Introdução à teoria de regularidade elíptica

Aula 2: Teoria de DeGiorgi

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General problem in the Calculus of Variation

Hilbert's 19th Problem

"Are the solutions of Lagrangians always analytic?"

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Hilbert's 19th Problem

"Are the solutions of Lagrangians always analytic?"

$$E(u) = \int_{\Omega} F(Du)dX \longrightarrow \text{min.}$$

- Convexity of F is a Necessary and Sufficient condition for Weak Lower Semicontinuity of E.

A PDE Approach for Hilbert's 19^{th} Problem

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$$0 = \frac{d}{dt}E(u + t\varphi)\Big|_{t=0} = \int_\Omega D\varphi \cdot F_j(Du)dX.$$

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- Recall

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We need to show a solution to the above equation is $C^{1,\alpha}$.

- Fix a direction μ. Deriving the above Equation in the μ direction gives
- Thus, u_μ satisfies an Equation

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A PDE Approach for Hilbert's 19$^{\text{th}}$ Problem

Goal

Establish Hölder Continuity for solutions to

$$\operatorname{div}(a_{ij}(X)Du) = 0,$$

when a_{ij} is only known to the bounded measurable and elliptic.

The Theorem

Theorem (De Giorgi-Nash-Moser)

Let a_{ij} be a uniform elliptic matrix and u an H^1 (distributional) solution to

$$\mathrm{div}\left(a_{ij}(X)Du\right) = 0 \text{ in } B_1.$$

Then u is Hölder continuous in $B_{1/2}$. Furthermore,

$$\|u\|_{C^\alpha(B_{1/2})} \leq C\|u\|_{L^2(B_1)},$$

where C depends only on dimension and ellipticity.

Outline

✓ The Proof is Divided in two parts:

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2. An Oscillation Lemma

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1. An L^∞ estimate in terms of the L^2 norm.
2. An Oscillation Lemma: De Giorgi's famous Oscillation Lemma.

$L^2 \Rightarrow L^\infty$ Estimate

Lemma ($L^2 \Rightarrow L^\infty$)

Let u satisfy

$$\operatorname{div}\left(a_{ij}(X)Du\right) \geq 0.$$

There exists a $\delta > 0$, depending only on ellipticity, such that

$$\|u^+\|_{L^2(B_1)} \leq \delta \quad \textit{implies} \quad \|u^+\|_{L^\infty(B_{1/2})} \leq 1.$$

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Two Competing Inequalities

1. Sobolev Inequality:

$$\int_{B_1} |f|^p dX \leq C \left(\int_{B_1} |\nabla f|^2 dX \right)^{p/2},$$

for $p = \frac{2n}{n-2} > 2$.

2. Energy Estimate: If $v \geq 0$ satisfy $\operatorname{div}(a_{ij}(X)Dv) \geq 0$ in B_1, Then,

$$\int_{B_1} |D(\psi v)|^2 dX \leq C \sup |\nabla \psi|^2 \int_{B_1} (v)^2 dX, \quad \forall \psi \in C_0^\infty(B_1)$$

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⋆ Sobolev and Energy Inequalities *compete* in a different homogeneity ⋆

Family of Cut-offs

- Define
$$\psi_k(X) := \begin{cases} 1 & \text{in} \quad B_{\frac{1}{2}+2^{-k}} \\ 0 & \text{in} \quad B_1 \setminus B_{\frac{1}{2}+2^{-(k-1)}}. \end{cases}$$
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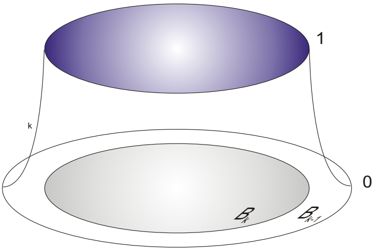

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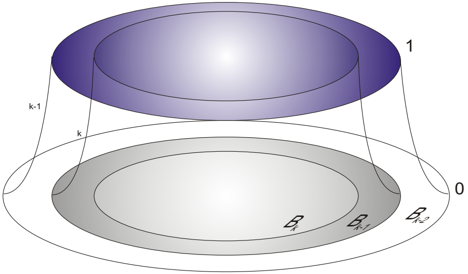

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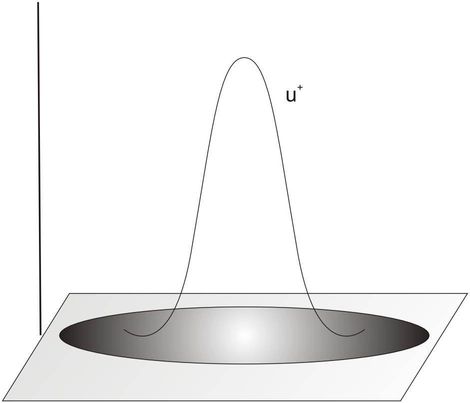

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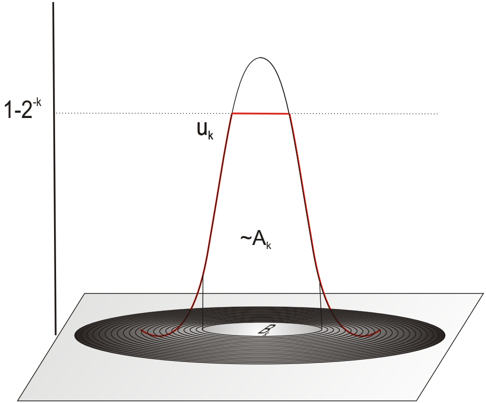

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- We want to show

$$\psi_k u_k \xrightarrow{k \to \infty} 0, \text{ provided } \|u^+\|_{L^2} \ll 1.$$

- Combing Sobolev Inequality and Energy Estimate, we reach the following Non-Linear Recursive Relation:

$$A_k \leq C \left[2^{2k} A_{k-1} \right]^{\mu+1}.$$

- Thus, if A_0 is small enough, indeed $A_k \xrightarrow{k \to \infty} 0$.

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Oscillation Lemma

Lemma (De Giorgi's oscillation lemma)

Let u be a solution to $\mathrm{div}\big(a_{ij}(X)Du\big) = 0$ in B_1. Assume $\operatorname{osc}_{B_1} u = 2$, then

$$\operatorname{osc}_{B_{1/2}} u \leq 2\lambda,$$

for some $\lambda < 1$ that depends only on ellipticity.

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Geometrical Idea of the Proof

1. We can assume $-1 \leq u \leq 1$.
2. If u is a solution, then u^+ is a subsolution .
3. If , previous Lemma guarantees

$$u^+ \leq 1/2 \text{ in } B_{1/2},$$

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3. If $\|u^+\|_{L^2(B_{3/4})} \le \delta/2$, previous Lemma guarantees

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How do you produce the above situation?

1. Assume $u^+ \equiv 0$ at least half of the time in $B_{3/4}$.
2. Cut the graph of u^+ at level $1/2$, i.e. define
$$v_1 := \min\{u^+, 1/2\}.$$
3. Because $\|u^+\|_{L^2}$ is under control, it needs some room to go from 0 to $1/2$.
4. Thus, Vol.$(\{v_1 = 1/2\})$ is a fixed proportion larger than Vol.$(\{u^+ = 0\})$ in $B_{3/4}$.
5. Consider the above part of the truncation and re-scale it to the normalized picture.
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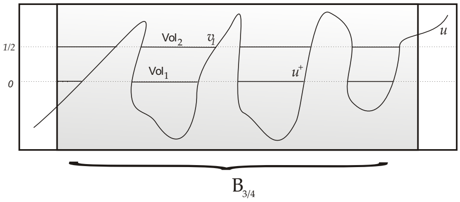
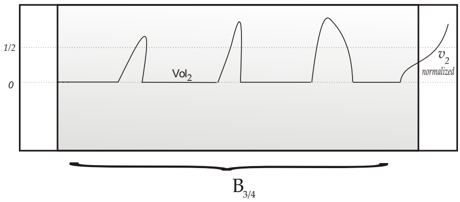


Vol$_2$ > Vol$_1$+



Closing

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