{2} \iint_{\omega \times (0,T)} |\varphi|^2 \, dx \, dt + \int_\Omega \varphi(x, 0) y^0(x) \, dx$$

over the Hilbert space

$$H = \{\, \varphi^0 : \text{ the solution } \varphi \text{ of (3) satisfies } \iint_{\omega \times (0,T)} |\varphi|^2 \, dx \, dt < \infty \,\}.$$

This ends the proof. □

As a consequence of this theorem, we also have the null boundary controllability of the heat equation, with controls in an arbitrarily small open subset of the boundary. See [26] for more details.

QUESTION 5: *Why does theorem 2.2 imply null boundary controllability?*

The previous controllability results also hold for linear parabolic equations with lower order terms depending on time and space.

For instance, the following system can be considered:

$$\begin{cases} y_t - \Delta y + a(x, t) y = v 1_\omega & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(x, 0) = y^0(x) & \text{in} \quad \Omega. \end{cases} \tag{15}$$

Here, we assume that $a \in L^\infty(Q)$. In this case, the adjoint system is

$$
\left\{
\begin{array}{lll}
-\varphi_t - \Delta\varphi + a(x,t)\varphi = 0 & \text{in} & Q, \\
\varphi = 0 & \text{on} & \Sigma, \\
\varphi(x,T) = \varphi^0(x) & \text{in} & \Omega.
\end{array}
\right. \tag{16}
$$

Again, the null controllability of (15), together with a L^2- estimate of the control, is equivalent to an observability inequality. Hence, in order to obtain a null controllability result for (15), what we have to do is to prove the estimate (11) for the solutions to (16).

The controllability properties of systems of this kind have been analyzed by several authors. Among them, let us mention the work of A.V. Fursikov and O.Yu. Imanuvilov (for instance, see [5],[12],[13]–[15] and [23]; more complicate linear heat equations involving first-order terms of the form $B(x,t)\cdot\nabla y$ have recently been considered in [24]). Their approach to the controllability problem is different and more general than the previous one and relies on appropriate (global) Carleman inequalities.

A general global Carleman inequality is an estimate of the form

$$
\iint_{\Omega\times(0,T)} \rho^{-2}\,|\varphi|^2\,dx\,dt \le C \iint_{\omega\times(0,T)} \rho^{-2}\,|\varphi|^2\,dx\,dt, \tag{17}
$$

where $\rho = \rho(x,t)$ is continuous, strictly positive and bounded from below. For an appropriate ρ that depends on Ω, ω, T and $\|a\|_{L^\infty(Q)}$, it is possible to deduce (17) and, consequently, also estimates of the form

$$
\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2\,dx\,dt \le C \iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt. \tag{18}
$$

This, together with the properties of the solutions of (16), leads to (11) and, therefore, implies the null controllability property for (15); see also [24, 11, 8] for some improved estimates.

QUESTION 6: *How can* (11) *be proved from* (18)*?*

Thus, at present we can affirm that, as in the case of the standard heat equation, (15) is both approximately and null controllable for any ω and any $T > 0$. Once more, null controllability implies approximate controllability for (15); this has been shown in [11].

An interesting question analyzed in [11] deals with explicit estimates of the *cost* in $L^2(Q)$ of the approximate, E-approximate (E is a finite-dimensional space) and null controllability of (15).

For instance, let us recall the results concerning approximate and null controllability. In the remainder of this Section, it will be assumed that C is a generic positive constant that only depends on Ω and ω.

Let us consider the linear state equation (15), where $a \in L^\infty(Q)$. For each $y^0 \in L^2(\Omega)$, $y^1 \in L^2(\Omega)$ and $\varepsilon > 0$, let us introduce the corresponding *set of admissible controls*

$$
\mathcal{U}_{\text{ad}}(y^0, y^1; \varepsilon) := \{\, v \in L^2(Q) : \text{ the solution of (15) satisfies (6)} \,\} \tag{19}
$$

and the following quantity, which measures the *cost of approximate controllability* or, more precisely, the cost of achieving (6):

$$\mathcal{C}(y^0, y^1; \varepsilon) := \inf_{v \in \mathcal{U}_{ad}(y^0, y^1; \varepsilon)} \|v\|_{L^2(Q)} \,. \tag{20}$$

Then, the question is: can we obtain "explicit" upper bounds for $\mathcal{C}(y^0, y^1; \varepsilon)$?

Taking into account that system (15) is linear, one can assume, without loss of generality, that $y^0 = 0$. Indeed,

$$\mathcal{C}(y^0, y^1; \varepsilon) = \mathcal{C}(0, z^1; \varepsilon) \,, \tag{21}$$

where $z^1 = y^1 - z(\cdot, T)$ and z is the solution of (15) with $v \equiv 0$.

Let us denote by $\|\cdot\|_\infty$ the usual norm in $L^\infty(Q)$. Then the following is satisfied:

THEOREM 2.3 *For any $y^1 \in H^2(\Omega) \cap H_0^1(\Omega)$, $\varepsilon > 0$, $T > 0$ and $a \in L^\infty(Q)$, one has:*

$$\mathcal{C}(0, y^1; \varepsilon) \le \exp\left[C\left[1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3} + \frac{\|a\|_\infty \|y^1\|_{L^2} + \|\Delta y^1\|_{L^2}}{\varepsilon}\right]\right] \|y^1\|_{L^2} \,. \tag{22}$$

Notice that (22) is only of interest when

$$\frac{\|\Delta y^1\|_{L^2}}{\lambda_1} > \varepsilon \,,$$

with λ_1 being the first eigenvalue of the dirichlet Laplacian $-\Delta$. Otherwise, we would have $\|y^1\|_{L^2} \le \varepsilon$ and then, taking $v \equiv 0$ in (15) for $y^0 = 0$, we would trivially obtain $y \equiv 0$ and

$$\|y(\cdot, T) - y^1\|_{L^2} \le \varepsilon \,.$$

In other words,

$$\mathcal{C}(0, y^1; \varepsilon) = 0 \ \text{ if } \ \frac{\|\Delta y^1\|_{L^2}}{\lambda_1} \le \varepsilon \,.$$

Furthermore, if instead of assuming $y^1 \in H^2(\Omega) \cap H_0^1(\Omega) = D(-\Delta)$ we assume that $y^1 \in D((-\Delta)^{\gamma/2})$ with $0 < \gamma \le 2$, other estimates similar to (22) can be established. See [11] for the details.

For the proof of (22), we first have to obtain sharp bounds on the cost of controlling to zero. Recall that (16) is the adjoint system of (15). Then we have the following explicit observability estimate:

LEMMA 2.1 *For any solution of* (16) *and for any $a \in L^\infty(Q)$, one has*

$$\|\varphi(\cdot, 0)\|_{L^2}^2 \le \exp\left(C\left(1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3}\right)\right) \iint_{\omega \times (0,T)} |\varphi|^2 \, dx \, dt \,. \tag{23}$$

The proof of (23) relies on global Carleman inequalities as in [15], but paying special attention to the constants arising in the integrations by parts. Once (23) is known, (22) can be proved easily.

QUESTION 7: *How can* (22) *be proved from* (23)*?*

As we have already seen, (23) implies the null controllability of (15). But it also provides an estimate for the associated cost $\mathcal{C}(y^0, 0)$. More precisely, one has:

THEOREM 2.4 *For each $y^0 \in L^2(\Omega)$, the set $\mathcal{U}_{\rm ad}(y^0, 0)$ is non-empty. Furthermore, one has*

$$\mathcal{C}(y^0, 0) \le \exp\left(C\left(1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3}\right)\right)\|y^0\|_{L^2}\,. \tag{24}$$

QUESTION 8: *How can* (24) *be proved from* (23)*?*

In the particular case in which $a \equiv \text{Const}$, (22) can be improved. More precisely, we can obtain a bound of the cost of approximate controllability of the order of $\exp(1/\sqrt{\varepsilon})$. Furthermore, it can be proved that this estimate is optimal in an appropriate sense; see [11] for the details.

REMARK 2.1 We can be more explicit on the way the constants C in (22) and (24) depend on Ω and ω: there exist "universal" constants $C_0 > 0$ and $m \ge 1$ such that C can be taken of the form

$$C = \exp\left(C_0\|\psi\|_{C^2}^m\right),$$

where $\psi \in C^2(\overline{\Omega})$ is any function satisfying $\psi > 0$ in Ω, $\psi = 0$ on $\partial\Omega$ and $\nabla\psi \ne 0$ in $\overline{\Omega} \setminus \omega$. All this is a consequence of the particular form that must have ρ in order to ensure (17); see [11] for more details. □

The results of this Section can be extended to more general equations of the form

$$\begin{cases} y_t - \Delta y + \nabla\cdot(yB(x,t)) + a(x,t)y = v1_\omega & \text{in} \quad Q,\\ y = 0 & \text{on} \quad \Sigma,\\ y(x,0) = y^0(x) & \text{in} \quad \Omega, \end{cases} \tag{25}$$

where $a \in L^\infty(Q)$ and $B \in L^\infty(Q;\mathbb{R}^N)$.

To do that, it is sufficient to obtain suitable observability estimates for the solutions of adjoint systems of the form

$$\begin{cases} -\varphi_t - \Delta\varphi - B(x,t)\cdot\nabla\varphi + a(x,t)\varphi = 0 & \text{in} \quad Q,\\ \varphi = 0 & \text{on} \quad \Sigma,\\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega. \end{cases} \tag{26}$$

More precisely, we can deduce that

$$\|\varphi(\cdot,0)\|_{L^2}^2 \le \exp\left(C\left(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3}+T^2\|B\|_\infty^2\right)\right)\iint_{\omega\times(0,T)}|\varphi|^2\,dx\,dt \tag{27}$$

for any solution of (26) and for all $a\in L^\infty(Q)$, $B\in L^\infty(Q;\mathbb{R}^N)$.

Then, arguments similar to those above lead to an estimate of the cost of approximate controllability in the case of (26).

The situation is more complicate when the state equation is of the form

$$\begin{cases} y_t-\Delta y+B(x,t)\cdot\nabla y+a(x,t)y=0 & \text{in} \quad Q\\ y=0 & \text{on} \quad \Sigma\\ y(x,0)=y^0(x) & \text{in} \quad \Omega\,. \end{cases} \tag{28}$$

Indeed, if B is only assumed to be in $L^\infty(Q;\mathbb{R}^N)$, the adjoint systems take the form

$$\begin{cases} -\varphi_t-\Delta\varphi-\nabla\cdot(\varphi B(x,t))+a(x,t)\varphi=0 & \text{in} \quad Q\\ \varphi=0 & \text{on} \quad \Sigma,\\ \varphi(x,T)=\varphi^0(x) & \text{in} \quad \Omega \end{cases} \tag{29}$$

and, therefore, the usual Carleman inequalities do not suffice. These questions have been considered and solved in [8], using some ideas from [24]. We omit the details.

To end this Section, let us make some comments on the convergence rate of algorithms devised to construct "good" controls.

It is rather natural to build approximate controls by *penalizing* a suitable optimal control problem. This has been done systematically, for instance, in the works by R. Glowinski [17] and R. Glowinski et al. [18]. This method has also been used to prove the approximate controllability for some linear and semilinear heat equations in [30] and [10], respectively.

Let us briefly describe the procedure in the case of the linear heat equation. First of all, without loss of generality, we set $y^0=0$. Given $y^1\in L^2(\Omega)$, we introduce the functional

$$F_k(v)=\frac{1}{2}\iint_{\omega\times(0,T)}|v|^2\,dx\,dt+\frac{k}{2}\|y(\cdot,T)-y^1\|_{L^2}^2\,, \tag{30}$$

which is well defined in $L^2(\omega\times(0,T))$ for all $k>0$, where y is the solution of (2) with $y^0=0$.

It was proved in [30] that F_k has a unique minimizer $v_k\in L^2(\omega\times(0,T))$ for all $k>0$ and that the associated states y_k satisfy

$$y_k(\cdot,T)\to y^1 \text{ in } L^2(\Omega) \text{ as } k\to\infty. \tag{31}$$

In view of (31), to compute a control v satisfying (6), it is sufficient to take $v=v_k$ for a sufficiently large $k=k(\varepsilon)$.

Using the results above, it is easy to get explicit estimates of the rate of convergence in (31) (we refer to [11] for the details of the proof):

THEOREM 2.5 *In the previous conditions, there exists $C > 0$ such that*

$$\|y_k(\cdot, T) - y^1\| \leq \frac{C}{\log k} \tag{32}$$

and

$$\|v_k\|_{L^2(Q)} \leq \frac{C\sqrt{k}}{\log k} \tag{33}$$

as $k \to \infty$.

QUESTION 9: *How can* (32) *and* (33) *be proved?*

Notice that (32) and (33) provide logarithmic (and therefore very slow) convergence rates. This fact agrees with the extremely high cost (exponentially depending on $1/\varepsilon$) of approximate controllability.

The methods of this Section can also be applied to obtain estimates on the cost of controllability when the control acts on an open subset of the boundary of Ω.

3 Basic results for the linear wave equation

Let us now consider the linear controlled wave equation

$$\begin{cases} y_{tt} - \Delta y = v 1_\omega & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x), \quad y_t(x,0) = y^1(x) & \text{in} \quad \Omega. \end{cases} \tag{34}$$

In (34), we have used the same notation as in Section 2. Again, $y = y(x,t)$ is the state and $v = v(x,t)$ is the control. For any $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $v \in L^2(\omega \times (0,T))$, (34) possesses exactly one solution $y \in C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$.

Roughly speaking, the *controllability* problem for (34) consists on *describing the set of reachable final states*

$$R(T; y^0, y^1) := \{\, (y(\cdot, T), y_t(\cdot, T)) : v \in L^2(\omega \times (0,T)) \,\}.$$

As in the case of the heat equation, we may distinguish several degrees of controllability:

(a) *Approximate controllability:* System (34) is said to be approximately controllable at time T if $R(T; y^0, y^1)$ is dense in $H_0^1(\Omega) \times L^2(\Omega)$ for every $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

(b) *Exact controllability:* System (34) is said to be exactly controllable at time T if $R(T; y^0, y^1) = H_0^1(\Omega) \times L^2(\Omega)$ for every $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

(c) *Null controllability:* System (34) is said to be null controllable at time T if $(0,0) \in R(T;(y^0,y^1))$ for every $(y^0,y^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

The previous controllability properties can also be formulated in other function spaces in which the wave equation is well posed.

Since we are now dealing with solutions of the wave equation, for any of these properties to hold, the control time T has to be sufficiently large due to the finite speed of propagation. On the other hand, since (34) is linear and reversible in time, null and exact controllability are equivalent notions. As we have seen, the situation is completely different in the case of the heat equation.

QUESTION 10: *Why do we need large T for any kind of controllability of the wave equation? Why are null controllability and exact controllability equivalent properties?*

Clearly, every exactly controllable system is approximately controllable too. However, (34) may be approximately but not exactly controllable.

Let us now briefly discuss the *approximate controllability problem* for the wave equation.

Again, it is easy to see that approximate controllability is equivalent to a specific *unique continuation property*. More precisely, let us introduce the *adjoint system*

$$\left\{\begin{array}{lll} \varphi_{tt} - \Delta\varphi = 0 & \text{in} & Q, \\ \varphi = 0 & \text{on} & \Sigma, \\ \varphi(x,T) = \varphi^0(x), \quad \varphi_t(x,T) = \varphi^1(x) & \text{in} & \Omega. \end{array}\right. \tag{35}$$

Then, (34) is approximately controllable with controls that depend continously on the data if and only if the following unique continuation property is fulfilled:

If φ solves (35) *and $\varphi = 0$ in $\omega \times (0,T)$, then necessarily $\varphi \equiv 0$, i.e. $(\varphi^0,\varphi^1) = (0,0)$.*

In fact, that the previous uniqueness property implies approximate controllability can be checked at least in two ways:

(a) Applying the Hahn-Banach theorem; see [29].

(b) Using the variational approach developed in [31].

Both approaches have been considered in the context of the heat equation. They will not be revisited here, for reasons of space.

QUESTION 11: *Which are the detailed arguments?*

In view of a well known consequence of *Holmgren's uniqueness theorem*, it can be easily seen that, for any non-empty open set $\omega \subset \Omega$, the previous unique continuation property holds if T

is large enough (depending on Ω and ω). We refer to Chapter 1 in [29] and [4] for a discussion on this problem.

Therefore, the following result holds:

THEOREM 3.1 *Let $\omega \subset \Omega$ be a non-empty open set. There exists $T_1 > 0$, only depending on Ω and ω, such that, for any $T > T_1$, the linear system* (34) *is approximately controllable at time T.*

When approximate controllability holds, the following (apparently stronger) property is also satisfied:

> *Let E be a finite dimensional subspace of $H_0^1(\Omega) \times L^2(\Omega)$ and let us denote by π_E the corresponding orthogonal projector. Then, for any $(y^0, y^1), (z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and any $\varepsilon > 0$, there exists $v \in L^2(Q)$ such that the solution of* (34) *satisfies*
>
> $$\text{(36)} \qquad \begin{cases} \|(y(\cdot, T) - z^0, y_t(\cdot, T) - z^1)\|_{H_0^1 \times L^2} \le \varepsilon, \\ \pi_E(y(\cdot, T), y_t(\cdot, T)) = \pi_E(z^0, z^1). \end{cases}$$

In other words, if $T > 0$ is large enough to ensure approximate controllability, for any finite dimensional subspace $E \subset H_0^1(\Omega) \times L^2(\Omega)$, we also have E-approximate controllability.

QUESTION 12: *Why does approximate controllability imply E-approximate controllability for any finite-dimensional space $E \subset H_0^1(\Omega) \times L^2(\Omega)$?*

The previous results hold for wave equations with analytic coefficients too. However, the problem is not completely solved in the frame of the wave equation with lower order potentials $a \in L^\infty(Q)$ of the form

$$y_{tt} - \Delta y + a(x, t) y = v 1_\omega \text{ in } Q.$$

We refer to [1],[38] and [35] for some deep results in this direction.

Let us now consider the *exact controllability problem.*

It was shown by J.L. Lions in [29] using the so called H.U.M. that exact controllability holds (with controls $v \in L^2(Q)$) if and only if

$$\text{(37)} \qquad \|(\varphi(\cdot, 0), \varphi_t(\cdot, 0))\|^2_{L^2 \times H^{-1}} \le C \iint_{\omega \times (0,T)} |\varphi|^2 \, dx \, dt$$

for any solution φ to the adjoint system (35).

This is an observability inequality, playing in this context the role played by (11) in Section 2. It provides an estimate of the *total energy* of the solution (35) by means of a measurement in the control region $\omega \times (0, T)$.

Notice that the energy

$$E(t) = \|(\varphi(\cdot, t), \varphi_t(\cdot, t))\|^2_{L^2 \times H^{-1}}$$

of any solution to (35) is conserved. Thus, (37) is equivalent to the so called *inverse inequality*

$$\|(\varphi^0, \varphi^1)\|^2_{L^2 \times H^{-1}} \leq C \iint_{\omega \times (0,T)} |\varphi|^2 \, dx \, dt. \tag{38}$$

QUESTION 13: *Why is* (37), *i.e.* (38) *equivalent to the exact controlability of* (34)*?*

When (37) holds, one can minimize the functional W, with

$$W(\varphi^0, \varphi^1) = \frac{1}{2} \iint_{\omega \times (0,T)} |\varphi|^2 \, dx \, dt + \langle (\varphi(\cdot, 0), \varphi_t(\cdot, 0)), (y^1, -y^0) \rangle, \tag{39}$$

in the space $L^2(\Omega) \times H^{-1}(\Omega)$. Indeed, the following result is easy to prove:

LEMMA 3.1 *Assume that* (37) *holds and* $(y^0, y^1) \in H^1_0(\Omega) \times L^2(\Omega)$ *is given. Then* W *possesses a unique minimizer* $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ *in* $L^2(\Omega) \times H^{-1}(\Omega)$*. The control* $v = \widehat{\varphi} 1_\omega$ *, where* $\widehat{\varphi}$ *is the solution to* (35) *corresponding to the final data* $(\widehat{\varphi}^0, \widehat{\varphi}^1)$*, is such that the associated state satisfies*

$$y(x,T) \equiv y_t(x,T) \equiv 0. \tag{40}$$

QUESTION 14: *How can lemma 3.1 be proved?*

As a consequence, the exact controllability problem is reduced to the analysis of the inequality (38). Let us now indicate what is known about this inequality:

• Using multiplier techniques in the spirit of C. Morawetz, L.F. Ho proved in [22] that, if one considers subsets of Γ of the form

$$\Gamma(x^0) = \{ \, x \in \Gamma : (x - x^0) \cdot n(x) > 0 \, \}$$

for some $x^0 \in \mathbb{R}^N$ (by $n(x)$ we denote the outward unit normal to Ω at $x \in \Gamma$) and $T > 0$ is large enough, the following boundary observability inequality holds:

$$\|(\varphi(\cdot, 0), \varphi_t(\cdot, 0))\|^2_{H^1_0 \times L^2} \leq C \iint_{\Gamma(x^0) \times (0,T)} \left| \frac{\partial \varphi}{\partial n} \right|^2 \, d\Gamma \, dt \tag{41}$$

for every couple $(\varphi^0, \varphi^1) \in H^1_0(\Omega) \times L^2(\Omega)$.

This is the observability inequality that is required to solve a boundary controllability problem similar to the one we are considering here.

Later on, (41) was proved in [28],[29] for any

$$T > T(x^0) = 2\|x - x^0\|_{L^\infty} \, . \tag{42}$$

In fact, this is the optimal observability time that one may obtain by means of multipliers.

Proceeding as in Vol. 1 of [29], one can easily prove that (41) implies (37) when ω is a neighborhood of $\Gamma(x^0)$ in Ω and $T > T(x^0)$. Consequently, the following result holds:

THEOREM 3.2 *Assume that $x^0 \in \mathbb{R}^N$, ω is a neighborhood of $\Gamma(x^0)$ in Ω and* (42) *is satisfied. Then* (34) *is exactly controllable at time T.*

More recently, A. Osses has introduced in [33] a new multiplier which is basically a rotation of the one in [29]. In this way, he proved that the class of subsets of the boundary for which observability holds is considerably larger.

• C. Bardos, G. Lebeau and J. Rauch [2] proved that, in the class of C^∞ domains, the observability inequality (37) holds if and only if the couple (ω, T) satisfies the following *geometric control condition* in Ω:

> *Every ray of geometric optics that propagates in Ω and is reflected on its boundary Γ enters ω at a time $t < T$.*

This result was proved with *microlocal analysis techniques.* Recently, the microlocal approach has been greatly simplified by N. Burq [3] by using the microlocal defect measures introduced by P. Gerard [16]. In [3], the geometric control condition was shown to be sufficient for exact controllability for domains Ω of class C^3 and equations with C^2 coefficients.

Therefore, one has:

THEOREM 3.3 *Let Ω be of class C^3, let $\omega \subset \Omega$ be a non-empty open set and let us assume that the couple (ω, T) satisfies the previous geometric condition. Then* (34) *is exactly controllable at time T.*

• Let us finally indicate that other methods have also been developed to address controllability problems for wave equations: Moment problems, the use of fundamental solutions, controllability via stabilization, Carleman estimates, etc. We will not present them here; for more details, we refer to the survey paper by D.L. Russell [36] and also to the works of J.-P. Puel [34] and X. Zhang [39].

As in the case of the heat equation, it is also natural to study the cost of the approximate of the wave equation or, in other words, the minimal size of a control needed to reach the ε-neighborhood of a final state which is not exactly reachable. The same can be said in the context of null controllability. These questions were considered by G. Lebeau in [25], with techniques which are not the same we used in Section 2.

References

[1] S. Alinhac, Non unicité du problème de Cauchy, Annals of Mathematics, **117** (1983), 77–108.

[2] C. Bardos, G. Lebeau and J. Rauch, Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, SIAM J. Cont. Optim., **30** (1992), 1024–1065.

[3] N. Burq, Contrôle de l'équation des ondes dans des ouverts peu réguliers, Asymptotic Analysis, **14** (1997), 157–191.

[4] T. Cazenave, On the propagation of confined waves along the geodesics, J. Math. Anal. Appl., **146** (1990), 591–603.

[5] D. Chae, O.Yu. Imanuvilov and S.M. Kim, Exact controllability for semilinear parabolic equations with Neumann boundary conditions, J. Dynamical and Control Systems, **2** (1996), 449–483.

[6] J.-M. Coron, *Control and nonlinearity, Control and nonlinearity,* Mathematical Surveys and Monographs, 136. American Mathematical Society, Providence, RI, 2007.

[7] G. DaPrato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge 1992.

[8] A. Doubova, E. Fernández-Cara, M. González-Burgos and E. Zuazua, *On the controllability of parabolic systems with a nonlinear term involving the state and the gradient,* SIAM J. Control Optim. 41 (2002), no. 3, 798–819.

[9] C. Fabre, J.P. Puel and E. Zuazua, Contrôlabilité approchée de l'équation de la chaleur linéaire avec des contrôles de norme L^∞ minimale, C.R. Acad. Sci. Paris, **316** (1993), 679–684.

[10] L.A. Fernández and E. Zuazua, *Approximate controllability for the semilinear heat equation involving gradient terms,* J. Optim. Theory Appl. 101 (1999), no. 2, 307–328.

[11] E. Fernández-Cara and E. Zuazua, The cost of approximate controllability for heat equations: The linear case, Advances Diff. Eqs., Vol. 5 (4–6), April-June 2000, p. 465–514.

[12] A.V. Fursikov, Exact boundary zero controllability of three dimensional Navier-Stokes equations, Journal of Dynamical and Control Systems, **1** (3) (1995), 325–350.

[13] A.V. Fursikov and O.Yu. Imanuvilov, Local exact controllability of the Boussinesq equation, SIAM J. Cont. Opt., **36** (2) (1998), 391–421.

[14] A.V. Fursikov and O.Yu. Imanuvilov, *On exact boundary zero-controllability of two-dimensional Navier-Stokes equations,* Acta Applicandae Mathematicae, **37** (1994), 67–76.

[15] A.V. Fursikov and O.Yu. Imanuvilov, *Controllability of evolution equations*, Lecture Notes Series # 34, Research Institute of Mathematics, Global Analysis Research Center, Seoul National University, 1996.

[16] P. Gérard, Microlocal defect measures, Comm. P.D.E., **16** (1991), 1761–1794.

[17] R. Glowinski, Ensuring well-posedness by analogy; Stokes problem and boundary control for the wave equation, J. Compt. Physics, **103** (2) (1992), 189–221.

[18] R. Glowinski, C.H. Li and J.L. Lions, A numerical approach to the exact boundary controllability of the wave equation (I). Dirichlet controls: Description of the numerical methods, Japan J. Appl. Math., **7** (1990), 1–76.

[19] R. Glowinski, J.L. Lions, Exact and approximate controllability for distributed parameter systems, Acta Numerica 1994, p. 269–378.

[20] R. Glowinski, J.-L. Lions, J. He, *Exact and approximate controllability for distributed parameter systems. A numerical approach,* Encyclopedia of Mathematics and its Applications, 117. Cambridge University Press, Cambridge, 2008.

[21] M.D. Gunzburger, *Perspectives in flow control and optimization,* Advances in Design and Control, 5, SIAM, Philadelphia, PA, 2003.

[22] L.F. Ho, Observabilité frontière de l'équation des ondes, C.R. Acad. Sci. Paris, **302** (1986), 443–446.

[23] O.Yu. Imanuvilov, Boundary controllability of parabolic equations, Russian Acad. Sci. Sb. Math., **186** (1995), 109–132 (in Russian).

[24] O.Yu. Imanuvilov and M. Yamamoto, *Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations,* Publ. Res. Inst. Math. Sci. 39 (2003), no. 2, 227–274.

[25] G. Lebeau, *Contrôle analytique I: estimations a priori,* Duke Math. J., **68** (1) (1992), 1–30.

[26] G. Lebeau and L. Robbiano, Contrôle exact de l'équation de la chaleur, Comm. P.D.E., **20** (1995), 335–356.

[27] G. Leugering and E. Zuazua, *On exact controllability of generic trees,* in "Contrôle de Systèmes Gouvernés par des Equations aux Dérivées Partielles", ESAIM Proceedings, Vol. 8, F. Conrad and M. Tucsnak eds., pp. 95–105.

[28] J.L. Lions, Exact controllability, stabilizability and perturbations for distributed systems, SIAM Review, **30** (1988), 1–68.

[29] J.L. Lions, *Contrôlabilité Exacte, Stabilisation et Perturbations de Systèmes Distribués, Tomes 1 & 2.* Masson, RMA **8** & **9**, Paris 1988.

[30] J.L. Lions, Remarques sur la contrôlabilité approchée, in *Jornadas Hispano-Francesas sobre Control de Sistemas Distribuidos*, University of Málaga, Spain, 1991, pp. 77–87.

[31] J.L. Lions, Remarks on approximate controllability, J. Anal. Math., **59** (1992), 103–116.

[32] J.L. Lions and B. Malgrange, Sur l'unicité rétrograde dans les problèmes mixtes paraboliques, Math. Scan., **8** (1960), 277–286.

[33] A. Osses, *A rotated multiplier applied to the controllability of waves, elasticity, and tangential Stokes control,* SIAM J. Control Optim. 40 (2001), no. 3, 777–800

[34] J.-P. Puel, *Global Carleman inequalities for the wave equations and applications to controllability and inverse problems,* in "Control of Solids and Structures: Mathematical Modelling and Engineering Applications", Udine (Italy), June 2004.

[35] L. Robbiano and C. Zuily, Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients, Invent. Math., **131** (1998), 493–539.

[36] D.L. Russell, Controllability and stabilizability theory for linear partial differential equations. Recent progress and open questions, SIAM Review **20** (1978), 639–739.

[37] D.L. Russell, A unified boundary controllability theory for hyperbolic and parabolic partial differential equations, Studies in Appl. Math., **52** (1973), 189–221.

[38] D. Tataru, Unique continuation for solutions to PDE's: between Hörmander's theorem and Holmgren's theorem, Comm. PDE, **20** (6–7) (1996), 855–884.

[39] X. Zhang, *Explicit observability estimate for the wave equation with potential and its application,* R. Soc. Lond. Proc. Ser. A, Math. Phys. Eng. Sci. 456 (2000), no.1997, 1101–1115.

[40] E. Zuazua, Finite dimensional null controllability for the semilinear heat equation, J. Math. Pures Appl., **76** (1997), 570–594.

[41] E. Zuazua *Controllability and observability of partial differential equations: some results and open problems,* Handbook of differential equations: evolutionary equations, Vol. III, 527–621, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2007.