II segundo ENAMA

Encontro Nacional de Análise Matemática e Aplicações

Minicurso 3

Non-variational Semilinear Elliptic Systems

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Realização

UFPB UFCG

João Pessoa, 05 a 07 de novembro de 2008

II ENAMA

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Os organizadores do II ENAMA desejam expressar sua gratidão aos órgãos e instituições que apoiaram e tornaram possível a realização deste evento: CNPq, CAPES, UFPB, UFCG, Banco do Brasil e Prefeitura Municipal de João Pessoa. Agradecem também a todos participantes do evento, bem como aos colaboradores pelo entusiasmo e esforço, que tanto contribuíram para o sucesso deste evento.

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Non-variational Semilinear Elliptic Systems

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September 8, 2008

1 Introduction

In these lectures we survey some results on the existence of solutions for the system of semilinear elliptic equations of the type

$$-\Delta u = f(x, u, v), \quad -\Delta v = g(x, u, v) \text{ in } \Omega. \tag{1.1}$$

Most of the material presented here is taken from our recent paper [70]. Details of the proofs can be seen in the papers listed in the References. The list of papers in the References at the end of these lectures were taken from [70].

On the above equations u and v are real-valued functions $u, v : \overline{\Omega} \to \mathbb{R}$, where Ω is some domain in R^N, $N \geq 3$, and $\overline{\Omega}$ its closure. In order to simplify the statements we assume that Ω is a smooth domain, although most of the results can be obtained under less regularity of the domain. We do not discuss the case $N = 2$, where the imbedding theorems of Trudinger-Moser allow the treatment of nonlinearities which have a growth faster than the polynomial growth required by the Sobolev imbeddings.

Although we concentrate in the case of the Laplacian many results can be extended to general second order elliptic operators. Of course, there is the problem of Maximum Principles for systems, which poses interesting questions.

In the present lectures the nonlinearity of the problem appears only in the real-valued functions $f, g : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Problems involving the p-Laplacian and fully nonlinear operators have been extensively studied recently.

Let us just mention two classes of variational systems that have been object of much research recently.

The system(1.1) is *variational* if either one of the following conditions holds:

(I) There is a real-valued differentiable function $F(x,u,v)$ for $(x,u,v) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$ such that $\dfrac{\partial F}{\partial u} = f$ and $\dfrac{\partial F}{\partial v} = g$. In this case, the system is said to be *gradient*.

(II) There is a real-valued differentiable function $H(x,u,v)$ for $(x,u,v) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$ such that $\dfrac{\partial H}{\partial u} = g$ and $\dfrac{\partial H}{\partial v} = f$. In this case, the system is said to be *Hamiltonean*.

The terminology *variational* comes from the fact that in both cases, the above system is the Euler-Lagrange equations of a functional naturally associated to the system. Indeed, if we work with functions u and v in $H_0^1(\Omega)$, the functional associated to the gradient system is

$$\Phi(u,v) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla v|^2) - \int_\Omega F(x,u,v). \tag{1.2}$$

while the one associated to a Hamiltonean system is

$$\Phi(u,v) = \int_\Omega \nabla u . \nabla v - \int_\Omega H(x,u,v), \tag{1.3}$$

provided F and H have the appropriate growth in order to get their integrability. Namely,

$$F(x,u,v) \le C(1 + |u|^p + |v|^q), \ \forall x \in \Omega, \ u,v \in \mathbb{R}$$

with $p,q \le \frac{2N}{N-2}$, if the dimension $N \ge 3$, and

$$H(x,u,v) \le C(1 + |u|^p + |v|^q), \forall x \in \Omega, \ u,v \in \mathbb{R}$$

with $p,q \le \frac{2N}{N-2}$.

This restriction on the powers of u and v in the Hamiltonean case is too restrictive. It has been proved that more general values of p,q can be allowed, see [26] and [84].

These two types of variational systems can be treated using Critical Point Theory.

However, if the system is not variational we use topological methods, via Leray-Schauder degree theory. The difficulty is obtaining a priori bounds for the solutions. There are several methods to tackle this question. We will comment some of them, including the use of Moving Planes and Hardy type inequalities. However the most successful one in our framework seems to be the Blow-up method; here we follow [42]. This method leads naturally to Liouville type theorems, that is, theorems asserting that certain systems have no non-trivial solution in the whole space R^N or in a half-space R^N_+. In Section 2, we present some results on Liouville theorems for systems.

The well-known notion of criticality for the Dirichlet problem

$$-\Delta u = f(u), \text{ in } \Omega, \ v \ = \ 0 \text{ on } \partial\Omega,$$

that is $p = \frac{N+2}{N-2}$, if $|f(u)| \sim |u|^p$ as $u \to \infty$, is replaced in the case of systems by the so-called critical hyperbola. In [26] we studied the system

$$\left\{ \begin{array}{rcll} -\Delta u & = & f(v), & \text{in} \quad \Omega, \\ -\Delta v & = & g(u), & \text{in} \quad \Omega, \\ u & = & 0, \ v \ = \ 0 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{1.4}$$

where $|f(v)| \le |v|^p$, and $|g(u)| \le |u|^q$, as $u, v \to +\infty$.

There we observed that a priori bounds on positive solutions could be obtained by the technique used in [39] if

(p, q) satisfy $\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}$.

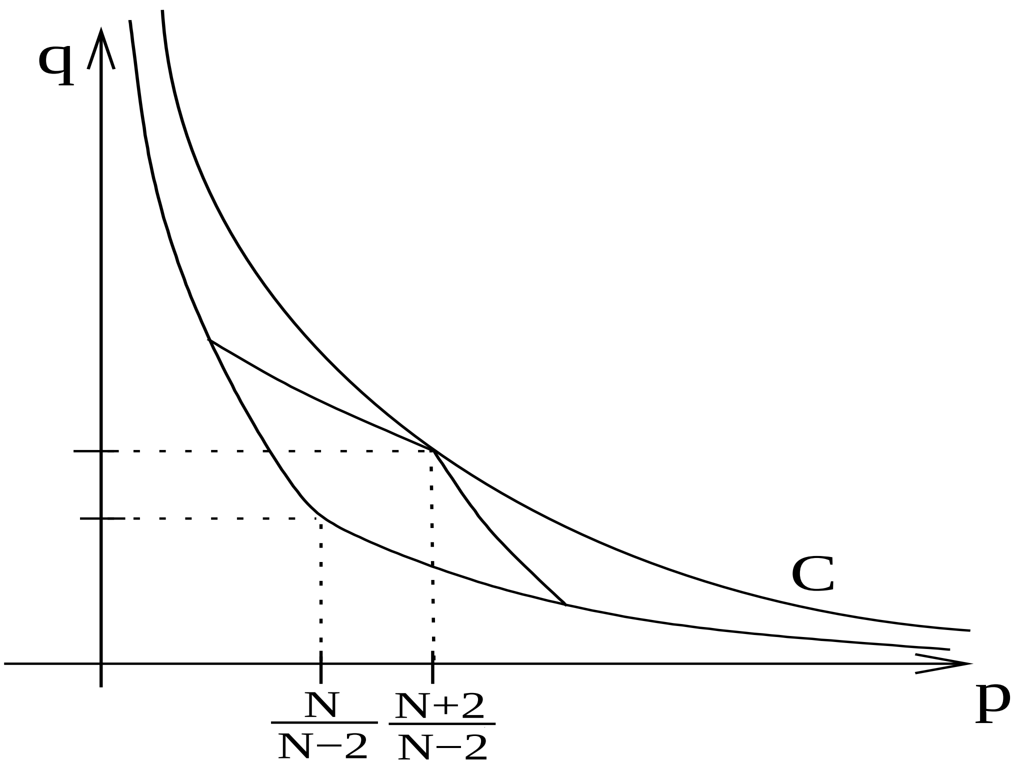

Figure 1: $\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}$.

In the above picture the higher curve C is the critical hyperbola. The lower parabola is the one obtained by Souto [95], related to the Liouville results, and the two little curves are connected with the work of Busca-Manasevich [18].

We call a system (1.4) to be *sub-critical* if the powers p, q are the coordinates of a point below the critical hyperbola.

For $N \geq 3$, the "critical hyperbola" plays an important role on the existence of non-trivial solutions. For instance, for the model problem (1.4) with $(p, q) \in \mathbb{R}^2$ on and above this curve, one finds the typical problems of non-compactness, and non-existence of solutions, as it was proved in [100], [26], [77], using Pohozaev type arguments.

2 Nonvariational elliptic systems.

In this section we study some systems of the general form (1.1) that do not fall in the categories of the variational systems mentioned before. So we shall treat them by Topological Methods. We will discuss here the existence of positive solutions. The main tool is the following result, due to Krasnoselskii [73], see also [12], [39].

Theorem 2.1 *(* **Krasnoselski** *) Let $\mathcal{C}$ be a cone in a Banach space X and $T : \mathcal{C} \to \mathcal{C}$ a compact mapping such that $T(0) = 0$. Assume that there are real numbers $0 < r < R$ and $t > 0$ such that*

(i) $x \neq tTx$ for $0 \leq t \leq 1$ and $x \in \mathcal{C}, ||x|| = r$, *and*

(ii) There exists a compact mapping $H : \overline{B}_R \times [0, \infty) \to \mathcal{C}$ (where $B_\rho = \{x \in \mathcal{C} : ||x|| < \rho\}$) such that

(a) $H(x, 0) = Tx$ for $||x|| = R$,

(b) $H(x, t) \neq x$ for $||x|| = R$ *and* $t \geq 0$

(c) $H(x, t) = x$ has no solution $x \in \overline{B}_R$ for $t \geq t_0$

Then

$$i_c(T, B_r) = 1, \ i_c(T, B_R) = 0, \ i_c(T, U) = -1,$$

where $U = \{x \in \mathcal{C} : r < ||x|| < R\}$, and i_c denotes the Leray-Schauder index. As a consequence T has a fixed point in U.

When applying this result the main difficulty arises in the verification of condition (b), which is nothing more than an *a priori bound* on the solutions of the system. It is well known that the existence of a priori bounds depends on the growth of the functions f and g as u and v go to infinity. It is known that, when treating the variational systems in dimension $N \geq 3$, the nonlinearities were restricted to have polynomial growth. This was a requirement in order to get the associated functional defined, as well as a Palais-Smale condition. Here similar restrictions appear in order to get a priori bounds.

A priori bounds for positive solutions of superlinear elliptic equations (the scalar case), namely

$$-\Delta u = f(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{2.1}$$

was first considered by Brézis-Turner [21] using an inequality due to Hardy. The same technique was used in [27] to obtain a priori bounds for solutions of systems

$$\begin{aligned} -\Delta u &= f(x,u,v,\nabla u,\nabla v), \\ -\Delta v &= g(x,u,v,\nabla u,\nabla v) \text{ in } \Omega, \\ u &= v = 0 \text{ on } \partial\Omega \end{aligned} \tag{2.2}$$

under the following set of conditions:

(f_1) $f, g : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times R^N \times R^N \longrightarrow \mathbb{R}$ is continuous,

(f_2) $\liminf_{t\to\infty} \frac{f(x,s,t,\xi,\eta)}{t} > \lambda_1$ uniformly in $(x,s,t,\xi,\eta) \in \Omega \times \mathbb{R} \times \mathbb{R} \times R^N \times R^N$,

(f_3) There exist $p \geq 1$ and $\sigma \geq 0$ such that

$$|f(x,s,t,\xi,\eta)| \leq C(|t|^p + |s|^{p\sigma} + 1)$$

(g_2) $\liminf_{t\to\infty} \frac{g(x,s,t,\xi,\eta)}{t} > \lambda_1$ uniformly in $(x,s,t,\xi,\eta) \in \Omega \times \mathbb{R} \times \mathbb{R} \times R^N \times R^N$,

(g_3) There exist $q \geq 1$ and $\sigma' \geq 0$ such that

$$|g(x,s,t,\xi,\eta)| \leq C(|s|^q + |t|^{q\sigma'} + 1).$$

In the work [27], instead of the critical hyperbola, two other hyperbolas appeared, due to the limitations coming from the method. Namely

$$\frac{1}{p+1} + \frac{N-1}{N+1}\frac{1}{q+1} = \frac{N-1}{N+1},$$

$$\frac{N-1}{N+1}\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-1}{N+1}.$$

This is precisely like in the scalar case in [21] where the exponent $\frac{N+1}{N-1}$ appeared instead of $\frac{N+2}{N-2}$. Observe that the intersection of the two above hyperbolas is the Brézis-Turner exponent $\frac{N+1}{N-1}$.

Theorem 2.2 *Let Ω be a smooth bounded domain in R^N, with $N \geq 4$. Assume that the conditions f_1,f_2,f_3,g_2,g_3 hold with p, q being the coordinates of a point below both of the above hyperbolas. Suppose that σ, σ' are given by*

$$\sigma = \frac{L}{\max(L,K)}, \quad \sigma' = \frac{K}{\max(L,K)},$$

where

$$K = \frac{p}{p+1} - \frac{2}{N} > 0, \quad L = \frac{q}{q+1} - \frac{2}{N} > 0.$$

Then the positive solutions of the system (2.2) are bounded in L^∞.

Remarks on the proof of Theorem 2.2. As said above the proof relies on an inequality of Hardy, namely

$$\|\frac{u}{\varphi_1}\|_{L^q} \leq C\|Du\|_{L^q}, \quad \forall\ u \in W_0^{1,q}.$$

Here $q > 1$ and φ_1 is the eigenfunction associated to the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. In [21] they introduced an interpolation of the Hardy inequality ($q = 2$) with Sobolev inequality

$$\|u\|_{2^*} \leq C\|Du\|_{L^2},, \ \forall\ \ u \in H_0^1,$$

obtaining the inequality below

$$\|\frac{u}{\varphi_1^\tau}\|_{L^q} \leq C\|Du\|_{L^2}, \ \ \forall\ \ u \in H_0^1,$$

where $\frac{1}{q} = \frac{1}{2} - \frac{1-\tau}{N}$. For the purpose of proving Theorem 2.2 one needs a Hardy-type inequality which follows from the inequalities above, namely

Proposition 2.1 *Let $r_0 \in (1,\infty]$, $r_1 \in [1,\infty)$ and $u \in L^{r_0}(\Omega) \bigcap W_0^{1,r_1}$. Then for all $\tau \in [0,1]$ we have*

$$\frac{u}{\varphi_1^\tau} \in L^r(\Omega), \text{ where } \frac{1}{r} = \frac{1-\tau}{r_0} + \frac{\tau}{r_1}.$$

Moreover

$$\|\frac{u}{\varphi_1^\tau}\|_{L^r} \leq C\|u\|_{L^{r_0}}^{1-\tau}\|u\|_{W^{1,r_1}}^{\tau},$$

where the constant C depends only on τ, r_0 and r_1.

In [39], moving planes techniques and Pohozaev type identities were used to obtain a priori bounds for positive solutions of the scalar equation (2.1). This method was extended by Clement-deFigueiredo-Mitidieri [26] to Hamiltonian systems of the type (1.4). Although one obtains the right growth for the nonlinear terms, namely $f(s) \sim s^q, g(s) \sim s^p$ with any p, q below the critical hyperbola, the method does not generalizes for other second order elliptic operators, and there are restrictions on the type of regions Ω.

Another interesting approach to obtaining a priori bounds can be seen in Quittner-Souplet [87] using weighted Lebesgue spaces.

The Blow-up Method.

The other technique used to obtain a priori bounds for solutions of systems is the *blow-up method*, first used in [64] to treat scalar equations as (2.1). Since there is some symmetry regarding the assumptions on the behavior of the nonlinearities with respect to the unknowns u, v, we change henceforth in this section the notations of these variables, and use u_1, u_2. So, let us consider the system in the form:

$$\left\{ \begin{array}{rcll} -\Delta u_1 &=& f(x, u_1, u_2) & \text{in} \quad \Omega \\ -\Delta u_2 &=& g(x, u_1, u_2) & \text{in} \quad \Omega \\ u_1 = u_2 &=& 0 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{2.3}$$

where u_1, u_2 are consequently real-valued functions defined on a smooth bounded domain Ω in R^N, $N \geq 3$, and f and g are real-valued functions defined in $\overline{\Omega} \times R \times R$.

We then write the system as follows, assuming that the leading parts of f and g involve just pure powers of u_1 and u_2.

$$\left\{ \begin{array}{rcl} -\Delta u_1 &=& a(x)u_1^{\alpha_{11}} + b(x)u_2^{\alpha_{12}} + h_1(x, u_1, u_2) \\ -\Delta u_2 &=& c(x)u_1^{\alpha_{21}} + d(x)u_2^{\alpha_{22}} + h_2(x, u_1, u_2), \end{array} \right. \tag{2.4}$$

where the α's are nonnegative real numbers, $a(x), b(x), c(x), d(x)$ are nonnegative continuous functions on $\overline{\Omega}$, and h_1, h_2 are locally bounded functions (the lower order terms) such that

$$\left\{ \begin{array}{rcl} \lim\limits_{|(u_1,u_2)| \to \infty} \left(a(x)u_1^{\alpha_{11}} + b(x)u_3^{\alpha_{12}}\right)^{-1} |h_1(x, u_1, u_2)| &=& 0 \\ \lim\limits_{|(u_1,u_2)| \to \infty} \left(c(x)u_1^{\alpha_{21}} + d(x)u_2^{\alpha_{22}}\right)^{-1} |h_2(x, u_1, u_2)| &=& 0. \end{array} \right. \tag{2.5}$$

Now we treat the system (2.4) using the blow-up method. So let us assume, by contradiction, that there exists a sequence $(u_{1,n}, u_{2,n})$ of positive solutions of (2.4) such that at least one of the sequences $u_{1,n}$ and $u_{2,n}$ tends to infinity in the L^∞-norm. Passing to subsequences if necessary, we may suppose that

$$||u_{1,n}||_\infty^{\beta_2} \geq ||u_{2,n}||_\infty^{\beta_1},$$

where β_1, β_2 are positive constants to be chosen later. Let $x_n \in \Omega$ be a point where $u_{1,n}$ assumes its maximum: $u_{1,n}(x_n) = \max_{x\in\Omega} u_{1,n}(x)$. Then the sequence $\lambda_n = u_{1,n}(x_n)^{-\frac{1}{\beta_1}}$ is such that $\lambda_n \to 0$. The functions

$$v_{i,n}(x) = \lambda_n^{\beta_i} u_{i,n}(\lambda_n x + x_n),$$

satisfy $v_{1,n}(0) = 1, \quad 0 \leq v_{i,n} \leq 1$ in Ω. One also verifies that the functions $v_{1,n}$ and $v_{2,n}$ satisfy

$$\left\{ \begin{array}{rcl} -\Delta v_{1,n} &=& a(\cdot)\lambda_n^{\beta_1+2-\beta_1\alpha_{11}}\, v_{1,n}^{\alpha_{11}} + b(\cdot)\lambda_n^{\beta_1+2-\beta_2\alpha_{12}}\, v_{2,n}^{\alpha_{12}} + \widetilde{h}_1(\cdot) \\ -\Delta v_{2,n} &=& c(\cdot)\lambda_n^{\beta_2+2-\beta_1\alpha_{21}}\, v_{1,n}^{\alpha_{21}} + d(\cdot)\lambda_n^{\beta_2+2-\beta_2\alpha_{22}}\, v_{2,n}^{\alpha_{22}} + \widetilde{h}_2(\cdot), \end{array} \right. \tag{2.6}$$

in the domain $\Omega_n = \dfrac{1}{\lambda_n}(\Omega - x_n)$, where the dot stands for $\lambda_n x + x_n$.
The idea of the method is then to pass to the limit as $n \to \infty$ in (2.6) and obtain a system either in R^N or in R^N_+, which can be proved that it has only the trivial solution. This would contradict the fact that the limit of $v_{1,n}$ has value 1 at the origin. By compactness the sequence (x_n) or a subsequence of it converges to a point x_0. We observe that the limiting system is defined in R^N or in R^N_+, accordingly to this limit point (x_0) being a point in Ω or in $\partial\Omega$. In the next proposition we make precise these statements.

Proposition 2.2 *The sequences $(v_{1,n})$ and $(v_{2,n})$ converge in $W^{2,p}_{\text{loc}}$, with $2 \leq p < \infty$ to functions $v_1, v_2 \in C^2(G)\bigcap C^0(\overline{G})$, satisfying the limiting system of (2.6) in $G = R^N$ or in $G = R^N_+$, provided all the powers of λ_n in (2.6) are non-negative. This limiting system is obtained by removing the terms in (2.6) where the powers of λ_n are strictly positive, the terms where the coefficients vanishes at x_0, and the lower order terms.*

In [79] and [41] two special classes of systems were studied, (i) weakly coupled and (ii) strongly coupled. The terminology is explained by the type of system obtained after the passage to the limit. We next analyze these two classes, and later we present more general results obtained recently in [42].

Definition 1. System (2.4) is *weakly coupled* if there are positive numbers β_1, β_2 such that

$$\begin{array}{lcl} \beta_1 + 2 - \beta_1\alpha_{11} = 0 &,& \beta_1 + 2 - \beta_2\alpha_{12} > 0 \\ \beta_2 + 2 - \beta_1\alpha_{21} > 0 &,& \beta_2 + 2 - \beta_2\alpha_{22} = 0 \end{array} \tag{2.7}$$

Definition 2. System (2.4) is *strongly coupled* if there are positive numbers β_1, β_2 such that

$$\begin{aligned} \beta_1 + 2 - \beta_1\alpha_{11} > 0 \quad &, \quad \beta_1 + 2 - \beta_2\alpha_{12} = 0 \\ \beta_2 + 2 - \beta_1\alpha_{21} = 0 \quad &, \quad \beta_2 + 2 - \beta_2\alpha_{22} > 0 \end{aligned} \tag{2.8}$$

Remark 2.1 *It follows that if the system (2.4) is weakly coupled then necessarily we should have*

$$\beta_1 = \frac{2}{\alpha_{11} - 1} \quad \text{and} \quad \beta_2 = \frac{2}{\alpha_{22} - 1}. \tag{2.9}$$

which requires that $\alpha_{11} > 1, \alpha_{22} > 1$ *and*

$$\alpha_{12} < \frac{\alpha_{22} - 1}{\alpha_{11} - 1}\alpha_{11} \quad \text{and} \quad \alpha_{21} < \frac{\alpha_{11} - 1}{\alpha_{22} - 1}\alpha_{22}, \tag{2.10}$$

Remark 2.2 *If the system (2.4) is strongly coupled then*

$$\beta_1 = \frac{2(\alpha_{12} + 1)}{\alpha_{12}\alpha_{21} - 1} \quad \text{and} \quad \beta_2 = \frac{2(\alpha_{21} + 1)}{\alpha_{12}\alpha_{21} - 1}. \tag{2.11}$$

which requires that $\alpha_{12}\alpha_{21} > 1$ *and*

$$\alpha_{11} < \frac{\alpha_{21} + 1}{\alpha_{12} + 1}\alpha_{12} \quad \text{and} \quad \alpha_{22} < \frac{\alpha_{12} + 1}{\alpha_{21} + 1}\alpha_{21}, \tag{2.12}$$

Remark 2.3 *We observe that the requirements that* $\alpha_{11}, \alpha_{22} > 1$ *and* $\alpha_{12}\alpha_{21} > 1$ *are known as super-linearity conditions.*

Weakly Coupled System. After the blow-up, the limiting system becomes, using a scaling of the solutions v_1, v_2:

$$\begin{aligned} -\Delta w_1 &= w_1^{\alpha_{11}} \\ -\Delta w_2 &= w_2^{\alpha_{22}} \quad , \quad \text{in} \quad R^N, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} -\Delta w_1 &= w_1^{\alpha_{11}}, \\ -\Delta w_2 &= w_2^{\alpha_{22}} \quad \text{in} \quad R^N_+ \\ w_1 = w_2 &= 0 \quad \text{on} \quad x_N = 0. \end{aligned} \tag{2.14}$$

The existence or not of positive solutions for such systems is the object of the so-called Liouville type theorems. They will be discussed in the next section. For the time being we anticipate that

(i) the equations in system (2.13) have only the trivial solution if $0 < \alpha_{11}, \alpha_{22} < \frac{N+2}{N-2}$
(ii) the equations in system (2.14) have only the trivial solution if $1 < \alpha_{11}, \alpha_{22} < \frac{N+1}{N-3}$, if the dimension $N > 3$, see Section 3.

So the following result holds.

Theorem 2.3 *Let (2.4) be a weakly coupled system with continuous coefficients* a, b, c, d, *exponents* $\alpha's \geq 0$, *and such that* $a(x), d(x) \geq c_0 > 0$ *for* $x \in \overline{\Omega}$. *Assume also that* $0 < \alpha_{11}, \alpha_{22} < (N+2)/(N-2)$. *Then there is a constant* $C > 0$ *such that*

$$||u_1||_{L^\infty}, ||u_2||_{L^\infty} \leq C$$

for all positive solutions $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *of system (2.4).*

Strongly Coupled System. As in the case of a weakly coupled system, the limiting systems are

$$-\Delta\omega_1 = \omega_2^{\alpha_{12}}, \quad -\Delta\omega_2 = \omega_1^{\alpha_{21}} \text{ in } R^N \tag{2.15}$$

and

$$-\Delta\omega_1 = \omega_2^{\alpha_{12}}, \quad -\Delta\omega_2 = \omega_1^{\alpha_{21}} \text{ in } (R^N)^+ \tag{2.16}$$

with

$$w_1 = w_2 = 0 \text{ on } x_N = 0$$

So a contradiction comes if the exponents are such that (2.15) and (2.16) have only the trivial solution $\omega_1 = \omega_2 \equiv 0$. In summary, the following result holds.

Theorem 2.4 *Let (2.4) be a strongly coupled system with continuous coefficients* a, b, c, d, *exponents* $\alpha's \geq 0$, *and such that* $b(x), c(x) \geq c_0 > 0$ *for* $x \in \overline{\Omega}$. *Assume that the following conditions hold:*

(L1) The exponents α_{12} *and* α_{21} *are such that the only non- negative solution of*

$$-\Delta\omega_1 = \omega_2^{\alpha_{12}}, \quad -\Delta\omega_2 = \omega_1^{\alpha_{21}} \text{ in } R^N$$

is $w_1 = \omega_2 \equiv 0$.

(L2) The only non-negative solution of

$$-\Delta\omega_1 = \omega_2^{\alpha_{12}}, \quad -\Delta\omega_2 = \omega_1^{\alpha_{21}} \text{ in } R^N_+$$

with $\omega_1(x', 0) = \omega_2(x', 0) = 0$ *is* $\omega_1 = \omega_2 \equiv 0$. *Then there is a constant* $C > 0$ *such that*

$$||u_1||_{L^\infty}, ||u_2||_{L^\infty} \leq C$$

for all non-negative solutions (u_1, u_2) *of system (2.4).*

Remark 2.4 *Which conditions should be required on the exponents α_{12} and α_{21} in such a way that (L1) and (L2) holds? Again these are Liouville type theorems for systems, which will be described in the next section.*

A more complete analysis of the blow-up process.

Now we proceed to do a more complete analysis of the system (2.4) using the blow up explained above and considering other types of limiting systems. We follow [42].

By looking at system (2.6), we see that in order to arrive at some system in R^N or R^N_+, by using the blow up method, it is necessary that all the exponents in the λ_n are greater than or equal to 0. When one of these exponents is positive, then the corresponding term vanishes in the limit (as $n \to \infty$), while if the exponent is zero than that term remains after the limiting process. The many possibilities would be better understood by the analysis of the figure below, Figure 2.

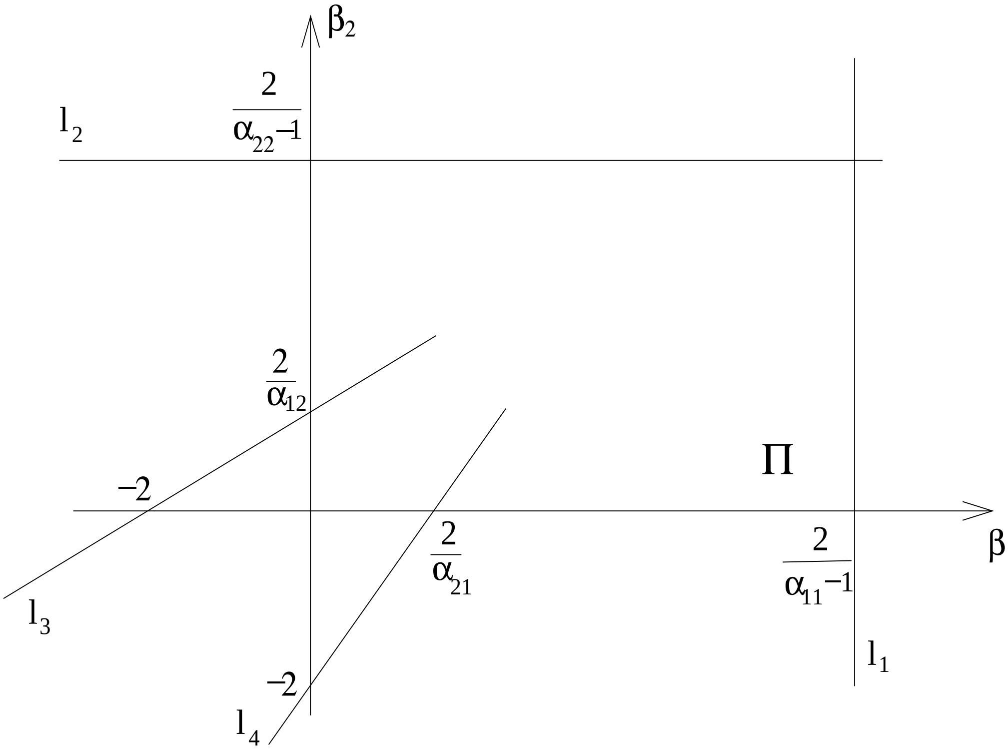

Figure 2: *Admissible couples (β_1, β_2) lie to the left of or on l_1, below or on l_2, below or on l_3, and above or on l_4.*

In the (β_1, β_2)-plane we denote $\overrightarrow{\beta} = (\beta_1, \beta_2) \in R^2_+$, and introduce the lines, whose expressions come from the exponents of λ_n in (2.6),

$$l_1 = \left\{ \vec{\beta} \mid \beta_1 + 2 - \beta_1 \alpha_{11} = 0 \right\}, \quad l_2 = \left\{ \vec{\beta} \mid \beta_2 + 2 - \beta_2 \alpha_{22} = 0 \right\},$$

$$l_3 = \left\{ \vec{\beta} \mid \beta_1 + 2 - \beta_2 \alpha_{12} = 0 \right\}, \quad l_4 = \left\{ \vec{\beta} \mid \beta_2 + 2 - \beta_1 \alpha_{21} = 0 \right\},$$

In order to have exponents of the λ_n greater or equal to 0, we have to consider points $(\beta_1, \beta_2) \in R^2_+$, which are to the left of or on l_1, below or on l_2 (note that l_1 and l_2 can be empty, and then they introduce no restriction), below or on l_3, and above or on l_4. Those points are called *admissible*. We divide the systems studied in three classes, which are determined by the exponents $\alpha_{i,j}$:

Case A.*The intersection of l_1 and l_2 is admissible.* Then we set $(\beta_1, \beta_2) = l_1 \bigcap l_2$. In this case we shall assume that the functions $a(x)$ and $d(x)$ are bounded below on $\overline{\Omega}$ by a positive constant.

Case B.*The intersection of l_3 and l_4 is admissible.* Then we set $(\beta_1, \beta_2) = l_3 \bigcap l_4$. In this case we shall assume that the functions $b(x)$ and $c(x)$ are bounded below on $\overline{\Omega}$ by a positive constant. Further, we have to assume that $\alpha_{12}, \alpha_{21} > 1$. (This last requirement comes from a restriction in some Liouville theorems).

Case C.*None of $l_1 \bigcap l_2$ and $l_3 \bigcap l_4$ is admissible.* Then either $l_1 \bigcap l_3$ or $l_2 \bigcap l_4$ is admissible and we take this intersection point to be our (β_1, β_2). In this case we shall assume that the function $b(x)$ (resp $c(x)$) is bounded below on $\overline{\Omega}$ by a positive constant.

Now we can state the main result of this section:

Theorem 2.5 *Assume that system (2.4) satisfies the conditions above and that the pair (β_1, β_2), which corresponds to the type of the system (A, B or C), satisfies the condition*

$$\min\{\beta_1, \beta_2\} \geq \frac{N-2}{2}. \tag{2.17}$$

Then all positive solutions of the system (2.4) are bounded in L^∞.

Remarks on the proof of Theorem 2.5. The proof in all three cases consists in verifying that the limiting systems have only the trivial, coming then to a contradiction. In the sequel we use G to denote either R^N or R^N_+

In Case A we choose $(\beta_1, \beta_2) = l_1 \bigcap l_2$, that is

$$\beta_1 = \frac{2}{\alpha_{11} - 1}, \ \ \beta_2 = \frac{2}{\alpha_{22} - 1}.$$

For the limiting systems there are three possibilities:
(a) if none of the lines l_3 and l_4 passes through $l_1 \bigcap l_2$ we get

$$\begin{aligned} -\Delta w_1 &= w_1^{\alpha_{11}} \\ -\Delta w_2 &= w_2^{\alpha_{22}} \quad , \quad \text{in} \ \ G, \end{aligned} \tag{2.18}$$

and as a consequence of hypothesis (2.17) we obtain

$$\max\{\alpha_{11}, \alpha_{22}\} < \frac{N+2}{N-2},$$

which then implies that system (2.18) has only the trivial solution. (This is precisely the weakly coupled case discussed before).
(b) If exactly one of the lines l_3 and l_4 (say l_3)passes through $l_1 \bigcap l_2$ we get

$$\begin{aligned} -\Delta v_1 &= a_0 v_1^{\alpha_{11}} + b_0 v_2^{\alpha_{12}} \\ -\Delta v_2 &= d_0 v_2^{\alpha_{22}} \quad , \quad \text{in} \ \ G, \end{aligned} \tag{2.19}$$

where $a_0 > 0, \ b_0 \geq 0, \ d_0 > 0$. So as above system (2.19) has only the trivial solution.
(c) If all four lines meet (this also contains one of the possibilities of Case B) we get

$$\begin{aligned} -\Delta v_1 &= a_0 v_1^{\alpha_{11}} + b_0 v_2^{\alpha_{12}} \\ -\Delta v_2 &= c_0 v^{\alpha_{21}} + d_0 v_2^{\alpha_{22}} \quad , \quad \text{in} \ \ G, \end{aligned} \tag{2.20}$$

with all four coefficients positive. In order to see that such a system under the assumption (2.17) has only the trivial solution, one requires a new Liouville theorem, which is proved in [42]. As in [18] our proof uses a change to polar coordinates and some monotonicity argument.

In Case B we choose $(\beta_1, \beta_2) = l_3 \bigcap l_4$, that is,

$$\beta_1 = \frac{2(1 + \alpha_{12})}{\alpha_{12}\alpha_{21} - 1}, \quad \beta_2 = \frac{2(1 + \alpha_{21})}{\alpha_{12}\alpha_{21} - 1}.$$

For the limiting systems there are two possibilities:

(a) if none of the lines l_1 and l_2 passes through $l_3 \bigcap l_4$ we get (after scaling)

$$\begin{aligned} -\Delta v_1 &= v_2^{\alpha_{12}} \\ -\Delta v_2 &= v_1^{\alpha_{21}} \quad , \quad \text{in} \quad G, \end{aligned} \tag{2.21}$$

From results of the next section it follows that system (2.21), under the hypothesis (2.17), has only the trivial solution. (This is precisely the strongly coupled case discussed above.)
(b) If one of the lines l_1 and l_2 (say l_1)passes through $l_3 \bigcap l_4$ we get

$$\begin{aligned} -\Delta v_1 &= a_0 v_1^{\alpha_{11}} + b_0 v_2^{\alpha_{12}} \\ -\Delta v_2 &= c_0 v_1^{\alpha_{21}} \quad , \quad \text{in} \quad G, \end{aligned} \tag{2.22}$$

where $b_0, c_0 > 0$, and $a_0 \geq 0$, and the results on Section 3 give that $v_1 = v_2 \equiv 0$.

In Case C there are two possibilities: (a) the line l_3 meets l_1 in a point above l_4, (b) the line l_4 meets l_2 in a point below l_3. In both cases (β_1, β_2) is chosen as this point of intersection. In case (a) we take

$$\beta_1 = \frac{2}{\alpha_{11} - 1}, \quad \beta_2 = \frac{2\alpha_{11}}{\alpha_{12}(\alpha_{11} - 1)}.$$

The limiting system is

$$\begin{aligned} -\Delta v_1 &= a_0 v_1^{\alpha_{11}} + b_0 v_2^{\alpha_{12}} \\ -\Delta v_2 &= 0 \quad , \quad \text{in} \quad G, \end{aligned} \tag{2.23}$$

which is easily treated by the Liouville results of the next section.

3 Liouville Theorems

The classical Liouville Theorem from Function Theory says that every bounded entire function is constant. In terms of a differential equation one has: if $(\partial/\partial\overline{z})f(z) = 0$ and $|f(z)| \leq C$ for all $z \in \mathbb{C}$ then $f(z)$ =const. Hence results with a similar contents are nowadays called Liouville theorems. For instance, a superharmonic function defined in the whole plane R^2, which is bounded below, is constant. Also, all results discussed in this section have this nature. For completeness, we survey also results on a single equation, namely

$$-\Delta u = u^p \tag{3.1}$$

If the equation is considered in R^2, then a non-negative solution of (3.1) is necessarily identically zero. The case when $R^N, N \geq 3$ is quite distinct. We discuss this case next.

Theorem 3.1 *Let u be a non-negative C^2 function defined in the whole of R^N, such that (3.1) holds in R^N. If $0 < p < (N+2)/(N-2)$, then $u \equiv 0$.*

This result was proved by Gidas-Spruck [65] in the case $1 < p < (N+2)/(N-2)$. A simpler proof using the method of moving parallel planes was given by Chen-Li [25], and it is valid in the whole range of p. A very elementary proof valid for $p \in [1, \frac{N}{N-2})$ was given by Souto [95]. In fact, his proof is valid for the case of u being a non-negative supersolution, i.e.

$$-\Delta u \geq u^p \quad \text{in } R^N, \tag{3.2}$$

with p in the same restricted range.

Theorem 3.2 *Let $u \in C^2(R^N_+) \cap C^0(R^N_+)$ be a non- negative function such that*

$$\begin{cases} -\Delta u = u^p & \text{in } R^N_+ \\ u(x', 0) = 0 \end{cases} \tag{3.3}$$

If $1 < p \leq (N+2)/(N-2)$ then $u \equiv 0$.

Remark 3.1 *This is Theorem 1.3 of [64], plus Remark 2 on page 895 of the same paper. It is remarkable that in the case of the half-space the exponent $(N+2)/(N-2)$ is not the right one for theorems of Liouville type. Indeed, Dancer [30] has proved the following result.*

Theorem 3.3 *Let $u \in C^2(R^N_+) \cap C^0(R^N_+)$ be a non- negative bounded solution of (3.3). If $1 < p < (N+1)/(N-3)$ for $N \geq 4$ and $1 < p < \infty$ for $N = 3$, then $u \equiv 0$.*

Remark 3.2 *If $p = (N+2)/(N-2), N \geq 3$, then (3.1) has a two-parameter family of bounded positive solutions:*

$$U_{\varepsilon, x_0}(x) = \left[\frac{\varepsilon \sqrt{N(N-2)}}{\varepsilon^2 + |x - x_0|^2} \right]^{\frac{N-2}{2}},$$

which are called instantons.

Next we state some results on supersolutions still in the scalar case.

Theorem 3.4 *Let $u \in C^2(R^N)$ be a non-negative supersolution of (3.2). If $1 \le p \le \dfrac{N}{N-2}$, then $u \equiv 0$.*

This result is proved in Gidas [62] for $1 < p \le N/(N-2)$. The case $p = 1$ is included in Souto [95]. See [7] and [74] for Liouville theorems for equations defined in cones.

Liouvile for systems defined in the whole of R^N.

We start considering systems of the form

$$-\Delta u = v^p, \quad -\Delta v = u^q. \tag{3.4}$$

In analogy with the scalar case just discussed, here the dividing line between existence and non-existence of positive solutions (u, v) defined in the whole of R^N should be the *critical hyperbola*, [26], [84], introduced before . We recall that, such hyperbola associated to problems of the form (3.4) is defined by

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}, \quad p, q > 0 \tag{3.5}$$

Continuing the analogy with the scalar case, one may *conjecture* that (3.4) has no bounded positive solutions defined in the whole of R^N if

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}, \quad p, q > 0. \tag{3.6}$$

To our knowledge, this conjecture has not been settled in full so far. Why such a conjecture? In answering it, let us remind some facts, already contained in the previous sections. The critical hyperbola appeared in the study of existence of positive solutions for superlinear elliptic systems of the form

$$-\Delta u = g(v), \quad -\Delta v = f(u) \tag{3.7}$$

subject to Dirichlet boundary conditions in a bounded domain Ω of R^N. If $g(v) \sim v^p$ and $f(u) \sim u^q$ as $u, v \to \infty$, then system (3.7) is said to be sub-critical if p, q satisfy (3.6). For such systems [in analogy with sub-critical scalar equations, $-\Delta u = f(u)$, $f(u) \sim u^p$ and $1 < p < (N+2)/(N-2)$] one can establish a priori bounds of positive solutions, prove a Palais-Smale condition and put through an existence theory by a topological or a variational method. This sort of work initiated in [26] and [84] has been continued. Recall that the problem in the critical scalar case, (that is,

$-\Delta u = |u|^{2^*-2}u$ in $\Omega,\ u = 0$ on $\partial\Omega$,) has no solution $u \neq 0$ if Ω is a star-shaped bounded domain in $R^N, N \geq 3$. In the case of systems, the critical hyperbola appears in the statement: if Ω is a bounded star-shaped domain in $R^N, N \geq 3$, the Dirichlet problem for the system below has no non-trivial solution:

$$-\Delta u = |v|^{p-1}v, \quad -\Delta v = |u|^{q-1}u$$

if, p, q satisfy (3.5). This follows from an identity of Pohozaev-type, see Mitidieri [77]; also Pucci-Serrin [86] for general forms of Pohozaev-type identities.

Next we describe several Liouville-type theorem for systems.

Theorem 3.5 *Let $p, q > 0$ satisfying (3.6). Then system (3.4) has no nontrivial radial positive solutions of class $C^2(R^N)$.*

Remark 3.3 *This result settles the conjecture in the class of radial functions. It was proved in [77] for $p, q > 1$, and for p, q in the full range by Serrin-Zou [91]. The proof explores the fact that eventual positive radial solutions of (3.4) have a definite decay at ∞; this follows from an interesting observation (cf. Lemma 6.1 in [77]): If $u \in C^2(R^N)$ is a positive radial superharmonic function, then*

$$ru'(r) + (N-2)u(r) \geq 0, \text{for all } r > 0.$$

Theorem 3.5 is sharp as far as the critical hyperbola is concerned. Indeed, there is the following existence result of Serrin-Zou [91].

Theorem 3.6 *Suppose that $p, q > 0$ and that*

$$\frac{1}{p+1} + \frac{1}{q+1} \leq 1 - \frac{2}{N} \tag{3.8}$$

Then there exist infinitely many values $\xi = (\xi_1, \xi_2) \in R^+ \times R^+$ such that system (3.4) admits a positive radial solution (u, v) with central values $u(0) = \xi_1,\ v(0) = \xi_2$. Moreover $u, v \to 0$ as $|x| \to \infty$. So the solution is in fact a ground state for (3.4).

Let us now mention some results on the nonexistence of positive solutions (or super-solutions) of (3.4), without the assumption of being radial.

Theorem 3.7 *Let $u, v \in C^2(R^N)$ be non-negative super-solutions of (3.4), that is*

$$-\Delta u \geq v^p, \qquad -\Delta v \geq u^q \quad \text{in } R^N, \tag{3.9}$$

where $p, q > 0$ and

$$\frac{1}{p+1} + \frac{1}{q+1} \geq \frac{N-2}{N-1}. \tag{3.10}$$

Then $u = v \equiv 0$.

This result is due to Souto [95]. The idea of his interesting proof is to reduce the problem to a question concerning a scalar equation. Suppose, by contradiction, that u and v are positive solutions of (3.9) in R^N. Introduce a function $\omega = uv$. So

$$\Delta\omega \leq 2\nabla u \nabla v - u^{q+1} - v^{p+1}. \tag{3.11}$$

Using the inequality

$$a \cdot b \leq \frac{1}{4}|a+b|^2 \quad a, b \in R^N$$

we get that

$$2\nabla u \nabla v \leq \frac{1}{2}\omega^{-1}|\nabla\omega|^2.$$

On the other hand, choose $r > 0$ such that $\frac{1}{r} = \frac{1}{p+1} + \frac{1}{q+1}$. Then by Young's inequality

$$\omega^r = u^r v^r \leq \frac{r}{q+1}u^{q+1} + \frac{r}{p+1}v^{p+1} \leq u^{q+1} + u^{p+1}.$$

So

$$\Delta\omega \leq \frac{1}{2}\omega^{-1}|\nabla\omega|^2 - \omega^r. \tag{3.12}$$

Replacing ω by f^2 in (3.12) one obtains

$$-\Delta f \geq \frac{1}{2}f^{2r-1} \ \text{ in } \ R^N,$$

with $f > 0$ in R^N. Using Theorem 3.4, we see that this is a contradiction, since $2r - 1 \leq N/(N-2)$. It is of interest to observe that Souto's hyperbola intersects the bisector of the first quadrant precisely at the Serrin exponent $\frac{N}{N-2}$.

In order to state the next results, we assume that $pq > 1$ and introduce the following notations

$$\alpha = \frac{2(p+1)}{pq-1}, \qquad \beta = \frac{2(q+1)}{pq-1}. \tag{3.13}$$

Theorem 3.8 *Suppose that $p, q > 1$ and*

$$\max\{\alpha, \beta\} \geq N - 2. \tag{3.14}$$

Then system (3.9) has no nontrivial super-solution of class $C^2(R^N)$.

Remark. The above result is Corollary 2.1 in [78]. Under hypothesis (3.14), it is proved in [90] that system (3.4) has no nontrivial solution, provided a weaker condition than $p, q > 1$ holds, namely $pq > 1$. In [90] it is also proved that (3.4) has no nontrivial solution in R^N, if $pq \leq 1$.

In order to illustrate some useful technique, let us comment briefly the proof given in [78], which uses spherical means. Let $v \in C(R^N)$, then the *spherical mean* of v at x of radius ρ is

$$M(v; x, \rho) = \frac{1}{\text{meas}[\partial B_\rho(x)]} \int_{\partial B_\rho(x)} v(y) d\sigma(y).$$

Changing coordinates we obtain

$$M(v; x, \rho) = \frac{1}{\omega_N} \int_{|\nu|=1} v(x + \rho\nu) d\omega \tag{3.15}$$

where ω_N denotes the surface area of the unit sphere of R^N and ν ranges over this unit sphere. Then, one has Darboux formula

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{N-1}{\rho} \frac{\partial}{\partial \rho} \right) M(v; x, \rho) = \Delta_x M(v; x, \rho). \tag{3.16}$$

Now let us use these ideas for the functions u and v in system (3.9):

$$\Delta_x M(u; x, \rho) = \frac{1}{\omega_N} \int_{|\nu|=1} \Delta_x u(x + \rho\nu) d\omega \leq -\frac{1}{\omega_N} \int_{|\nu|=1} [v(x + \rho\nu)]^p d\omega$$

and using Jensen's inequality we obtain

$$\Delta_x M(u; x, \rho) \leq -[M(v; x, \rho)]^p. \tag{3.17}$$

Using the notation

$$M(u(x);x,\rho) = u^{\#}(\rho)\ ,\ \ M(v(x);x,\rho) = v^{\#}(\rho)$$

we obtain

$$-\Delta_\rho u^{\#} \geq (v^{\#})^p,\ \ -\Delta_\rho v^{\#} \geq (u^{\#})^q,\ \text{where,}\ \Delta_\rho = \left(\frac{\partial^2}{\partial\rho^2} + \frac{N-1}{\rho}\,\frac{\partial}{\partial\rho}\right) \tag{3.18}$$

The proof of Theorem 3.8 will be concluded by the use of the following results, see [78].

Proposition 3.1 *If $v \in C^2(R^N)$, then $M(v;x,\rho)$ is also $C^2(R^N)$ in the variable x and $C^2([0,\infty))$ in the variable ρ. Moreover,*

$$\left(\frac{d}{d\rho}v^{\#}\right)(0) = 0, \quad \text{and} \left(\frac{d}{d\rho}v^{\#}\right)(\rho) \leq 0,$$

that is $v^{\#}(\rho)$ is non-increasing.

Proposition 3.2 *If $u \in C^2(R^N)$ is a positive radial superharmonic function, then*

$$ru'(r) + (N-2)u(r) \geq 0 \quad \text{for} \quad r > 0. \tag{3.19}$$

Proposition 3.3 *Let $u(\rho),\ v(\rho)$ be two C^2 functions defined and non-increasing in $[0,\infty)$, such that $u'(0) = v'(0) = 0$ and*

$$-\Delta_\rho u \geq v^p, \quad -\Delta_\rho v \geq u^q. \tag{3.20}$$

Suppose that $p,q > 1$ and that (3.14) holds. Then $u = v \equiv 0$.

The next result extends, as compared with the previous results, the region under the critical hyperbola where the Liouville theorem holds.

Theorem 3.9 *A) If $p > 0$ and $q > 0$ are such that $p,q \leq (N+2)/(N-2)$, but not both equal to $(N+2)/(N-2)$, then the only non-negative solution of (3.4) is $u = v = 0$.*

B) If $\alpha = \beta = (N+2)/(N-2)$, then u and v are radially symmetric with respect to some point of R^N.

This theorem is due to deFigueiredo-Felmer [40]. The proof uses the method of Moving Planes. A good basic reference of this method is [14]. The idea in the proof of the above theorem is to use Kelvin transform in the solutions u, v of (3.4), which a priori have no known (or prescribed) behavior at infinite. By means of Kelvin's u and v are transformed in new unknowns w and z satisfying

$$\begin{aligned} -\Delta w &= \tfrac{1}{|x|^{N+2-p(N-2)}} z^p(x), \\ -\Delta z &= \tfrac{1}{|x|^{N+2-q(N-2)}} w^q(x) \end{aligned} \tag{3.21}$$

which now have a definite decay at ∞, provided (p, q) satisfy the conditions of Theorem 3.9. It is precisely at this point that we cannot take $p > \frac{N+2}{N-2}$, because then one would loose the right type of monotonicity of the coefficients necessary to put the moving plane method to work. So having this correct monotonicity of the coefficients the method of moving planes can start. This result has been extended by Felmer [55] to systems with more than two equations.

The next result is due to Busca-Manasevich [18] and extends further, as compared with Theorem 3.9, the region of values of p, q where the Liouville theorem for system (3.4) holds

Theorem 3.10 *Suppose that* $p, q > 1$ *and*

$$\min\{\alpha, \beta\} > \frac{N-2}{2}. \tag{3.22}$$

Then system (3.4) has no nontrivial solution of class $C^2(R^N)$.

If some behavior of u and v at ∞ is known, the Liouville theorem can be established for all (p, q) below the critical hyperbola, as in the next result.

Theorem 3.11 *Let* $p > 0$ *and* $q > 0$ *satisfying (3.6) then there are no positive solutions of (3.4) satisfying*

$$u(x) = o(|x|^{-\frac{N}{q+1}}), \quad v(x) = o(|x|^{-\frac{N}{p+1}}), \text{ as } |x| \to \infty. \tag{3.23}$$

The above result is due to Serrin-Zou [91], where the next result is also proved.

Theorem 3.12 *Let* $N = 3$, *and* $p, q > 0$ *satisfying (3.6). Then there are no positive solutions of (3.4) for which either* u *or* v *has at most algebraic growth at infinity.*

Remark 3.4 *Observe that Theorem 3.11 extends Theorem 3.5, since radial positive solutions have a decay at infinity.*

The proof of Theorem 3.11 is based on an interesting L^2 estimate of the gradient of a superharmonic function, namely,

Lemma 3.1 *Let $\omega \in C^2(R^N)$ be positive, superharmonic (i.e. $-\Delta\omega \geq 0$ in R^N) and*

$$\omega(x) = o(|x|^{-\gamma}) \quad \text{as} \quad |x| \to \infty. \tag{3.24}$$

Then

$$\int_{B_{2R}\setminus B_R} |\nabla\omega|^2 = o(R^{N-2-2\gamma}) \quad \text{as} \quad R \to \infty, \tag{3.25}$$

where B_R is the ball of radius R in R^N centered at the origin.

Another basic ingredient in the proof of Theorem 3.11 is an identity of Pohožaev-type, a special case of a general identity in [86]. As a matter of fact, the formula below is exactly the one in Corollary 2.1 of [77], taking there $h = x$ and $\Omega = B_\rho$

Lemma 3.2 *Let (u, v) be a positive solution of (3.4) and let a_1 and a_2 be constants such that $a_1 + a_2 = N - 2$. Then*

$$\int_{B_\rho} \left\{ \left(\tfrac{N}{p+1} - a_1\right) v^{p+1} + \left(\tfrac{N}{q+1} - a_2\right) u^{q+1} \right\} = \rho \int_{\partial B_\rho} \left\{ \tfrac{v^{p+1}}{p+1} + \tfrac{u^{q+1}}{q+1} \right\}$$
$$+\rho \int_{\partial B_\rho} \left(2\tfrac{\partial u}{\partial r}\ \tfrac{\partial v}{\partial r} - \nabla u \cdot \nabla v\right) + \int_{\partial B_\rho} \left(a_1 \tfrac{\partial u}{\partial r} v + a_2 u \tfrac{\partial v}{\partial r}\right). \tag{3.26}$$

Proof of Theorem 3.11. using these two lemmas. Choose a_1 and a_2 in such a way that

$$\frac{N}{p+1} - a_1 = \frac{N}{q+1} - a_2 = \delta, \ a_1 + a_2 = N - 2.$$

Next, dividing (3.26) by ρ and integrating with respect to ρ between some R and $2R$ and estimating we get

$$\delta \ln 2 \int_{B_R} (u^{q+1} + v^{p+1}) \leq \int_{B_{2R}\setminus B_R} \left(\frac{u^{q+1}}{q+1} + \frac{v^{p+1}}{p+1} \right)$$
$$+ \int_{B_{2R}\setminus B_R} |\nabla u . \nabla v| + cR^{-1} \int_{B_{2R}\setminus B_R} (v|\nabla u| + u|\nabla v|), \tag{3.27}$$

where c is a constant $< N-2$. Now using the hypothesis (3.23), we see that the first integral in the right side of (3.27) is $o(1)$. Next one uses Lemma 3.1 with $\omega = u$, $\gamma = \dfrac{N}{q+1}$ and $\omega = v, \gamma = \dfrac{N}{p+1}$. With that we can estimate the second and third integrals using Cauchy -Schwarz and get that they are $o(R^{N-2-\frac{N}{p+1}-\frac{N}{q+1}})$ which is $o(1)$. Then if follows from (3.27), letting $R \to \infty$, that $u = v = 0$.

Liouville theorems for systems defined in half-spaces

Now we look at the system below and state some results on the nonexistence of non-trivial solutions and also of supersolutions.

$$\begin{cases} -\Delta u &= v^p \quad \text{in} \quad R_+^N \\ -\Delta v &= u^q \quad \text{in} \quad R_+^N \\ u, v &\geq 0 \quad \text{in} \quad R_+^N \\ u, v &= 0 \quad \text{on} \quad \partial R_+^N \end{cases} \tag{3.28}$$

Theorem 3.13 *Let $p, q > 1$ satisfying*

$$\max(\alpha, \beta) \geq N - 3. \tag{3.29}$$

Then the system (3.28) has only the trivial solution.

Remark 3.5 *This result is due to Birindelli-Mitidieri [16], where, instead of a half- space, more general cones are considered. It is also proved there that system (3.28) has no supersolutions if*

$$\max(\alpha, \beta) \geq N - 1,$$

where α, β are defined in (3.13).

A Liouville theorem for a full system.

Now we consider the following system

$$\begin{aligned} -\Delta u_1 &= u_1^{\alpha_{11}} + u_2^{\alpha_{12}} \\ -\Delta u_2 &= u_1^{\alpha_{21}} + u_2^{\alpha_{22}}, \quad \text{in} \quad R^N, \end{aligned} \tag{3.30}$$

which has appeared in Section 2, when we proved a priori bounds for solutions of systems in Case A. That was when all 4 lines l_1, l_2, l_3, l_4 intersect at (β_1, β_2). This implies that

$$\beta_1 = \frac{2}{\alpha_{11} - 1}, \quad \beta_2 = \frac{2}{\alpha_{22} - 1}$$

$$\alpha_{12} = \frac{\alpha_{11}(\alpha_{22} - 1)}{\alpha_{11} - 1}, \quad \alpha_{21} = \frac{\alpha_{22}(\alpha_{11} - 1)}{\alpha_{22} - 1}$$

Theorem 3.14 *System (3.30) has only the trivial solution if the following condition hold:*

$$\alpha_{11}, \alpha_{22} < \frac{N+2}{N-2}. \tag{3.31}$$

Remark. In terms of (β_1, β_2), the condition of the Theorem 3.14 reads as follows:

$$\min(\beta_1, \beta_2) > \frac{N-2}{2}.$$

This result is due to deFigueiredo-Sirakov [42], and it relies on the following three theorems, which are also proved in [42]. The first one is an extension of a result by Dancer [30], proved for the scalar case. The third one is an extension of a result in [18].We state those results in the special case of system (3.30), although they are valid for more general functions in the right hand sides of the system.

Theorem 3.15 *Suppose that u_1, u_2 is a nonnegative bounded classical solution of system (3.30) in R^N_+ such that $u_1 = u_2 = 0$ on ∂R^N_+. Then*

$$\frac{\partial u_i}{\partial x_N} > 0, \quad \text{in } R^N_+, \ \text{ for } i = 1, 2.$$

Theorem 3.16 *Suppose that system (3.30) has a nontrivial nonnegative bounded classical solution defined in R^N_+, such that $u_1 = u_2 = 0$ on ∂R^N_+. Then the same problem has a positive solution in R^{N-1} (the limit as $x_N \to \infty$ in R^N_+.*

Theorem 3.17 *Let $u_i(t, \theta)$, i=1,2 be a C^2-function defined in $\mathbb{R} \times S^{N-1}$ satisfying*

$$\frac{\partial^2 u_i}{\partial t^2} + \Delta_\theta u_i - \delta_i \frac{\partial u_i}{\partial t} - \nu_i u_i + u_1^{\alpha_{i1}} + u_2^{\alpha_{i2}} = 0,$$

in $\mathbb{R} \times S^{N-1}$, with $u_i \to 0$ as $t \to -\infty$.
Suppose that $\delta_i \geq 0, \ \max\{\delta_1, \delta_2\} > 0, \ \nu_i > 0, \ i = 1, 2$ are constants.
Assume also that there exists $t_0 \in \mathbb{R}$ such that $\frac{\partial u_i}{\partial t} > 0$ in $(-\infty, t_0) \times S^{N-1}, \ i = 1, 2$.
Then $\frac{\partial u_i}{\partial t} > 0$ in $\mathbb{R} \times S^{N-1}$, for i=1,2.

Remarks on the proof of Theorem 3.14
We follow [42]. Assume first $G = R^N_+$. Then it follows from Theorem 3.16 that if $(u_1, u_2) \neq (0, 0)$ then there exists a nontrivial solution of system (3.30) in R^{N-1}. So if we prove that system (3.30) has only the trivial solution in R^N under hypothesis (3.31), then it has no nontrivial solution in R^N_+, under the hypothesis $\min\{\beta_1, \beta_2\} > \frac{N-3}{2}$, which is a consequence of (3.31).

So from now on we suppose $G = R^N$ and distinguish two cases,

$$\max\{\beta_1, \beta_2\} \geq N - 2 \ \text{(Case 1)} \quad \text{and} \quad \max\{\beta_1, \beta_2\} < N - 2 \ \text{(Case 2)}.$$

In Case 1 (say $\beta_1 \geq N - 2$) we have $\alpha_{11} \leq \frac{N}{N-2}$. But the first equality in system (3.30) implies $-\Delta u_1 \geq a_0 u_1^{\alpha_{11}}$ in R^N, so $u_1 \equiv 0$ in R^N, by the results about non-existence of supersolutions for scalar equations. Then the second equation in (3.30) becomes $-\Delta u_2 = d_0 u_2^{\alpha_{22}}$ in R^N. So $u_2 \equiv 0$ in R^N, because $\alpha_{22} < \frac{N+2}{N-2}$.
In Case 2 we write system (3.30)in polar coordinates $(r, \theta) \in \mathbb{R} \times S^{N-1}$ and make the change of variables, as in [18],

$$t = \ln |x| \in \mathbb{R}, \quad \theta = \frac{x}{|x|} \in S^{N-1},$$

and set

$$w_i(t, \theta) = e^{\beta_i t} u_i(e^t, \theta).$$

Then system (3.30) transforms into

$$\begin{cases} -L_1 w_1 = a_0 e^{(\beta_1+2-\alpha_{11}\beta_1)t} w_1^{\alpha_{11}} + b_0 e^{(\beta_1+2-\alpha_{12}\beta_2)t} w_2^{\alpha_{12}} \\ \\ -L_2 w_2 = c_0 e^{(\beta_2+2-\alpha_{21}\beta_1)t} w_1^{\alpha_{21}} + d_0 e^{(\beta_2+2-\alpha_{22}\beta_2)t} w_2^{\alpha_{22}} \end{cases} \tag{3.32}$$

in $\mathbb{R} \times S^{N-1}$, where

$$L_i = \frac{\partial^2}{\partial t^2} + \Delta_\theta - \delta_i \frac{\partial}{\partial t} - \nu_i, \quad i = 1, 2,$$

and

$$\delta_i = 2\beta_i - (N - 2), \qquad \nu_i = \beta_i(N - 2 - \beta_i), \quad i = 1, 2.$$

Observe that the system is autonomous (the exponents in equation (3.32) are the equations of the lines $l's$) and both δ_i and ν_i are positive. So we can apply Theorem 3.17. It follows that $\frac{\partial w_i}{\partial t} > 0$ in $R \times S^{N-1}$. Then

$$\beta_i u_i + r \frac{\partial u_i}{\partial r} > 0, \ \ i = 1, 2.$$

Observing that all the above arguments can be carried out for any translations of (u_1, u_2), we get

$$\beta_i u_i(x) + \nabla u_i(x).(x - x_0) > 0$$

for all $x, x_0 \in R^N$. This implies $\nabla u_i(x) \equiv 0$ (write $x_0 = x - \tau e,\ \tau > 0,\ e \in S^{N-1}$, divide by τ, let $\tau \to \infty$, and observe that this holds for any $e \in S^{N-1}$). And from equation (3.30) we conclude.

Final remarks on Liouville Theorem For Systems.

(i) The conjecture on the validity of a Liouville theorem in the whole of R^N for all p and q below the critical hyperbola, and $p, q > 0$, seems to be unsettled at this moment. In dimension $N = 3$, the conjecture has been proved in [90], see Theorem 3.12 above, provided one supposes that u or v has at most algebraic growth.
(ii) Liouville theorems for systems of inequalities in the whole of R^N are given in Theorems 3.7 and 3.8. Is inequality

$$\max\{\alpha, \beta\} \geq N - 2$$

in (3.14) sharp? Observe that if $p = q$, (3.14) yields $p \leq N/(N-2)$, which is the value obtained in Theorem 3.4.
(iii) Observe that a Liouville theorem for a system of inequalities in R_+^N is stated in the Remark right after Theorem 3.13. Compare this result with the following theorem of [95].

Theorem 3.18 *Let $u, v \in C^2(R_+^N) \cap C^0(\overline{R_+^N})$ be non-negative solutions of (3.28) with $u = v = 0$ on ∂R_+^N. If $1 \leq p, q \leq \dfrac{N+2}{N-2}$ then $u = v \equiv 0$.*

(iv) Liouville-type theorems for systems of p-Laplacians have been studied recently by Mitidieri-Pohozaev.
(v) Liouville theorems for equations with a weight have been considered in Berestycki, Capuzzo Dolcetta- Nirenberg [13].
(vi) There is a recent work of P. Souplet [94], where he proves this Liouville theorem in dimension 4.

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