II segundo ENAMA

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Minicurso 2

Tsirelson's space and some applications to Holomorphy

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II ENAMA

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Os organizadores do II ENAMA desejam expressar sua gratidão aos órgãos e instituições que apoiaram e tornaram possível a realização deste evento: CNPq, CAPES, UFPB, UFCG, Banco do Brasil e Prefeitura Municipal de João Pessoa. Agradecem também a todos participantes do evento, bem como aos colaboradores pelo entusiasmo e esforço, que tanto contribuíram para o sucesso deste evento.

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Tsirelson's space and some applications to Holomorphy

Pilar Rueda

Abstract

Tsirelson's space T has provided several interesting counterexamples in the frame of Banach Spaces Theory. However, the behavior of T, that can be considered as pathological, turns to provide nice properties in the Infinite Dimensional Holomorphy. These notes are devoted to Tsirelson's space, which is presented following the construction given by Figiel and Johnson, that was used by mathematicians as Casazza and Shura or Lindenstrauss and Tzafriri. The main application we present of Tsirelson space in Infinite Dimensional Holomorphy is due to Alencar, Aron and Dineen. They used Tsirelson's space to get the first example of a reflexive space of holomorphic functions defined on a Banach space of infinite dimension.

1 Introduction

Until 1974 all known Banach spaces had copies of c_0 or ℓ_p for some $1 \leq p < \infty$. However, that year Tsirelson constructed a reflexive Banach space with an unconditional basis that had neither isomorphic copies of ℓ_p, for any $1 \leq p < \infty$, nor of c_0. Tsirelson's work allowed many mathematicians as Figiel, Johnson, Odell, Schechtman, Argyros, Deliyanni..., to construct new counterexamples in the theory of Banach spaces. This examples were called modified Tsirelson's spaces, mixed Tsirelson's spaces...

Tsirelson's space has also found applications to Holomorphy. In the seventies properties as reflexivity of spaces of polynomials and holomorphic functions on a Banach space X were being studied although there were no known examples yet of infinite dimensional Banach spaces X with such a property. In 1984 Alencar, Aron and Dineen used Tsirelson's space to get the first example of a reflexive space of holomorphic functions defined on an infinite dimensional Banach space. Later on, Prieto [18] proved that the space of all holomorphic functions on the original Tsirelson's space T^* that are of bounded type is reflexive. Rueda [19] and García, Maestre and Rueda [8] generalized Prieto's result to weighted spaces of holomorphic functions defined on a balanced open subset of T^*. Tsirelson's space has been (and is still) used in the frame of Holomorphy and that has occasioned our personal interest on this space.

The purpose of this course is to introduce Tsirelson space paying attention to those properties that yield mainly to the example given by Alencar-Aron-Dineen, Prieto and García-Maestre-Rueda.

The construction we are going to follow is the one by Figiel and Johnson [7], that was used by mathematicians as Casazza and Shura [3] or Lindenstrauss and Tzafriri [13], and that was given the name of Tsirelson's space T although the original Tsirelson's space [21] is (an identification of) the dual T^* of T. Both spaces T and T^* share the most important properties as reflexivity, both have unconditional basis and do not contain copies of c_o or any ℓ_p.

These notes are based mainly in some of the results that appear in [3], [6], [13] concerning Tsirelson's space. We refer to these excellent books for further references. Some properties on T have been selected: on one hand, the "nice" ones as reflexivity or the containment of an unconditional basis and those "pathological" properties as the non containment of copies of any ℓ_p and, on the other hand, those properties that yield to the applications in Infinite Dimensional Holomorphy. The aim of proportionating self-contained and detailed proofs around Tsirelson's space has motivated the selection of the results, although the basics on Banach space Theory is presented without proofs. As these notes are addressed mainly to people that are involved with Holomorphy, in the applications some previous results concerning just Holomorphy and not Tsirelson's space are supposed to be known and so, they are used without proofs. Despite of this and for the sake of completeness, we have reproduced some known proofs and completed or modified other ones.

A first draft of these notes was done in 2002/03 during a Seminar that I taught in the Departamento de Análisis Matemático of the Universidad de Valencia, where the suggestions and comments of the friends that attended the course there helped me to improve it. Thanks are given to all of them. In particular, I would like to thank Vicente Montesinos who proved Proposition 9. His result allowed to avoid a less pleasant argument that used that

the closed linear hull of any block sequence in T generates a complemented subspace of T [3, Proposition II.6]. This current version forms the material for a mini-course that will be taught in II Enama, II Encontro Nacional de Análise Matemática e Aplicaçoes, taking place in the Universidade Federal da Paraíba, João Pessoa, in 2008. I would like to thank Geraldo Botelho and Daniel Pellegrino for their kind invitation to visit their respective Universities and to give this course in the II Enama.

2 Tsirelson's space

2.1 Preliminaries on Schauder bases

Let X be a complex Banach space and let X^* be its topological dual. Let $(x_n)_n$ be a sequence in X. The sequence $(x_n)_n$ is called a *Schauder basis* of X if for every $x \in X$ there exists a unique sequence of scalars $(a_n)_n$ such that $x = \sum_{n=1}^{\infty} a_n x_n$. The sequence $(x_n)_n$ is called a *basic sequence* if it is a Schauder basis of its closed linear span. It is well known that if $(x_n)_n$ is a Schauder basis of X then the projections $q^m : X \longrightarrow X$ defined by $q^m(\sum_{n=1}^{\infty} a_n x_n) = \sum_{n=1}^{m} a_n x_n$ are bounded linear operators and $c := \sup_m \|q^m\| < \infty$. The constant c is called the *basis constant* of $(x_n)_n$.

A Schauder basis $(x_n)_n$ of X is called:

- *equivalent* to a Schauder basis $(y_n)_n$ of a Banach space Y if $\sum_{n=1}^{\infty} a_n x_n$ converges in X if and only if $\sum_{n=1}^{\infty} a_n y_n$ converges in Y, for any scalar sequence $(a_n)_n$. By [4, Theorem V.5] $(x_n)_n$ and $(y_n)_n$ are equivalent if and only if there is an isomorphism between X and Y that maps each x_n to y_n.

- *unconditional* if the convergence of the series $\sum_{n=1}^{\infty} b_n x_n$ implies the convergence of the series $\sum_{n=1}^{\infty} a_n x_n$ whenever $|a_n| \leq |b_n|$ for all n.

- *normalized* if $\|x_n\| = 1$ for all n (and so $(\frac{x_n}{\|x_n\|})_n$ is a normalized Schauder basis of X).

- *shrinking* if the coordinates functionals $(x_n^*)_n$, given by $x_m^*(\sum_{n=1}^{\infty} a_n x_n) = a_m$, form a Schauder basis of X^*.

- *boundedly complete* if the series $\sum_{n=1}^{\infty} a_n x_n$ converges in X whenever $\sup_m \| \sum_{n=1}^{m} a_n x_n\| < \infty$.

Let us recall the following facts about a Banach space X with a Schauder basis $(x_n)_n$:

• X is reflexive if and only if $(x_n)_n$ is shrinking and boundedly complete (see [9] or [13, Theorem 1.b.5]).

• Assume that $(x_n)_n$ is unconditional, then X does not have a copy of c_o if and only if $(x_n)_n$ is boundedly complete (see [13, Theorem 1.c.10]).

• Assume that $(x_n)_n$ is unconditional. If X does not have copies of c_o or ℓ_1 then X is reflexive (see [13, Theorem 1.c.12]).

A sequence $(y_n)_n$ in X is called a *block sequence* of $(x_n)_n$ if each y_n can be written as $y_n = \sum_{j=p_n+1}^{p_{n+1}} a_j x_j$, for some sequence of scalars $(a_n)_n$ and some increasing sequence of integers $p_1 < p_2 < \cdots$.

Given a sequence $(x_n)_n$ in X, let $\mathrm{supp}((x_n)_n)$ denote the support of $(x_n)_n$, that is

$$\mathrm{supp}((x_n)_n) := \{n \in \mathbb{N} : x_n \neq 0\}.$$

2.2 Definition and basic properties

In order to construct Tsirelson's space, we consider the space T_0 of all complex sequences that are eventually zero. Let $e_n = (0, \ldots, 1, 0, \ldots)$, with the 1 in the nth position. For each $x \in T_0$, $x = (a_1, a_2, \ldots)$, we define recurrently the following norms:

$$\begin{aligned} \|x\|_0 &= \max_n |a_n|, \\ \|x\|_{m+1} &= \max\{\|x\|_m, \quad \frac{1}{2} \max_{\substack{k \le n_1 < \cdots < n_{k+1} \\ k=1,2,\ldots}} \sum_{j=1}^{k} \| \sum_{n=n_j+1}^{n_{j+1}} a_n e_n \|_m\}. \end{aligned}$$

Lemma 1. *It follows:*

1. *For every $m = 0, 1, 2, \ldots$, $\| \ \|_m$ is a norm.*
2. *For every $m = 0, 1, 2, \ldots$, $\|x\|_m \le \|x\|_{m+1}$ for all $x \in T_0$.*
3. *For every $m = 0, 1, 2, \ldots$, $\|x\|_m \le \|x\|_{\ell_1}$ for all $x \in T_0$.*

Proof: 1. It is easy to prove by induction that if $\|x\|_m = 0$ then $x = 0$ and that $\|\lambda x\|_m = |\lambda| \|x\|_m$. To prove the triangular inequality it suffices to proceed by induction and use that $\max(a+b, a'+b') \le \max(a, a') + \max(b, b')$ for $a, a', b, b' > 0$.

2. It is trivial.

3. We proceed by induction. Clearly $\|x\|_0 \leq \sum_{j=1}^{\infty} |a_j|$ for all $x \in T_0$. If we assume that $\|x\|_m \leq \sum_{j=1}^{\infty} |a_j|$ for all $x \in T_0$ then for each $k \leq n_1 < \cdots < n_{k+1}$, $k = 1, 2, \ldots$

$$\sum_{j=1}^{k} \| \sum_{n=n_j+1}^{n_{j+1}} a_n e_n \|_m \leq \sum_{j=1}^{k} \sum_{n=n_j+1}^{n_{j+1}} |a_n| \leq \sum_{n=1}^{\infty} |a_n|.$$

Hence

$$\|x\|_{m+1} \leq \sum_{n=1}^{\infty} |a_n|.$$

■

We define

$$\|x\| = \lim_m \|x\|_m.$$

By the above Lemma $\| \ \|$ exists and it is easy to show that it defines a norm on T_0. *Tsirelson's space* T is defined as the completion of $(T_0, \| \ \|)$. We shall call *original Tsirelson's space* the dual T^* of T.

Proposition 2. *The sequence $(e_n)_n$ is a 1-unconditional Schauder basis of T.*

PROOF: Let us start proving that: if $|a_n| < |b_n|$ for every n (assuming that $a_n = 0$ if $b_n = 0$) then $\| \sum_{n=1}^{\infty} a_n e_n \|_m \leq \| \sum_{n=1}^{\infty} b_n e_n \|_m$, for all $(b_1, b_2, \ldots) \in T_0$ and all $m = 0, 1, 2, \ldots$. We proceed by induction. For $m = 0$ is trivial. If we assume it is true for m, then

$$\begin{aligned}
\| \sum_{n=1}^{\infty} a_n e_n \|_{m+1} &= \max\{ \| \sum_{n=1}^{\infty} a_n e_n \|_m, \quad \frac{1}{2} \max_{\substack{k \leq n_1 < \cdots < n_{k+1} \\ k=1,2,\ldots}} \sum_{j=1}^{k} \| \sum_{n=n_j+1}^{n_{j+1}} a_n e_n \|_m \} \\
&\leq \max\{ \| \sum_{n=1}^{\infty} b_n e_n \|_m, \quad \frac{1}{2} \max_{\substack{k \leq n_1 < \cdots < n_{k+1} \\ k=1,2,\ldots}} \sum_{j=1}^{k} \| \sum_{n=n_j+1}^{n_{j+1}} b_n e_n \|_m \} \\
&= \| \sum_{n=1}^{\infty} b_n e_n \|_{m+1}.
\end{aligned}$$

Now, make $m \to \infty$. It follows that if $|a_n| < |b_n|$, for all n, then $\| \sum_{n=1}^{\infty} a_n e_n \| \leq \| \sum_{n=1}^{\infty} b_n e_n \|$, for all $(b_1, b_2, \ldots) \in T_0$. As T is the closed linear span of the

e_n's the result follows from the simple fact that the linear space of all elements of the form $\sum_{n=1}^{\infty} a_n e_n$ is a closed linear space (see [13, Proposition 1.a.3]). ∎

Proposition 3. *For every* $x = \sum_{n=1}^{\infty} a_n e_n \in T$,

$$\|x\| = \max\{\max_n |a_n|, \quad \frac{1}{2} \sup_{\substack{k \le n_1 < \cdots < n_{k+1} \\ k=1,2,\ldots}} \sum_{j=1}^{k} \| \sum_{n=n_j+1}^{n_{j+1}} a_n e_n \|\}.$$

Proof: Let us denote $|||x|||$ the term of the right hand. It suffices to be proved for $x = \sum_{n=1}^{\infty} a_n e_n \in T_0$. Actually, in that case, if $x \in T$,

$$\|x\| = \lim_n \|x_n\| = \lim_n |||x_n||| = |||x|||,$$

for some sequence $(x_n)_n$ in T_0 that converges to x.

Since $\|x\| \ge \max_n |a_n|$ and

$$\|x\| \ge \|x\|_{m+1} \ge \frac{1}{2} \max_{\substack{k \le n_1 < \cdots < n_{k+1} \\ k=1,2,\ldots}} \sum_{j=1}^{k} \| \sum_{n=n_j+1}^{n_{j+1}} a_n e_n \|_m$$

for all m, when $m \to \infty$ it follows

$$\|x\| \ge \frac{1}{2} \max_{\substack{k \le n_1 < \cdots < n_{k+1} \\ k=1,2,\ldots}} \sum_{j=1}^{k} \| \sum_{n=n_j+1}^{n_{j+1}} a_n e_n \|.$$

Then $\|x\| \ge |||x|||$.

If we assume that there is $x \in T_0$ such that $\|x\| > |||x|||$ then

$$\begin{aligned} \|x\| > |||x||| \quad &\ge \quad \frac{1}{2} \max_{\substack{k \le n_1 < \cdots < n_{k+1} \\ k=1,2,\ldots}} \sum_{j=1}^{k} \| \sum_{n=n_j+1}^{n_{j+1}} a_n e_n \| \\ &\ge \quad \frac{1}{2} \max_{\substack{k \le n_1 < \cdots < n_{k+1} \\ k=1,2,\ldots}} \sum_{j=1}^{k} \| \sum_{n=n_j+1}^{n_{j+1}} a_n e_n \|_m \end{aligned}$$

for all m. Since $\|x\| = \lim_m \|x\|_m$ there exists a natural number l_0 such that for all $l \ge l_0$

$$\|x\|_l > \frac{1}{2} \max_{\substack{k \le n_1 < \cdots < n_{k+1} \\ k=1,2,\ldots}} \sum_{j=1}^{k} \| \sum_{n=n_j+1}^{n_{j+1}} a_n e_n \|_m$$

for all m. In particular,

$$\|x\|_l > \frac{1}{2} \max_{\substack{k \leq n_1 < \cdots < n_{k+1} \\ k=1,2,\ldots}} \sum_{j=1}^{k} \| \sum_{n=n_j+1}^{n_{j+1}} a_n e_n \|_{l-1}.$$

Then $\|x\|_l = \|x\|_{l-1}$. Since

$$\|x\|_{l-1} = \|x\|_l > \frac{1}{2} \max_{\substack{k \leq n_1 < \cdots < n_{k+1} \\ k=1,2,\ldots}} \sum_{j=1}^{k} \| \sum_{n=n_j+1}^{n_{j+1}} a_n e_n \|_{l-2}$$

then $\|x\|_{l-1} = \|x\|_{l-2}$. In such a way and after a finite recurrence we get that $\|x\|_l = \|x\|_{l-1} = \cdots = \|x\|_0 = \max_n |a_n|$ for all $l \geq l_0$. Then $\|x\| = \max_n |a_n|$ what contradicts that $\|x\| > |||x||| \geq \max_n |a_n|$. ∎

Proposition 4. *For every natural number k and every normalized sequence of blocks $(y_i)_{i=1}^k$ such that $y_i = \sum_{n=p_i+1}^{p_{i+1}} a_n e_n$ with $1 \leq i \leq k$ and $k \leq p_1 < p_2 < \cdots < p_{k+1}$ we have:*

$$\frac{1}{2} \sum_{i=1}^{k} |b_i| \leq \| \sum_{i=1}^{k} b_i y_i \| \leq \sum_{i=1}^{k} |b_i| \tag{1}$$

for all scalar sequence $(b_i)_{i=1}^k$.

Proof: Since $k \leq p_1 < \cdots < p_{k+1}$ it follows from Proposition 3

$$\begin{aligned} \| \sum_{j=1}^{k} b_j y_j \| &\geq \frac{1}{2} \sum_{j=1}^{k} \| \sum_{i=p_j+1}^{p_{j+1}} b_j a_i e_i \| = \frac{1}{2} \sum_{j=1}^{k} |b_j| \| \sum_{i=p_j+1}^{p_{j+1}} a_i e_i \| \\ &= \frac{1}{2} \sum_{j=1}^{k} |b_j| \|y_j\| = \frac{1}{2} \sum_{j=1}^{k} |b_j|. \end{aligned}$$

On the other hand,

$$\| \sum_{i=1}^{k} b_i y_i \| \leq \sum_{i=1}^{k} |b_i| \|y_i\| = \sum_{i=1}^{k} |b_i|.$$

∎

The above proposition says that any sequence $(y_i)_{i=1}^k$ of unitary vectors in T with increasing disjoint finite supports (in the sense that $\max\{i : i \in \operatorname{supp} y_j\} < \min\{i : i \in \operatorname{supp} y_{j+1}\}$ for all j) whose first k coordinates are 0 fulfills (1). Next proposition shows a dual analog in T^*. Let (e_n^*) be the dual basis of $(e_n)_n$.

Proposition 5. *Let $(y_n^*)_{n=1}^k$ be a sequence of unitary vectors in T^* of the form $y_i^* = \sum_{n=p_i+1}^{p_{i+1}} a_n e_n^*$, $1 \le i \le k$ and $k \le p_1 < p_2 < \cdots < p_{k+1}$ (with increasing disjoint finite supports whose first k coordinates are 0). Then*

$$\sup_{1\le i\le k} |b_i| \le \| \sum_{i=1}^k b_i y_i^* \|^* \le 2 \sup_{1\le i\le k} |b_i|$$

for all scalar sequence $(b_i)_{i=1}^k$.

Proof: Let z_j^* denote the restriction of y_j^* to the subspace S_j generated by $e_{p_j+1}, \ldots, e_{p_{j+1}}$. Let $y_j \in S_j$ be such that $z_j^*(y_j) = 1$. Since the supports are disjoint it follows that $y_j^*(y_i) = \delta_{ij}$.

Consider the subspace F generated by $y_1, \ldots, y_k$ endowed with the norms $\| \ \|_{\ell_1}$ and $\| \ \|$. We know by the above proposition that $\frac{1}{2}\| \ \|_{\ell_1} \le \| \ \| \le \| \ \|_{\ell_1}$ in F. Taking duals in F^* we get that

$$\begin{aligned} \|x^*\|^* &= \sup_{\|x\|\le 1, x\in F} |x^*(x)| \\ &\le \sup_{\|x\|_{\ell_1}\le 2, x\in F} |x^*(x)| \\ &= \sup_{\|\frac{1}{2}x\|_{\ell_1}\le 1, x\in F} 2|x^*(\frac{1}{2}x)| \\ &= 2\|x^*\|_\infty \end{aligned}$$

and

$$\|x^*\|_\infty = \|x^*\|_{\ell_1}^* = \sup_{\|x\|_{\ell_1}\le 1} |x^*(x)| \le \sup_{\|x\|\le 1} |x^*(x)| = \|x^*\|^*$$

for all $x^* \in F^*$. ∎

2.3 T does not contain ℓ_p spaces

Let us now prove that Tsirelson's space T contains neither copies of c_0 nor of ℓ_p, for any $1 < p < \infty$. We start proving next result on equivalent sequences whose proof is taken from [4, Theorem V.9].

Proposition 6. *([4, Theorem V.9]) Let $(x_n)_n$ be a basic sequence in a Banach space X and let $(x_n^*)_n$ be the sequence of coordinates functionals. If $(y_n)_n$ is a sequence in X such that $\sum_{n=1}^{\infty} \|x_n^*\|\|x_n - y_n\| < 1$ then $(y_n)_n$ is a basic sequence equivalent to $(x_n)_n$.*

PROOF: By Hahn-Banach extension Theorem we extend each x_n^* to the whole X. Define $S : X \longrightarrow X$ as $S(x) = \sum_{n=1}^{\infty} x_n^*(x)(x_n - y_n)$. Since

$$\|S\| \leq \sum_{n=1}^{\infty} \|x_n^*\|\|x_n - y_n\| < 1$$

then $I - S$ is an invertible operator with $(I - S)(x_n) = y_n$. ∎

Before proving next theorem observe the following remark:

Let Y be either c_0 or ℓ_p, $1 \leq p < \infty$, and let $(u_n)_n$ the canonical basis of Y. For each increasing sequence of natural numbers $(n_k)_k$, the mapping $S_{(n_k)} : Y \longrightarrow Y$ given by $S_{(n_k)}(\sum_{n=1}^{\infty} a_n u_n) = \sum_{k=1}^{\infty} a_k u_{n_k}$ is an isomorphism into whose inverse is $S_{(n_k)}^{-1}(\sum_{k=1}^{\infty} a_{n_k} u_{n_k}) = \sum_{k=1}^{\infty} a_{n_k} u_k$, and that maps each u_k to u_{n_k}. Therefore $(u_{n_k})_k$ is equivalent to $(u_n)_n$, for all sequence $(u_{n_k})_k$. Moreover, each $S_{(n_k)}$ is an isometry. Indeed, if $Y = c_0$ then

$$\|\sum_{k=1}^{\infty} a_k u_{n_k}\|_\infty = \sup_k |a_k| = \|\sum_{k=1}^{\infty} a_k u_k\|_\infty,$$

whereas if $Y = \ell_p$, $1 \leq p < \infty$, then

$$\|\sum_{k=1}^{\infty} a_k u_{n_k}\|_p = (\sum_{k=1}^{\infty} |a_k|^p)^{1/p} = \|\sum_{k=1}^{\infty} a_k u_k\|_p.$$

The proof of T not containing copies of c_0 nor of ℓ_p, $1 < p < \infty$, is based on the Selection Principle of Bessaga-Pelczyński that asserts that given a Banach space X with a Schauder basis $(x_n)_n$, if $(x_n^*)_n$ is the sequence of coordinates functionals and $(y_n)_n$ is a sequence in X such that $\liminf_k \|y_k\| > 0$ and $\lim_k x_n^* y_k = 0$ for all n then $(y_n)_n$ has a subsequence which is equivalent to a basic block sequence of $(x_n)_n$. We will reproduce his proof while proving Theorem 7.

Theorem 7. *Tsirelson's space T does not contain subspaces that are isomorphic to c_0 or ℓ_p, for all $1 < p < \infty$.*

Proof: Let Y denote either c_0 or ℓ_p, $1 < p < \infty$. Let us assume that there exists an isomorphism into $S : Y \longrightarrow T$. Let $c, C > 0$ be constants such that

$$c\|y\| \leq \|S(y)\| \leq C\|y\|,$$

for all $y \in Y$. Consider the canonical basis $(u_n)_n$ of Y. Then $y_n := S(u_n)$ is a basic sequence of T equivalent to $(u_n)_n$. Writing $y_m = \sum_{n=1}^{\infty} a_n^m e_n$ we have that, for each fixed n,

$$a_n^m = e_n^*(S(u_m)) = {}^tS(e_n^*)(u_m)$$

converges to 0 when $m \to \infty$. Indeed, from ${}^tS(e_n^*) \in Y^*$, if ${}^tS(e_n^*) = \sum_{m=1}^{\infty} b_m^n u_m^*$ then ${}^tS(e_n^*)(u_m) = b_m^n \longrightarrow 0$ when $m \to \infty$.

Moreover, for each increasing sequence $(n_k)_k$, $S \circ S_{(n_k)} \circ S^{-1}|_{\overline{LIN(y_k)_{k=1}^{\infty}}}$ is an isomorphism into that maps each y_k to y_{n_k}. Then $(y_k)_k$ is equivalent to $(y_{n_k})_k$. For each m

$$\|\sum_{k=1}^{m} a_k y_{n_k}\| \leq C\|\sum_{k=1}^{m} a_k u_{n_k}\| = C\|\sum_{k=1}^{m} a_k u_k\| \leq \frac{C}{c}\|\sum_{k=1}^{m} a_k y_k\|.$$

Then

$$\|\sum_{k=1}^{m} a_k y_k\| \geq \frac{c}{C}\|\sum_{k=1}^{m} a_k y_{n_k}\| \tag{2}$$

for all increasing sequence of natural numbers $(n_k)_k$ and all natural number m.

" Let K be the basic constant of $(e_n)_n$ that is, $K := \sup_n \|q^n\|$ where $q^n(\sum_{k=1}^{\infty} b_k e_k) = \sum_{k=1}^{n} b_k e_k$.

Let $p_1 > 1$ be such that

$$\|y_1 - \sum_{n=1}^{p_1} a_n^1 e_n\| \leq \frac{c}{8K2}.$$

Let $0 < \epsilon_1 < \frac{c}{8K2}$. For p_1, as $\sum_{n=1}^{p_1} a_n^m e_n \longrightarrow 0$ when $m \to \infty$, there exists n_1 so that

$$\|\sum_{n=1}^{p_1} a_n^{n_1} e_n\| \leq \epsilon_1.$$

Let now $p_2 > p_1 + 1$ such that

$$\|\sum_{n=p_2+1}^{\infty} a_n^{n_1} e_n\| < (\frac{c}{8K2} - \epsilon_1).$$

Then

$$\|y_{n_1} - \sum_{n=p_1+1}^{p_2} a_n^{n_1} e_n\| = \|\sum_{n=1}^{p_1} a_n^{n_1} e_n + \sum_{n=p_2+1}^{\infty} a_n^{n_1} e_n\| \leq \epsilon_1 + (\frac{c}{8K2} - \epsilon_1) = \frac{c}{8K2}.$$

Let $0 < \epsilon_2 < \frac{c}{8K2^2}$. For p_2, since $\sum_{n=1}^{p_2} a_n^m e_n \longrightarrow 0$ when $m \to \infty$ there exists $n_2 > n_1$ so that

$$\|\sum_{n=1}^{p_2} a_n^{n_2} e_n\| \leq \epsilon_2.$$

Let now $p_3 > p_2 + 2$ be such that

$$\|\sum_{n=p_3+1}^{\infty} a_n^{n_2} e_n\| < (\frac{c}{8K2^2} - \epsilon_2).$$

Then

$$\|y_{n_2} - \sum_{n=p_2+1}^{p_3} a_n^{n_2} e_n\| = \|\sum_{n=1}^{p_2} a_n^{n_2} e_n + \sum_{n=p_3+1}^{\infty} a_n^{n_2} e_n\| \leq \epsilon_2 + (\frac{c}{8K2^2} - \epsilon_2) = \frac{c}{8K2^2}.$$

By induction we construct sequences $(n_j)_j$ and $(p_j)_j$ with $p_{j+1} > p_j + j$ such that

$$\|y_{n_j} - \sum_{n=p_j+1}^{p_{j+1}} a_n^{n_j} e_n\| \leq \frac{c}{8K2^j}$$

for all j. Let us denote

$$z_j := \sum_{n=p_j+1}^{p_{j+1}} a_n^{n_j} e_n, \quad j \geq 1.$$

Then $(z_j)_j$ is a block sequence of $(e_n)_n$ with $\mathrm{supp}(z_j) > j$. So, if $p < q$ then

$$\|\sum_{j=1}^{p} a_j z_j\| \leq K\|\sum_{j=1}^{q} a_j z_j\|$$

for all scalar sequence $(a_n)_n$. Hence, $(z_j)_j$ is a basic sequence with basic constant less than or equal to K. Therefore, given $x = \sum_{i=1}^{\infty} z_j^*(x) z_j$ it follows that

$$K \geq \sup_i \|q^i\| \geq \|q^j\| = \sup_{\|y\|\leq 1} \|q^j(y)\| \geq \|q^j(\frac{x}{\|x\|})\| = \frac{1}{\|x\|}\|q^j(x)\|$$

and then

$$\|z_j^*(x)z_j\| = \|q^j(x) - q^{j-1}(x)\| \leq 2K\|x\|.$$

From $\|y_n\| \geq c$ for all n, it follows that

$$c \leq \|y_{n_j}\| \leq \|y_{n_j} - z_j\| + \|z_j\| \leq \frac{c}{8K2^j} + \|z_j\|.$$

Then

$$\|z_j\| \geq c - \frac{c}{8K2^j} \geq \frac{c}{2}.$$

Hence

$$|z_j^*(x)| = \|z_j^*(x)z_j\|\frac{1}{\|z_j\|} \leq 2K\frac{2}{c}\|x\| = \frac{4K}{c}\|x\|.$$

Thus $\|z_j^*\| \leq \frac{4K}{c}$ and then

$$\sum_{j=1}^{\infty} \|z_j^*\|\|z_j - y_{n_j}\| \leq \frac{4K}{c}\sum_{j=1}^{\infty} \|z_j - y_{n_j}\| \leq \frac{4K}{c}\sum_{j=1}^{\infty} \frac{c}{8K2^j} \leq \frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{2^j} < 1.$$

From Proposition 6 it follows that $(y_{n_j})_j$ is a basic sequence equivalent to $(z_j)_j$. Then there exists a constant $M > 0$ such that

$$\|\sum_{j=1}^{\infty} a_j z_j\| \leq M\|\sum_{j=1}^{\infty} a_j y_{n_j}\|$$

for all scalar sequence $(a_j)_j$."

From (2), taking the sequence $(n_{m-1+j})_j$, it follows that for any m and any finite sequence $(a_j)_{j=1}^m$

$$\begin{aligned} C\sum_{j=1}^{m}|a_j| &= C\sum_{j=1}^{m}|a_j|\frac{\|y_j\|}{\|y_j\|} \geq \|\sum_{j=1}^{m} a_j y_j\| \\ &\geq \frac{c}{C}\|\sum_{j=1}^{m} a_j y_{n_{m-1+j}}\| \geq \frac{c}{CM}\|\sum_{j=1}^{m} a_j z_{n_{m-1+j}}\| \\ &\geq \frac{1}{2}\frac{c}{CM}\sum_{j=1}^{m}|a_j|, \end{aligned}$$

where we have used Proposition 4 in the last inequality. That is,

$$C\sum_{j=1}^{m}|a_j| \geq \|\sum_{j=1}^{m} a_j y_j\| \geq \frac{c}{2CM}\sum_{j=1}^{m}|a_j|$$

for all sequence $(a_j)_{j=1}^m$ and all m. So, $(y_j)_j$ is equivalent to the canonical basis of ℓ_1 which is absurd. ∎

The quoted part of the proof constitutes the proof of Bessaga-Pelczyński Selection Principle (see also [4]).

Next step is to prove that T does not contain a copy of ℓ_1. We need the following result due to James [10]:

Theorem 8. *(James) If X is a Banach space that contains a subspace isomorphic to ℓ_1 then, for any $\epsilon > 0$ there exists a sequence $(u_n)_n$ in X with $\|u_n\| \leq 1$ such that*

$$(1-\epsilon)\sum_{j=1}^{\infty}|b_j| \leq \|\sum_{j=1}^{\infty} b_j u_j\| \leq \sum_{j=1}^{\infty}|b_j| \tag{3}$$

for all sequence $(b_j)_j \in \ell_1$.

Proof: Let $(x_n)_n$ be a basic sequence in X which is equivalent to the canonical basis of ℓ_1. Then there exist m, $M > 0$ such that

$$m\sum_{j=1}^{\infty}|a_j| \leq \|\sum_{j=1}^{\infty} a_j x_j\| \leq M\sum_{j=1}^{\infty}|a_j|$$

for all sequence $(a_j)_j \in \ell_1$.

For each n we define

$$E_n = \{(a_j)_j \in \ell_1 : \quad \|(a_j)_j\|_{\ell_1} = 1,\ \mathrm{supp}(a_j)_j < \infty \text{ and } a_1 = a_2 = \cdots = a_n = 0\}$$

$$K_n = \inf\{\|\sum_{j=1}^{\infty} a_j x_j\| : \quad (a_j)_j \in E_n\}.$$

Since $E_{n+1} \subset E_n$, the sequence $(K_n)_n$ is increasing. Since $m \leq K_n \leq M$ for all n, there exists $K := \lim_n K_n$. Moreover, $K \geq K_j$ for all j and $m \leq K \leq M$.

Let $0 < \theta < 1 < \theta'$.

Chose $p_1 > 1$ such that $K_{p_1} > \theta K$. Since $K_{p_1} < \theta' K$ there exists a sequence of the form $(0, \ldots, 0, a_{p_1+1}, \ldots, a_{p_2}, 0, \ldots) \in E_{p_1}$ such that

$$\|\sum_{j=p_1+1}^{p_2} a_j x_j\| < \theta' K.$$

Since $K_{p_2} < \theta' K$ there exists a sequence $(0, \ldots, 0, a_{p_2+1}, \ldots, a_{p_3}, 0, \ldots) \in E_{p_2}$ such that

$$\|\sum_{j=p_2+1}^{p_3} a_j x_j\| < \theta' K.$$

By induction we construct for each j scalars $a_{p_j+1}, \ldots, a_{p_{j+1}}$ such that

$$\sum_{n=p_j+1}^{p_{j+1}} |a_n| = 1 \qquad \text{and} \qquad \|\sum_{n=p_j+1}^{p_{j+1}} a_n x_n\| < \theta' K.$$

Let

$$y_j := \sum_{n=p_j+1}^{p_{j+1}} a_n x_n.$$

Then $(y_n)_n$ is a block sequence of $(x_n)_n$ and $\|y_j\| < \theta' K$. Define

$$u_j := \frac{y_j}{\theta' K}.$$

Then $\|u_j\| < 1$ and

$$\|\sum_{j=1}^{\infty} b_j u_j\| \le \sum_{j=1}^{\infty} |b_j| \|u_j\| \le \sum_{j=1}^{\infty} |b_j|$$

for all sequence $(b_j)_j \in \ell_1$.

On the other hand, let $(b_j)_j \in \ell_1$ and $m \ge 1$. Since

$$\sum_{j=1}^{m} \sum_{n=p_j+1}^{p_{j+1}} |b_j a_n| = \sum_{j=1}^{m} (|b_j| \sum_{n=p_j+1}^{p_{j+1}} |a_n|) = \sum_{j=1}^{m} |b_j|$$

then

$$z := \frac{1}{\sum_{j=1}^{m} |b_j|} (0, \ldots, b_1 a_{p_1+1}, \ldots, b_1 a_{p_2}, \ldots, b_m a_{p_m+1}, \ldots, b_m a_{p_{m+1}}, 0, \ldots) \in E_{p_1}.$$

Hence

$$\|\sum_{j=1}^{m} b_j y_j\| = \|\sum_{j=1}^{m} \sum_{n=p_j+1}^{p_{j+1}} b_j a_n x_n\| \frac{\sum_{j=1}^{m} |b_j|}{\sum_{j=1}^{m} |b_j|} \ge K_{p_1} \sum_{j=1}^{m} |b_j| > \theta K \sum_{j=1}^{m} |b_j|.$$

Thus,

$$\|\sum_{j=1}^{m} b_j u_j\| > \frac{\theta}{\theta'} \sum_{j=1}^{m} |b_j|,$$

for all m. Therefore,

$$\|\sum_{j=1}^{\infty} b_j u_j\| > \frac{\theta}{\theta'} \sum_{j=1}^{\infty} |b_j|.$$

It suffices to chose θ and θ' such that $\theta/\theta' > 1 - \epsilon$. ∎

Note that in the proof of the Theorem of James (Theorem 8) we have started with a basic sequence in X equivalent to the canonical basis of ℓ_1. When $X = T$, if we assume that T contains a copy of ℓ_1 then we can assume that this sequence is a block sequence of $(e_n)_n$. Indeed, we have the following result due to Vicente Montesinos. We shall say that a basic sequence $(u_n)_n$ is $(1-\epsilon)$-equivalent to the canonical basis of ℓ_1 if it fulfils (3).

Proposition 9. *(V. Montesinos) Let X be a Banach space with a Schauder basis $(t_n)_n$ and let $0 < \epsilon < 1$. If X has an isomorphic copy of ℓ_1, then it has a basic sequence which is $(1-\epsilon)$-equivalent to the canonical basis of ℓ_1 and equivalent to a block sequence of $(t_n)_n$.*

Proof: By Theorem 8 if a Banach space X has an isomorphic copy of ℓ_1, then it has a basic sequence $(u_n)_n$ $(1-\epsilon)$-equivalent to the canonical basis of ℓ_1. Let $(t_n^*)_n$ be the sequence of coordinates functionals associated to $(t_n)_n$. For each $n = 1, 2, \ldots$, let S_n denote the linear span of $\{u_n, u_{n+1}, u_{n+2}, \ldots\}$ and, for $m \geq n$ let S_n^m denote the linear span of $\{u_n, u_{n+1}, \ldots, u_m\}$. Since $ker(t_1^*) \cap S_1 \neq \{0\}$ there exist $p_1 \in \mathbb{N}$ and $b_1 \in X$, $\|b_1\| = 1$, such that

$$b_1 \in ker(t_1^*) \cap S_1^{p_1}.$$

Since $\bigcap_{i=1}^{2} ker(t_i^*) \cap S_{p_1+1} \neq \{0\}$ there exist $p_2 > p_1$ and $b_2 \in X$, $\|b_2\| = 1$, such that

$$b_2 \in \bigcap_{i=1}^{2} ker(t_i^*) \cap S_{p_1+1}^{p_2}.$$

By induction, we construct a normalized sequence $(b_n)_n$ in X such that $\lim_k t_n^*(b_k) = 0$ for all n. Applying Bessaga-Pelczyński Selection Principle, $(b_n)_n$ has a subsequence, $(s_n)_n$, which is equivalent to a basic block sequence of $(t_n)_n$. Being $(s_n)_n$ a normalized block sequence of $(u_n)_n$, $(s_n)_n$ is $(1-\epsilon)$-equivalent to the canonical basis of ℓ_1. ∎

We can then conclude the following fact: if T contains a copy of ℓ_1 then T has a basic block sequence of $(e_n)_n$ equivalent to the canonical basis of ℓ_1. If we start the proof of Theorem 8 with such a block sequence, then what we eventually show is that if T contains a copy of ℓ_1 then T has a basic block sequence of $(e_n)_n$ $(1-\epsilon)$-*equivalent* to the canonical basis of ℓ_1. We will use this fact in the proof of Theorem 10.

Theorem 10. *Tsirelson's space T has no subspaces isomorphic to ℓ_1.*

PROOF: Taken from [13, Example 2.e.1]. Assume that T has a subspace isomorphic to ℓ_1. By the above comment, there exists a basic block sequence $(v_j)_{j=0}^{\infty}$ of $(e_n)_n$, with $\|v_j\| \leq 1$, for all $j = 0, 1, \ldots$, and such that for all sequence $(b_j)_{j=0}^{\infty} \in \ell_1$

$$\frac{8}{9}\sum_{j=0}^{\infty}|b_j| \leq \|\sum_{j=0}^{\infty} b_j v_j\| \leq \sum_{j=0}^{\infty}|b_j|.$$

In particular,

$$\|v_0 + \frac{1}{r}(v_1 + \cdots + v_r)\| \geq \frac{16}{9}, \quad r = 1, 2, \ldots \tag{4}$$

Consider $k \leq p_1 < p_2 < \cdots < p_{k+1}$ and for each $j = 1, 2, \ldots, k$ let

$$q_j(\sum_{n=1}^{\infty} b_n e_n) := \sum_{n=p_j+1}^{p_{j+1}} b_n e_n.$$

Using the characterization of the norm of T it follows that

$$\|x\| \geq \frac{1}{2}\sum_{j=1}^{k}\|q_j(x)\|,$$

for all $x \in T$.

Let $n_0 := \max\{i : \ i \in \mathrm{supp}(v_0)\}$ and let $z = v_0 + \frac{1}{r}(v_1 + \cdots + v_r)$. Let us show that $\sum_{j=1}^{k}\|q_j(z)\| \leq \frac{7}{2}$. We distinguish two cases:

If $k \geq n_0$ then $q_j(v_0) = 0$ for all $j = 1, \dots, k$. Then,

$$\sum_{j=1}^{k} \|q_j(z)\| = \sum_{j=1}^{k} \|q_j(\frac{1}{r}(v_1 + \cdots + v_r))\| \leq 2\|\frac{1}{r}(v_1 + \cdots + v_r)\| \leq 2 \leq \frac{7}{2}.$$

If $k < n_0$ we define

$$\delta := \{i \geq 1 : \ \|q_j(v_i)\| \neq 0 \text{ for at least two values of } j\}$$

$$\sigma := \{i \geq 1 : \ \|q_j(v_i)\| \neq 0 \text{ for at most a value of } j\}.$$

Since there are k blocks determined by $p_1 < \cdots < p_{k+1}$ and the v_i have disjoint supports then the cardinal of δ is $|\delta| \leq k-1$. Moreover, if $i \in \sigma$ and j_0 is the only integer between 1 and k such that $\|q_{j_0}(v_i)\| \neq 0$ (if there is no such j then $\|q_j(v_i)\| = 0$ for all $j = 1, \dots, k$) then $\text{supp}(v_i) \subset \{p_{j_0} + 1, \dots, p_{j_0+1}\}$. Hence $q_{j_0}(v_i) = v_i$ and $q_j(v_i) = 0$ for all $j \neq j_0$. Thus

$$\begin{aligned}
\sum_{j=1}^{k} \|q_j(z)\| &\leq \sum_{j=1}^{k} \|q_j(v_0)\| + \frac{1}{r} \sum_{i=1}^{r} \sum_{j=1}^{k} \|q_j(v_i)\| \\
&= \sum_{j=1}^{k} \|q_j(v_0)\| + \frac{1}{r} (\sum_{i \in \delta} \sum_{j=1}^{k} \|q_j(v_i)\| + \sum_{i \in \sigma} \sum_{j=1}^{k} \|q_j(v_i)\|) \\
&\leq 2\|v_0\| + \frac{1}{r}(2 \sum_{i \in \delta} \|v_i\| + \sum_{i \in \sigma} \|v_i\|) \\
&\leq 2 + \frac{1}{r}(2|\delta| + |\sigma|) = 2 + \frac{1}{r}(|\delta| + r) \\
&\leq 2 + \frac{1}{r}(k - 1 + r) = 3 + \frac{k-1}{r} < 3 + \frac{n_0 - 1}{r}.
\end{aligned}$$

Taking $r \geq 2n_0$ we have that

$$\sum_{j=1}^{k} \|q_j(z)\| \leq 3 + \frac{n_0 - 1}{2n_0} \leq \frac{7}{2}.$$

Then we can conclude that for $z = \sum_{n=1}^{\infty} a_n e_n$ we get

$$\begin{aligned}
\|z\| &= \max\{\max_n |a_n|, \frac{1}{2} \sup_{\substack{k \leq p_1 < \cdots < p_{k+1} \\ k=1,2,\dots}} \sum_{j=1}^{k} \|q_j(z)\|\} \\
&\leq \max\{\max_n |a_n|, \frac{7}{4}\} = \frac{7}{4}. \qquad (5)
\end{aligned}$$

Indeed, as the v_j have disjoint supports, for each n such that $a_n \neq 0$ there exists a j so that $n \in \text{supp}(v_j)$, then

if $j \geq 1$ we have that $|a_n| \leq \frac{1}{r}\|v_j\| \leq \frac{1}{r} \leq 1 \leq \frac{7}{4}$,

if $j = 0$ we have that $|a_n| \leq \|v_0\| \leq 1 \leq \frac{7}{4}$.

Hence, from (4) and (5) it follows that $16/9 \leq 7/4$ which is absurd. ∎

Corollary 11. *Tsirelson's space T is reflexive.*

Proof: Since T has an unconditional basis, but it does not have copies of c_0 or ℓ_1 then it is reflexive (see [13, Theorem 1.c.12]). ∎

3 Applications to Infinite Dimensional Holomorphy

3.1 Preliminaries

Let X be a complex Banach space and let U be an open subset of X. Let $\mathcal{P}(^nX)$ denote the space of all continuous n-homogeneous polynomials on X. Let $\mathcal{P}_w(^nX)$ ($\mathcal{P}_{wsc}(^nX)$) be the subspace of $\mathcal{P}(^nX)$ of all continuous polynomials that are weakly continuous on bounded sets (resp. weakly sequentially continuous). Clearly $\mathcal{P}_w(^nX) \subset \mathcal{P}_{wsc}(^nX) \subset \mathcal{P}(^nX)$. We consider these spaces with the usual supremum norm. Given $P \in \mathcal{P}(^nX)$, by $\check{P}$ we mean the continuous symmetric n-linear mapping associated to P.

Let $\mathcal{H}(U)$ be the space of all holomorphic functions on U. If $f \in \mathcal{H}(U)$ we denote by $\sum_{n=0}^{\infty} \frac{\widehat{d^n f(0)}}{n!}$ the Taylor series expansion of f at the origin. A semi-norm p on $\mathcal{H}(U)$ is *ported* by the compact set $K \subset U$ if for every open set V, $K \subset V \subset U$, there exists a constant $c(V) > 0$ such that $p(f) \leq c(V) \sup_{x \in V} |f(x)|$ for all $f \in \mathcal{H}(U)$. The *Nachbin* or *ported* topology on $\mathcal{H}(U)$ is the locally convex topology generated by the semi-norms ported by the compact subsets of U, and it is denoted by τ_ω. Another important topology we shall consider on $\mathcal{H}(U)$ is the τ_δ topology generated by the countable open covers, that is, a semi-norm p on $\mathcal{H}(U)$ is τ_δ *continuous* if for each increasing countable open cover of U, $(U_n)_n$, there exist a constant $C > 0$ and a natural number n_0 such that $p(f) \leq C \sup_{x \in U_{n_0}} |f(x)|$, for all $f \in \mathcal{H}(U)$. The τ_δ topology is then defined as the locally convex topology generated by all τ_δ continuous semi-norms. The τ_δ topology is related to the Nachbin topology τ_ω in the sense that τ_δ is the barrelled topology associated with τ_ω [17] (see also

[6, Corollary 3.37(b)]). Moreover, we can consider a fundamental system of semi-norms for τ_δ with "good Taylor series convergence" properties. Indeed, the τ_δ topology is generated by all semi-norms p on $\mathcal{H}(U)$ satisfying that $p_{|\mathcal{P}(^m X)}$ is τ_ω continuous for all m and that $p(\sum_{n=0}^{\infty} \frac{\widehat{d^n f(0)}}{n!}) = \sum_{n=0}^{\infty} p(\frac{\widehat{d^n f(0)}}{n!})$ for all $\sum_{n=0}^{\infty} \frac{\widehat{d^n f(0)}}{n!} \in \mathcal{H}(U)$. It is worth mentioning that $\tau_\delta = \tau_\omega = \|.\|$ on $\mathcal{P}(^m X)$, for all m (see [5, Example 1.36]). Moreover, if X is a Banach space with a Schauder basis and U is a balanced open subset of X then $\tau_\omega = \tau_\delta$ on $\mathcal{H}(U)$ (see [15] and [6, Corollary 4.16]).

Let $\mathcal{H}_b(U)$ denote the space of holomorphic functions of bounded type on U, that is, the space of all holomorphic functions that are bounded on U-bounded sets. Recall that a bounded subset $A \subset U$ is U-bounded if its distance to the boundary of U is positive (when $U = X$, U bounded sets are just all bounded sets). The space $\mathcal{H}_b(U)$ is endowed with the topology τ_b of uniform convergence on U-bounded sets. It is well known that $(\mathcal{H}_b(U), \tau_b)$ is a Fréchet space.

3.2 Reflexive spaces of holomorphic functions

Let us recall that the dual T^* of T is called the original Tsirelson's space. In 1984 Alencar, Aron and Dineen [1] prove that for any balanced open subset U of the original Tsirelson's space T^* the space $(\mathcal{H}(U), \tau_\omega)$ is reflexive. That was the first known example in the frame of infinite dimensional Banach spaces. However, examples of non normable locally convex spaces were known (e.g. E Fréchet nuclear with the DN property) for which the correspondent space of entire functions was reflexive. On the other hand, the space $\mathcal{H}(\ell_p)$ is not reflexive for any $1 \leq p \leq \infty$ (see [20]). Let us mention what Casazza and Shura [3, Chapter XI] commented about Infinite Dimensional Holomorphy and, in particular, about the example by Alencar, Aron and Dineen: "It is curious to us that this is the only area of analysis wherein the ℓ_p spaces are pathological, while Tsirelson's construction yields an example of a space with 'good' properties". In 1992 Prieto [18] reinforced this appreciation by showing that $(\mathcal{H}_b(T^*), \tau_b)$ is also reflexive. García, Maestre and Rueda used Prieto's techniques to generalize this to weighted spaces of holomorphic functions. In particular, they proved Prieto's result for $\mathcal{H}_b(U)$ where U is any balanced open subset of T^*.

Let us start, in a similar way as in [1], by showing that any Banach space X for which $(\mathcal{H}(X^*), \tau_\omega)$ is reflexive has to share many of the properties of

T. For example, as X is a subspace of $(\mathcal{H}(X^*), \tau_\omega)$, X should be reflexive. Moreover, if we assume that X has a copy of some ℓ_p, $1 < p < \infty$, then let $S : \ell_p \longrightarrow X$ be an isomorphism into. Consider its transpose ${}^tS : X^* \longrightarrow \ell_p' =: \ell_q$ and let $(u_n)_n$ be the canonical basis of ℓ_p. For each $x \in X^*$ and each $(x_n)_n \in \ell_p$ we have that

$$ {}^tS(x)((x_n)_n) = {}^tS(x)(\sum_{n=1}^{\infty} x_n u_n) = \sum_{n=1}^{\infty} x_n \, {}^tS(x)(u_n) = \sum_{n=1}^{\infty} x_n x(S(u_n)). \quad (6) $$

Since $\ell_p' = \ell_q$ it follows that $(x(S(u_n)))_n \in \ell_q$ and then,

$$ \sum_{n=1}^{\infty} |x(S(u_n))|^q < \infty. \quad (7) $$

Let $l := [q] + 1$ and $Q_n := S(u_n)^l \in \mathcal{P}({}^lX^*)$. By (7) it follows that

$$ M := \sum_{n=1}^{\infty} |Q_n(x)| = \sum_{n=1}^{\infty} |x(S(u_n))|^l < \infty. \quad (8) $$

Using that the pointwise limit of a sequence of continuous homogeneous polynomials is continuous, we get that $\sum_{n=1}^{\infty} Q_n \in \mathcal{P}({}^lX^*)$. These calculations will allow to prove that $\mathcal{P}({}^lX^*)$ has a copy of ℓ_∞.

Indeed, if we start with $(x_n)_n \in \ell_\infty$, from (6) and (8) the mapping $(x_n)_n \in \ell_\infty \mapsto P_{(x_n)} \in \mathcal{P}({}^lX^*)$ given by

$$ P_{(x_n)}(x) := \sum_{n=1}^{\infty} x_n(\, {}^tS(x)(u_n))^l $$

is well defined. Let $y_n \in X^*$ be such that ${}^tS(y_n) = u_n^*$. Since $\|y_n\| \leq C\| \, {}^tS(y_n)\| = C\|u_n^*\| = C$ for some $C > 0$, and $P_{(x_n)}(y_n) = x_n$, then

$$ \sup_n |x_n| = \frac{C^l}{C^l} \sup_n |P_{(x_n)}(y_n)| \leq C^l \|P_{(x_n)}\| \leq C^l \sup_n |x_n| M. $$

Hence $\mathcal{P}({}^lX^*)$ should have a copy of ℓ_∞ which is impossible by reflexivity. Thus, X should be a reflexive Banach space without copies of ℓ_p, $1 < p < \infty$.

3.2.1 Weakly sequentially continuous polynomials

In order to prove the result by Alencar, Aron and Dineen we need the following Lemma. With the aim of giving as many details as possible we include the proof which is taken from [6, Lemma 1.9(b)]:

Lemma 12. *[6, Lemma 1.9(b)] Let X be a Banach space and let $(x_j)_{j=1}^k$ be a finite sequence in X. Then*

$$\sup_{\substack{|\lambda_j|\le 1\\ j=1,\dots,k}} |P(\sum_{j=1}^k \lambda_j x_j)|^2 \ge \sum_{j=1}^k |P(x_j)|^2,$$

for all $P \in \mathcal{P}(^m X)$.

Proof: Let $P \in \mathcal{P}(^m X)$. We start by proving [6, Lema 1.9] that for any x, $y \in X$ then

$$\sup_{|\lambda|\le 1} |P(x+\lambda y)|^2 \ge |P(x)|^2 + |P(y)|^2. \tag{9}$$

Define

$$g(\lambda) := P(x+\lambda y) = \sum_{j=0}^m \binom{m}{j} \check{P}(x^j, y^{m-j})\lambda^{m-j}.$$

Denoting $a_{m-j} := \binom{m}{j} \check{P}(x^j, y^{m-j})$ we have $a_0 = P(x)$ and $a_m = P(y)$. Then, by the Maximum Modulus Principle

$$\begin{aligned}
\sup_{|\lambda|\le 1} |P(x+\lambda y)|^2 &= \sup_{|\lambda|=1} |P(x+\lambda y)|^2 \;=\; \sup_{|\lambda|=1} |g(\lambda)|^2 \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sup_{|\lambda|=1} |g(\lambda)|^2\, d\theta \;\ge\; \frac{1}{2\pi}\int_0^{2\pi} |g(e^{i\theta})|^2\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (\sum_{j=0}^m a_j e^{ij\theta})(\sum_{k=0}^m \overline{a_k} e^{-ik\theta})\, d\theta \\
&= \sum_{0\le j,k\le m} a_j \overline{a_k} \frac{1}{2\pi}\int_0^{2\pi} e^{i(j-k)\theta}\, d\theta \\
&= \sum_{j=0}^m |a_j|^2 \;\ge\; |a_0|^2 + |a_m|^2 \\
&= |P(x)|^2 + |P(y)|^2.
\end{aligned}$$

Hence if $|\mu| = 1$,

$$\sup_{|\lambda|\le 1} |P(\mu x + \lambda y)| \ge (|P(\mu x)|^2 + |P(y)|^2)^{1/2} = (|P(x)|^2 + |P(y)|^2)^{1/2},$$

and using again the Maximum Modulus Principle,

$$\sup_{\substack{|\lambda|\le 1\\|\mu|\le 1}} |P(\mu x + \lambda y)| = \sup_{\substack{|\lambda|\le 1\\|\mu|=1}} |P(\mu x + \lambda y)| \ge (|P(x)|^2 + |P(y)|^2)^{1/2}.$$

Apply now induction: if we assume that given $x_1, \ldots, x_{k-1} \in X$

$$\sup_{\substack{|\lambda_j|\le 1\\ j=1,\ldots,k-1}} |P(\sum_{j=1}^{k-1} \lambda_j x_j)|^2 \ge \sum_{j=1}^{k-1} |P(x_j)|^2,$$

then for $x_1, \ldots, x_k \in X$ it follows that

$$\begin{aligned} \sup_{\substack{|\lambda_j|\le 1\\ j=1,\ldots,k}} |P(\sum_{j=1}^{k} \lambda_j x_j)|^2 &= \sup_{\substack{|\lambda_j|\le 1\\ j=1,\ldots,k-1}} \sup_{|\lambda_k|\le 1} |P((\sum_{j=1}^{k-1} \lambda_j x_j) + \lambda_k x_k)|^2 \\ &\ge \sup_{\substack{|\lambda_j|\le 1\\ j=1,\ldots,k-1}} |P(\sum_{j=1}^{k-1} \lambda_j x_j)|^2 + |P(x_k)|^2 \\ &\ge \sum_{j=1}^{k-1} |P(x_j)|^2 + |P(x_k)|^2 = \sum_{j=1}^{k} |P(x_j)|^2. \end{aligned}$$

■

Theorem 13. *(Alencar, Aron, Dineen [1]) Every continuous polynomial on T^* is weakly sequentially continuous at the origin.*

Proof: It is an adaptation of the proof of [1, Theorem 6]. Assume that there exist m and $P \in \mathcal{P}({}^m T^*)$ which is not weakly sequentially continuous at the origin. Let $\delta > 0$ and $(x_n)_n \in T^*$ weakly convergent to 0 such that $|P(x_n)| > \delta$ for all n. We can assume that $\|x_n\| \le 1$ for all n.

Let $(t_n)_n$ be the Schauder basis of T^*, that is, $t_n = e_n^*$. For each natural number N we write $x_N = \sum_{j=1}^{\infty} a_j^N t_j$.

Take $N_1 := 1$. Since

$$\delta < |P(x_1)| = |P(\sum_{i=1}^{\infty} a_i^1 t_i)| = \lim_n |P(\sum_{i=1}^{n} a_i^1 t_i)|$$

there exists $n_1 > 2$ such that

$$|P(\sum_{i=1}^{n_1} a_i^1 t_i)| > \delta.$$

Being $(x_N)_N$ weakly convergent to 0, given m the sequence $e_m(x_N) = a_m^N$ converges to 0 when $N \to \infty$. Then $\sum_{i=1}^{n_1} a_i^N t_i \longrightarrow 0$ when $N \to \infty$. Let $N_2 > N_1$ be such that

$$\| \sum_{i=1}^{n_1} a_i^n t_i \| \leq \frac{1}{2}$$

for all $n \geq N_2$. By induction we construct strictly increasing sequences $(n_j)_j$ and $(N_j)_j$ such that

$$|P(\sum_{i=1}^{n_j} a_i^{N_j} t_i)| > \delta \quad \text{and} \quad \| \sum_{i=1}^{n_j} a_i^{N_k} t_i \| \leq \frac{1}{2^j}$$

for all $k > j$. Let $q^{n_j}(x_{N_j}) := \sum_{i=1}^{n_j} a_i^{N_j} t_i$. By the above Lemma, given a natural number l,

$$\begin{aligned} \sup_{\substack{|\lambda_j| \leq 1 \\ j=2^l+1,\ldots,2^{l+1}}} \left| P\left(\sum_{j=2^l+1}^{2^{l+1}} \lambda_j q^{n_j}(x_{N_j}) \right) \right|^2 \quad &\geq \quad \sum_{j=2^l+1}^{2^{l+1}} |P(q^{n_j}(x_{N_j}))|^2 \\ &> \quad \sum_{j=2^l+1}^{2^{l+1}} \delta^2 \;=\; \delta^2(2^{l+1} - 2^l) \\ &= \quad \delta^2 2^l. \end{aligned}$$

Then there exists a sequence $(\lambda_j)_{j=2^l+1}^{2^{l+1}}$, with $|\lambda_j| \leq 1$, such that

$$\left| P\left(\sum_{j=2^l+1}^{2^{l+1}} \lambda_j q^{n_j}(x_{N_j}) \right) \right| > \delta(\sqrt{2})^l. \tag{10}$$

On the other hand, if we call

$$q_{n_{j-1}}^{n_j}(x_{N_j}) := q^{n_j}(x_{N_j}) - q^{n_{j-1}}(x_{N_j}) = \sum_{i=n_{j-1}+1}^{n_j} a_i^{N_j} t_i,$$

then the vectors $\{q_{n_{j-1}}^{n_j}(x_{N_j})\}_{j=2^l+1}^{2^{l+1}}$ have increasing disjoint supports.

If $2^l + 1 \leq j \leq 2^{l+1}$ then $n_{j-1} \geq j-1 \geq 2^l$. Hence

$$q^{2l}(q_{n_{j-1}}^{n_j}(x_{N_j})) = 0,$$

that is, the first 2^l coordinates are 0. This allows us to apply Proposition 5 and we get

$$\begin{aligned} \| \sum_{j=2^l+1}^{2^{l+1}} \lambda_j q_{n_{j-1}}^{n_j}(x_{N_j}) \| &\leq 2 \sup_{2^l+1 \leq j \leq 2^{l+1}} |\lambda_j| \| q_{n_{j-1}}^{n_j}(x_{N_j}) \| \\ &\leq 2 \sup_{2^l+1 \leq j \leq 2^{l+1}} \| \sum_{i=n_{j-1}+1}^{n_j} a_i^{N_j} t_i \| \\ &\leq 2 \sup_{2^l+1 \leq j \leq 2^{l+1}} \| x_{N_j} \| \leq 2. \end{aligned}$$

Then,

$$\begin{aligned} \left\| \sum_{j=2^l+1}^{2^{l+1}} \lambda_j q^{n_j}(x_{N_j}) \right\| &\leq \left\| \sum_{j=2^l+1}^{2^{l+1}} \lambda_j q^{n_{j-1}}(x_{N_j}) \right\| + \left\| \sum_{j=2^l+1}^{2^{l+1}} \lambda_j q_{n_{j-1}}^{n_j}(x_{N_j}) \right\| \\ &\leq \sum_{j=2^l+1}^{2^{l+1}} |\lambda_j| \frac{1}{2^{j-1}} + 2 \leq 3. \end{aligned}$$

Hence

$$\left| P \left(\sum_{j=2^l+1}^{2^{l+1}} \lambda_j q^{n_j}(x_{N_j}) \right) \right| \leq \|P\| 3^m.$$

Thus, using (10) it follows that

$$\delta(\sqrt{2})^l < \|P\| 3^m$$

for any l, which is absurd. It has been proved that any continuous polynomial is weakly sequentially continuous at the origin. ∎

Theorem 14. *(Alencar, Aron, Dineen [1]) Let X be a Banach space and let $m \in \mathbb{N}$. All continuous polynomials of degree less than or equal to m are weakly sequentially continuous if and only if all of them are weakly sequentially continuous at the origin.*

Proof: The proof is taken from [6, Lema 2.32]. Let us just prove the non trivial implication. Assume that every continuous polynomial of degree $\leq m$ is weakly sequentially continuous at 0. Let $P = \sum_{n=0}^{m} P_n$, $P_n \in \mathcal{P}(^nX)$. Let $(x_j)_j$ weakly convergent to some x in X. For each j we have that

$$|P(x_j)-P(x)| = |\sum_{n=0}^{m}(P_n(x_j)-P_n(x))| \leq \sum_{n=0}^{m}\left|\sum_{k=1}^{n}\binom{n}{k}\check{P}_n(x^{n-k},(x_j-x)^k)\right|.$$

For each n and k, $1 \leq k \leq n$, the mapping

$$y \mapsto \binom{n}{k}\check{P}_n(x^{n-k}, y^k)$$

is a k-homogeneous polynomial, $k \leq m$ and, by hypothesis, weakly sequentially continuous at 0. Then,

$$\binom{n}{k}\check{P}_n(x^{n-k},(x_j-x)^k) \longrightarrow 0$$

when $j \to \infty$, for all $1 \leq k \leq n \leq m$. Hence $P(x_j) \longrightarrow P(x)$ when $j \to \infty$. Thus P is weakly sequentially continuous at x. ∎

Corollary 15. *$\mathcal{P}(^nT^*) = \mathcal{P}_{wsc}(^nT^*)$, for all n.*

3.2.2 Reflexivity on spaces of polynomials

Using weakly sequentially continuity of polynomials, Ryan [20] proved the following result about reflexivity on certain spaces of polynomials.

Theorem 16. *(Ryan [20]) Let X be a reflexive Banach space. If $\mathcal{P}(^nX) = \mathcal{P}_{wsc}(^nX)$ then $\mathcal{P}(^nX)$ is reflexive.*

Proof: We follow the proof of [6, Proposition 2.30]. By [11] a Banach space is reflexive if and only if every continuous linear functional attains its supremum on the unit sphere. So, since $\mathcal{P}(^nX)$ is a dual space it follows that $\mathcal{P}(^nX)$ is reflexive if and only if every $P \in \mathcal{P}(^nX)$ attains its norm. Let $P \in \mathcal{P}(^nX)$ and choose a sequence $(x_n)_n$, $\|x_n\| \leq 1$, such that $(P(x_n))_n$ converges to $\|P\|$. Being the closed unit ball of X weakly sequentially compact by the Eberlein-Smulian Theorem (see [4, Theorem III]), there exists a subsequence $(x_{n_k})_k$ weakly convergent to some x in X. By assumption P is weakly sequentially continuous and then $(P(x_{n_k}))_k$ converges to $P(x)$ too. Hence $P(x) = \|P\|$. Thus, $\mathcal{P}(^nX)$ is reflexive. ■

Corollary 17. *(Alencar, Aron, Dineen [1]) The space $\mathcal{P}(^nT^*)$ is reflexive, for all n.*

Proof: It follows from Corollaries 11, 15 and Theorem 16. ■

We have seen that $\mathcal{P}(^nT^*) = \mathcal{P}_{wsc}(^nT^*)$, for all n. Let us go further by proving that $\mathcal{P}_w(^nT^*) = \mathcal{P}_{wsc}(^nT^*)$, for all n.

Theorem 18. *(Aron, Hervés, Valdivia [2]) Let X be a Banach space. If X does not have copies of ℓ_1 then every weakly sequentially continuous polynomial on X is weakly continuous on bounded sets.*

Proof: The proof follows [6, Proposition 2.36]. Let P be a weakly sequentially continuous polynomial on X. Using Rosenthal's ℓ_1-Theorem, every bounded sequence on X contains a weakly Cauchy subsequence (the result can be found in [13, Theorem 2.e.5] or [4]). Moreover, every bounded subset is sequentially dense in its weak closure. Then if A is a bounded subset of X and x belongs to the weak closure of A, $\overline{A}^{\sigma(X,X^*)}$, then there exists a sequence $(x_j)_j \subset A$ weakly convergent to x. Being P weakly sequentially continuous it follows that $P(x_j)$ converges to $P(x)$. Then

$$P(\overline{A}^{\sigma(X,X^*)}) \subset \overline{P(A)}.$$

Hence P is weakly continuous on bounded sets. ■

The converse to this result is also true and is due to Diestel (see [6, Proposition 2.36]).

Corollary 19. $\mathcal{P}(^nT^*) = \mathcal{P}_{wsc}(^nT^*) = \mathcal{P}_w(^nT^*)$ *for all n.*

3.2.3 Reflexivity on spaces of holomorphic functions of bounded type

By using a projective description of the space $\mathcal{H}_b(U)$ and the linearization of spaces of bounded holomorphic mappings given by Mujica [14, Theorem 2.1], Mujica and Valdivia [16] proved next result, that yields to the reflexivity of $(\mathcal{H}_b(U), \tau_b)$, for all balanced open subset $U \subset T^*$.

Theorem 20. *(Mujica, Valdivia [16, Corollary 4.2]) Let X be a reflexive Banach space. If every continuous polynomial on X is weakly continuous on bounded sets then $(\mathcal{H}_b(U), \tau_b)$ is reflexive, for all balanced open subset U of X.*

Let us now show another approach to the reflexivity of $\mathcal{H}_b(T^*)$ by means of Schauder decompositions. We will follow the work of Prieto [18] and García, Maestre and Rueda [8].

Given a locally convex space E, a sequence $(E_n)_n$ of subspaces of E is said to be a *Schauder decomposition* of E if any x in E can be written in a unique way as $x = \sum_{n=1}^{\infty} x_n$, with $x_n \in E_n$, and the projections $x = \sum_{n=1}^{\infty} x_n \mapsto \sum_{n=1}^{m} x_n$, $m \in N$, are continuous. Schauder bases on Banach spaces can be considered as Schauder decompositions of dimension one.

Schauder decompositions are a powerful tool to lift properties from the spaces forming the decomposition to the whole space. Next result, due to Kalton [12, Theorem 3.2], shows that the reflexivity property can be lifted for some special Schauder decompositions.

A Schauder decomposition $(E_n)_n$ of E is *shrinking* if $((E_n)'_\beta)_n$ is a Schauder decomposition of E'_β, where E'_β denotes the strong dual of E. The Schauder decomposition $(E_n)_n$ is called *boundedly complete* if the series $\sum_{n=1}^{\infty} x_n$, $x_n \in E_n$, converges in E whenever the sequence $(\sum_{n=1}^{m} x_n)_m$ is bounded in E. The Schauder decomposition $(E_n)_n$ is said *Taylor Series convergent* (T.S. τ-convergent for short) if for any sequence $(x_n)_n$, $x_n \in E_n$, $n \in \mathbb{N}$, the series $\sum_{n=1}^{\infty} x_n$ converges in E whenever the series $\sum_{n=1}^{\infty} p(x_n)$ converges for all continuous semi-norm p on E. Finally, $(E_n)_n$ is called *unconditional* if, for each $x = \sum_{n=1}^{\infty} x_n$ in E, $x_n \in E_n$ for all n, the series $\sum_{n=1}^{\infty} x_{\sigma(n)}$ converges to x, for all permutation σ of the natural numbers.

The key to study the reflexivity of spaces of holomorphic functions is the following result due to Kalton [12, Theorem 3.2]:

Theorem 21. *(Kalton [12, Theorem 3.2]) A locally convex space E with a* *boundedly complete shrinking Schauder decomposition $(E_n)_n$ is semi-reflexive if and only if each space E_n is semi-reflexive.*

We recall that a locally convex space is reflexive if and only if it is semi-reflexive and barrelled. For instance, any Fréchet space is barrelled.

Using Kalton's result and the fact that $(\mathcal{P}(^nX))_n$ is a boundedly complete shrinking Schauder decomposition of the Fréchet space $(\mathcal{H}_b(U), \tau_b)$, Prieto [18] and García, Maestre and Rueda [8] showed the equivalence between the reflexivity of $\mathcal{H}_b(U)$ and the reflexivity of all $\mathcal{P}(^nX)$, $n \in \mathbb{N}$. Prieto proves it for $U = X$ whereas García, Maestre and Rueda got it for an arbitrary balanced open subset $U \subset X$. Actually, in [8] it is shown a much more general result involving weighted spaces of holomorphic functions.

Theorem 22. *Let X be a Banach space and let U be a balanced open subset of X. The space $(\mathcal{H}_b(U), \tau_b)$ is reflexive if and only if $(\mathcal{P}(^nX), \|.\|)$ is reflexive for all $n \in \mathbb{N}$.*

As a consequence they also proved Corollary 23 (for $U = X$ in [18] and arbitrary balanced open set $U \subset T^*$ in [8]).

Corollary 23. *The space $(\mathcal{H}_b(U), \tau_b)$ is reflexive, for all balanced open subset $U \subset T^*$.*

3.2.4 Reflexivity on spaces of holomorphic functions

Let us now prove that $(\mathcal{H}(U), \tau_\omega)$ is reflexive, for all balanced open subset U of T^*. We will follow the above ideas by looking for a Schauder decomposition of $(\mathcal{H}(U), \tau_\omega)$.

Lemma 24. *Let $(E_n)_n$ be a Schauder decomposition of a locally convex space (E, τ) which is T.S. τ-complete. Assume that τ is generated by a family of semi-norms p satisfying that*

(a) $p_{|E_n}$ is τ continuous in E_n, for all n, and

(b) $p(x) = \sum_{n=1}^{\infty} p(x_n)$ for all $x = \sum_{n=1}^{\infty} x_n \in E$.

Then $(E_n)_n$ is boundedly complete.

Proof: Let $(x_n)_n$, $x_n \in E_n$, $n \in \mathbb{N}$, such that $(\sum_{n=1}^{m} x_n)_m$ is τ bounded. We have to show that the sequence $(\sum_{n=1}^{m} x_n)_m$ converges to some x in E.

Let p be a continuous semi-norm. By assumption we can choose p satisfying (b). Then the sequence $(p(\sum_{n=1}^{m} x_n))_m = (\sum_{n=1}^{m} p(x_n))_m$ is bounded. Hence $\sum_{n=1}^{\infty} p(x_n)$ converges. Since $(E_n)_n$ is T.S. τ-complete we conclude that $\sum_{n=1}^{\infty} x_n$ converges in E. ∎

Theorem 25. *Let U be a balanced open subset of a Banach space X. Then $(\mathcal{P}(^nX), \|.\|)_n$ is a boundedly complete shrinking Schauder decomposition of $(\mathcal{H}(U), \tau_\delta)$.*

Proof: By [6, Proposition 3.36] and [5, Corollary 3.14] $(\mathcal{P}(^nX))_n$ is a shrinking Schauder decomposition of $\mathcal{H}(U)$. Moreover, since $(\mathcal{H}(U), \tau_\delta)$ is T.S. τ_δ complete (see the proof of [6, Corollary 3.53]) it follows from Lemma 24 that $(\mathcal{P}(^nX))_n$ is a boundedly complete Schauder decomposition of $\mathcal{H}(U)$. ∎

Actually, it is well-known that $(\mathcal{H}(U), \tau_\delta)$ is complete when U is a balanced open subset of a Banach space X (see [6, Corollary 3.53]).

Theorem 26. *(Dineen [6, Corollary 4.19]) Let X be a Banach space with a Schauder basis. Let U be a balanced open subset of X. The space $(\mathcal{H}(U), \tau_\omega)$ is reflexive if and only if $(\mathcal{P}(^nX), \|.\|)$ is reflexive for all n.*

Proof: By Theorem 25 and Kalton's result it follows that $(\mathcal{H}(U), \tau_\delta)$ is semi-reflexive if and only if $(\mathcal{P}(^nX), \|.\|)$ is reflexive for all n. Since the τ_δ topology is barrelled (see [6, Corollary 3.37]), and a locally convex space is reflexive if and only if it is semi-reflexive and barrelled, it follows that $(\mathcal{H}(U), \tau_\delta)$ is reflexive if and only if $(\mathcal{P}(^nX), \|.\|)$ is reflexive for all n. Moreover, since X has a Schauder basis, by [15] or [6, Corollary 4.16], $\tau_\omega = \tau_\delta$ on $\mathcal{H}(U)$. ∎

Corollary 27. *(Alencar, Aron, Dineen [1]) The space $(\mathcal{H}(U), \tau_\omega)$ is reflexive, for all balanced open subset U of T^*.*

Proof: It follows from Proposition 2 and Theorem 26. ∎

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