Stabilized Finite Element Methods in Porous Media Flow

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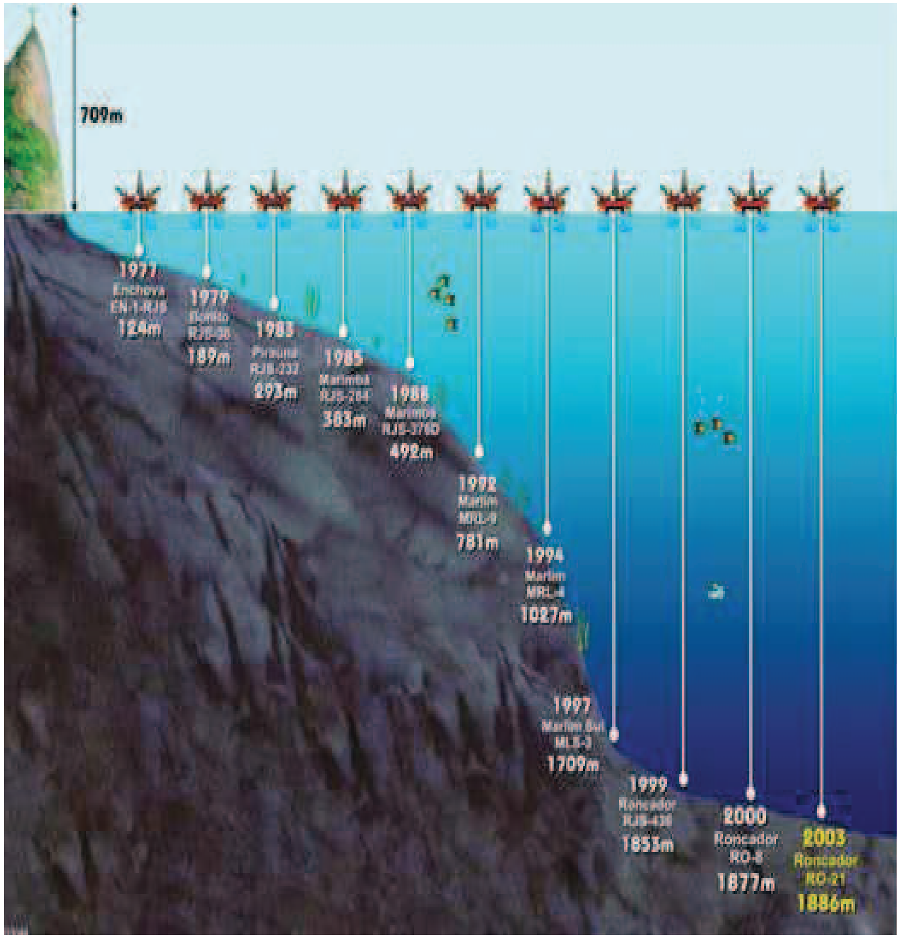
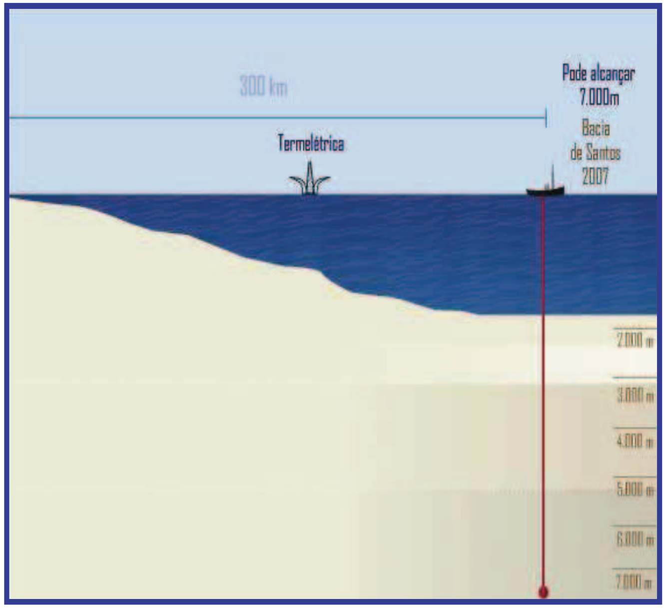
O Brasil

Exemplos do impacto positivo da C,T&I no sucesso da economia do Brasil atual



PETROBRAS

2007- Tupi – 7000 m



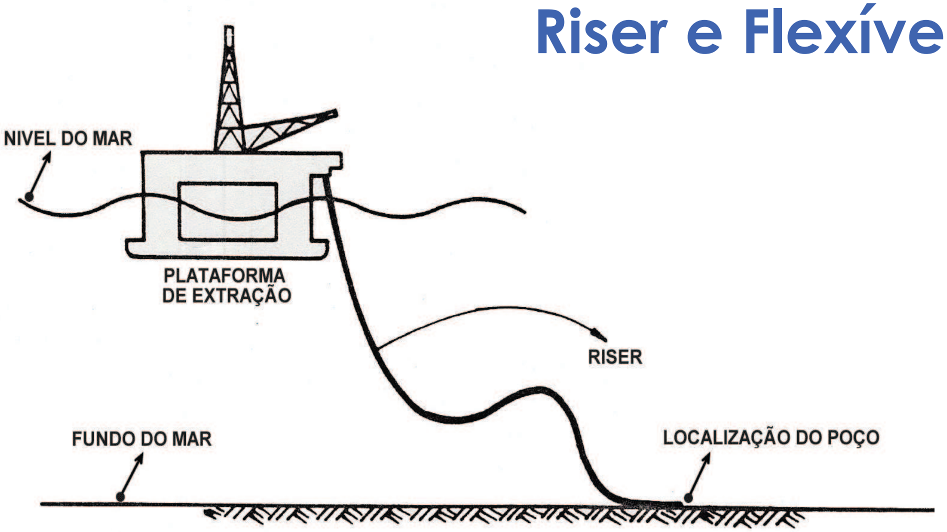
Líder em prospecção de óleo e gás em águas profundas

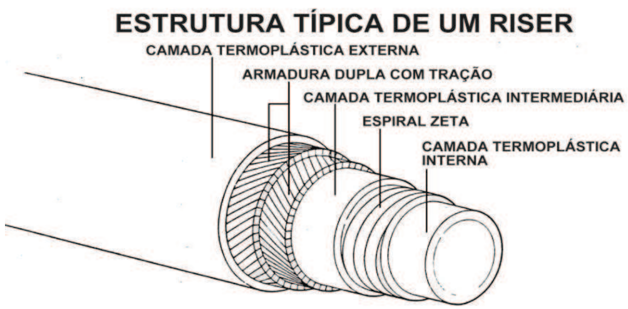
Record: 2.777 metros em 2007

- 174m em 1977 no campo Enchovia EN-1 RJS
- 189m em 1979 no campo Bonito RJS-36
- 293m em 1983 no campo Piraúna RJS-232
- 383m em 1985 no campo Marimbá RJS-284
- 492m em 1988 no campo Marimbá RJS-3760
- 781m em 1992 no campo Marlim MRL-9
- 1.027m em 1994 no campo Marlim MRL-4
- 1.709m em 1997 no campo Marlim MLS-3
- 1.853m em 1999 no campo Roncador RJS-436
- 1.877m em 2000 no campo Roncador RO-8
- 1.886m em 2003 no campo Roncador RO-21

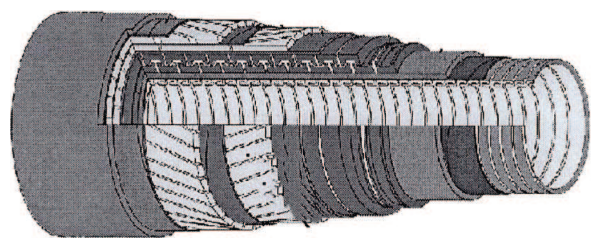
Risers and Flexibles

Riser e Flexíveis





Secção de um riser



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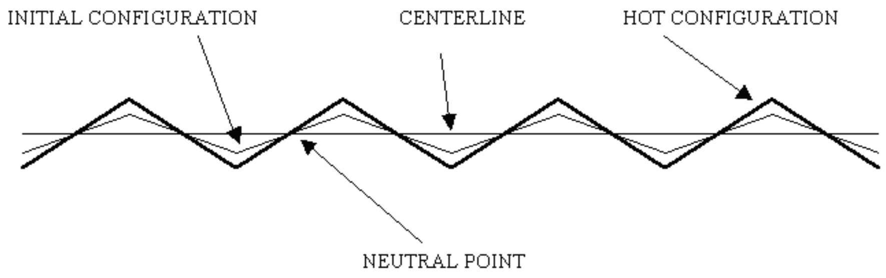
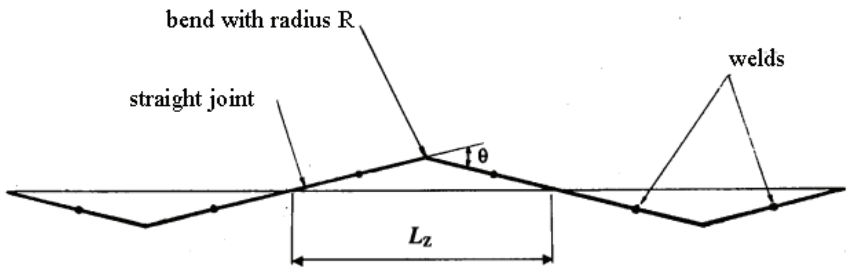
Zig-Zag Design Concept

Estudo da Sensibilidade do Duto em Zig-Zag



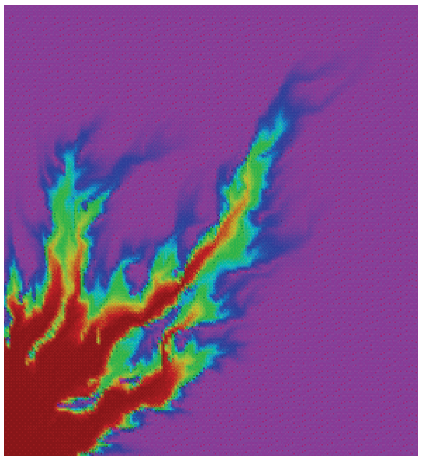


Porous Media Flow

Escoamento em Meios Porosos

Modelagem Matemática
Análise Numérica
Simulção Computacional

Aplicações a reservatórios de petróleo e águas subterrâneas,
tecidos biológicos e simulação de escoamento sanguíneos



Extração de Petróleo

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Introduction

Miscible displacement in porous media:

$$\phi \frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c - \mathrm{div}(D \nabla c) + \hat{f} c = g \quad \text{in} \quad \Omega \times (0, T),$$

$$D = D(\mathbf{u}) = \alpha_m \mathbf{I} + |\mathbf{u}| \{ \alpha_l E(\mathbf{u}) + \alpha_t E^{\perp}(\mathbf{u}) \}$$

$$E(\mathbf{u}) = \frac{1}{|\mathbf{u}|^2} \mathbf{u} \otimes \mathbf{u}, \qquad E^{\perp}(\mathbf{u}) = \mathbf{I} - E(\mathbf{u})$$

with the boundary and initial conditions

$$D \nabla c \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \times (0, T)$$

$$c(\mathbf{x}, 0) = c_0(\mathbf{x}) \quad \text{on} \quad \Omega$$

The velocity field $\mathbf{u}$ is given by the Darcy flow model

Outline

- Introduction
- On finite element formulations for Darcy flow
  - Discontinuous Galerkin Methods
  - Dual mixed formulation
  - Hybridization
- Stabilized formulations
  - Stabilized dual mixed formulation
  - Stabilized dual hybrid mixed formulation
  - Hybridizable mixed DG method
- Approximations
- Numerical results
- Concluding remarks

Model problem - Darcy flow

Given the hydraulic conductivity tensor $\mathbf{K}$*, with inverse* $\mathbf{A} = \mathbf{K}^{-1}$*, and a source term* $f : \Omega \longrightarrow \mathbb{R}$*, find the hydraulic potential* $p : \Omega \longrightarrow \mathbb{R}$ *and Darcy velocity* $\mathbf{u} : \Omega \longrightarrow \mathbb{R}^2$ *such that:*

$$\mathbf{A}\mathbf{u} = -\nabla p \quad \textit{in} \quad \Omega,$$

$$\operatorname{div} \mathbf{u} = f \quad \textit{in} \quad \Omega$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \textit{on} \quad \partial\Omega.$$

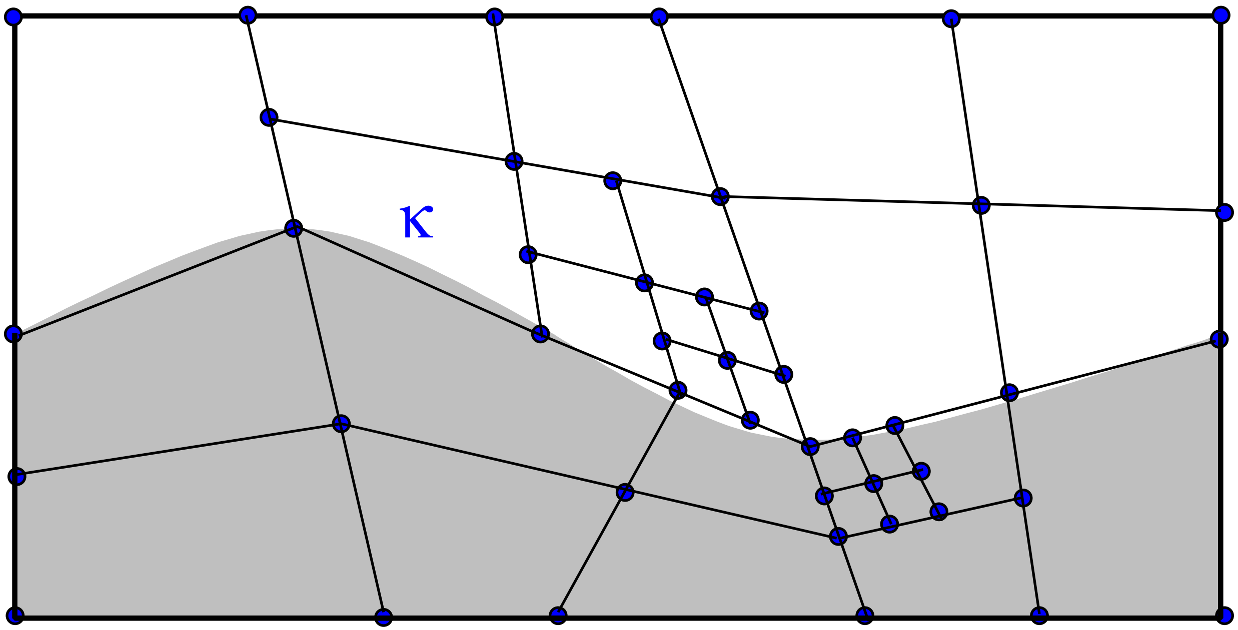
Equivalently

$$-\operatorname{div}(\mathbf{K}\nabla p) = f \quad \textit{in} \quad \Omega$$

$$\mathbf{K}\nabla p \cdot \mathbf{n} = 0 \quad \textit{on} \quad \partial\Omega.$$

Why discontinuous formulations?

- Adaptivity (unstructured and irregular meshes)
- Domain decomposition
- Conservation
- Stabilization



On Discontinuous FE formulations

Edges, averages and jumps

$$\text{Regular partition of } \Omega : \mathcal{T}_h = \{\mathcal{K}\} := \text{union of all elements } \mathcal{K}$$

$$\text{All edges : } \mathcal{E}_h = \{e : e \text{ is an edge of } \mathcal{K} \text{ for all } \mathcal{K} \in \mathcal{T}_h\}$$

$$\text{Interior edges : } \mathcal{E}_h^0 = \{e \in \mathcal{E}_h \; e \text{ is an interior edge}\}$$

$$\text{Boundary edges : } \quad \mathcal{E}_h^\partial = \mathcal{E}_h \cap \partial\Omega,$$

Averages and jumps:

$$\text{Scalar function : } \{\varphi\} = \frac{1}{2}(\varphi^1 + \varphi^2), \quad [\![\varphi]\!] = \varphi^1 \mathbf{n}^1 + \varphi^2 \mathbf{n}^2 \quad \text{on } e \in \mathcal{E}^0$$

$$\textbf{Vector function} : \{\mathbf{v}\} = \frac{1}{2}(\mathbf{v}^1 + \mathbf{v}^2), \quad [\![\mathbf{v}]\!] = \mathbf{v}^1 \cdot \mathbf{n}^1 + \mathbf{v}^2 \cdot \mathbf{n}^2 \quad \text{on } e \in \mathcal{E}^0.$$

Darcy flow in heterogeneous media

Broken space formulation

Find the hydraulic potential $p : \Omega \rightarrow \mathbb{R}$ *such that:*

$$-\mathrm{div}\,(K\nabla p) = f \quad \textit{in each} \quad \mathcal{K} \in \mathcal{T}_h$$

with interface conditions:

$$[\![p]\!]_{|e} = 0 \quad \textit{on each edge} \quad e \in \mathcal{E}_h^0.$$

$$[\![K\nabla p]\!]_{|e} = 0 \quad \textit{on each edge} \quad e \in \mathcal{E}_h^0.$$

and boundary condition:

$$K\nabla p \cdot \mathbf{n} = 0 \quad \textit{on each edge} \quad e \in \mathcal{E}_h^\partial = \mathcal{E}_h \cap \partial\Omega.$$

DGFEM - IP formulation

Find $p \in \mathcal{U}$ *such that*

$$b(p,q) = l(q) := \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} f q d\Omega \quad \forall \quad q \in \mathcal{U}$$

$$\mathcal{U} = \left\{ q \in L^2(\Omega) \ : q|_{\kappa} \in H^2(\kappa) \quad \forall \kappa \in \mathcal{T}_h \right\} ,$$

$$b(p,q) = \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} \mathbf{K} \nabla p \cdot \nabla q d\Omega + b_{\partial}(p,q) ,$$

$$b_{\partial}(p,q) = \int_{\mathcal{E}_h^0} \left(\lambda [\![p]\!] \{\mathbf{K}\nabla q \cdot \mathbf{n}\} - \{\mathbf{K}\nabla p \cdot \mathbf{n}\} [\![q]\!] + \beta [\![p]\!] [\![q]\!] \right) ds$$

- $\lambda = -1$ *for SIP formulation: Douglas Arnold 1982*
- $\lambda = 1$ *for NSIP formulation*, $\lambda = 0$ *for Neutral formulation*

Abstract Mixed Formulation

$$A\mathbf{u} + B^* p = g \quad \text{in} \;\; \Omega$$

$$B\mathbf{u} = f \quad \text{in} \;\; \Omega$$

Weak form: Find $\{\mathbf{u}, p\} \in W \times V$ such that

$$a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) - g(\mathbf{v}) + b(\mathbf{u},q) - f(q) = 0 \;\; \forall \{\mathbf{v},q\} \in W \times V$$

Babuška - Brezzi / Inf-sup conditions:

$$\sup_{\mathbf{v}\in W} \frac{a(\mathbf{u},\mathbf{v})}{\|\mathbf{v}\|_W} \geq \alpha \|\mathbf{u}\|_W \;\; \forall \mathbf{v} \in K_0$$

$$\sup_{\mathbf{v}\in W} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_W} \geq \beta \|q\|_V \;\; \forall q \in V$$

$$K_0 = \{\mathbf{v} \in W \,, \; b(\mathbf{v},q) = 0 \;\; \forall q \in V\}$$

References o mixed methods

- I. Babuška and A.K. Aziz (1972). *Lectures on the mathematical foundation of the fininte element method*. University of Maryland, Technical Note BN-748
- I. Babuška (1973). *The finite element method with Lagrange multiplyiers*. Numer. Math. 20:179-192
- F. Brezzi (1974). *On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multiplyers*. R.A.I.R.O Anal. Numer. R2:129-151
- J.T. Oden and J.N. Reddy (1976). *An introduction to the mathematical theory of the finite elements*. John Wiley& Sons
- P.A. Raviart and J.M. Thomas (1977). *A mixed finite element method for second order elliptic problems*. Lecture Notes in Mathematics 606. Springer-Verlag

GLS Stabilization

- Mixed Petrov-Galerkin method (Galerkin least-squares): Find $\{\mathbf{u}, p\} \in W \times V$ such that $\forall \{\mathbf{v}, q\} \in W \times V$

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) - g(\mathbf{v}) + b(\mathbf{u}, q) - f(q) +$$

$$\delta_1 (A\mathbf{u} + B^* p - g, A\mathbf{v} + B^* q)_h + \delta_2 (Bu - f, Bv)_h = 0$$

- Loula, Hughes, Franca and Miranda (1987). *Mixed Petrov-Galerkin method for the Timoshenko beam*. CMAME, 63:133-154
- Franca, Hughes, Loula and Miranda (1988). *A New family of stable elements for nearly incompressible elasticity based on a mixed Petrov-Galerkin method* . Numer. Math. 53:123-141

Dual mixed formulation for Darcy flow

Find $[\mathbf{u}, p] \in W \times V$ such that

$$(\mathbf{A}\mathbf{u}, \mathbf{v}) - (p, \mathrm{div}\mathbf{v}) = 0 \quad \forall \mathbf{v} \in W$$

$$-(\mathrm{div}\mathbf{u}, q) - (f, q) = 0 \quad \forall q \in V$$

For $V = L^2(\Omega)/\mathbb{R}$ and $W = \{\mathbf{v} \in H(\mathrm{div}) : \ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \ \partial\Omega\}$

with associated norms:

$$\|p\|^2 = (p, p) \ \ \forall p \ \in V,$$

$$\|\mathbf{u}\|_W^2 = \|\mathbf{u}\|_{H(\mathrm{div})}^2 = (\mathbf{u}, \mathbf{u}) + (\mathrm{div}\mathbf{u}, \mathrm{div}\mathbf{u}) \ \ \forall \mathbf{u} \ \in W$$

- Stability is proved since $\mathrm{div}(W) = V$
- Raviart & Thomas, 1977

Stabilization in $[H^1(\Omega)]^2 \times H^1(\Omega)$

C^0 Lagrangian formulation

$$\mathcal{U} = \{\mathbf{v} \in [H^1(\Omega)]^2, \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}\ ;\ \ \mathcal{Q} = H^1(\Omega)/\mathbb{R}.$$

$CGLS$: *Find* $\{\mathbf{u}, p\} \in \mathcal{U} \times \mathcal{Q}$ *such that*

$$\begin{aligned} (\mathbf{A}\mathbf{u}, \mathbf{v}) - (\mathrm{div}\mathbf{v}, p) - (\mathrm{div}\mathbf{u}, q) + (f, q)\ \ (\mathrm{Galerkin}) \\ -\frac{1}{2}\left(\mathbf{K}\left(\mathbf{A}\mathbf{u} + \nabla p\right), \mathbf{A}\mathbf{v} + \nabla q\right) + (|\mathbf{A}|(\mathrm{div}\mathbf{u} - f), \mathrm{div}\mathbf{v})\ \ (\mathrm{GLS}) \\ +(|\mathbf{K}|\mathrm{rot}(\mathbf{A}\mathbf{u}), \mathrm{rot}(\mathbf{A}\mathbf{v})) = 0 \quad \forall\ [\mathbf{v}, q] \in \bar{\mathcal{U}} \times \mathcal{Q} \end{aligned}$$

- Unconditional stability is proved
- M.R. Correa and A.F. Loula, *Unconditionally stable mixed finite element methods for Darcy flow*, CMAME, 197 (2008).

Stability. Babuška's lema

Given the stabilized bilinear form,

$$A\left(\left[\mathbf{u},p\right],\left[\mathbf{v},q\right]\right)=\left(\mathbf{A}\mathbf{u},\mathbf{v}\right)-\left(\mathrm{div}\mathbf{v},p\right)-\left(\mathrm{div}\mathbf{u},q\right)$$
$$-\frac{1}{2}\left(\mathbf{K}\left(\lambda\mathbf{u}+\nabla p\right),\mathbf{A}\mathbf{v}+\nabla q\right)$$
$$+(|\mathbf{A}|(\mathrm{div}\mathbf{u},\mathrm{div}\mathbf{v})+(|\mathbf{K}|\mathrm{rot}(\mathbf{A}\mathbf{u}),\mathrm{rot}(\mathbf{A}\mathbf{v}))$$

for any pair $[\mathbf{u},p]$, we can always find $\bar{\mathbf{v}}=\mathbf{u}$ and $\bar{q}=-p$ such that

$$\sup_{[\mathbf{v},q]\in\bar{\mathcal{U}}\times\mathcal{Q}}\frac{|A([\mathbf{u},p],[\mathbf{v},q])|}{\|\,[\mathbf{v},q]\,\|_{\bar{\mathcal{U}}\times\mathcal{Q}}}\geq\frac{|A([\mathbf{u},p],\{\bar{\mathbf{v}},\bar{q}\}\,|}{\|\,\{\bar{\mathbf{v}},\bar{q}\}\,\|_{\bar{\mathcal{U}}\times\mathcal{Q}}}\geq\alpha\|\,[\mathbf{u},p]\,\|_{\bar{\mathcal{U}}\times\mathcal{Q}}$$

with $\alpha>0$ and

$$\|\,[\mathbf{u},p]\,\|_{\bar{\mathcal{U}}\times\mathcal{Q}}=\left(\|\mathbf{u}\|^2+\|\nabla\mathbf{u}\|^2+\|\nabla p\|^2\right)^{1/2}$$

Numerical analysis

- Conforming approximations are $[H^1(\Omega)]^2 \times H^1(\Omega)$ stable
- Any C^0 Lagrangian interpolation is stable

Error estimates:

(Optimal for equal order C^0 Lagrangian approximations $l = k$):

$$\|p - p_h\| + h\|\nabla p - \nabla p_h\| \leq Ch^{k+1}\left(|\mathbf{u}|_{k+1} + |p|_{k+1}\right),$$

$$\|\mathbf{u} - \mathbf{u}_h\|_1 \leq Ch^k\left(|\mathbf{u}|_{k+1} + |p|_{k+1}\right),$$

$$\|\mathrm{div}\mathbf{u} - \mathrm{div}\mathbf{u}_h\| \leq Ch^k\left(|\mathbf{u}|_{k+1} + |p|_{k+1}\right).$$

- Optimal for pressure in $L^2(\Omega)$ and $H^1(\Omega)$ norms
- Optimal for velocity in $[L^2(\Omega)]^2$ and $[H^1(\Omega)]^2$ norms

Darcy flow in heterogeneous media

Mixed formulation in Broken space formulation.

Find the hydraulic potential $p : \Omega \to \mathbb{R}$ *and Darcy velocity* $\mathbf{u} : \Omega \to \mathbb{R}^2$ *such that:*

$$\operatorname{div} \mathbf{u} = f \quad \textit{in each} \quad \mathcal{K} \in \mathcal{T}_h$$

$$\mathbf{u} = -K \nabla p \quad \textit{in each} \quad \mathcal{K} \in \mathcal{T}_h$$

with interface and boundary conditions

$$[\![p]\!]_{|e} := p^1 \mathbf{n}^1 + p^2 \mathbf{n}^2 = 0 \quad \textit{on each edge} \quad e \in \mathcal{E}_h^0.$$

$$[\![\mathbf{u}]\!]_{|e} := \mathbf{u}^1 \cdot \mathbf{n}^1 + \mathbf{u}^2 \cdot \mathbf{n}^2 = 0 \quad \textit{on each edge} \quad e \in \mathcal{E}_h.$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \textit{on each edge} \quad e \in \mathcal{E}_h^{\partial} = \mathcal{E}_h \cap \partial\Omega.$$

Hybrid methods: Pian & Tong

Some references:

- P. Tong and T.H.H. Pian, *A variational principle and the convergence of a finite-element method based on assumed stress distribution*, 1969, Int. J. Solids Struct.Vol **5**, 463-472.
- E.C.P. Lima, *Finite elements for plate bending with assumed stress fields*, 1972, MSc Thesis, COPPE/UFRJ .
- P.A. Raviart and J.M. Thomas *Primal hybrid finite element methods for 2nd order elliptic equations*, 1977, Mathematics of Computation **31**, 391-413.
- D.N. Arnold and F. Brezzi, *Mixed and Nonconforming finite element methods: implementation, post-processing and error estimates*, 1985, RAIRO MMAN.Vol **19**, 7-32.

Hybridization

On a regular finite element mesh $\mathcal{T}_h$, on each element $\mathcal{K}$ we define

$$Q_{\mathcal{K}} = \{q \in L^2(\mathcal{K}) \; \forall \mathcal{K} \in \mathcal{T}_h\},$$

$$U_{\mathcal{K}} = \{\mathbf{v} \in L^2(\mathcal{K}) \times L^2(\mathcal{K}), \mathrm{div}\mathbf{v} \in L^2(\mathcal{K}) \; \forall \mathcal{K} \in \mathcal{T}_h\},$$

And consider a local weak form of the equation $\mathbf{Au} + \nabla p = 0$:

$$(\mathbf{Au} + \nabla p, \mathbf{v})_{\mathcal{K}} = \int_{\mathcal{K}} \mathbf{Au} \cdot \mathbf{v} d\Omega - \int_{\mathcal{K}} p \mathrm{div} \mathbf{v} d\Omega + \int_{\partial \mathcal{K}} p \mathbf{v} \cdot \mathbf{n} ds = 0$$

in each element $\mathcal{K}$

Local problems

Given $p = \bar{p}$ on $\partial\mathcal{K}$, we can solve the local problem:

LP: For each $\mathcal{K} \in \mathcal{T}_h$, find $[\mathbf{u}, p] \in U_{\mathcal{K}} \times Q_{\mathcal{K}}$, such that

$$a_{\mathcal{K}}\left([\mathbf{u}, p], [\mathbf{v}, q]\right) = f_{\mathcal{K}}([\mathbf{v}, q]) \; \forall \, [\mathbf{v}, q] \in U_{\mathcal{K}} \times Q_{\mathcal{K}}.$$

with

$$a_{\mathcal{K}}\left([\mathbf{u}, p], [\mathbf{v}, q]\right) = (\mathbf{A}\mathbf{u}, \mathbf{v})_{\mathcal{K}} - (p, \mathrm{div}\mathbf{v})_{\mathcal{K}} - (\mathrm{div}\mathbf{u}, q)_{\mathcal{K}},$$

$$f_{\mathcal{K}}([\mathbf{v}, q]) = -(f, q)_{\mathcal{K}} - c_{\mathcal{K}}(\bar{p}, \mathbf{v}),$$

$$c_{\mathcal{K}}(\bar{p}, \mathbf{v}) = \int_{\partial\mathcal{K}} \bar{p}\mathbf{v} \cdot \mathbf{n} ds,$$

FEM: Local Raviart-Thomas or BDM finite element on each $\mathcal{K}$.

Dual hybrid mixed formulation

Arnold & Brezzi (1985): Find $[\mathbf{u}, p] \in U \times Q$ and $\lambda \in M$, such that

$$a\left([\mathbf{u}, p], [\mathbf{v}, q]\right) + c(\lambda, \mathbf{v}) = f([\mathbf{v}, q]), \ \forall [\mathbf{v}, q] \in U \times Q,$$

$$c(\mu, \mathbf{u}) = 0 \ \forall \mu \in M,$$

with $M = \{\mu \in L^2(e) \ \forall e \in \mathcal{E}_h^0\}$, $U = \prod_{\mathcal{K}} U_{\mathcal{K}}$, $Q = \prod_{\mathcal{K}} Q_{\mathcal{K}}$,

$$a\left([\mathbf{u}, p], [\mathbf{v}, q]\right) = \sum_{\mathcal{K}} a_{\mathcal{K}}\left([\mathbf{u}, p], [\mathbf{v}, q]\right) \ \ \forall \ \ [\mathbf{u}, p], [\mathbf{v}, q] \ \in U \times Q$$

$$c(\mu, \mathbf{v}) = \sum_{\mathcal{K}} c_{\mathcal{K}}(\mu, \mathbf{v}) = \int_{\mathcal{E}^0} \mu [\![\mathbf{v}]\!] ds \ \ \forall \ \mu \in M, \ \forall \mathbf{v} \in U$$

$$f([\mathbf{v}, q]) = \sum_{\mathcal{K}} (f, q) - \int_{\mathcal{E}_h^\partial} g\mathbf{v} \cdot \mathbf{n} ds \ \forall \mathbf{v} \in U, \ \forall q \in Q$$

Stabilized DHM formulation

SDHM: Find $[\mathbf{u}, p] \in U \times Q$ and $\lambda \in M$ such that

$$a_\delta\left([\mathbf{u},p],[\mathbf{v},q]\right)+c(\lambda,\mathbf{v})+\sum_{\mathcal{K}}\int_{\partial\mathcal{K}}\beta(p-\lambda)q ds = f_\delta([\mathbf{v},q]) \;\; \forall\,[\mathbf{v},q] \in U \times Q$$

$$c(\mu,\mathbf{u}) + \sum_{\mathcal{K}}\int_{\partial\mathcal{K}}\beta(\lambda - p)\mu ds = 0 \;\forall\; \mu \in M$$

$$a_\delta\left([\mathbf{u}_h,p_h],[\mathbf{v}_h,q_h]\right) = a\left([\mathbf{u}_h,p_h],[\mathbf{v}_h,q_h]\right)$$

$$+\,\delta_1\sum_{\mathcal{K}}\left(|\mathbf{A}|\mathrm{div}\mathbf{u},\mathrm{div}\mathbf{v}\right)_{\mathcal{K}} + \delta_2\sum_{\mathcal{K}}\left(\mathbf{K}\left(\mathbf{A}\mathbf{u}+\nabla p\right),\mathbf{A}\mathbf{v}+\nabla q\right)_{\mathcal{K}}$$

$$+\,\delta_3\sum_{\mathcal{K}}\left(|\mathbf{K}|\mathrm{rot}(\mathbf{A}\mathbf{u}),\mathrm{rot}(\mathbf{A}\mathbf{v})\right)_{\mathcal{K}}$$

$$f_\delta([\mathbf{v},q]) = f([\mathbf{v},q]) + \textstyle\sum_{\mathcal{K}}\delta_1(|\mathbf{A}|f,\mathrm{div}\mathbf{v})_{\mathcal{K}}$$

Hybridizable mixed DG method

Solving first the multiplier equation we get

$$\lambda = \{p\} - \frac{1}{2\beta}[\![\mathbf{u}]\!].$$

yielding the stabilized mixed discontinuous Galerkin formulation

SMDG: Find $[\mathbf{u}, p]$ such that

$$a_\delta\left([\mathbf{u}, p], [\mathbf{v}, q]\right) + \int_{\mathcal{E}_h^0} \left(\{p\}[\![\mathbf{v}]\!] + [\![\mathbf{u}]\!]\{q\}\right) ds$$
$$+ \int_{\mathcal{E}_h^0} \frac{\beta}{2}[\![p]\!] \cdot [\![q]\!] ds - \int_{\mathcal{E}_h^0} \frac{1}{2\beta}[\![\mathbf{u}]\!][\![\mathbf{v}]\!] ds = f_\delta([\mathbf{v}, q]) \;\; \forall\, [\mathbf{v}, q] \in U \times Q$$

Setting $\delta_i = 0$, for $i = 1, 2, 3$, **SMDG** recovers the LDG-H (Local Discontinuous Galerkin - Hybridizable formulation).

B. Cockburn and Coworkers, Math. Comp. 2010.

Finite element approximations

$\mathbf{SDHM}_h$: Find $[\mathbf{u}_h, p_h] \in U_h^m \times Q_h^l$ and $\lambda_h \in M_h^n$ such that

$$a_\delta\left([\mathbf{u}_h, p_h], [\mathbf{v}_h, q_h]\right) + \int_{\mathcal{E}_h^0} \frac{\beta}{2} [\![p_h]\!] \cdot [\![q_h]\!] ds + \int_{\mathcal{E}_h^0} \lambda_h [\![\mathbf{v}_h]\!] ds$$

$$+ \int_{\mathcal{E}_h^0} 2\beta\left(\{p_h\} - \lambda_h\right)\{q_h\} ds = f_\delta([\mathbf{v}_h, q_h]) \;\; \forall\, [\mathbf{v}_h, q_h] \in U_h^m \times Q_h^l$$

$$\int_{\mathcal{E}_h^0} 2\beta\left((\lambda_h - \{p_h\}) + [\![\mathbf{u}_h]\!]\right)\mu_h ds = 0 \;\; \forall\, \mu_h \in M_h.$$

$$Q_h^l = \{q \in Q : \quad q|_K \in \mathcal{P}^l(K) \;\; \forall K \in \mathcal{T}_h\},$$

$$U_h^m = \{\mathbf{v} \in U : \quad \mathbf{v}|_K \in \mathcal{P}^m(K) \times \mathcal{P}^m(K) \;\; \forall K \in \mathcal{T}_h\},$$

$$M_h^n = \{\mu \in M : \quad \mu|_e \in \mathcal{P}^n(e) \;\; \forall e \in \mathcal{E}^0\}$$

Hybridizable DG FEM

For $n \geq l$ and $n \geq m$ we can solve explicitly the finite element equation for the multiplier λ_h, obtaining

$$\lambda_h = \{p_h\} - \frac{1}{2\beta}[\![\mathbf{u}_h]\!],$$

and yielding the following HDG method

$\mathbf{SMDG}_h$: Find $[\mathbf{u}_h, p_h] \in U_h^m \times Q_h^l$ such that $\forall\ [\mathbf{v}_h, q_h] \in U_h \times Q_h$

$$a_\delta\left([\mathbf{u}_h, p_h], [\mathbf{v}_h, q_h]\right) + \int_{\mathcal{E}_h^0} \left(\{p_h\}[\![\mathbf{v}_h]\!] + [\![\mathbf{u}_h]\!]\{q_h\}\right) ds +$$

$$\int_{\mathcal{E}_h^0} \frac{\beta}{2}[\![p_h]\!] \cdot [\![q_h]\!] ds - \int_{\mathcal{E}_h^0} \frac{1}{2\beta}[\![\mathbf{u}_h]\!][\![\mathbf{v}_h]\!] ds = f_\delta([\mathbf{v}_h, q_h])$$

Convergence studies

Darcy flow in homogeneous media:

Given $\mathbf{K} = \mathbf{I}$, $[\mathbf{u}, p]$ in $\Omega = [-1,1] \times [-1,1]$ and $f(x,y) = \cos(\pi x)\cos(\pi y)$ such that:

$$p = \frac{1}{2\pi^2}\cos\pi x \cos\pi y; \quad \mathbf{u} = \frac{1}{2\pi}\left[\begin{array}{c} \sin\pi x \cos\pi y \\ \cos\pi x \sin\pi y \end{array}\right]$$

Equal order polynomial spaces for all fields: $l = n = m$

$$\text{DHM (No stabilization)}: \ \delta_1 = \delta_2 = \delta_3 = \beta = 0 \ \text{(Unstable)}$$

$$\text{LDG} - \text{H(Penalty)}: \ \delta_1 = \delta_2 = \delta_3 = 0; \ \beta = 1 \ \text{(Not optimal)}$$

$$\text{SDHM}: \delta_1 = 0.5; \quad \delta_2 = -0.5; \quad \delta_3 = 0.5; \quad \beta = 0$$

Convergence study

Homogeneous medium

Given $\mathbf{K} = \mathbf{I}$, $[\mathbf{u}, p]$ in $\Omega = [-1,1] \times [-1,1]$ such that:

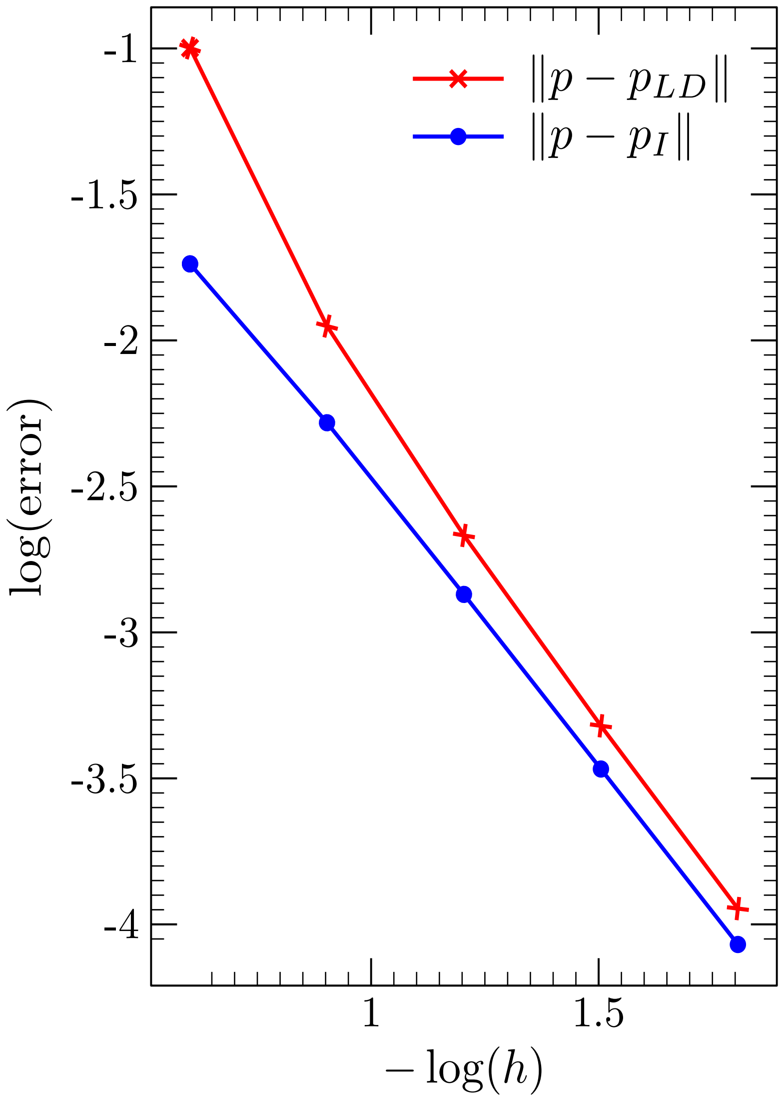
$$\begin{aligned} \mathrm{div}\mathbf{u} &= \cos(\pi x)\cos(\pi y) && \text{in} \quad \Omega \\ \mathbf{u} &= -\nabla p && \text{in} \quad \Omega \\ \mathbf{u}\cdot\mathbf{n} &= 0 && \text{on} \quad \partial\Omega \end{aligned}$$

Exact solution:

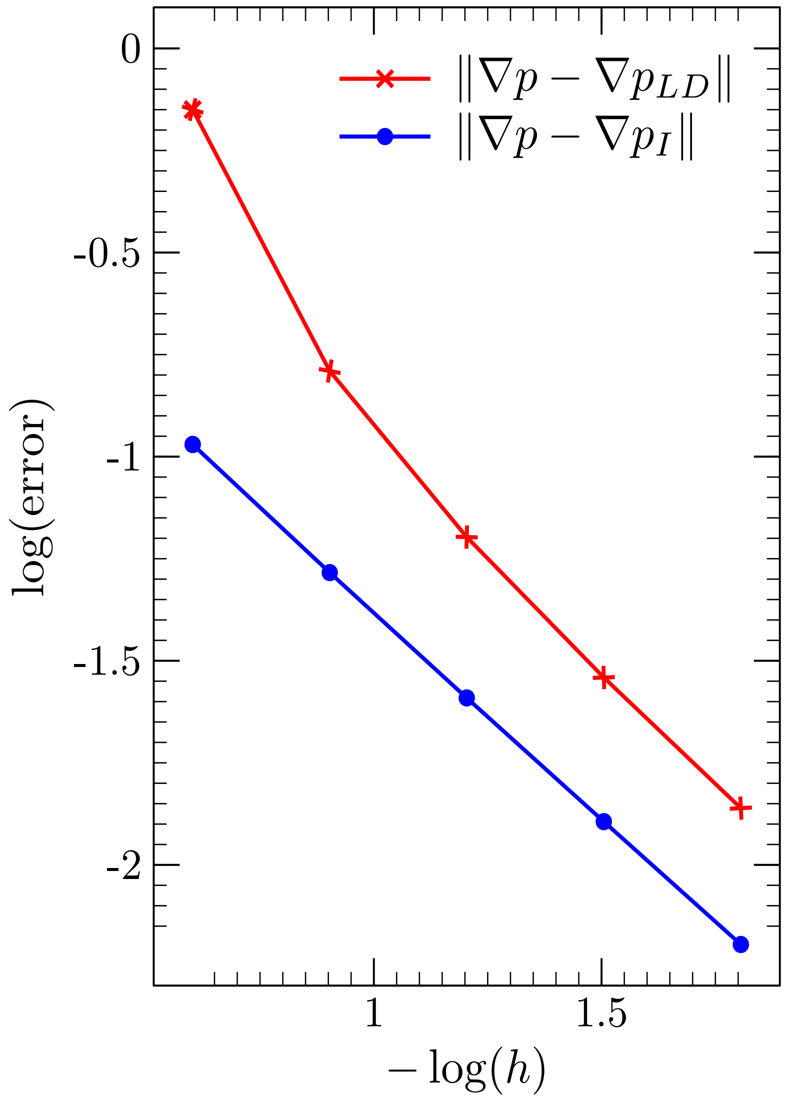
$$p = \frac{1}{2\pi^2}\cos\pi x \cos\pi y$$

$$\mathbf{u} = \frac{1}{2\pi}\begin{bmatrix} \sin\pi x \cos\pi y \\ \cos\pi x \sin\pi y \end{bmatrix}$$

LDG-H ($l = 1$)

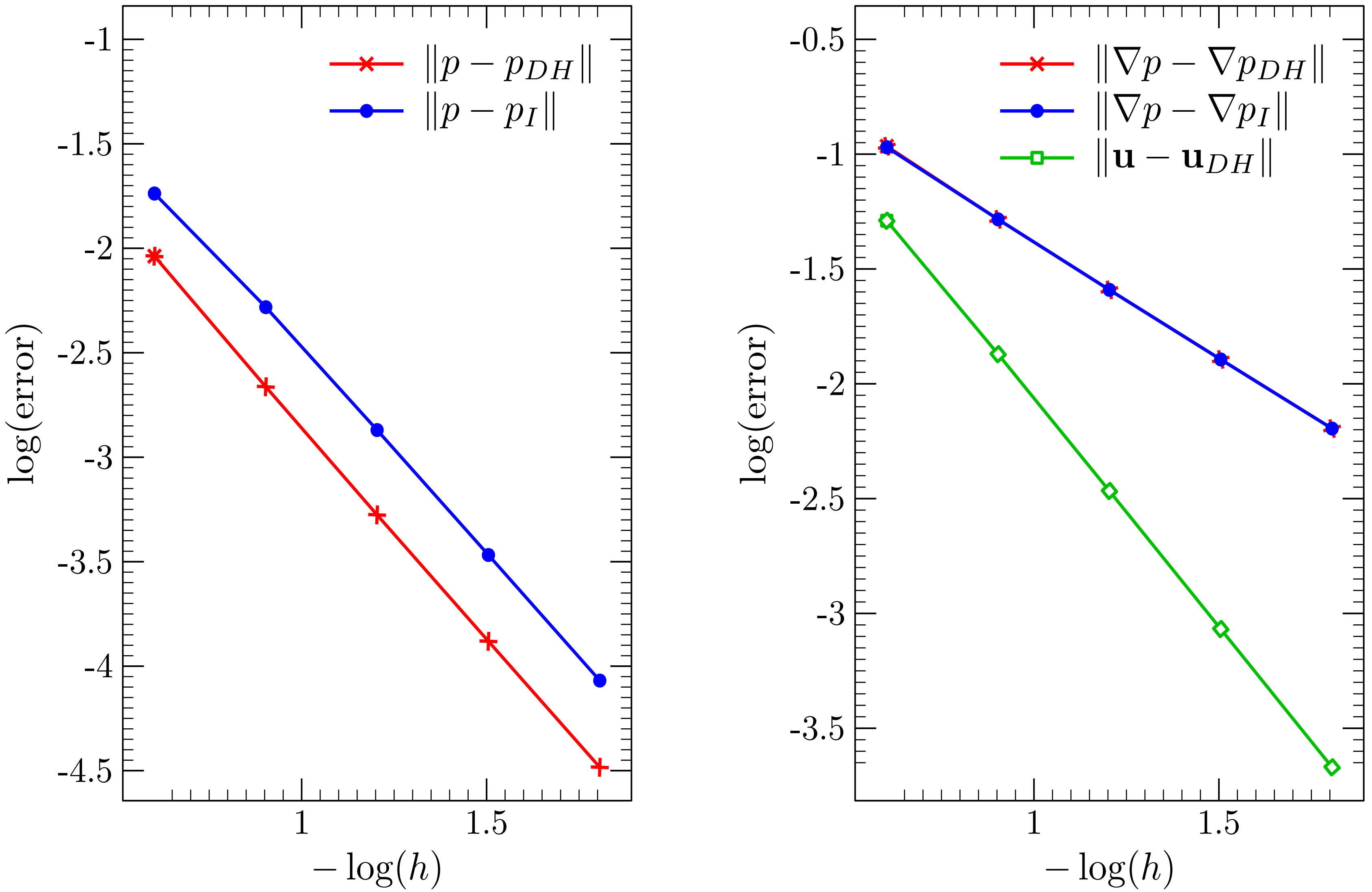


$\|p - p_{LD}\|$
$\|p - p_I\|$
log(error)
$-\log(h)$
-1
-1.5
-2
-2.5
-3
-3.5
-4
1
1.5

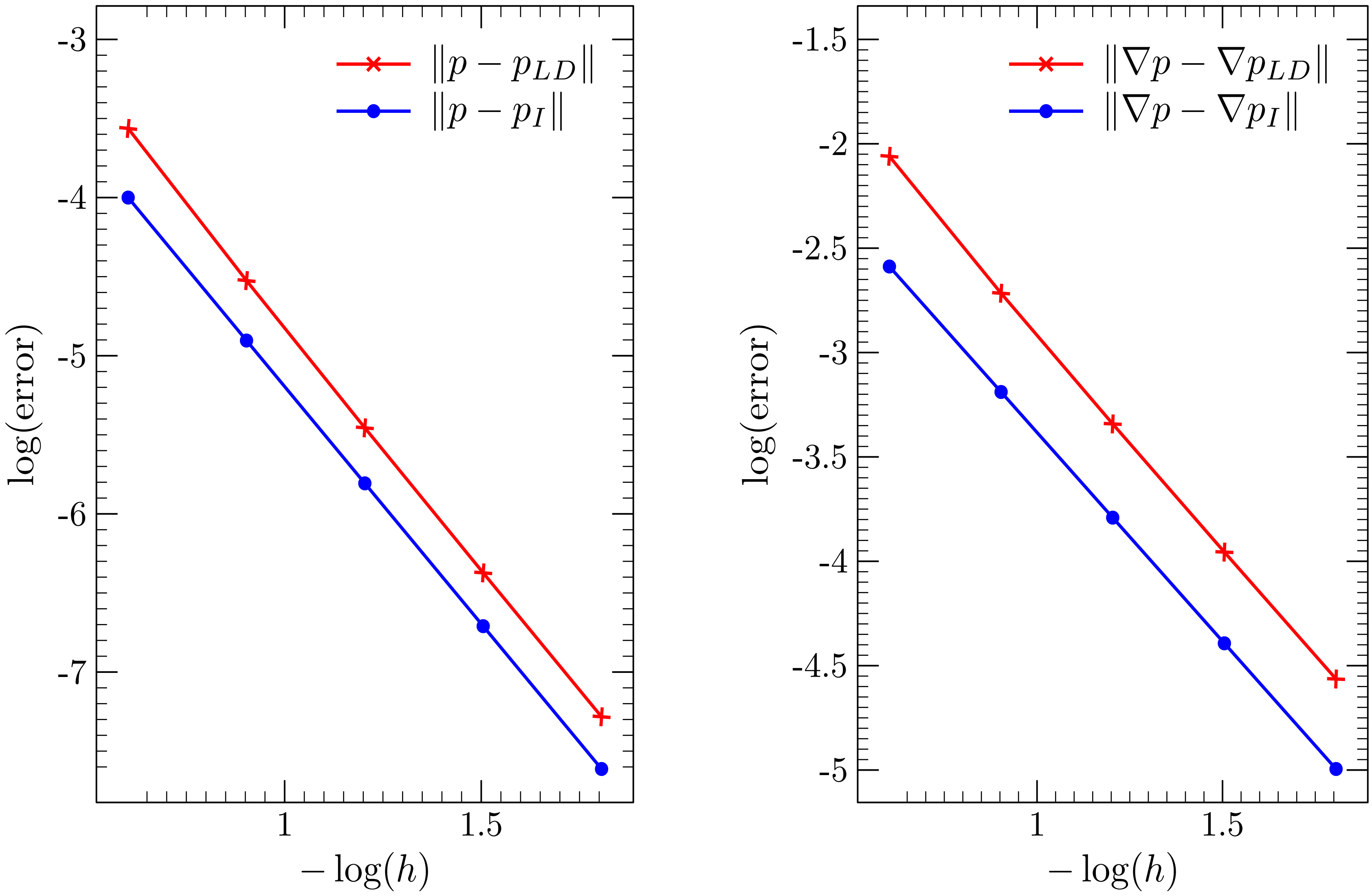


$\|\nabla p - \nabla p_{LD}\|$
$\|\nabla p - \nabla p_I\|$
log(error)
$-\log(h)$
0
-0.5
-1
-1.5
-2
1
1.5

SDHM. h-convergence for $l = 1$

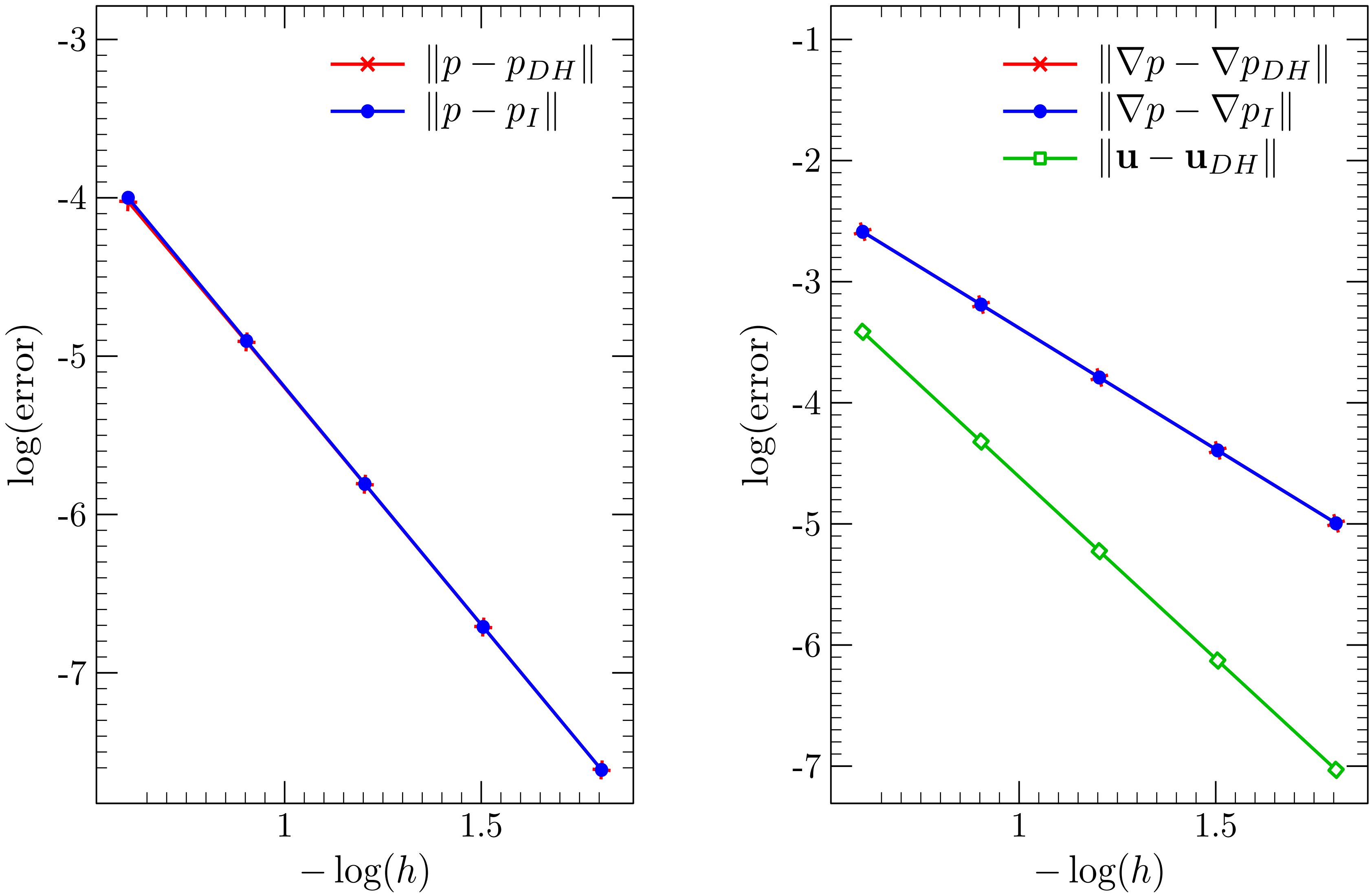


LDG-H ($l = 2$)

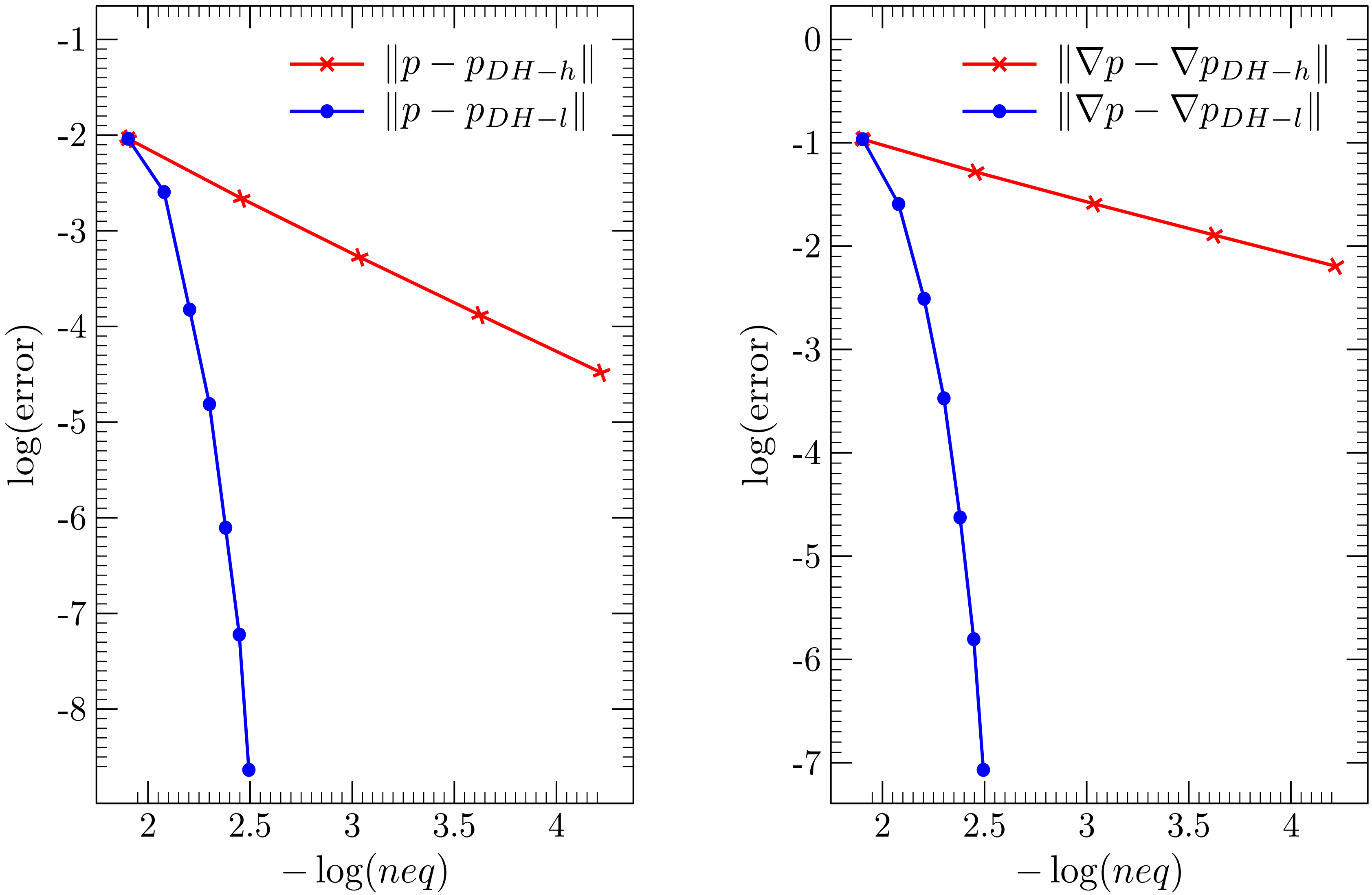


$\|p - p_{LD}\|$
$\|p - p_I\|$
log(error)
$-\log(h)$
-3
-4
-5
-6
-7
1
1.5
$\|\nabla p - \nabla p_{LD}\|$
$\|\nabla p - \nabla p_I\|$
log(error)
$-\log(h)$
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
-5
1
1.5

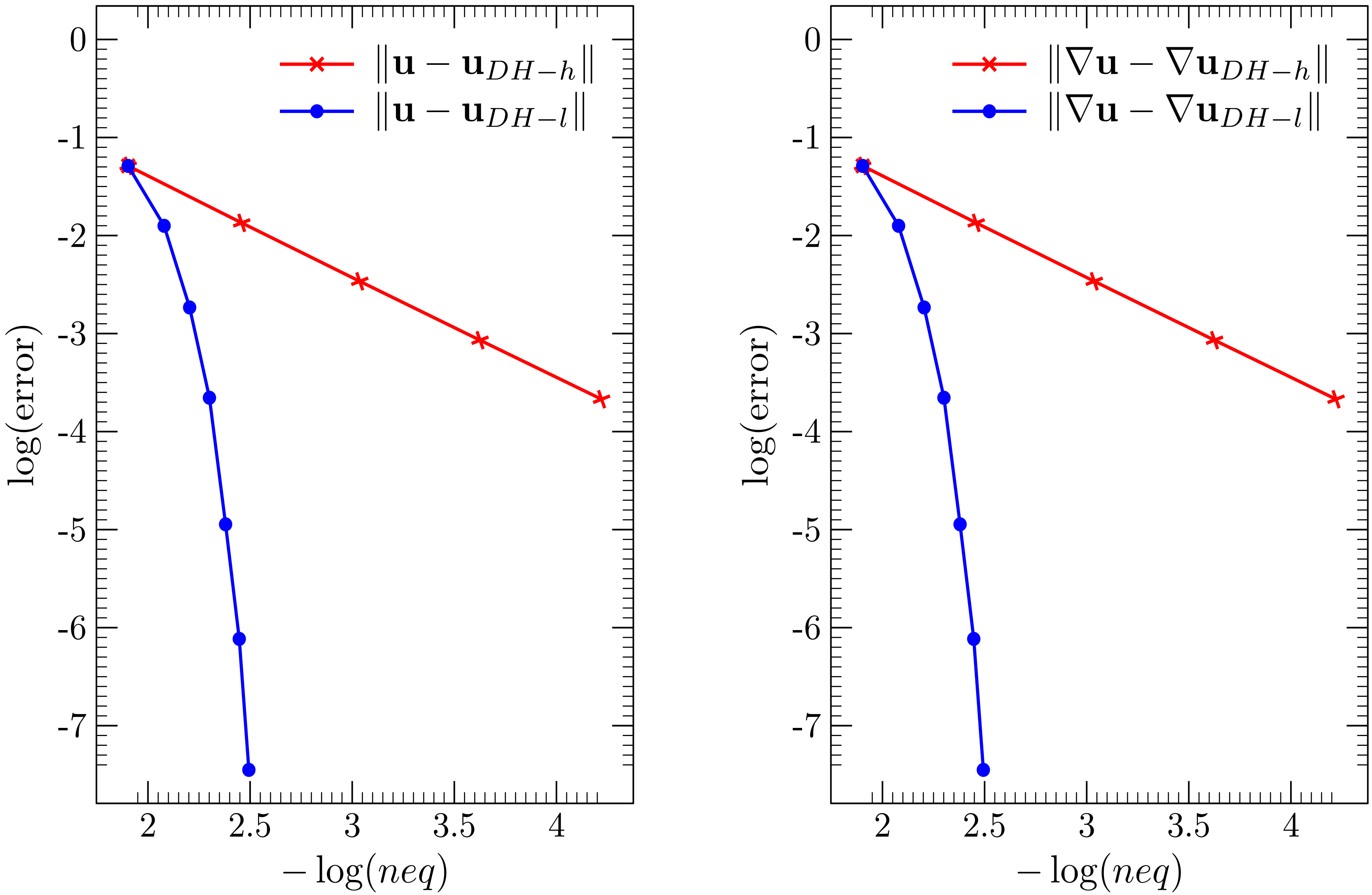
SDHM. h-convergence for $l = 2$



SDHM. h x l convergence



SDHM. h x l convergence



Heterogeneous Anisotropic Media

Domain: $\Omega = (-2, 2)^2$, Sub domain: $\Omega_I = (-1, 1)^2$

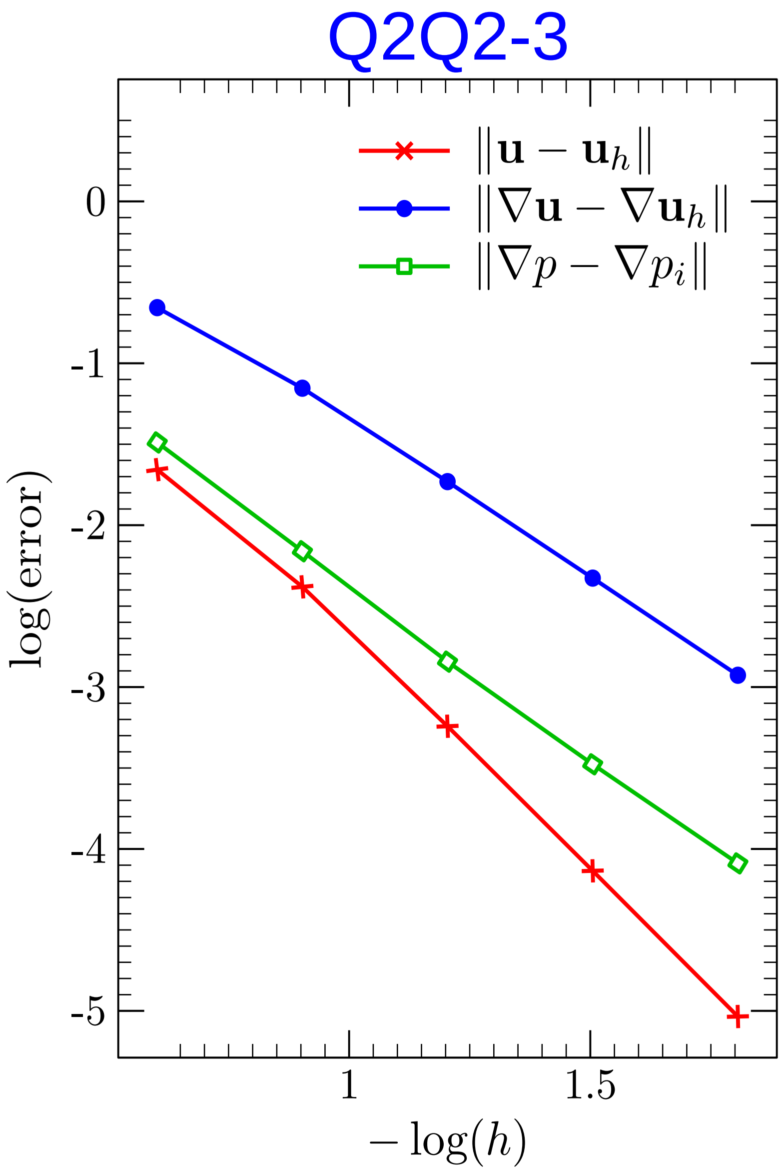
Hydraulic Conductivity:

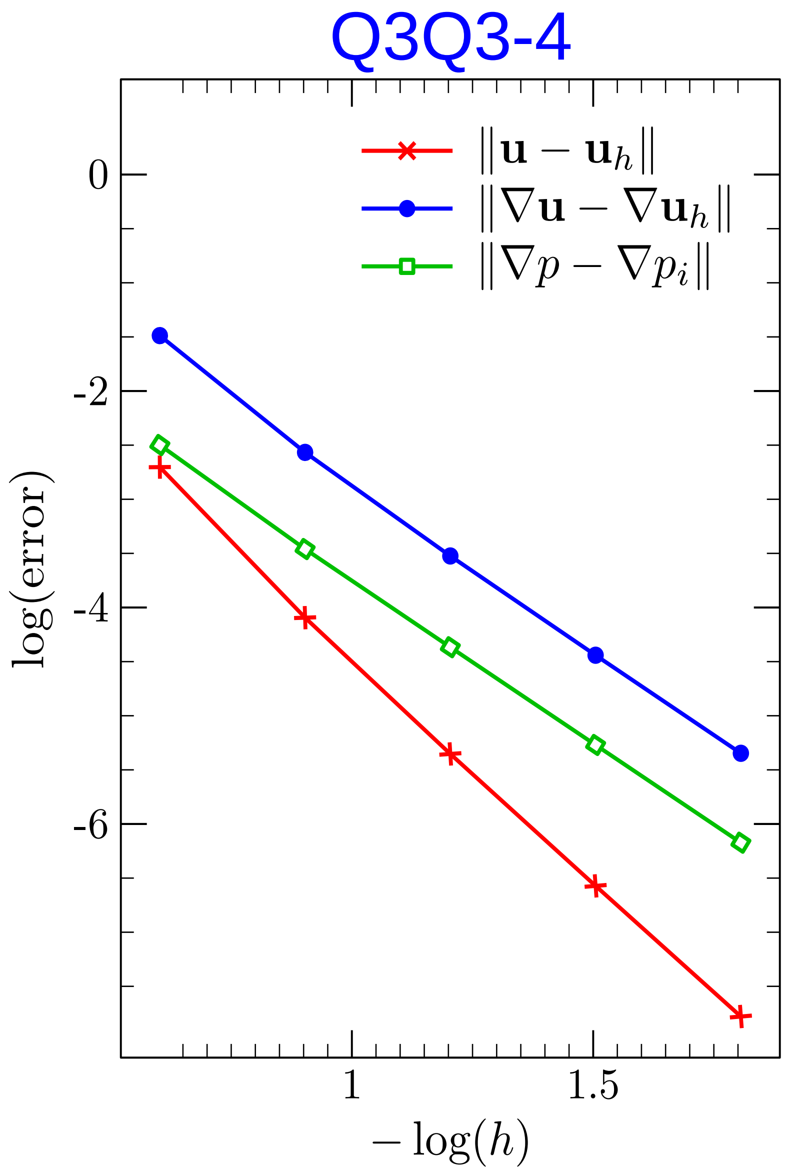
$$\mathbf{K} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad x \in \Omega_I \qquad \mathbf{K} = \alpha \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad x \in \Omega / \Omega_I$$

Exact Solution:

$$p = \begin{cases} \sin(\pi x)\sin(\pi y), & x \in \Omega_I \\ 2\sin(\pi x)\sin(\pi y), & x \in \Omega/\Omega_I \end{cases}$$

SDHM: Heterogeneous medium





Concluding remarks

On Dual Hybrid Mixed Methods

- Adaptivity/Flexibility
  - h-refinement
  - Polynomial refinement
  - Irregular meshes
- Local conservation

On Stabilized Dual Hybrid Methods

- Additional flexibility
  - DHM (Dual Hybrid Mixed)
  - LDG-H (Local DG Hybridizable)
- Unconditional stability

Obrigado

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