FINITE-TIME BLOWUP FOR A FAMILY OF NONLINEAR PARABOLIC EQUATIONS

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1. Introduction

These notes are based in particular on several joint works with Flávio Dickstein and Fred Weissler, whose overall objective is to shed some light on the mechanisms that lead certain solutions of nonlinear evolution PDEs to blow up in finite-time. More specifically, we study here finite-time blowup for the complex Ginzburg-Landau equation

$$e^{-i\theta}u_t = \Delta u + |u|^\alpha u, \tag{1.1}$$

where $\alpha > 0$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. In fact, changing u to $\overline{u}$ in (1.1) changes θ to $-\theta$, so we may as well assume

$$0 \le \theta \le \frac{\pi}{2}. \tag{1.2}$$

Equation (1.1) with $\theta = 0$ reduces to the nonlinear heat equation

$$u_t - \Delta u = |u|^\alpha u. \tag{1.3}$$

For $\theta = \pi/2$, equation (1.1) becomes the nonlinear Schrödinger equation

$$iu_t + \Delta u + |u|^\alpha u = 0. \tag{1.4}$$

Thus (1.1) is "intermediate" between the nonlinear heat and Schrödinger equations.

The equation (1.1) is a particular case of the more general complex Ginzburg-Landau equation

$$u_t = e^{i\theta}\Delta u + e^{i\gamma}|u|^\alpha u, \tag{1.5}$$

which has been studied in the context of a wide variety of applications. The nonlinear Schrödinger equation (1.4) is an important model in nonlinear optics and in the study of weakly nonlinear dispersive waves. (See [44].) The nonlinear heat

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equation (1.3) is also an important model, in particular in biology and chemistry. (See [11].) In the general case, equation (1.5) is used to model such phenomena as superconductivity, chemical turbulence and various types of fluid flows. (See [9] and the references cited therein.)

Local and global existence of solutions of (1.5), on both the whole space $\mathbb{R}^N$ and a domain $\Omega \subset \mathbb{R}^N$, are known under various boundary conditions and assumptions on the parameters, see e.g. [10, 12, 13, 17, 27, 32, 33, 34, 35].

On the other hand, when the equation (1.5) is neither the nonlinear heat equation (i.e. $\theta = \gamma = 0$) nor the nonlinear Schrödinger equation (i.e. $\theta = \gamma = \pi/2$), there are relatively few results concerning the existence of solutions which blow up in finite time. In [46], blowing-up solutions for the equation (1.5) on $\mathbb{R}^N$ are proved to exist, when the equation is "close" to the nonlinear heat equation (1.3), i.e. when $\theta = 0$ and $|\gamma|$ is small. A result in the same spirit is obtained in [42] when the equation is "close" to the nonlinear Schrödinger equation (1.4). The result in [46] was significantly extended in [21], where the authors give a rigorous justification of the numerical and formal arguments of [37, 38]. More precisely, they consider the equation (1.5) on $\mathbb{R}^N$ with $-\pi/2 < \theta, \gamma < \pi/2$ and prove the existence of blowing-up solutions when $\tan^2 \gamma + (\alpha + 2) \tan \gamma \tan \theta < \alpha + 1$. In [43], the authors prove finite-time blowup under a general negative energy condition when $\theta = \gamma$, $N = 1, 2$, $\alpha = 2$ and $|\theta| < \pi/4$. The calculations of [43] can be carried out in any space dimension and for more general values of α, and the condition $|\theta| < \pi/4$ takes the form $\cos^2 \theta > \frac{2}{\alpha+2}$ (still assuming $\theta = \gamma$). Note also that, under certain assumptions on the parameters, blowup for an equation similar to (1.5) on a bounded domain with Dirichlet or periodic boundary conditions, but with the nonlinearity $|u|^{\alpha+1}$ instead of $|u|^\alpha u$ is proved to occur in [28, 29, 36].

There are essentially two types of techniques for proving blowup in nonlinear evolution PDEs. One can look for an ansatz of an approximate blowing-up solution, and then show that the remainder becomes small with respect to the approximate solution, as time tends to the blow-up time of the approximate solution. The first difficulty in applying this method is to find the appropriate ansatz. Then, proving the smallness of the remainder is often quite involved technically. When this method is successful, it provides a precise description of how the corresponding solutions blow up. It may also explain the mechanism that makes these solution blow up. The drawback is that it applies only to initial values in some small neighborhood. This is the strategy employed in particular by Merle and Zaag [26] (for the nonlinear heat equation); Zaag [46], Masmoudi and Zaag [21] (for the complex Ginzburg-Landau equation); Martel and Merle [19], Martel, Merle and Raphaël [20] (for the generalized KdV equation); Merle and Raphaël [22, 23, 24], Raphaël [40], Raphaël and Szeftel [41], Merle, Raphaël and Szeftel [25] (for the nonlinear Schrödinger equation).

The second type of arguments often used for proving finite-time blowup consists in deriving a differential inequality on some quantity related to the solution, which can only hold on a finite-time interval. The major difficulty there is to guess the appropriate quantity to calculate. However, when such a method can be applied, it normally proves blowup under an explicit condition on the initial value, which usually yields a large class of initial values for which blowup occurs. The drawback is that it does not give any idea of what is the mechanism that leads to blowup.

This is the type of argument used by Kaplan [15], Levine [18], Ball [2] (for the nonlinear heat equation); Snoussi and Tayachi [43] (for the complex Ginzburg-Landau equation); Glassey [14], Zakharov [47], Kavian [16], Ogawa and Tsutsumi [30, 31] (for the nonlinear Schrödinger equation).

In what follows, we mostly use the second approach to study blowup for (1.1). In Section 2, we consider the endpoint $\theta = 0$ in (1.1), i.e. the nonlinear heat equation (1.3). We review the classical techniques, then we present some results that indicate that the mechanisms leading to finite-time blowup are not as simple as what appears at first sight. In Section 3, we recall the classical results for the other endpoint of equation (1.1), i.e. the nonlinear Schrödinger equation (1.4), and we discuss the possible mechanisms leading to blowup. Finally, we consider in Section 4 the equation (1.1) in the intermediate case $0 < \theta < \pi/2$. We obtain a general condition for finite-time blowup and we study how the blow-up time depends on the parameter θ.

Notation. Given a domain $\Omega \subset \mathbb{R}^N$, we denote by $L^p(\Omega)$, for $1 \le p \le \infty$, the usual (complex valued) Lebesgue spaces endowed with their standard norms. $H^1(\Omega)$, $H^1_0(\Omega)$ and $H^{-1}(\Omega)$ are the usual (complex valued) Sobolev spaces, endowed with their standard norms. (See e.g. [1] for the definitions and properties of these spaces.) We denote by $C^\infty_{\mathrm{c}}(\Omega)$ the set of (complex valued) functions that are compactly supported in Ω and of class C^∞. We denote by $C_0(\Omega)$ the closure of $C^\infty_{\mathrm{c}}(\Omega)$ in $L^\infty(\Omega)$. In particular, $C_0(\Omega)$ is the space of functions u that are continuous $\overline{\Omega} \to \mathbb{C}$ and such that $u(x) = 0$ for all $x \in \partial\Omega$ and $u(x) \to 0$ if $x \in \Omega$, $|x| \to \infty$. $C_0(\Omega)$ is endowed with the sup norm (i.e. the norm of $L^\infty(\Omega)$).

2. The nonlinear heat equation

The nonlinear heat equation (1.3) is probably the technically simplest case of (1.1). If we neglect the Laplacian in (1.3), then we obtain the ODE

$$z' = |z|^\alpha z. \tag{2.1}$$

The solution of (2.1) with the initial condition $z(0) = c \in \mathbb{C}$ is given by

$$z(t) = \frac{c}{[1 - t\alpha|c|^\alpha]^{\frac{1}{\alpha}}}, \tag{2.2}$$

for all $t \ge 0$ such that $t\alpha|c|^\alpha < 1$. If $c \ne 0$, it blows up at the finite time $T_{\max} = 1/\alpha|c|^\alpha$.

Remark 2.1. Note that if one changes the sign of the nonlinearity, i.e. if one considers the ODE $z' + |z|^\alpha z = 0$, then the solution with the initial condition $z(0) = c$ is given by $u(t) = c(1 + t\alpha|c|^\alpha)^{-\frac{1}{\alpha}}$. In particular, we see that all solutions are global and bounded as $t \to \infty$.

The above calculation clearly implies finite-time blowup for constant in space solutions of (2.3). Indeed, if $\Omega \subset \mathbb{R}^N$ and $c \in \mathbb{C}$, $c \ne 0$, then

$$u(t, x) = z(t),$$

for all $0 \le t < T_{\max}$ and $x \in \Omega$, where z is given by (2.2), is a solution of (1.3) on $[0, T_{\max}) \times \Omega$ which becomes singular (everywhere in space) at the time $T_{\max}$. These are admissible solutions if the equation is set on $\mathbb{R}^N$ or on a domain with the Neumann boundary condition.

Consider now the initial-boundary value problem for (1.3) with Dirichlet boundary condition. More precisely, let $\Omega \subset \mathbb{R}^N$ be a smooth domain (bounded or not), and consider the problem

$$\begin{cases} u_t = \Delta u + |u|^\alpha u, \\ u_{|\partial\Omega} = 0, \\ u(0, \cdot) = u_0(\cdot). \end{cases} \tag{2.3}$$

If Ω is unbounded, the boundary condition $u_{|\partial\Omega} = 0$ is interpreted as $u(t,x) = 0$ if $x \in \partial\Omega$ and $|u(t,x)| \to 0$ if $x \in \Omega$, $|x| \to \infty$. It is well known that the problem (2.3) is locally well posed in appropriate spaces. In particular, given any $u_0 \in C_0(\Omega)$, there exist $T = T(\|u_0\|_{L^\infty}) > 0$ and a unique function $u \in C([0,T], C_0(\Omega))$ which is C^1 in $t \in (0,T)$ and C^2 in $x \in \Omega$, satisfies the equation $u_t = \Delta u + |u|^\alpha u$ on $(0,T) \times \Omega$ and such that $u(0) = u_0$. Moreover, u can be extended to a maximal existence interval $[0, T_{\max})$ and satisfies the blowup alternative: either $T_{\max} = \infty$ (i.e. u is a global solution) or else $T_{\max} < \infty$ and $\|u(t)\|_{L^\infty} \to \infty$ as $t \uparrow T_{\max}$ (i.e. u blows up in finite time). In addition, if u_0 is real valued, then so is u; if $u_0 \ge 0$, then $u(t) \ge 0$ for all $0 \le t < T_{\max}$; and if $u_0 \ge 0$, $u_0 \not\equiv 0$, then $u(t) > 0$ for all $0 \le t < T_{\max}$. Furthermore, if $u_0 \in H^1_0(\Omega)$, then $u \in C([0,T_{\max}), H^1_0(\Omega))$; and if u_0 is radially symmetric, then so is $u(t)$ for all $0 \le t < T_{\max}$. See for example [39]. Note that in [39] real valued solutions are considered, but the same arguments apply to complex valued solutions.

Remark 2.2. Note that if one changes the sign of the nonlinearity, i.e. if one considers the equation $u_t = \Delta u - |u|^\alpha u$, then all solutions are global. This follows from the blowup alternative and the maximum principle (see Section 52 in [39]). Indeed, it follows from the maximal principle that $\|u(t)\|_{L^\infty} \le \|u_0\|_{L^\infty}$. Using the blowup alternative, we conclude that u is global.

Note that the fact that the solution of (2.3) vanishes on the boundary competes with the tendency of the nonlinear term to make the solution blow up. In fact, it is not difficult to show when Ω is bounded that if $\|u_0\|_{L^\infty}$ is sufficiently small, then the corresponding solution of (2.3) is global. (This is a consequence of the exponential decay of the heat semigroup. See e.g. Theorem 19.2 in [39].)

A general condition for blowup of positive solutions of (2.3) was obtained by Kaplan [15] in the case where Ω is bounded. It is based on the following calculation

$$\frac{d}{dt}\int_\Omega u\varphi_1 = -\lambda_1 \int_\Omega u\varphi_1 + \int_\Omega |u|^\alpha u\varphi_1 \ge -\lambda_1 \int_\Omega u\varphi_1 + \Big(\int_\Omega u\varphi_1\Big)^{\alpha+1},$$

where we used Jensen's inequality. Here φ_1 is the first eigenfunction of $-\Delta$ in $L^2(\Omega)$ with Dirichlet boundary condition, normalized by the condition $\int_\Omega \varphi_1 = 1$. Comparing with the solution of the ODE $z' = -\lambda_1 z + |z|^\alpha z$, $z(0) = z_0 \overset{\text{def}}{=} \int_\Omega u_0\varphi_1$, we deduce that

$$\int_\Omega u(t)\varphi_1 \ge \frac{\lambda_1 z_0}{[\lambda_1^\alpha - z_0^\alpha(1 - e^{-\lambda_1 t})]^{\frac{1}{\alpha}}},$$

for all $t < T_{\max}$ such that $t < T^\star$ where

$$T^\star = \begin{cases} \infty & \text{if } z_0 \le \lambda_1^{\frac{1}{\alpha}}, \\ -\frac{1}{\alpha\lambda_1}\log(1 - \frac{\varphi_1}{z_0^\alpha}) & \text{if } z_0 > \lambda_1^{\frac{1}{\alpha}}. \end{cases}$$

Thus we see that if $u_0 \in C_0(\Omega)$, $u_0 \geq 0$ satisfies

$$\int_\Omega u_0 \varphi_1 > \lambda_1^{\frac{1}{\alpha}},$$

then $T_{\max} < T^\star < \infty$, so that the corresponding solution of (2.3) blows up in finite time (by the blowup alternative).

A different condition was obtained by Levine [18], and it applies to sign-changing initial values and, more generally, to any complex valued solution with initial value in $C_0(\Omega) \cap H_0^1(\Omega)$. It is based on the energy identities

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |u|^2 = -I(u(t)), \tag{2.4}$$

$$\frac{d}{dt} E(u(t)) = -\int_\Omega |u_t|^2, \tag{2.5}$$

where

$$I(w) = \int_\Omega |\nabla w|^2 - \int_\Omega |w|^{\alpha+2}, \tag{2.6}$$

$$E(w) = \frac{1}{2} \int_\Omega |\nabla w|^2 - \frac{1}{\alpha+2} \int_\Omega |w|^{\alpha+2}. \tag{2.7}$$

The identity (2.4) (respectively, (2.5)) is obtained by multiplying the equation (2.3) by $\overline{u}$ (respectively, $\overline{u_t}$), integrating by parts and taking the real part. Note that

$$\begin{aligned} I(w) &= 2E(w) - \frac{\alpha}{\alpha+2} \int_\Omega |w|^{\alpha+2} \\ &= (\alpha+2)E(w) - \frac{\alpha}{2} \int_\Omega |\nabla w|^2. \end{aligned} \tag{2.8}$$

Levine's argument shows that if $E(u_0) < 0$, then the corresponding solution of (2.3) blows up in finite time. (Note that if $w \in C_0(\Omega) \cap H_0^1(\Omega)$, $w \neq 0$ and $u_0 = kw$, then $E(u_0) < 0$ for $|k|$ large.) Indeed, it follows from (2.4) and the first identity in (2.8) that

$$\frac{d}{dt} \int_\Omega |u|^2 = -4E(u(t)) + \frac{2\alpha}{\alpha+2} \int_\Omega |u|^{\alpha+2}. \tag{2.9}$$

Assuming $E(u_0) < 0$, we deduce from (2.5) that $E(u(t)) \leq E(u_0) < 0$, so that (2.9) yields

$$\frac{d}{dt} \int_\Omega |u|^2 \geq \frac{2\alpha}{\alpha+2} \int_\Omega |u|^{\alpha+2} \geq \frac{2\alpha}{\alpha+2} |\Omega|^{-\frac{2}{\alpha}} \Big(\int_\Omega |u|^2 \Big)^{\frac{\alpha+2}{2}}, \tag{2.10}$$

for all $0 < t < T_{\max}$. One concludes as above that $T_{\max} < \infty$. This is in fact the argument of Ball [2]. The original argument of Levine is slightly more involved, but it applies in any domain, bounded or not.

Theorem 2.3 (Levine [18])**.** *Let $u_0 \in C_0(\Omega) \cap H_0^1(\Omega)$ and let u be the corresponding solution of (2.3) defined on the maximal interval $[0, T_{\max})$. If $E(u_0) < 0$, then $T_{\max} < \infty$, i.e. u blows up in finite time.*

Proof. It follows from (2.4), the second identity in (2.8), and (2.5) that

$$\begin{aligned} \frac{d}{dt} \int_\Omega |u|^2 &\geq -2(\alpha+2)E(u(t)) \\ &= -2(\alpha+2)E(u_0) + 2(\alpha+2) \int_0^t \int_\Omega |u_t|^2. \end{aligned} \tag{2.11}$$

Assuming by contradiction that u is global, we deduce from (2.11) that

$$\|u(t)\|_{L^2} \underset{t\to\infty}{\longrightarrow} \infty. \tag{2.12}$$

Moreover, setting

$$f(t) = \int_0^t \|u(t)\|_{L^2}^2,$$

it follows from (2.11) and Cauchy-Schwarz that

$$\begin{aligned} ff'' &\geq 2(\alpha+2)\Big(\int_0^t \int_\Omega |u_t|^2\Big)\Big(\int_0^t \int_\Omega |u|^2\Big) \\ &\geq 2(\alpha+2)\Big(\int_0^t \int_\Omega |\overline{u} u_t|\Big)^2 \geq 2(\alpha+2)\Big(\int_0^t \Big|\int_\Omega \overline{u} u_t\Big|\Big)^2 \\ &\geq \frac{\alpha+2}{2}\Big(\int_0^t \frac{d}{dt}\int_\Omega |u|^2\Big)^2, \end{aligned}$$

which means

$$ff'' \geq \frac{\alpha+2}{2}(f'(t) - f'(0))^2. \tag{2.13}$$

Since $f'(t) \to \infty$ as $t \to \infty$ by (2.12) we deduce from (2.13) that for t large

$$ff'' \geq \frac{\alpha+4}{4} f'^2,$$

so that

$$(f^{-\frac{\alpha}{4}})'' \leq 0. \tag{2.14}$$

Equations (2.12) and (2.14) are clearly incompatible. $\square$

The above techniques, especially that of Kaplan, are closely related to the ODE (2.1). They seem to indicate that the solution of (2.3) blows up in finite time whenever the nonlinear term in the right-hand side of the equation dominates the linear part.

This is also what is suggested by the following classical result. (See Section 19.2, pp. 120–125 in [39].) Suppose Ω is bounded and

$$\alpha < \begin{cases} \infty & N = 1, 2; \\ \dfrac{4}{N-2} & N \geq 3. \end{cases} \tag{2.15}$$

It follows that there exists a positive, stationary solution $\phi \in C_0(\Omega) \cap H_0^1(\Omega)$ of (2.3), i.e.

$$\begin{cases} -\Delta\phi = |\phi|^\alpha \phi, \\ \phi_{|\partial\Omega} = 0. \end{cases}$$

If $0 \leq u_0 \leq \phi$ and $u_0 \neq \phi$, then the corresponding solution of (2.3) is global and converges (exponentially) to 0 as $t \to \infty$. On the other hand, if $u_0 \geq \phi$, $u_0 \neq \phi$, then the corresponding solution of (2.3) blows up in finite time. This can be interpreted as follows:

(i) If $u_0 = \phi$, there is a perfect balance between the linear and the nonlinear terms in the right-hand side of the equation, and the resulting solution is stationary.

(ii) If $0 \leq u_0 \leq \phi$ and $u_0 \neq \phi$ then, with respect to the previous case, the nonlinear term in the right-hand side of the equation becomes smaller than the linear term (at $t = 0$), and the resulting solution is global.

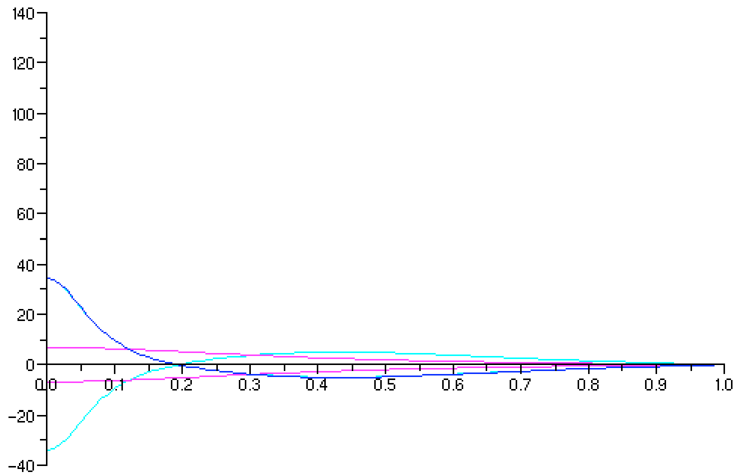

FIGURE 1. First movie: $u_0 = k\phi$ with $k = 1.01$ (click on the picture to play the movie)

(iii) If $u_0 \geq \phi$ and $u_0 \neq \phi$ then, with respect to the first case, the nonlinear term in the right-hand side of the equation becomes larger than the linear term (at $t = 0$), and the resulting solution blows up.

Note, however, that this interpretation may fail in the case of sign-changing stationary solutions. The two movies in figures 1 and 2 show what happens in the cubic, three dimensional case ($\alpha = 2$, $N = 3$), when Ω is the unit ball. The initial value u_0 has the form $u_0 = k\phi$, where ϕ is the one-node, radially symmetric stationary solution which is positive at the origin. The first movie corresponds to $k = 1.01$ and the second to $k = 0.99$. (Click on the picture to play the movie)

The movies suggest that multiplying the sign-changing stationary solution by a constant smaller than one, even though making the contribution of the nonlinear term smaller than the contribution of the linear term at $t = 0$, makes the corresponding solution blow up in finite time. In fact, such a property can be proved. More precisely, the following holds.

Theorem 2.4 ([5])**.** *Let Ω be the unit ball of $\mathbb{R}^N$, $N \geq 3$. It follows that there exists $0 < \underline{\alpha} < \frac{4}{N-2}$ with the following property. If $\underline{\alpha} < \alpha < \frac{4}{N-2}$ and if $\Phi \in C_0(\Omega)$ is a radially symmetric stationary solution of (2.3) which takes both positive and negative values, then there exist $0 < \underline{\lambda} < 1 < \overline{\lambda}$ such that if $\underline{\lambda} < \lambda < \overline{\lambda}$ and $\lambda \neq 1$, then the classical solution of (2.3) with the initial value $u_0 = \lambda\Phi$ blows up in finite time.*

Proof. Here is a sketch of the proof. For the details, see [5]. We linearize around the stationary solution Φ, i.e. we set

$$z^\lambda = \frac{u_{\lambda\Phi}(t) - \Phi}{\lambda - 1}$$

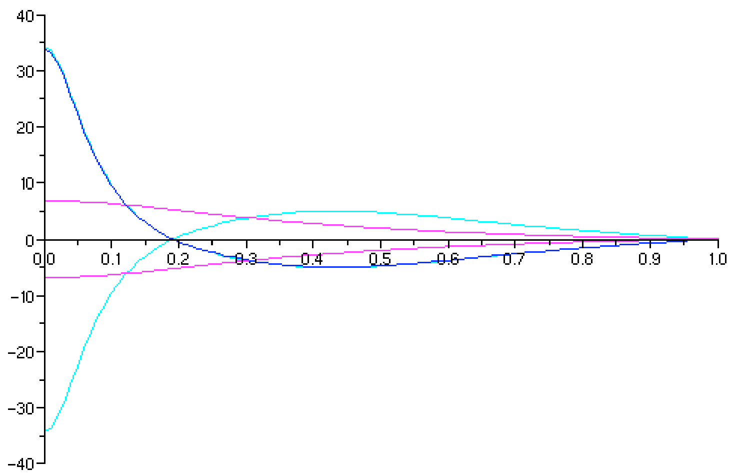

FIGURE 2. Second movie: $u_0 = k\phi$ with $k = 0.99$ (click on the picture to play the movie)

where $u_{\lambda\Phi}$ is the solution of (2.3) with the initial value $\lambda\Phi$. As $\lambda \to 1$, $z^\lambda \to z$ uniformly on bounded time intervals, where

$$\begin{cases} z_t = \Delta z + (\alpha+1)|\Phi|^\alpha z, \\ z(0) = \Phi. \end{cases}$$

Let λ_1 and $\varphi_1 > 0$ be the first eigenvalue and eigenvector of the operator $-\Delta - (\alpha+1)|\Phi|^\alpha$ on $L^2(\Omega)$ with Dirichlet boundary condition. If

$$\mathcal{I} = \int_\Omega \Phi\varphi_1 \neq 0, \tag{2.16}$$

(first Fourier coefficient of the expansion of Φ), then

$$z(t) \approx \mathcal{I}e^{-\lambda_1 t}\varphi_1 \text{ as } t \to \infty.$$

Therefore, if $\mathcal{I} \neq 0$, for exemple if $\mathcal{I} > 0$, then $z(T) > 0$ for T large. Thus, if $|\lambda - 1|$ is small, then $z^\lambda(T) > 0$, i.e.

$$u_{\lambda\Phi}(T) > \Phi \text{ if } \lambda > 1, \quad u_{\lambda\Phi}(T) < \Phi \text{ if } \lambda < 1.$$

In both cases, $u_{\lambda\Phi}$ blows up. (Φ is unstable from both above and below.)

We have finished if we can show (2.16). Φ is given, after scaling, by the solution w of (see figure 3)

$$\begin{cases} w'' + \dfrac{N-1}{r}w' + |w|^\alpha w = 0, \\ w(0) = 1, \quad w'(0) = 0. \end{cases}$$

If $\alpha \uparrow \alpha_\star = 4/(N-2)$, it follows that $w \to w_\star$, where

$$w_\star(r) = (1 + \frac{1}{N(N-2)}r^2)^{-\frac{N-2}{2}} > 0$$

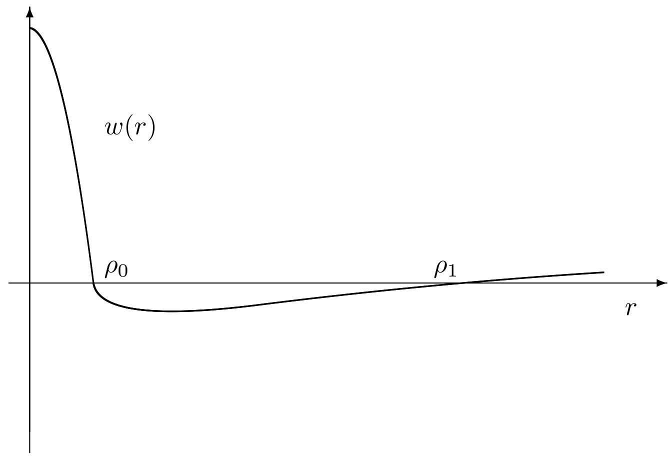

FIGURE 3. The function w

is the solution for $\alpha = \alpha_\star$. Now, if $\psi_\star$ is the first eigenvector (which exists) of $-\Delta - (\alpha_\star + 1) w_\star^{\alpha_\star}$ in $\mathbb{R}^N$ with $\psi_\star(0) = 1$, then obviously $\int_{\mathbb{R}^N} w_\star \psi_\star > 0$. A limiting argument shows that $\mathcal{I} > 0$ if α is close to $\alpha_\star$ (independently of the number of zeroes). □

Remark 2.5. Here are some comments on Theorem 2.4.

(i) The condition (2.16) is essential in the proof of Theorem 2.4.
(ii) One can obtain a similar result when $N = 3$ and α small, by a passage to the limit as above, but as $\alpha \to 0$ instead of $\alpha \to \alpha_\star$. See [6].
(iii) Note that in space dimension $N = 1$, we have $\mathcal{I} = 0$ for all $\alpha > 0$ and all sign-changing stationary solutions.
(iv) In space dimension $N \geq 2$, numerical experiments indicate that the property (2.16) holds for all $0 < \alpha < \frac{4}{N-2}$ and all sign-changing, radially symmetric stationary solutions of (2.3). However, the mechanism which is involved is not trivial, see figure 4, which represents $\Phi/\Phi(0)$ (blue curve), where Φ is the one-node stationary solution, and the corresponding first eigenvector (red curve), in the case $N = 3$, $\alpha = 2$.

Open problem 2.6. *Does property (2.16) hold for all $N \geq 2$, all $0 < \alpha < \frac{4}{N-2}$ and all sign-changing, radially symmetric stationary solutions of (2.3)?*

3. The nonlinear Schrödinger equation

Consider the nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u + |u|^\alpha u = 0 & (t,x) \in (0,T) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & x \in \mathbb{R}^N, \end{cases} \tag{3.1}$$

where $\alpha > 0$. The nonlinearity is called "focusing", while with the other sign it is called "defocusing". For the ODE

$$iz + |z|^\alpha z, \tag{3.2}$$

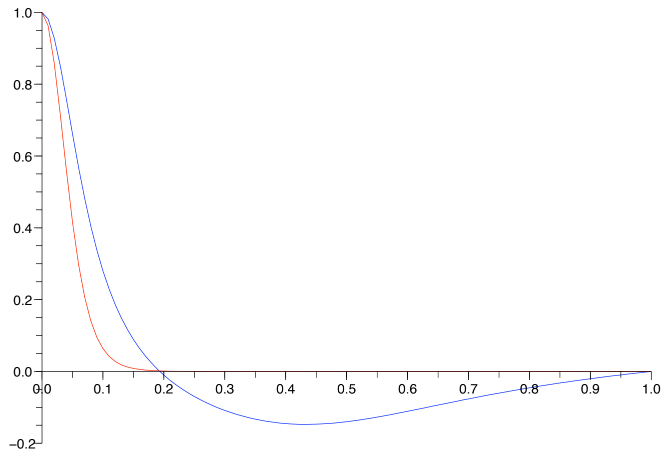

FIGURE 4. The functions Φ and φ_1

with the initial condition $z(0) = c \in \mathbb{C}$, there is no blowup. All solutions are global and bounded, given by

$$z(t) = ce^{it|c|^\alpha}.$$

If one changes the sign of the nonlinearity, i.e. if one considers the ODE $iz' - |z|^\alpha z = 0$, the solution is given by $z(t) = ce^{-it|c|^\alpha}$. Here again, all solutions are global and bounded, and there is no obvious distinction between the cases of the focusing and defocusing nonlinearities.

It is well known that the Cauchy problem (3.1) is locally well posed in $H^1(\mathbb{R}^N)$ provided

$$\alpha < \frac{4}{N-2}. \tag{3.3}$$

(It is also locally well posed if $\alpha = \frac{4}{N-2}$, but there is not the usual blowup alternative.) In particular, given any $u_0 \in H^1(\mathbb{R}^N)$, there exist $T = T(\|u_0\|_{H^1}) > 0$ and a unique solution $u \in C([0,T], H^1(\mathbb{R}^N)) \cap C^1([0,T], H^{-1}(\mathbb{R}^N)$ of (3.1). Moreover, u can be extended to a maximal existence interval $[0, T_{\max})$ and satisfies the blowup alternative: either $T_{\max} = \infty$ (i.e. u is a global solution) or else $T_{\max} < \infty$ and $\|u(t)\|_{H^1} \to \infty$ as $t \uparrow T_{\max}$ (i.e. u blows up in finite time). In addition, if $u_0 \in L^2(\mathbb{R}^N, |x|^2 dx)$, then $u \in C([0, T_{\max}, L^2(\mathbb{R}^N, |x|^2 dx))$. Moreover, if u_0 is radially symmetric, then so is $u(t)$ for all $0 \le t < T_{\max}$. In contrast with the heat equation, there is conservation of charge and energy,

$$\int_{\mathbb{R}^N} |u(t,x)|^2 dx = \int_{\mathbb{R}^N} |u_0|^2, \tag{3.4}$$

$$E(u(t)) = E(u_0), \tag{3.5}$$

for all $t \in [0, T_{\max})$, where the energy E is defined by (2.7). (See, e.g. [4] and the references therein for all the above properties.)

Finite-time blowup for (3.1) is known to occur if $\alpha \ge 4/N$.

Theorem 3.1 ([47, 14])**.** *Assume* (3.3) *and*

$$\alpha \geq \frac{4}{N}. \tag{3.6}$$

Let $u_0 \in H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2dx)$ and let u be the corresponding solution of (3.1) *defined on the maximal interval $[0, T_{\max})$. If $E(u_0) < 0$, then $T_{\max} < \infty$, i.e. u blows up in finite time.*

Proof. In view of the conservation laws (3.4) and (3.5), one cannot apply Levine's argument to prove blowup for (3.1). Instead, the proof of blowup is based on the variance identity (see e.g. Proposition 6.5.1 in [4] for details)

$$\begin{aligned} \frac{1}{2}\frac{d^2}{dt^2}\int_{\mathbb{R}^N} |x|^2|u|^2 &= -\frac{2N\alpha}{\alpha+2}\int_{\mathbb{R}^N} |u|^{\alpha+2} + 4\int_{\mathbb{R}^N} |\nabla u|^2 \\ &= 2N\alpha E(u_0) - (N\alpha - 4)\int_{\mathbb{R}^N} |\nabla u|^2. \end{aligned} \tag{3.7}$$

If $E(u_0) < 0$, then

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^N} |x|^2|u|^2 \leq -A, \tag{3.8}$$

with $A = -4N\alpha E(u_0) > 0$. Since $\int_{\mathbb{R}^N} |x|^2|u|^2 \geq 0$ for all $0 \leq t < T_{\max}$, we see that the solution cannot be global. □

Remark 3.2. Note that if

$$\alpha < \frac{4}{N},$$

then all solutions of (3.1) are global. Indeed, it follows easily from the conservation laws (3.4) and (3.5) together with Gagliardo-Nirenberg's inequality that $\sup_{0\leq t<T_{\max}} \|u(t)\|_{H^1} < \infty$. Thus $T_{\max} = \infty$ by the blowup alternative. (See e.g. Theorem 6.1.1 in [4] for details.)

Remark 3.3. Note that if one changes the sign of the nonlinearity, i.e. if one considers the equation $iu_t + \Delta u - |u|^\alpha u = 0$, then all solutions are global. Indeed, it follows from the conservation laws (3.4) and (3.5) that $\sup_{0\leq t<T_{\max}} \|u(t)\|_{H^1} < \infty$. (Note that if one changes the sign of the nonlinearity, this changes the sign of the term $|u|^{\alpha+2}$ in the energy.) Thus $T_{\max} = \infty$ by the blowup alternative.

The assumption $u_0 \in L^2(\mathbb{R}^N, |x|^2dx)$ in Theorem 3.1 is relaxed in [30, 31] if $N = 1$ and $\alpha = 4$, or if $N \geq 2$, $\alpha \leq 4$ and u_0 is radially symmetric. In particular, we have the following result.

Theorem 3.4 ([30])**.** *Suppose $N \geq 2$,* (3.3) *and $4/N \leq \alpha \leq 4$. Let $u_0 \in H^1(\mathbb{R}^N)$ be radially symmetric and let u be the corresponding solution of* (3.1) *defined on the maximal interval $[0, T_{\max})$. If $E(u_0) < 0$, then $T_{\max} < \infty$, i.e. u blows up in finite time.*

The proof of Theorem 3.4 uses an identity of the type (3.7) but with with the function $|x|^2$ replaced by a bounded function which coincides with $|x|^2$ on a large set. More precisely, fix a real-valued function $\Psi \in C^\infty(\mathbb{R}^N) \cap W^{4,\infty}(\mathbb{R}^N)$, let $u_0 \in H^1(\mathbb{R}^N)$ and let u be the corresponding solution of (3.1) defined on the maximal interval $[0, T_{\max})$. It follows that

$$\frac{1}{2}\frac{d^2}{dt^2}\int_{\mathbb{R}^N}\Psi|u|^2 = -\frac{1}{2}\int_{\mathbb{R}^N}\Delta^2\Psi|u|^2 - \frac{\alpha}{\alpha+2}\int_{\mathbb{R}^N}\Delta\Psi|u|^{\alpha+2} + 2\Re\int_{\mathbb{R}^N}\langle H(\Psi)\nabla\overline{u}, \nabla u\rangle, \quad (3.9)$$

for all $0 < t < T_{\max}$, where $H(\Psi)$ is the Hessian matrix $(\partial^2_{ij}\Psi)_{i,j}$. (See Kavian [16].) If both u and Ψ are radially symmetric, then (3.9) takes the form

$$\frac{1}{2}\frac{d^2}{dt^2}\int_{\mathbb{R}^N}\Psi|u|^2 = 2N\alpha E(u(t)) - (N\alpha-4)\int_{\mathbb{R}^N}|u_r|^2 - 2\int_{\mathbb{R}^N}(2-\Psi'')|u_r|^2 + \frac{\alpha}{\alpha+2}\int_{\mathbb{R}^N}(2N-\Delta\Psi)|u|^{\alpha+2} - \frac{1}{2}\int_{\mathbb{R}^N}\Delta^2\Psi|u|^2. \quad (3.10)$$

For a radially symmetric solution, the identity (3.10) resembles (3.7), except that there are extra terms that must be estimated. This is the purpose of the following lemma.

Lemma 3.5 (Lemma 5.3 in [7])**.** *Suppose $N \ge 2$ and $\alpha \le 4$. Given any $0 < a, A < \infty$, there exists a radially symmetric function $\Psi \in C^\infty(\mathbb{R}^N) \cap W^{4,\infty}(\mathbb{R}^N)$, such that $\Psi(x) > 0$ for $x \ne 0$ and*

$$-2\int_{\mathbb{R}^N}(2-\Psi'')|u_r|^2 + \frac{\alpha}{\alpha+2}\int_{\mathbb{R}^N}(2N-\Delta\Psi)|u|^{\alpha+2} - \frac{1}{2}\int_{\mathbb{R}^N}\Delta^2\Psi|u|^2 \le a, \quad (3.11)$$

for all radially symmetric $u \in H^1(\mathbb{R}^N)$ such that $\|u\|^2_{L^2} \le A$.

The proof of Lemma 3.5 is somewhat technical, and we refer the reader to [7] for its proof. We now can complete the proof of Theorem 3.4.

Proof of Theorem 3.4. Let u_0 and u be as in the statement and let $\Psi \in C^\infty(\mathbb{R}^N) \cap W^{4,\infty}(\mathbb{R}^N)$ be real-valued and radially symmetric. We deduce from (3.10), (3.5) and the assumtion $\alpha \ge 4/N$ that

$$\frac{1}{2}\frac{d^2}{dt^2}\int_{\mathbb{R}^N}\Psi|u|^2 \le 2N\alpha E(u_0) - 2\int_{\mathbb{R}^N}(2-\Psi'')|u_r|^2 + \frac{\alpha}{\alpha+2}\int_{\mathbb{R}^N}(2N-\Delta\Psi)|u|^{\alpha+2} - \frac{1}{2}\int_{\mathbb{R}^N}\Delta^2\Psi|u|^2. \quad (3.12)$$

We now observe that by (3.4), $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$. Applying Lemma 3.5 with $a = -N\alpha E(u_0) > 0$ and $A = \|u_0\|^2_{L^2}$, we deduce from (3.12) that

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^N}\Psi|u|^2 \le N\alpha E(u_0), \quad (3.13)$$

for all $0 \le t < T_{\max}$. Since $N\alpha E(u_0) < 0$ and $\int_{\mathbb{R}^N}\Psi|u|^2 \ge 0$, we conclude that $T_{\max} < \infty$. ∎

Remark 3.6. The assumptions that u_0 is radially symmetric and that $\alpha \le 4$ in Theorem 3.4 may seem unnatural. However, both these assumptions are necessary for the method we use. Indeed, the proof of Theorem 3.4 relies on the identity (3.9). Assuming that Ψ is radially symmetric, (3.9) takes the form

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}\int_{\mathbb{R}^N}\Psi|u|^2 &= 2N\alpha E(u(t))\\ -(N\alpha-4)\int_{\mathbb{R}^N}|\nabla u|^2 &+ 2\int_{\mathbb{R}^N}\Big(\frac{\Psi'}{r}-\Psi''\Big)(|\nabla u|^2-|u_r|^2) - 2\int_{\mathbb{R}^N}(2-\Psi'')|\nabla u|^2\\ &+ \frac{\alpha}{\alpha+2}\int_{\mathbb{R}^N}(2N-\Delta\Psi)|u|^{\alpha+2} - \frac{1}{2}\int_{\mathbb{R}^N}\Delta^2\Psi|u|^2.\end{aligned}$$

In order to complete the argument, we need at least an estimate of the form

$$\begin{aligned}-(N\alpha-4)\int_{\mathbb{R}^N}|\nabla u|^2 &+ 2\int_{\mathbb{R}^N}\Big(\frac{\Psi'}{r}-\Psi''\Big)(|\nabla u|^2-|u_r|^2)\\ -2\int_{\mathbb{R}^N}(2-\Psi'')|\nabla u|^2 &+ \frac{\alpha}{\alpha+2}\int_{\mathbb{R}^N}(2N-\Delta\Psi)|u|^{\alpha+2} \le F(\|u\|_{L^2}), \qquad (3.14)\end{aligned}$$

where F is bounded on bounded sets. Lemma 3.5 provides such an estimate for radially symmetric u under the assumption $\alpha \le 4$. It is not difficult to show that if $N\alpha > 4$, then there is no radially symmetric $\Psi \in C^4(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $\Psi \ge 0$, such that the estimate (3.14) holds for general u. (See Section 6 in [7].) Furthermore, one can also show that the assumption $\alpha \le 4$ is necessary in order that (3.14) holds for some $\Psi \in W^{4,\infty}(\mathbb{R}^N)\cap C^4(\mathbb{R}^N)$ and all radially symmetric u. (See Section 6 in [7].)

Open problem 3.7. *Can the finite variance assumption (i.e. $u_0 \in L^2(\mathbb{R}^N, |x|^2dx)$) be removed in Theorem 3.1?*

The above techniques do not give any clue as to what produces blowup in (3.1). We recall the following discussion from [8]. Given a function $u : [0,T]\times\mathbb{R}^N \to \mathbb{C}$, set

$$h(t) = \frac{\Re u \Im u_t - \Re u_t \Im u}{|u|^2}$$

It follows that

$$\Im \overline{u} u_t = |u|^2 h(t)$$

and, if we write $u = \rho e^{i\theta}$, then $h(t) = \theta_t$. In other words, h measures the speed of rotation of u.

Suppose first that u is a solution of the linear Schrödinger equation

$$iu_t + \triangle u = 0. \qquad (3.15)$$

Multiplying (3.15) by $\overline{u}$, integrating over $\mathbb{R}^N$, and taking the real part, we get that

$$\int_{\mathbb{R}^N}|u(t)|^2h(t)\,dx = -\|\nabla u(t)\|_{L^2}^2 = -\|\nabla u(0)\|_{L^2}^2.$$

Now suppose that u is a solution of the ODE (3.2). Multiplying (3.2) by $\overline{u}$ and taking the real part (without integrating) quickly yields that

$$h(t) = |u(t)|^\alpha. \qquad (3.16)$$

The competition between the two parts of (3.1), i.e. equations (3.15) and (3.2), is now evident. The linear equation (3.15) produces, on the average, a negative rotation, while the ordinary differential equation (3.2) produces a positive rotation at every point.

Finally, suppose that u is a solution of (3.1) with $4/N \le \alpha \le 4/(N-2)$ and that $u(0) \stackrel{\text{def}}{=} u_0 \in H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2 dx)$. Following the same steps as with equation (3.15) above, we arrive at the formula

$$\int_{\mathbb{R}^N} |u(t)|^2 h(t)\, dx = -2E(u_0) + \frac{\alpha}{\alpha+2} \|u(t)\|_{L^{\alpha+2}}^{\alpha+2}.$$

If u is a global solution and scatters as $t \to +\infty$, it follows easily that $\|u(t)\|_{L^{\alpha+2}} \to 0$ as $t \to \infty$. Moreover, since $E(u_0) < 0$ implies finite-time blow up and since u being a solution that scatters at $+\infty$ is an open condition on $u_0 \in H^1(\mathbb{R}^N)$, we see that $E(u_0) > 0$. Thus,

$$\int_{\mathbb{R}^N} |u(t)|^2 h(t)\, dx \to -2E(u_0) < 0,$$

as $t \to \infty$. In this case, the negative rotation induced by (3.15) wins. If, on the other hand, u blows up in finite time $T_{\max}$, then $\|u(t)\|_{L^{\alpha+2}} \to \infty$, as $t \uparrow T_{\max}$; and so,

$$\int_{\mathbb{R}^N} |u(t)|^2 h(t)\, dx \to \infty,$$

as $t \uparrow T_{\max}$. Here, the positive rotation induced by (3.16) wins.

4. The complex Ginzburg-Landau equation

Consider the complex Ginzburg-Landau equation

$$\begin{cases} e^{-i\theta} u_t = \Delta u + |u|^\alpha u & (t,x) \in (0,T) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & x \in \mathbb{R}^N, \end{cases} \tag{4.1}$$

where $\alpha > 0$ and

$$0 \le \theta < \frac{\pi}{2}.$$

For the ODE

$$e^{-i\theta} z' = |z|^\alpha z,$$

the solution with the initial condition $z(0) = c \in \mathbb{C}$ is given by

$$z(t) = c[1 - t\alpha |c|^\alpha \cos\theta]^{-\frac{1}{\alpha}(1 + i\tan\theta)},$$

for all $t \ge 0$ such that $t\alpha|c|^\alpha \cos\theta < 1$ and blows up at the finite time

$$T_{\max} = \frac{1}{\alpha |c|^\alpha \cos\theta},$$

provided $c \neq 0$.

Remark 4.1. Note that if one changes the sign of the nonlinearity, i.e. if one considers the ODE $e^{-i\theta} z' + |z|^\alpha z = 0$, then the solution with the initial condition $z(0) = c$ is given by $z(t) = c[1 + t\alpha|c|^\alpha \cos\theta]^{-\frac{1}{\alpha}(1+i\tan\theta)}$. In particular, we see that all solutions are global and bounded as $t \to \infty$.

Since $\theta < \pi/2$, it is easy to show that the initial value problem (4.1) is locally well posed in $C_0(\mathbb{R}^N)$ and in $C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ in exactly the same terms as the heat equation. See Section 2 in [7]. More precisely, given any $u_0 \in C_0(\mathbb{R}^N)$, there exist $T = T(\|u_0\|_{L^\infty}) > 0$ and a unique function $u \in C([0,T], C_0(\mathbb{R}^N))$ which is C^1 in $t \in (0,T)$ and C^2 in $x \in \mathbb{R}^N$, satisfies the equation $e^{-i\theta} u_t = \Delta u + |u|^\alpha u$ on $(0,T) \times \mathbb{R}^N$ and such that $u(0) = u_0$. Moreover, u can be extended to a maximal existence interval $[0, T_{\max})$ and satisfies the blowup alternative: either

$T_{\max} = \infty$ (i.e. u is a global solution) or else $T_{\max} < \infty$ and $\|u(t)\|_{L^\infty} \to \infty$ as $t \uparrow T_{\max}$ (i.e. u blows up in finite time). Furthermore, if $u_0 \in H^1(\mathbb{R}^N)$, then $u \in C([0, T_{\max}), H^1(\mathbb{R}^N))$.

The main feature of the equation (4.1), with respect to the more general Ginzburg-Landau equation (1.5) is that its solutions satisfy energy identities. More precisely,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 = -\cos\theta I(u(t)), \tag{4.2}$$

$$\frac{d}{dt}E(u(t)) = -\cos\theta \int_\Omega |u_t|^2, \tag{4.3}$$

where I and E are defined by (2.6). In fact, negative energy solutions of (4.1) blow up in finite time, as the following result shows.

Theorem 4.2 ([7], Theorem 1.1)**.** *Let $u_0 \in C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ and let $u \in C([0, T_{\max}), C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N))$ be the corresponding maximal solution of* (4.1)*. If $E(u_0) < 0$, then $T_{\max} < \infty$, i.e. u blows up in finite time.*

Proof. In view of the identities (4.2) and (4.3), one can reproduce Levine's calculations for the heat equation. Arguing as in the proof of Theorem 2.3, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega |u|^2 &\geq -2(\alpha+2)\cos\theta E(u(t))\\ &= -2(\alpha+2)\cos\theta E(u_0) + 2(\alpha+2)\cos^2\theta \int_0^t \int_\Omega |u_t|^2.\end{aligned} \tag{4.4}$$

Assuming by contradiction that u is global, we deduce from (4.4) that

$$\|u(t)\|_{L^2}^2 \geq [2(\alpha+2)\cos\theta E(u_0)]t \underset{t\to\infty}{\longrightarrow} \infty. \tag{4.5}$$

Moreover, setting

$$f(t) = \int_0^t \|u(t)\|_{L^2}^2,$$

it follows from (4.4) and Cauchy-Schwarz that

$$\begin{aligned} ff'' &\geq 2(\alpha+2)\cos^2\theta \Big(\int_0^t\int_\Omega |u_t|^2\Big)\Big(\int_0^t\int_\Omega |u|^2\Big)\\ &\geq 2(\alpha+2)\cos^2\theta\Big(\int_0^t\int_\Omega |\overline{u}u_t|\Big)^2 \geq 2(\alpha+2)\cos^2\theta\Big(\int_0^t\Big|\int_\Omega \overline{u}u_t\Big|\Big)^2\\ &\geq \frac{\alpha+2}{2}\cos^2\theta\Big(\int_0^t \frac{d}{dt}\int_\Omega |u|^2\Big)^2 = \frac{\alpha+2}{2}\cos^2\theta(f'(t)-f'(0))^2.\end{aligned} \tag{4.6}$$

One can conclude as in the proof of Theorem 2.3 provided

$$\frac{\alpha+2}{2}\cos^2\theta > 1.$$

However, the above condition can be removed. One slightly modifies (4.6) as follows.

$$\begin{aligned} ff'' &\geq 2(\alpha+2)\cos^2\theta\Big(\int_0^t\int_\Omega |u_t|^2\Big)\Big(\int_0^t\int_\Omega |u|^2\Big)\\ &\geq 2(\alpha+2)\cos^2\theta\Big(\int_0^t\Big|\int_\Omega \overline{u}u_t\Big|\Big)^2.\end{aligned} \tag{4.7}$$

Next, we observe that, multiplying the equation (4.1) by $\overline{u}$,

$$\int_{\mathbb{R}^N} \overline{u} u_t = -e^{i\theta} I(u),$$

so that

$$\Big| \int_{\mathbb{R}^N} \overline{u} u_t \Big| = |I(u)|.$$

Since $I(u) < 0$ by (2.8), we conclude that

$$\Big| \int_{\mathbb{R}^N} \overline{u} u_t \Big| = -I(u) = \frac{1}{2\cos\theta} \frac{d}{dt} \int_\Omega |u|^2, \tag{4.8}$$

where we used (4.2) in the last identity. We deduce from (4.7) and (4.8) that

$$f f'' \ge \frac{\alpha+2}{2} (f'(t) - f'(0))^2, \tag{4.9}$$

and we conclude as in the proof of Theorem 2.3. ☐

Remark 4.3. Here are some comments on Theorem 4.2.

(i) One can refine the conclusion of Theorem 4.2. More precisely, one can obtain the following estimate of the maximal existence time

$$T_{\max} \le \frac{\|u_0\|_{L^2}^2}{\alpha(\alpha+2)(-E(u_0))\cos\theta}. \tag{4.10}$$

(See Theorem 1.1 in [7].)

(ii) One can quantify the end of the proof as follows. Set

$$K = \Big[1 - \Big(\frac{\alpha+4}{2\alpha+4} \Big)^{\frac{1}{2}} \Big]^{-1} > 1, \tag{4.11}$$

and

$$\tau = \sup\{ t \in [0, T_{\max});\ \|u(s)\|_{L^2}^2 \le K \|u_0\|_{L^2}^2 \text{ for } 0 \le s \le t \} \le T_{\max}. \tag{4.12}$$

If $\tau < T_{\max}$, we deduce from (4.4) that

$$\|u(s)\|_{L^2}^2 \le K \|u_0\|_{L^2}^2 \le \|u(t)\|_{L^2}^2 \quad 0 \le s \le \tau \le t < T_{\max}, \tag{4.13}$$

so that $\frac{\alpha+2}{2}(f'(t) - f'(0))^2 \ge \frac{\alpha+4}{4} f'(t)^2$ for $\tau \le t < T_{\max}$; and so $(f^{-\frac{\alpha}{4}})'' \le 0$ for $\tau \le t < T_{\max}$. Therefore,

$$f(t)^{-\frac{\alpha}{4}} \le f(\tau)^{-\frac{\alpha}{4}} + (t-\tau)(f^{-\frac{\alpha}{4}})'(\tau) = f(\tau)^{-\frac{\alpha}{4}} \Big[1 - \frac{\alpha}{4} (t-\tau) f(\tau)^{-1} h'(\tau) \Big],$$

for $\tau \le t < T_{\max}$. Since $f(t)^{-\frac{\alpha}{4}} \ge 0$, we deduce that for every $\tau \le t < T_{\max}$,

$$\frac{\alpha}{4}(t-\tau) f(\tau)^{-1} h'(\tau) \le 1,$$

i.e.

$$(t-\tau)\|u(\tau)\|_{L^2}^2 \le \frac{4}{\alpha} \int_0^\tau \|u(s)\|_{L^2}^2 ds \le \frac{4}{\alpha} \tau \|u(\tau)\|_{L^2}^2, \tag{4.14}$$

where we used (4.13) in the last inequality. Thus $t \le \frac{\alpha+4}{\alpha}\tau$ for all $\tau \le t < T_{\max}$. Thus we conclude that

$$T_{\max} \le \frac{\alpha+4}{\alpha} \tau. \tag{4.15}$$

Fix an initial value $u_0 \in C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ such that $E(u_0) < 0$ and, given $\theta \in [0, \pi/2)$, let $u^\theta \in C([0, T^\theta_{\max}), C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N))$ be the corresponding solution of (4.1), so that u^θ blows up in finite time by Theorem 4.2.

If $\alpha < 4/N$, then the solution of (3.1) (i.e. (4.1) for $\theta = \pi/2$) is global, so we may expect that the blow-up time $T^\theta_{\max}$ of u^θ goes to infinity as $\theta \to \pi/2$. This is indeed the case, as the following result shows.

Theorem 4.4 ([7], Theorem 1.2)**.** *Fix an initial value $u_0 \in C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ and, for $0 \le \theta < \pi/2$ let $u^\theta \in C([0, T^\theta_{\max}), C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N))$ denote the corresponding maximal solution of* (4.1)*. If*

$$0 < \alpha < \frac{4}{N},$$

then there exists a constant $c = c(N, \alpha, \|u_0\|_{L^2}, E(u_0)) > 0$ such that

$$T^\theta_{\max} \ge \frac{c}{\cos\theta}, \tag{4.16}$$

for all $0 \le \theta < \frac{\pi}{2}$.

Proof. Global existence for (3.1) with $\alpha < 4/N$ follows from the conservation of charge and energy and Gagliardo-Nirenberg's inequality. Similarly, Theorem 4.4 follows from energy identities and Gagliardo-Nirenberg's inequality. We give here a sketch of the proof and refer the reader to [7] for details.

If $T^\theta_{\max} = \infty$, there is nothing to prove. We then assume $T^\theta_{\max} < \infty$. Since $\alpha < 4/N$, it is not difficult to show that the Cauchy problem (4.1) is locally well posed in $L^2(\mathbb{R}^N)$ (see [45]), from which it follows easily that

$$\|u^\theta(t)\|_{L^2} \uparrow \infty \quad \text{as} \quad t \uparrow T^\theta_{\max}. \tag{4.17}$$

Therefore, if we set

$$S^\theta = \sup\{t \in [0, T^\theta_{\max});\ \|u^\theta(s)\|^2_{L^2} \le 2\|u_0\|^2_{L^2} \text{ for } 0 \le s \le t\},$$

then $S^\theta < T^\theta_{\max}$ and

$$\|u^\theta(S^\theta)\|^2_{L^2} = 2\|u_0\|^2_{L^2}. \tag{4.18}$$

Since $E(u^\theta(t)) \le E(u_0)$ by (4.3) and

$$\|u^\theta(t)\|^2_{L^2} \le 2\|u_0\|^2_{L^2}, \tag{4.19}$$

for $0 \le t \le S^\theta$, it follows from Gagliardo-Nirenberg's inequality that there exists a constant K such that

$$\|\nabla u^\theta(t)\|^2_{L^2} + \|u^\theta(t)\|^{\alpha+2}_{L^{\alpha+2}} \le K, \tag{4.20}$$

for all $0 \le \theta < \pi/2$ and $0 \le t \le S^\theta$; and so

$$|I(u^\theta(t))| \le K. \tag{4.21}$$

Applying (4.2) and (4.21), we deduce that

$$\|u^\theta(S^\theta)\|^2_{L^2} \le \|u_0\|^2_{L^2} + 2K\cos\theta S^\theta. \tag{4.22}$$

It now follows from (4.22) and (4.18) that

$$S^\theta \ge \frac{\|u_0\|^2_{L^2}}{2K\cos\theta}, \tag{4.23}$$

which proves the desired estimate. □

Remark 4.5. Note that, under the assumptions of Theorem 4.4 and if, in addition, $E(u_0) < 0$, there exist $c, C > 0$ such that

$$\frac{c}{\cos\theta} \le T^\theta_{\max} \le \frac{C}{\cos\theta},$$

for all $-\pi/2 < \theta < \pi/2$. This follows from (4.16) and (4.10).

As observed above (Theorem 3.1), if $4/N \le \alpha < 4/(N-2)$ and if u_0 has negative energy and finite variance, then the corresponding solution of the nonlinear Schrödinger equation (3.1) blow up in finite time. Thus we may expect that for such initial values u_0, the blow-up time of u^θ remains bounded as $\theta \to \pi/2$.

It seems relevant to try the variance identity. For (3.1), multiplying the equation by $-i|x|^2\overline{u}$, integrating by parts and taking the real part yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} |x|^2|u|^2 = 2\Im\int_{\mathbb{R}^N} \overline{u}x\cdot\nabla u. \tag{4.24}$$

The identity (3.7) follows by taking the time derivative and replacing u_t in the right-hand side by using the equation. For (4.1), multiplying the equation by $e^{i\theta}|x|^2\overline{u^\theta}$, integrating by parts and taking the real part yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} |x|^2|u^\theta|^2 &= 2\sin\theta\Im\int_{\mathbb{R}^N} \overline{u^\theta}x\cdot\nabla u^\theta \\ &\quad + \cos\theta\int_{\mathbb{R}^N} (-2\Re(\overline{u^\theta}x\cdot\nabla u^\theta) - |x|^2|\nabla u^\theta|^2 + |x|^2|u^\theta|^{\alpha+2}).\end{aligned} \tag{4.25}$$

Next, one must take the time derivative of (4.25). With respect to (4.24), there are three more terms whose time derivative must be calculated. However, some of the numerous terms that appear in the calculations combine and one can write the resulting identity in the following form. (See Section 7 in [7].)

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}\int_{\mathbb{R}^N} |x|^2|u^\theta|^2 &= 2N\alpha E(u^\theta(t)) - (N\alpha-4)\int_{\mathbb{R}^N}|\nabla u^\theta|^2 \\ &\quad + \cos\theta\frac{d}{dt}\int_{\mathbb{R}^N}\Big\{-2|x|^2|\nabla u^\theta|^2 + \frac{\alpha+4}{\alpha+2}|x|^2|u^\theta|^{\alpha+2} + 2N|u^\theta|^2\Big\} \\ &\quad - 2\cos^2\theta\int_{\mathbb{R}^N}|x|^2|u^\theta_t|^2.\end{aligned} \tag{4.26}$$

The identity follows from formal calculations that can be justified by standard techniques assuming u_0 is sufficiently regular, and certainly if $u_0 \in C^\infty_{\mathrm{c}}(\mathbb{R}^N)$. In particular,

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}\int_{\mathbb{R}^N} |x|^2|u^\theta|^2 &\le 2N\alpha E(u_0) \\ &\quad + \cos\theta\frac{d}{dt}\int_{\mathbb{R}^N}\Big\{-2|x|^2|\nabla u^\theta|^2 + \frac{\alpha+4}{\alpha+2}|x|^2|u^\theta|^{\alpha+2} + 2N|u^\theta|^2\Big\}.\end{aligned} \tag{4.27}$$

Integrating twice in time, one gets to an estimate of the form

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^2|u^\theta|^2 &\le C + Ct + 2N\alpha E(u_0)t^2 \\ &\quad + \cos\theta\int_0^t\int_{\mathbb{R}^N}\Big\{-2|x|^2|\nabla u^\theta|^2 + \frac{\alpha+4}{\alpha+2}|x|^2|u^\theta|^{\alpha+2} + 2N|u^\theta|^2\Big\}.\end{aligned} \tag{4.28}$$

The factor of $\cos\theta$ in (4.28) (due to the term $|x|^2|u^\theta|^{\alpha+2}$) seems very difficult to control, so the only hope seems to be that it can be controlled by non-weighted terms. This can be done with the following estimate, similar to some results in [3].

Lemma 4.6 (Lemma 7.1 in [7])**.** *Suppose $N \ge 2$ and $4/N \le \alpha \le 4$. Given any $M > 0$, there exists a constant C such that*

$$\int |x|^2 |w^{\alpha+2} \le \int |x|^2 |\nabla w|^2 + C \int |w|^{\alpha+2} + C, \tag{4.29}$$

for all smooth, radially symmetric w such that $\|w\|_{L^2} \le M$.

Let $K > 1$ and $0 < \tau^\theta \le T^\theta_{\max}$ be defined by (4.11) and (4.12). We deduce from (4.28) and Lemma 4.6 that there exists a constant C (independent of $0 \le \theta < \pi/2$) such that

$$\int_{\mathbb{R}^N} |x|^2 |u^\theta|^2 \le C + Ct + 2N\alpha E(u_0) t^2 + C\cos\theta \int_0^t \int_{\mathbb{R}^N} |u^\theta|^{\alpha+2}, \tag{4.30}$$

for $0 \le t < \tau^\theta$. On the other hand, it follows from (4.2) and (2.8) that

$$\frac{d}{dt}\int_{\mathbb{R}^N} |u^\theta|^2 = \cos\theta\Big(-2E(u^\theta) + \frac{2\alpha}{\alpha+2}\int_{\mathbb{R}^N} |u^\theta|^{\alpha+2}\Big) \ge \cos\theta \frac{2\alpha}{\alpha+2}\int_{\mathbb{R}^N} |u^\theta|^{\alpha+2},$$

so that

$$\cos\theta \int_0^{\tau^\theta} \int_{\mathbb{R}^N} |u^\theta|^{\alpha+2} \le \frac{(K-1)(\alpha+2)}{2\alpha} \|u_0\|^2_{L^2}. \tag{4.31}$$

It now follows from (4.30) and (4.31) that, for C possibly larger but still independent of θ,

$$\int_{\mathbb{R}^N} |x|^2 |u^\theta|^2 \le C + Ct + 2N\alpha E(u_0) t^2, \tag{4.32}$$

for $0 \le t < \tau^\theta$. Since the left-hand side of (4.32) is nonnegative, we deduce that there exists T independent of θ such that $\tau^\theta \le T < \infty$ for all $0 \le \theta < \pi/2$. Finally, it follows from Remark 4.3 (ii) that

$$T^\theta_{\max} \le \frac{\alpha+4}{\alpha}\tau^\theta,$$

for all $0 \le \theta < \pi/2$, so that $T^\theta_{\max}$ remains bounded as $\theta \to \pi/2$.

The assumptions that u_0 is radially symmetric and $\alpha \le 4$ seem unnatural. These conditions come from Lemma 4.6, which is essential in the argument. It is interesting to note that both radial symmetry and the bound $\alpha \le 4$ are essential in Lemma 4.6. (See Section 7 in [7].)

Note that these assumptions are precisely those made by Ogawa and Tsutsumi in [30], where the authors eliminate the finite variance assumption of [14, 47]. At this stage one might as well proceed as in [30] and use a truncated variance. Doing so, one obtains the following result.

Theorem 4.7 (Theorem 1.5 in [7])**.** *Suppose*

$$N \ge 2, \quad \frac{4}{N} \le \alpha \le 4, \tag{4.33}$$

and fix a radially symmetric initial value $u_0 \in H^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$. Given any $0 \le \theta < \pi/2$, let $u^\theta \in C([0, T^\theta_{\max}), C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N))$ denote the corresponding maximal solution of (4.1). If $E(u_0) < 0$, then there exists $\overline{T} < \infty$ such that $T^\theta_{\max} \le \overline{T}$ for all $0 \le \theta < \frac{\pi}{2}$.

Proof. The proof of Theorem 4.7 is based on the following identity, valid for all radially symmetric $u_0 \in H^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$ and all radially symmetric $\Psi \in C^4(\mathbb{R}^N) \cap W^{4,\infty}(\mathbb{R}^N)$. (See formula (5.22) in [7].)

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}\int_{\mathbb{R}^N}\Psi|u^\theta|^2 &= 2N\alpha E(u^\theta(t)) - (N\alpha-4)\int_{\mathbb{R}^N}|u^\theta_r|^2 - 2\int_{\mathbb{R}^N}(2-\Psi'')|u^\theta_r|^2 \\ &\quad + \frac{\alpha}{\alpha+2}\int_{\mathbb{R}^N}(2N-\Delta\Psi)|u^\theta|^{\alpha+2} - \frac{1}{2}\int_{\mathbb{R}^N}\Delta^2\Psi|u^\theta|^2 \\ &\quad + \cos\theta\frac{d}{dt}\int_{\mathbb{R}^N}\Big\{-2\Psi|\nabla u^\theta|^2 + \frac{\alpha+4}{\alpha+2}\Psi|u^\theta|^{\alpha+2} + \Delta\Psi|u^\theta|^2\Big\} \\ &\qquad - 2\cos^2\theta\int_{\mathbb{R}^N}\Psi|u^\theta_t|^2.\end{aligned}$$

Since $E(u^\theta(t))$ is nonincreasing and $N\alpha \le 4$, we deduce (assuming $\Psi \ge 0$) that

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}\int_{\mathbb{R}^N}\Psi|u^\theta|^2 &\le 2N\alpha E(u_0) - 2\int_{\mathbb{R}^N}(2-\Psi'')|u^\theta_r|^2 \\ &\quad + \frac{\alpha}{\alpha+2}\int_{\mathbb{R}^N}(2N-\Delta\Psi)|u^\theta|^{\alpha+2} - \frac{1}{2}\int_{\mathbb{R}^N}\Delta^2\Psi|u^\theta|^2 \\ &\quad + \cos\theta\frac{d}{dt}\int_{\mathbb{R}^N}\Big\{-2\Psi|\nabla u^\theta|^2 + \frac{\alpha+4}{\alpha+2}\Psi|u^\theta|^{\alpha+2} + \Delta\Psi|u^\theta|^2\Big\}. \end{aligned} \tag{4.34}$$

One controls the first "problematic" terms in (4.34) by using Lemma 3.5. More precisely, let $K > 1$ and $0 < \tau^\theta \le T^\theta_{\max}$ be defined by (4.11) and (4.12). Applying Lemma 3.5 with $a = -N\alpha E(u_0) > 0$ and $A = K\|u_0\|^2_{L^2}$, we deduce from (4.34) that

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}\int_{\mathbb{R}^N}\Psi|u^\theta|^2 &\le N\alpha E(u_0) \\ &\quad + \cos\theta\frac{d}{dt}\int_{\mathbb{R}^N}\Big\{-2\Psi|\nabla u^\theta|^2 + \frac{\alpha+4}{\alpha+2}\Psi|u^\theta|^{\alpha+2} + \Delta\Psi|u^\theta|^2\Big\}, \end{aligned} \tag{4.35}$$

for all $0 \le t < \tau^\theta$. We next integrate (4.35) twice in time and we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}\Psi|u^\theta|^2 &\le C + Ct + N\alpha E(u_0)t^2 \\ &\quad + \cos\theta\int_0^t\int_{\mathbb{R}^N}\Big\{-2\Psi|\nabla u^\theta|^2 + \frac{\alpha+4}{\alpha+2}\Psi|u^\theta|^{\alpha+2} + \Delta\Psi|u^\theta|^2\Big\}, \end{aligned} \tag{4.36}$$

for all $0 \le t < \tau^\theta$, where the constant C is independent of $0 \le \theta < \pi/2$. One then concludes as above, see the argument following (4.30). □

Remark 4.8. The assumptions that u_0 is radially symmetric and that $\alpha \le 4$ in Theorem 4.7 may seem unnatural. However, both these assumptions are necessary for the method we use, see the comments in Remark 3.6 above.

Open problem 4.9. *Suppose $4/N \le \alpha < 4/(N-2)$. Fix an initial value $u_0 \in H^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$ and, given $0 \le \theta < \pi/2$, let $u^\theta \in C([0, T^\theta_{\max}), C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N))$ denote the corresponding maximal solution of (4.1). If $E(u_0) < 0$ (and, possibly, $u_0 \in L^2(\mathbb{R}^N, |x|^2dx)$), is it true that $\limsup_{\theta\uparrow\pi/2} T^\theta_{\max} < \infty$?*

Remark 4.10. If one changes the sign of the nonlinearity in (4.1), i.e. if one considers the equation $e^{-i\theta} u_t = \Delta u - |u|^\alpha u$, then the corresponding Cauchy problem is locally well posed in $C_0(\mathbb{R}^N)$ and in $C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$. Note that the factor of $|u|^{\alpha+2}$ comes with a positive sign in both the corresponding quantities I and E. If the initial value belongs to $C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, then the energy identities (4.2) and (4.3) yield a control of $\|u(t)\|_{H^1}$, which is uniform in $0 \le t < T_{\max}$. Using a standard parabolic bootstrap argument, it is not difficult to deduce that if $\alpha < 4/(N-2)$ ($\alpha < \infty$ if $N = 1, 2$), then the L^∞ norm of the solution is also controlled, so that the solution is global by the blowup alternative. Thus we see that all solutions with initial value in $C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ are global if $\alpha < 4/(N-2)$. Note that these estimates make use of the energies, so they are not valid for initial values that are only in $C_0(\mathbb{R}^N)$.

In view of the above remark, we emphasize the following open problems.

Open problem 4.11. *Consider the equation $e^{-i\theta} u_t = \Delta u - |u|^\alpha u$ and suppose $N \ge 3$ and $\alpha \ge 4/(N-2)$. Given any $u_0 \in C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ let $u \in C([0, T_{\max}), C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N))$ be the the maximal solution corresponding to the initial value u_0. Is u global?*

Open problem 4.12. *Consider the equation $e^{-i\theta} u_t = \Delta u - |u|^\alpha u$. Given any $u_0 \in C_0(\mathbb{R}^N)$ let $u \in C([0, T_{\max}), C_0(\mathbb{R}^N))$ be the the maximal solution corresponding to the initial value u_0. Is u global?*

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