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# Mini-Curso III

# Countable Groups of Isometries on Banach Spaces

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# Countable groups of isometries on Banach spaces

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**Abstract**

A group $G$ is representable in a Banach space $X$ if $G$ is isomorphic to the group of isometries on $X$ in some equivalent norm. We prove that a countable group $G$ is representable in a separable real Banach space $X$ in several general cases, including when $G \simeq \{-1,1\} \times H$, $H$ finite and $\dim X \geq |H|$, or when $G$ contains a normal subgroup with two elements and $X$ is of the form $c_0(Y)$ or $\ell_p(Y)$, $1 \leq p < +\infty$. This is a consequence of a result inspired by methods of S. Bellenot and stating that under rather general conditions on a separable real Banach space $X$ and a countable bounded group $G$ of isomorphisms on $X$ containing $-Id$, there exists an equivalent norm on $X$ for which $G$ is equal to the group of isometries on $X$. [1]

## 1 Introduction

The theory of representation of groups studies the possible representations of a given group $G$ as some group of isometries, or of isomorphisms, on a Hilbert space (or more generally on a Banach space). In this mini-course we shall ask a more restrictive question: what groups $G$ may be seen as **the** group of isometries on a

[1]This minicourse is inspired from an article with the same title and written in collaboration with E. Medina Galego from IME - USP.

Banach space $X$? This question may be formulated by the following definition given by K. Jarosz in [6].

**Definition 1** *(K. Jarosz, 1988) A group $G$ is representable in a Banach space $X$ if there exists an equivalent norm on $X$ for which the group of isometries on $X$ is isomorphic to $G$.*

In [6], Jarosz stated as an open question which groups were representable in a given Banach space. The difference with the classical theory of representation of groups on linear spaces is that here we require an isomorphism with the group of isometries on a Banach space, and not just some group of isometries or isomorphisms.

**Fact 2** *A group which is representable in a real Banach space must always contain a normal subgroup with two elements.*

*Proof :* Indeed $\{-Id, Id\}$ is always a normal subgroup of the group of isometries on a real Banach space. $\square$

Conversely:

**Theorem 3** *(J. Stern, 1979) For any group $G$ which contains a normal subgroup with two elements, there exists a real Hilbert space $H$ such that $G$ is representable in $H$. Furthermore if $G$ is countable then $H$ may be chosen to be separable.*

For an arbitrary Banach space $X$ it remains open which groups are representable in $X$. Jarosz proved that $\{-1, 1\}$ is representable in any real Banach space (that is any space may be renormed so that the only isometries are $-Id$ and $Id$), and that the unit circle $C$ is representable in any complex space (the separable real case had been solved previously by S. Bellenot [1]). He also proved that for any countable group $G$, $\{-1, 1\} \times G$ is representable in $C([0, 1])$, and that for any group $G$ there exists a complex space $X$ such that $C \times G$ is representable in $X$. These results led Jarosz to the following conjecture:

**Conjecture 4** *(Jarosz, 1988) The group $\{-1, 1\} \times G$ is representable in $X$ for any group $G$ and any real space $X$ such that $\dim X \geq |G|$.*

In this mini-course we give a much more general answer to the question of representability by partially answering the conjecture of Jarosz:

- the group $\{-1, 1\} \times G$ is representable in $X$ whenever $G$ is a finite group and $X$ a separable real space $X$ such that $\dim X \geq |G|$, Theorem 30,

- the group $G$ is representable in $X$ whenever $G$ is a countable group admitting a normal subgroup with two elements and $X$ is a separable real Banach space with a symmetric decomposition either isomorphic to $c_0(Y)$ or to $\ell_p(Y)$ for some $Y$ and $1 \leq p < +\infty$, or to a dual space, Theorem 32,

- the group $\{-1, 1\} \times G$ is representable in $X$ whenever $G$ is a countable group and $X$ an infinite-dimensional separable real Banach space containing a complemented subspace with a symmetric basis, Theorem 34.

As an application of our results we obtain that a countable group $G$ is representable in $c_0$ (resp. $C([0, 1])$, $\ell_p$ for $1 \leq p < +\infty$, $L_p$ for $1 \leq p < +\infty$) if and only if it contains a normal subgroup with two elements, Corollary 33.

Our method is to ask, given a group $G$ of linear isomorphisms on a real Banach space $X$, whether there exists an equivalent norm on $X$ for which $G$ is the group of isometries on $X$. Once the problem of representability is reduced to representing a given group as some group of isomorphisms on a given Banach space, it is much simpler to address, and this leads to Theorem 30, Theorem 32, and Theorem 34. In other words, we explore in which respect the question of representability of groups in Banach spaces belongs to the renorming theory or rather may be reduced to the purely isomorphic theory.

If a group of isomorphisms is the group of isometries on a real (resp. complex) Banach space in some equivalent norm, then it must be bounded, contain $-Id$ (resp. $\lambda Id$ for all $\lambda \in C$), and be closed for the convergence of $T$ and $T^{-1}$ in the strong operator topology. Therefore the question is:

**Question 5** *Let $X$ be a real (resp. complex) Banach space and let $G$ be a group of isomorphisms on $X$ which is bounded, contains $-Id$ (resp. $\lambda Id$ for all $\lambda \in C$), and is closed for the convergence of $T$ and $T^{-1}$ in the strong operator topology. Does there exist an equivalent norm on $X$ for which $G$ is the group of isometries on $X$?*

A positive answer was obtained by Y. Gordon and R. Loewy [4] when $X = \mathbb{R}^n$ and $G$ is finite; this answered a question by J. Lindenstrauss. In this paper, we extend the methods of Bellenot to considerably improve this result:

- Let $X$ be a separable real dual Banach space. Then for any countable bounded group $G$ of isomorphisms on $X$ which contains $-Id$ and is separated by some point with discrete orbit, there exists an equivalent norm on $X$ for which $G$ is equal to the group of isometries on $X$, Theorem 26.

- Let $X$ be a separable real Banach space. Then for any finite group $G$ of isomorphisms on $X$ which contains $-Id$, there exists an equivalent norm on $X$ for which $G$ is equal to the group of isometries on $X$, Theorem 27.

Therefore for separable real spaces and finite groups, the question of representability really does not belong to renorming theory. Also, note that a countable group of isomorphisms on $X$ which is equal to the group of isometries in some equivalent norm must always be discrete for the convergence of $T$ and $T^{-1}$ in the strong operator topology and admit a separating point, Lemma 28. It remains unknown however whether this implies the existence of a separating point with discrete orbit, that is, if the implication in Theorem 26 is an equivalence for countable groups.

We deduce Theorem 30, Theorem 32 and Theorem 34 essentially from Theorem 27 and Theorem 26. We also prove that Theorem 30 and Theorem 34 are optimal in the sense that there exists a real space in which representable finite groups are exactly those of the form $\{-1,1\} \times G$, Proposition 35, and a real space containing a complemented subspace with a symmetric basis in which representable countable groups are exactly those of the form $\{-1,1\} \times G$, Proposition 36. On the other hand we have the classical examples of $c_0$, $C([0,1])$, $\ell_p, 1 \leq p < +\infty$ and $L_p, 1 \leq p < +\infty$ for which Corollary 33 states that representable countable groups are exactly those which admit a normal subgroup with two elements, and we also provide an intermediary example of a space in which the class of representable finite groups is strictly contained in between the above two classes, Proposition 37.

In a first part of this minicourse we shall present some results of the isometric theory about renormings of Banach, mainly about locally uniformly rotund (LUR) renormings. We shall also recall a few topological properties of the group of isometries on a Banach space. This first part should be seen as a very incomplete introduction to the theory of renormings, concentrating on the results which are needed for the proofs of our main theorems. Our reference for renorming theory is the book of R. Deville, G. Godefroy and V. Zizler [2], and for classical results in infinite dimensional Banach space theory, the book of J. Lindenstrauss and L.

Tzafriri [9]. In a second part we shall sketch the proof of Theorem 30, Theorem 32, Theorem 34.

## 2 Renorming theory and topological properties of isometries

Questions of differentiability have led to study the set of possible equivalent norms on a given Banach space. A typical question in this area is, given a Banach space $X$ with a specific norm, whether there exists an equivalent norm on $X$ with some additional property. The unit ball of $X$ is denoted $B_X$ and the unit sphere $S_X$. Any norm on a space $X$ is by definition convex, i.e. for any $x, y$ in $B_X$, $\frac{x+y}{2} \in B_X$. Different stronger notions of convexity have been considered. We list some of them here, starting from the weaker and ending with the stronger form of convexity.

**Definition 6** *Let $X$ be a Banach space and $\|.\|$ be a norm on $X$. Then*

$$X \text{ strictly convex } \Leftrightarrow$$
$$\forall x \neq y \in S_X, \frac{\|x+y\|}{2} < 1, \tag{1}$$

$$X \text{ locally uniformly convex (LUR)} \Leftrightarrow$$
$$\forall x_0 \in S_X, \{x_n\} \subset S_X, (\lim \|x_0 + x_n\| = 2) \Rightarrow \lim x_n = x_0, \tag{2}$$

$$X \text{ uniformly convex } \Leftrightarrow$$
$$\forall \{x_n, y_n\} \subset S_X, (\lim \|x_n + y_n\| = 2) \Rightarrow \lim \|x_n - y_n\| = 0, \quad (3).$$

**Example 7** *The norm on $c_0$ is not strictly convex. There exists a space with a norm which is strictly convex but not LUR. The space $c_0$ admits an equivalent norm which is LUR but not uniformly convex. The $\ell_p$-norm, $1 < p < +\infty$ is uniformly convex.*

*Proof :* Exercise (hint: you may use results which will be given later on). □

Property (3) is a very strong form of convexity. Indeed having an equivalent uniformly convex norm is equivalent to the space being superreflexive (see [2] for details). Property (2) turns out to be the strongest convexity property which is satisfied, up to equivalence of norm, by most natural spaces. First we give a list of properties which are equivalent to the LUR property:

**Proposition 8** *Let $X$ be a Banach space, $\|.\|$ a norm on $X$, and $x_0 \in S_X$. Then the following are equivalent:*

- *The norm $\|.\|$ is LUR at $x_0$.*
- *Whenever $\{x_n\} \subset X$ is such that $\lim_n \|x_n\| = 1$ and $\lim_n \|x_0 + x_n\| = 2$, then $\lim_n x_n = x_0$.*
- *Whenever $\{x_n\} \subset X$ is such that $\lim_n 2(\|x_0\|^2 + \|x_n\|^2) - \|x_n + x_0\|^2 = 0$, then $\lim_n x_n = x_0$.*

*Proof :* See [2] p 42 Proposition 1.2. □

The most important result for us in the theory of LUR renormings will be the following.

**Theorem 9** *(M. Kadec 1967) Let $X$ be a real separable Banach space. Then $X$ admits an equivalent LUR norm.*

*Proof :* The theorem is a direct consequence of the next two lemmas and the fact that the norm on $\ell_2$ is uniformly convex and therefore LUR. Recall that a dual norm on a dual space $X^*$ is an equivalent norm $\|.\|^*$ on $X^*$ which is the dual norm of some equivalent norm $\|.\|$ on $X$.

**Lemma 10** *Let $X$ and $Y$ be two Banach spaces such that $Y$ is a dual space which admits a dual and LUR norm, and such that there exists $T : Y \to X$ a bounded linear operator, weak* to weak continuous with $TY$ norm dense in $X$. Then $X$ admits a LUR norm.*

*Proof :* (sketch) Let $\|.\|$ be the original norm on $X$ and $|.|$ be an equivalent dual LUR norm on $Y$. For $x \in X$ and $n \in \mathbb{N}$, define

$$|x|_n^2 = \inf\{\|x - Ty\|^2 + \frac{1}{n}|y|^2; y \in Y\},$$

and

$$|||x|||^2 = \sum_{n=1}^{\infty} 2^{-n}|x|_n^2.$$

Check that this defines an equivalent LUR norm on $X$. □

**Lemma 11** *If $X$ is a separable Banach space, then there is a bounded linear weak* to weak continuous operator $T : X^* \to \ell_2$ such that $T^* : \ell_2^* \simeq \ell_2 \to X$ is weak* to weak continuous and $T^*(\ell_2^*)$ is norm dense in $X^*$.*

*Proof :* (sketch) Let $\{x_i\}_{i=1}^{\infty} \subset S_X$ be dense in $S_X$. Define $T$ by $Tf(i) = 2^{-i}f(x_i), i = 1, 2, \ldots$. □ □

For the spaces $c_0$ and $\ell_1$ it is possible to verify directly Theorem 9 by constructing an LUR norm which also respects the symmetries of these spaces.

**Example 12** *Day's norm on $c_0$, defined for $x = (x_n)_n$ by*

$$\|x\| = \sup((\sum_{k=1}^{n} x_{i_k}^2/4^k)^{1/2}, n \in \mathbb{N}, i_1, \ldots, i_k \text{ distinct})$$

*is an LUR norm.*

*Proof :* See [2] p 69 Theorem 7.3. □

**Example 13** *The norm defined on $\ell_1$ by*

$$\|x\| = (|x|_1^2 + |Ix|_2^2)^{1/2},$$

*where $|.|_1$ (resp. $|.|_2$) is the canonical norm on $\ell_1$ (resp. $\ell_2$) and $I$ the canonical "identity"map of $\ell_1$ into $\ell_2$ is an LUR norm.*

*Proof :* See [2] p 72 Theorem 7.4. □

The theorem does not generalize to nonseparable spaces:

**Example 14** *The space $\ell_\infty$ does not admit an equivalent LUR norm. However it admits an equivalent strictly convex norm.*

*Proof :* See [2] p 48 Theorem 2.6 and p 74 Theorem 7.10. □

We shall also use the following result from [8].

**Theorem 15** *(G. Lancien, 1993) Let $X$ be a separable dual Banach space. Then $X$ admits an equivalent norm which does not diminish the group of isometries.*

## 2.1 Topological properties of the group of isometries on a Banach space

In this subsection a few basic topological properties of the group of isometries on a Banach space are recalled.

**Definition 16** *If $X$ is a Banach space, the strong operator topology on $L(X)$ is defined by*

$$f_n \to^s f \Leftrightarrow \forall x \in X, \lim_n f_n x = fx.$$

*If $T_n, n \in \mathbb{N}$ and $T$ belong to $GL(X)$ then we say that $T_n$ converges to $T$ in the strong operator topology for the convergence of $T$ and $T^{-1}$ if $T_n \to^s T$ and $T_n^{-1} \to^s T^{-1}$.*

**Proposition 17** *The group of isometries on a Banach space $X$ is closed for the convergence of $T$ and $T^{-1}$ in the strong operator topology.*

*Proof :* Exercise. □

**Corollary 18** *A group $G$ of isomorphisms on a Banach space $X$ which is the group of isometries on $X$ for some equivalent norm must be bounded, contain $-Id$ and be closed for the convergence of $T$ and $T^{-1}$ in the strong operator topology.*

**Example 19** *The group of rational rotations on $\mathbb{R}^2$ is not the group of isometries on $\mathbb{R}^2$ in some equivalent norm.*

*Proof :* This group is not closed for the convergence of $T$ and $T^{-1}$ in the strong operator topology. □

**Example 20** *The group of rotations on $\mathbb{R}^2$ is bounded, contains $-Id$ and is closed for the convergence of $T$ and $T^{-1}$ in the strong operator topology. However it is not the group of isometries on $\mathbb{R}^2$ in some equivalent norm.*

*Proof :* An equivalent norm invariant by rotations is a multiple of the euclidean norm and therefore the associated group of isometries must contain the symmetries with respect to the axes. □

# 3 Representation of countable groups on separable real Banach spaces

## 3.1 $G$-pimple norms on separable Banach spaces

We recall the result of Bellenot from [1].

**Theorem 21** *(Bellenot,1986) Any separable real Banach space $X$ may be renormed to admit only trivial isometries, i.e. so that the only isometries on $X$ are $-Id$ and $Id$.*

In this subsection we extend the construction of Bellenot from $\{-Id, Id\}$ to countable groups of isometries. So in the following, $X$ is real separable, $G$ is a countable group of isometries on $X$, and under certain conditions on $G$, we construct an equivalent norm on $X$ for which $G$ is the group of isometries on $X$.

Let us give an idea of our construction. Bellenot renorms $X$ with an LUR norm and then defines, for $x_0$ in $X$ of norm $1$, a new unit ball (the "pimple" ball) obtained by adding two small cones in $x_0$ and $-x_0$. Any isometry in the new norm must preserve the cones and therefore send $x_0$ to $\pm x_0$. Repeating this for a sequence $(x_n)_n$ with dense linear span, chosen carefully so that one can add the cones "independantly", and so that the sizes of the cones are "sufficiently" different, any isometry sends $x_n$ to $\pm x_n$. Finally, if each $x_n$ was chosen "much closer" to $x_0$ than to $-x_0$, any isometry fixing $x_0$ must fix each $x_n$ and therefore any isometry is equal to $Id$ or $-Id$.

In our case one should obviously put cones of same size in each $gx_0, g \in G$, defining a "G-pimple ball"; therefore any isometry preserves the orbit $Gx_0$. Then one repeats a similar procedure as above, adding other cones in $gx_n, g \in G$ for a sufficiently dense sequence $(x_n)_n$, so that any isometry preserves $Gx_n$ for all $n$. These $x_n$'s for $n \geq 1$ are called of type $1$. Finally, a last step is added to only allow as isometries isomorphisms whose restriction to $Gx_0$ is a permutation which corresponds to the action of some $g \in G$ on $Gx_0$. This is technically more complicated and is obtained by adding cones at some points of $\overline{span} Gx_0$ which code the structure of $G$ and are called of type $2$.

The reader may get a geometric feeling of this proof by looking at the group $G = \{\pm Id, \pm R\}$ of $\mathbb{R}$-linear isometries on $\mathbb{C}$ where $R$ is the rotation of angle $\pi/2$. By adding cones on the unit ball at $\pm 1$ and $\pm i$, one allows the isometries in $G$ but also symmetries with respect to the axes. A way of correcting this is to add one well-placed smaller cone next to each element of $\{\pm 1, \pm i\}$ so that the only isometries in the new norm are those of $G$.

**Definition 22** *Let $X$ be a real Banach space with norm $\|.\|$, let $G$ be a group of isometries on $X$ such that $-Id \in G$, and let $(x_k)_{k\in K}$ be a possibly finite sequence of unit vectors of $X$. Let $\Lambda = (\lambda_k)_{k\in K}$ be such that $1/2 < \lambda_k < 1$ for all $k \in K$. The $\Lambda, G$-*pimple at $(x_k)_k$ for $\|.\|$ *is the equivalent norm on $X$ defined by*

$$\|y\|_{\Lambda,G} = \inf\{\sum [[y_i]]_{\Lambda,G} : y = \sum y_i\},$$

*where $[[y]]_{\Lambda,G} = \lambda_k \|y\|$, whenever $y \in Vect(g.x_k)$ for some $k \in K$ and $g \in G$, and $[[y]]_{\Lambda,G} = \|y\|$ otherwise.*

In other words, the unit ball for $\|.\|_{\Lambda,G}$ may be seen as the convexification of the union of the unit ball for $\|.\|$ with line segments between $gx_k/\lambda_k$ and $-gx_k/\lambda_k$ for each $k \in K$ and $g \in G$.

Some observations are in order. First of all $(\inf_{k\in K} \lambda_k) \|.\| \leq \|.\|_{\Lambda,G} \leq \|.\|$. Any $g \in G$ remains an isometry in the norm $\|.\|_{\Lambda,G}$. In [1] Bellenot had defined the notion of $\lambda$-pimple at $x_0 \in X$, which corresponds to $(\lambda), \{-Id, Id\}$-pimple in our terminology. We recall a crucial result from [1].

**Proposition 23** *(Bellenot [1]) Let $(X, \|.\|)$ be a real Banach space and let $\|x_0\| = 1$ so that*

- *(1) $\|.\|$ is LUR at $x_0$, and*
- *(2) there exists $\epsilon > 0$ so that if $\|y\| = 1$ and $\|x_0 - y\| < \epsilon$, then $y$ is an extremal point (i.e. an extremal point of the ball of radius $\|y\|$).*

*Then given $\delta > 0$, $B > 0$ and $0 < m < 1$, there exists a real $0 < \lambda_0 < 1$ of the form $\lambda_0 = \max(m, \lambda_0(\epsilon, \delta, B, \lambda(x_0, \eta(\epsilon, \delta, B)))) < 1$, so that whenever $\lambda_0 \leq \lambda < 1$ and $\|.\|_\lambda$ is the $\lambda$-pimple at $x_0$, then*

- *(3) $m\|.\| \leq \|.\|_\lambda \leq \|.\|$,*
- *(4) if $1 = \|y\| > \|y\|_\lambda$ then $\|x_0 - y\| < \delta$ or $\|x_0 + y\| < \delta$,*
- *(5) $x_\lambda = \lambda^{-1}x_0$ is the only isolated extremal point of $\|.\|_\lambda$ which satisfies $\|x/\|x\| - x_0\| < \epsilon$,*
- *(6) if $w$ is a vector so that $x_\lambda$ and $x_\lambda + w$ are endpoints of a maximal line segment in the unit sphere of $\|.\|_\lambda$, then $B \geq \|w\| \geq \lambda^{-1} - 1$.*

For more details we refer to [1]. We generalize this result to $(\Lambda, G)$-pimples in a natural manner which for the $\Lambda$ part is inspired from [1]. Write $\Lambda \leq \Lambda'$ to mean $\lambda_k \leq \lambda'_k$ for all $k \in K$, if $\Lambda = (\lambda_k)_k$ and $\Lambda' = (\lambda'_k)_k$.

**Proposition 24** *Let $(X, \|.\|)$ be a real Banach space, let $G$ be a group of isometries on $X$ containing $-Id$ and let $(x_k)_{k\in K}$ be a possibly finite sequence of unit vectors of $X$. Assume*

- *(1)' $\|.\|$ is strictly convex on $X$ and LUR in $x_k$ for each $k \in K$, and*
- *(2)' for all $k \in K$, $c_k := \inf\{\|x_j - gx_k\| : j \in K, g \in G, (j, g) \neq (k, Id)\} > 0$.*

*Then given $\delta > 0$, $B = (b_k)_k > 0$ and $0 < m < 1$, there exists $\Delta = (\delta_k)_k$ with $\delta_0 \leq \delta$ and for all $k \geq 1$, $\delta_k \leq \min(\delta_{k-1}, c_k/4, 1 - \lambda(x_k, c_k))$, and $0 < \Lambda_0 = (\lambda_{0k})_k < 1$ with for all $k$, $\lambda_{0k} = \max(m, \lambda'_0(\epsilon_k, \delta_k, b_k, \lambda(x_0, \eta(\epsilon_k, \delta_k, b_k)))) < 1$, so that whenever $\Lambda_0 \leq \Lambda < 1$ and $\|.\|_{\Lambda,G}$ is the $\Lambda, G$-pimple at $(x_k)_k$, then*

- *(3)' $m\|.\| \leq \|.\|_{\Lambda,G} \leq \|.\|$,*
- *(4)' if $1 = \|y\| > \|y\|_{\Lambda,G}$ then $\exists g \in G, k \in K : \|gx_k - y\| < \delta_k$*
- *(5)' $x_{k,\lambda} = \lambda_k^{-1} x_k$ is the only isolated extremal point of $\|.\|_{\Lambda,G}$ which satisfies $\|x/\|x\| - x_k\| < \epsilon_k$,*
- *(6)' if $w$ is a vector so that $x_{k,\lambda}$ and $x_{k,\lambda} + w$ are endpoints of a maximal line segment in the unit sphere of $\|.\|_{\Lambda,G}$, then $b_k \geq \|w\| \geq \lambda_k^{-1} - 1$.*

*Proof :* Proposition 23 corresponds to the case $G = \{-Id, Id\}$ and $K$ a singleton. We shall deduce the general case from Proposition 23 and from the fact that for well-chosen $\Lambda$, the closed unit ball of the $\Lambda, G$-pimple at $(x_k)_k$ is equal to $B_0$, the union over $k \in K$ and $g \in G$ of the closed unit balls $B_{k,g}$ of the $\lambda_k$-pimples $\|.\|_{\lambda_k,g}$ at $gx_k$. Let $B$ denote the closed unit ball for $\|.\|$.

Note that by (1)', Proposition 23 (1)(2) apply in any $x_k, k \in K$, for any $\epsilon > 0$. Let $\epsilon_k = c_k/2$. Let $\lambda_{0k} \geq \max(m, \lambda'_0(\epsilon_k, \delta_k, b_k, \lambda(x_k, \eta(\epsilon_k, \delta_k, b_k))))$ given by Proposition 23 in $x_k$ for $\epsilon = \epsilon_k$, with $1 - \lambda_{0k}^{-1} \leq c_k/6$ for all $k \in K$ and with $\lim_{k\to+\infty} \lambda_{0k} = 1$ if $K$ is infinite. The limit condition on $\lambda_{0k}$ ensures that $B_0$ is closed. Assuming $x, y \in B_0$ and $\frac{x+y}{2} \notin B_0$ let $(k, g)$ and $(l, h)$ be such that $x \in B_k^g$ and $y \in B_l^h$. By convexity of $B_k^g$ and $B_l^h$, either $k \neq l$ (e.g. $k < l$), or $k = l$ and $g \neq \pm h$, and $x \in B_k^g \setminus B$, $y \in B_l^h \setminus B$, i.e. $\|x\|_{\lambda_k,g} < \|x\|$ and

$\|y\|_{\lambda_l,h} < \|y\|$. Therefore by (4) applied to $x$ for the $\lambda_k$-pimple at $gx_k$, and up to replacing $g$ by $-g$ if necessary, $\|gx_k - x\| < \delta_k$. Likewise $\|hx_l - y\| < \delta_l$. Then

$$\left\| \frac{x+y}{2} - \frac{gx_k + hx_l}{2} \right\| < \frac{\delta_k + \delta_l}{2} \leq \delta_k.$$

Since $\|gx_k - hx_l\| \geq c_k$ by (2)', it follows by LUR of $\|.\|$ in $gx_k$ that

$$\left\| \frac{gx_k + hx_l}{2} \right\| \leq \lambda(gx_k, c_k) = \lambda(x_k, c_k),$$

and

$$\left\| \frac{x+y}{2} \right\| \leq \delta_k + \lambda(x_k, c_k) \leq 1,$$

a contradiction. Therefore $B_0$ is closed convex and $B_0$ is equal to the closed unit ball of the $\Lambda, G$-pimple at $(x_k)_k$. Equivalently

$$\|.\|_{\Lambda,G} = \inf_{k \in K, g \in G} \|.\|_{\lambda_k,g}.$$

In fact, since whenever $x \in B_k^g \setminus B$ and $y \in B_l^h \setminus B$ with $B_k^g \neq B_l^h$ and $k \leq l$, and up to replacing $g$ by $-g$ or $h$ by $-h$ if necessary, we have

$$\|x - y\| \geq \|gx_k - hx_l\| - \|x - gx_k\| - \|y - hx_l\| \geq c_k - \delta_k - \delta_l \geq c_k/3,$$

it follows that for any $x$ such that $\|x\|_{\Lambda,G} < \|x\|$, there exists a unique $(g, \lambda_k)$ such that $\|x\|_{\lambda_k,g} < \|x\|$, and $\|x\|_{\Lambda,G} = \|x\|_{\lambda_k,g}$.

We now prove (3)'-(6)'. (3)' is obvious from (3) for each $\|.\|_{\lambda_k,g}$. For (4)' assume $1 = \|y\| > \|y\|_{\Lambda,G}$ then as we have just observed, there exist $g, k$ such that $1 = \|y\| > \|y\|_{\lambda_k,g}$, so from (4), $\|gx_k - y\| < \delta_k$ or $\|-gx_k - y\| < \delta_k$.

To prove (5)' note that if $\|x/\|x\| - x_{k,\Lambda}\| < \epsilon_k$ then whenever $g \neq Id$ or $k \neq l$,

$$\begin{aligned} \|x/\|x\| - gx_l\| &> \|gx_l - x_k\| - \|x_k - x_{k.\Lambda}\| - \epsilon_k \\ &\geq c_k - (1 - \lambda_k^{-1}) - \epsilon_k \geq c_k/2 - (1 - \lambda_{k0}^{-1}) \geq \delta_k. \end{aligned}$$

Therefore by (4)' $\|x\| = \|x\|_{\lambda_l,g}$ whenever $g \neq Id$ or $k \neq l$, and so $\|x\|_{\Lambda,G} = \|x\|_{\lambda_k}$. Now if $x$ is an isolated extremal point of $\|.\|_{\Lambda,G}$, it is therefore an isolated extremal point of $\|.\|_{\lambda_k}$ and by (5), $x = x_{k,\Lambda}$.

The proof of (6)' is a little bit longer. Write $S_k^g$ the unit sphere for $\|.\|_{\lambda_k,g}$, $S_\Lambda^G$ the unit sphere for $\|.\|_{\Lambda,G}$, $S$ the unit sphere for $\|.\|$, $S'$ the set of points of $S$ on which $\|.\|_{\Lambda,G} = \|.\|$. As we know, $S_\Lambda^G = S' \cup (\cup_{k,g}(S_k^g \setminus S))$.

As we noted before, whenever $x \in S_k^g \setminus S$, $y \in S_l^h \setminus S$, with $S_k^g \neq S_l^h$, it follows that $\|x - y\| \geq c_{\min(k,l)}/3$. So for $x \in S_k^g \setminus S$, $\|x - y\| \geq \frac{1}{3}\min\{c_i, i \leq k\}$ whenever $y$ belongs to some $S_l^h \setminus S$, with $S_k^g \neq S_l^h$. Therefore a line segment in $S_{\Lambda,G}$ containing points both in $S_k^g \setminus S$ and $S_l^h \setminus S$ with $S_k^g \neq S_l^h$ must have a subsegment included in $S$, but this contradicts the strict convexity of $\|.\|$.

We deduce that if $[x_{k,\Lambda}, x_{k,\Lambda} + w]$ is a maximal line segment in $S_{\Lambda,G}$, it is a line segment in $S_k^{Id}$. It is now enough to prove that it cannot be extended in $S_k^{Id}$, then by (6) applied for $\|.\|_{\lambda_k,Id}$, $b_k \geq \|w\| \geq \lambda_k^{-1} - 1$.

But for any strict extension $[x_{k,\lambda}, y]$ of $[x_{k,\Lambda}, x_{k,\Lambda}+w]$ in $S_k^{Id}$, either $[x_{k,\lambda}, y] \subset S_k^{Id} \setminus S \subset S_\Lambda^G$ and the maximality in $S_\Lambda^G$ is contradicted, or there exists a sequence $(y_n)_n$ of distinct points converging to $x_{k,\Lambda} + w$ in $[x_{k,\lambda}, y]$ with $y_n \in S$ for all $n$, but this again contradicts the strict convexity of $\|.\|$. □

**Theorem 25** *Let $X$ be a separable real Banach space with an LUR-norm $\|.\|$ and let $G$ be a countable group of isometries on $X$ such that $-Id \in G$. Assume that there exists a unit vector $x_0$ in $X$ which separates $G$ and such that the orbit $Gx_0$ is discrete. Then $X$ admits an equivalent norm $|||.|||$ such that $G$ is the group of isometries on $X$ for $|||.|||$.*

*Proof :* Since $Gx_0$ is discrete and $x_0$ separates $G$, let $\alpha \in ]0,1[$ be such that $\|x_0 - gx_0\| \geq \alpha$, for all $g \neq Id$.

Let $V_0 = \overline{span}\{gx_0, g \in G\}$ and let $y_0 = x_0$. If $V_0 \neq X$ then it is possible to pick a possibly finite sequence $(y_n)_{n\geq 1}$ such that, if $V_n := \overline{span}\{gy_k, k \leq n, g \in G\}$, we have that $y_n \notin V_{n-1}$ for all $n \geq 1$ and $\cup_n V_n$ is dense in $X$.

Let $(u_n)_{n\geq 1}$ be a (possibly finite) enumeration of $\{gx_0, g \in G \setminus \{\pm Id\}\} \cup \{y_k, k \geq 1\}$. Then define a (possibly finite) sequence $(x_n)_n$ of unit vectors of $X$ by induction as follows. Assume $x_0, \ldots, x_{n-1}$ are given.

If $u_n = y_k$ for some $k \geq 1$ then let $E = span(V_{k-1}, y_k)$. Pick some $z_n \in E$ such that $\|z_n\| \in [\alpha/10, \alpha/5]$ and $d(z_n, V_{k-1}) = \alpha/10$, and let $x_n = a_n x_0 + z_n$ where $a_n > 0$ is such that $\|x_n\| = 1$. Such an $x_n$ will be called of type 1.

If $u_n$ is of the form $gx_0$ then we shall pick some $\alpha_n \in [\alpha/10, \alpha/5]$ and define $z_n = \alpha_n gx_0$, $x_n = a_n x_0 + z_n$ with $a_n > 0$ and $\|x_n\| = 1$. Such an $x_n$ will be called of type $2$. The choice of $\alpha_n$ will be made more precise later. Let us first observe a few facts.

By construction, $X$ is the closed linear span of $\{gx_n, g \in G, n\}$ (actually only $x_0$ and $x_n$'s of type $1$ are required for this). Note that for all $n$, $\|z_n\| \leq \alpha/5$ and therefore $a_n \in [1-\alpha/5, 1+\alpha/5]$; and obviously $x_0$ may also be written $x_0 = a_0 x_0 + z_0$ with these conditions. We now evaluate $\|x_n - gx_m\|$ for all $(n, Id) \neq (m, g)$.

If $g \neq Id$ then $\|x_n - gx_m\| = \|a_n x_0 + z_n - g a_m x_0 - z_m\|$ therefore

$$\|x_n - gx_m\| \geq \|x_0 - gx_0\| - |1-a_n| - |1-a_m| - \|z_n\| - \|z_m\| \geq \alpha/5.$$

If $g = Id$, without loss of generality assume $n > m$. If $x_n$ is of type $1$ then, if $k$ is such that $x_n$ is associated to $y_k$, the vector $gx_m$ is in $V_{k-1}$ and $\|x_n - gx_m\| \geq d(x_n, V_{k-1}) = \alpha/10$. If $x_n$ is of type $2$ and $x_m$ of type $1$ then $\|x_n - gx_m\| = \|x_m - g^{-1}x_n\| \geq d(x_m, V_0) \geq \alpha/10$.

It now remains to study the more delicate case when $x_n$ and $x_m$ both are of type $2$, or one is of type $2$ and the other is $x_0$. We describe how to choose the $x_n$'s of type $2$, i.e. how to choose each corresponding $\alpha_n \in [\alpha/10, \alpha/5]$ in the definition of $x_n$ to obtain good estimates for $\|x_n - x_m\|$ in that case. This will be done by induction. To simplify the notation, we shall denote $(x'_n)_{n\in N}$ the subsequence $(x_{k_n})_{n\in N}$ corresponding to the $x_k$'s of type $2$, with $N = \{1, \ldots, |G|-2\}$ or $N = \mathbb{N}$ according to the cardinality of $G$, and we shall write $x'_n = b_n x_0 + \beta_n g_n x_0$, where $g_n$ is the associated element of $G \setminus \{\pm Id\}$, $b_n = a_{k_n}$ and $\beta_n = \alpha_{k_n}$. Write $x'_0 = x_0$.

Let $\forall m \geq 1, I^0_m = [\alpha/10, \alpha/5]$. For $\beta \in [\alpha/10, \alpha/5]$, let $x'_m(\beta) = b_m(\beta)x_0 + \beta g_m x_0$ where $b_m(\beta) > 0$ is such that $\|x'_m(\beta)\| = 1$.

We observe that $\|x'_m(\beta) - x'_m(\gamma)\| \geq \frac{\alpha}{2}|\beta - \gamma|$. Indeed if $x'_m(\beta) - x'_m(\gamma) = (\beta - \gamma)\epsilon$ with $\|\epsilon\| < \alpha/2$ and $\beta \neq \gamma$, then

$$(b_m(\beta) - b_m(\gamma))x_0 = (\gamma - \beta)(g_m x_0 - \epsilon),$$

so $g_m x_0 - \epsilon = \pm \|g_m x_0 - \epsilon\| \, x_0$. If for example $\pm = -$ in this equality, then

$$\|g_m x_0 + x_0\| = \|\epsilon + (1 - \|g_m x_0 - \epsilon\|)x_0\| \leq 2\,\|\epsilon\| < \alpha,$$

and by separation, $g_m = -Id$, a contradiction. Similarly the case $\pm = +$ would imply $g_m = Id$.

Now for all $m \geq 1$ divide $I_m^0 = [\alpha/10, \alpha/5]$ in three successive intervals of equal length $\alpha/30$. Since

$$\|x'_m(\beta) - x'_m(\gamma)\| \geq \frac{\alpha}{2}|\beta - \gamma| \geq \frac{\alpha^2}{60}$$

whenever $\beta$ is in the first and $\gamma$ in the last interval, it follows that there exists an interval $I_m^1 \subset I_m^0$ of length $\alpha/30$ (which is either the first or the last subinterval), such that

$$\beta \in I_m^1 \Rightarrow \|x'_m(\beta) - x'_0\| \geq \frac{\alpha^2}{120}.$$

We then pick $\beta_1$ in $I_1^1$ and fix $x'_1 = x'_1(\beta_1)$. Therefore we have ensured

$$\|x'_1 - x'_0\| \geq \frac{\alpha^2}{120}.$$

Assume selected $\beta_1, \ldots, \beta_{n-1}$, $x'_1, \ldots, x'_{n-1}$ associated, and for $0 \leq i \leq n-1$ and $m \geq i$, decreasing in $i$ intervals $I_m^i$ of length $\frac{\alpha}{10.3^i}$. For any $m \geq n-1$, dividing $I_m^{n-1}$ in three subintervals and picking the first or the last, we find by the same reasoning as above $I_m^n \subset I_m^{n-1}$ of length $\frac{\alpha}{10.3^n}$ with

$$\beta \in I_m^n \Rightarrow \left\|x'_m(\beta) - x'_{n-1}\right\| \geq \frac{\alpha^2}{40.3^n}.$$

We then pick $\beta_n$ in $I_n^n$ and fix $x'_n = x'_n(\beta_n)$. Therefore for all $k < n$, $\beta_n \in I_n^n \subset I_n^{k+1}$ and we have ensured

$$\forall 0 \leq k < n, \|x'_n - x'_k\| \geq \frac{\alpha^2}{40.3^{k+1}}.$$

We have finally proved that for all $k$,

$$\inf\{\|x_n - gx_k\|, n \geq k, g \in G, (n, g) \neq (k, Id)\} \geq \frac{\alpha^2}{40.3^{k+1}},$$

and so

$$\inf\{\|x_n - gx_k\|, n, g \in G, (n, g) \neq (k, Id)\} \geq \frac{\alpha^2}{40.3^{k+1}},$$

therefore (2)' in Proposition 24 is satisfied; and (1)' is clearly satisfied since $\|.\|$ is LUR.

We then define $|||.|||$ as the $\Lambda, G$-pimple at $(x_n)_n$ for $\Lambda = (\lambda_n)_n$ associated to $\epsilon_n, b_n$ so that Proposition 24 applies and such that $b_n > b_{n+1}$, $1/2 \leq \lambda_n < \lambda_{n+1}$

and $\lambda_n^{-1} - 1 > 2b_{n+1}$ for all $n$. This is possible by induction and the expression of $\Lambda_0$ in Proposition 24.

Observe that $E = \{gx_n/\lambda_n, g \in G, n\}$ is the set of isolated extremal points of $|||.|||$. Indeed for a point $x$ of $S_{\Lambda,G}$ either $\|x/\|x\| - gx_k\| < \epsilon_k$ for some $g, k$, in which case by (5)' $x = \lambda_k^{-1} gx_k$ if it is an isolated extremal point; or $\|x/\|x\| - gx_k\| \geq \epsilon_k > \delta_k$ for all $g, k$ then by (4)' $\|.\| = |||.|||$ in a neighborhood of $x$ and then $x$ is not an isolated extremal point since $\|.\|$ is LUR at $x$.

Therefore any isometry $T$ for $|||.|||$ maps $E$ onto itself. If $n < m$, $g \in G$, then $T$ cannot map $\lambda_n^{-1}x_n$ to $\lambda_m^{-1}gx_m$. Indeed if $w$ (resp. $w'$) is a vector so that $\lambda_n^{-1}x_n$ and $\lambda_n^{-1}x_n + w$ (resp. $\lambda_m^{-1}gx_m$ and $\lambda_m^{-1}gx_m + w'$) are endpoints of a maximal line segment in the unit sphere of $|||.|||$, then since $g$ is an isometry for $|||.|||$ we may assume $g = Id$, and then by (6)',

$$|||w||| \geq \frac{1}{2}\|w\| \geq \frac{1}{2}(\lambda_n^{-1} - 1) > b_{n+1} \geq b_m \geq \|w'\| \geq |||w'|||.$$

It follows that for each $n$, the orbit $Gx_n$ is preserved by $T$.

We finally prove that $T$ belongs necessarily to $G$. Without loss of generality we may assume that $Tx_0 = x_0$ and then by density it is enough to prove that $Tgx_n = gx_n$ for all $g \in G$ and any $x_n$ of type $1$ or equal to $x_0$.

Let $g \in G$, $g \neq \pm Id$. Let $x'$ be the associated vector of type $2$ of the form $x' = ax_0 + \beta gx_0$. Then

$$Tx' = ax_0 + \beta Tgx_0 = h(ax_0 + \beta gx_0)$$

for some $h \in G$. So $|a|\,\|x_0 - hx_0\| = \beta\,\|Tgx_0 - hgx_0\|$ and

$$\|x_0 - hx_0\| \leq \frac{\alpha/5}{1-\alpha/5}(1 + 2|||Tgx_0|||) \leq \frac{\alpha}{4}(1 + 2|||x_0|||) < \alpha,$$

therefore by separation $h = Id$. It follows immediately that

$$Tgx_0 = gx_0.$$

and this holds for any $g \in G$. Finally if $x_n$ is of type $1$, and $g \in G$, then

$$Tgx_n = T(a_n gx_0 + gz_n) = a_n gx_0 + Tgz_n,$$

and since $T(gx_n)$ is of the form $hx_n$ for some $h \in G$,

$$Tgx_n = a_n hx_0 + hz_n.$$

Therefore $a_n \|gx_0 - hx_0\| = \|Tgz_n - hz_n\|$ and by similar computations as above,

$$\|gx_0 - hx_0\| \leq \frac{3\alpha/5}{1-\alpha/5} < \alpha,$$

whence again by separation $g = h$ and

$$Tgx_n = gx_n.$$

□

## 3.2 Representable groups of linear isomorphisms

In this subsection, we give sufficient conditions for a group of isomorphims on a Banach space $X$ to be representable in $X$.

**Theorem 26** *Let $X$ be a separable real dual Banach space and $G$ be a countable bounded group of isomorphisms on $X$, containing $-Id$, and such that some point separates $G$ and has discrete orbit. Then $X$ admits an equivalent norm for which $G$ is the group of isometries on $X$.*

*Proof :* We may assume that every $g$ in $G$ is an isometry on $X$ by using the equivalent norm $\sup_{g \in G} \|gx\|$. Then by Theorem 15 $X$ may be renormed with an LUR norm without diminishing the group of isometries. We are then in position to apply Theorem 25. □

**Theorem 27** *Let $X$ be a separable real Banach space and $G$ be a finite group of isomorphisms such that $-Id \in G$. Then $X$ admits an equivalent norm for which $G$ is the group of isometries on $X$.*

*Proof :* By Theorem 9 we may assume that the norm $\|.\|$ on $X$ is LUR. Then we define an equivalent norm $\|.\|_G$ on $X$ by

$$\|x\|_G = (\sum_{g \in G} \|gx\|^2)^{1/2}.$$

Since this is the $l_2$-sum of the LUR norm $\|.\|$ with an equivalent norm, it is classical to check that it is also LUR, see [2] Fact 2.3, and obviously any $g \in G$ becomes an isometry for $\|.\|_G$. To apply Theorem 25 it therefore only remains to find some $x_0$ such that $x_0 \neq gx_0$ for all $g \neq Id$. But if such an $x_0$ didn't exist then $Ker(Id - g)$ would have non-empty interior for some $g \neq Id$, but by linearity this would actually imply that $g = Id$. □

Note that the condition in Theorem 26 that some point separates $G$ and has discrete orbit implies directly that $G$ is closed (and discrete) in the strong operator topology and therefore also for the convergence of $T$ and $T^{-1}$ in the strong operator topology. Conversely to Theorem 26:

**Lemma 28** *Let $X$ be a separable real Banach space and $G$ be a group of isomorphisms which is the group of isometries in some equivalent norm on $X$. If $G$ is countable then $G$ is discrete for the convergence of $T$ and $T^{-1}$ in the strong operator topology, and $G$ admits a separating point. If $X$ is finite dimensional and $G$ is countable then $G$ is finite.*

*Proof :* The existence of a separating point is a consequence of the Theorem of Baire. Indeed for any $g \in G$, $g \neq Id$, the set of points which separate $g$ from $Id$, i.e. the set $X \setminus Ker(g - Id)$, is dense open, therefore the set of separating points is a $G_\delta$ dense set.

To prove that $G$ is discrete we may assume that the norm is such that $G$ is the group of isometries on $X$. It is classical to check that $G$ is Polish. Indeed since $X$ is separable, the unit ball $L_1(X)$ of $L(X)$ with the (relative) strong topology is Polish [7], page 14. We define $\phi : G \to L_1(X) \times L_1(X)$ by $\phi(T) = (T, T^{-1})$ and note that $\phi(G)$ is closed in $L_1(X) \times L_1(X)$ (this follows immediately from the fact that if $(T_n)_{n \in \mathbb{N}}$ converges to $T$ in $L_1(X)$ and $(L_n)_{n \in \mathbb{N}}$ converges to $L$ in $L_1(X)$, then $T_n L_n$ converges to $TL$ in $L_1(X)$). Hence $\phi(G)$ is a Polish space, and as $\phi$ is a bijection onto the image, $G$ is a Polish space with the induced topology by $\phi$. We then conclude using the fact that every countable Polish group is a discrete space. Indeed if $G$ is a countable Polish group, then by [7], Theorem 6.2, $G$ is not a perfect space, that is, $G$ has an isolated point, therefore by the group property all points are isolated.

Finally if $X$ is finite dimensional then the strong topology on $L_1(X)$ coincides with the usual one for which $L_1(X)$ is compact. So $\phi(G)$ as a discrete subset of $L_1(X) \times L_1(X)$ is therefore finite. □

Note however that it seems to remain unknown whether a group $G$ of isomorphisms, which is the group of isometries on a real Banach space $X$ in some equivalent norm, and which is countable, must have some separating point with discrete orbit.

The next question remains open in general (i.e. for a space which is not isomorphic to a dual space):

**Question 29** *Let $X$ be a separable real Banach space and let $G$ be an infinite countable bounded group of isomorphisms on $X$ such that $-Id \in G$, and some point separates $G$ and has discrete orbit. Does $X$ admit an equivalent norm for which $G$ is the group of isometries on $X$?*

## 3.3 Representation of countable groups in Banach spaces

Jarosz conjectured that any group of the form $\{-1,1\} \times G$ (or $C \times G$ in the complex case) could be represented in any Banach space $X$ provided $\dim X \geq |G|$. From Theorem 27 and Theorem 25 we obtain rather general answers to his question for countable groups and separable real spaces.

**Theorem 30** *Let $G$ be a finite group and $X$ be a separable real Banach space such that $\dim X \geq |G|$. Then $\{-1,1\} \times G$ is representable in $X$.*

*Proof :* The group $\{-1,1\} \times G$ may be canonically represented as a group of isometries on $\ell_2(G)$: denoting $(e_g)_{g \in G}$ the canonical basis of $\ell_2(G)$, associate to any $(\epsilon, g)$ in $\{-1,1\} \times G$ the isometry $T_{\epsilon,g}$ defined on $\ell_2(G)$ by

$$T_{\epsilon,g}(\sum_{h \in G} \lambda_h e_h) = \epsilon \sum_{h \in G} \lambda_h e_{gh}.$$

Since $\dim X \geq |G|$, the space $X$ is isomorphic to the $l_2$ direct sum $l_2(G) \oplus_2 Y$, for some space $Y$. By associating to any $(\epsilon, g)$ in $\{-1,1\} \times G$ the isometry $A_{\epsilon,g}$ defined on $\ell_2(G) \oplus_2 Y$ by

$$A_{\epsilon,g}(t,y) = (T_{\epsilon,g}(t), \epsilon y),$$

we see that $\{-1,1\} \times G$ is isomorphic to a group of isometries on $\ell_2(G) \oplus_2 Y$ containing $-Id$. Therefore Theorem 27 applies to deduce that $\{-1,1\} \times G$ is isomorphic to the group of isometries on $X$ in some equivalent norm. $\square$

By Lemma 28 an infinite countable group is representable in a real space $X$ only if $X$ is infinite dimensional. For finite groups, it seems to remain open whether the condition on the dimension is necessary in Theorem 30. This is not the case when $|G|$ is an odd prime. Indeed, letting $p = |G|$, $G$ is then isomorphic to $\mathbb{Z}/p\mathbb{Z}$ and so $\{-1, 1\} \times G$ is isomorphic to $\mathbb{Z}/2p\mathbb{Z}$ and therefore may be represented as the group $\{e^{ik\pi/p}Id, 0 \leq k \leq 2p-1\}$ of isometries on $\mathbb{C}$; so $\{-1, 1\} \times G$ is representable in $\mathbb{R}^2$, and by the proof of Theorem 30, in any separable real space of dimension at least 2. For other values of $|G|$ the question remains open:

**Question 31** *For arbitrarily large $n \in \mathbb{N}$, does there exist a group $G$ with $|G| = n$, such that $\{-1, 1\} \times G$ is representable in a separable real Banach space $X$ if and only if $\dim X \geq n$?*

A group which is representable in a Banach space necessarily admits a normal subgroup with two elements. Recall that reciprocally any (resp. countable) group which admits a normal subgroup with two elements is representable in a (resp. the separable) Hilbert space [10]. The next theorem shows that this extends to a wide class of spaces, including the classical spaces $c_0$, $C([0, 1])$, $\ell_p, 1 \leq p < +\infty$, and $L_p, 1 \leq p < +\infty$.

A basis $(s_n)_s$ of a Banach space is said to be 1-symmetric if for any permutation $\sigma$ and any eventually zero sequence of coefficients $a_n$, $\|\sum_n a_n s_n\| = \left\|\sum_n a_n s_{\sigma(n)}\right\|$. This is the case of the standard basis of $c_0$ or $\ell_p, 1 \leq p < +\infty$. A Banach space $X$ is said to have a symmetric decomposition if it is isomorphic to a space of the form $(\sum \oplus Y)_S$, for some space $S$ with a 1-symmetric basis $(s_n)_n$, i.e. an equivalent norm on $X$ is given by $\|(y_n)_n\| = \|\sum_n \|y_n\| s_n\|$.

**Theorem 32** *Let $G$ be a countable group which admits a normal subgroup with two elements and $X$ be an infinite-dimensional separable real Banach space with a symmetric decomposition which either is isomorphic to $c_0(Y)$ or to $l_p(Y)$ for some $Y$ and $1 \leq p < +\infty$, or to a dual space. Then $G$ is representable in $X$.*

*Proof :* We first assume that $G$ is infinite. Let $\{1, j\}$ be a normal subgroup of $G$ with two elements, therefore $j$ commutes with any element of $G$. Let $G'$ be a subset of $G$ containing 1 and such that $G = G' \cup jG'$ and $G' \cap jG' = \emptyset$. For $g \in G$ let $\epsilon_g = 1$ if $g \in G'$ and $\epsilon_g = -1$ otherwise, and let $|g|$ denote the unique element of $\{g, jg\} \cap G'$.

Write $X = (\sum \oplus Y)_S$ and index the decomposition on $G'$, i.e write an element of $X$ as $(y_g)_{g\in G'}$. We associate to any $g$ in $G$ the isomorphism $T_g$ defined on $X$ by

$$T_g((y_h)_{h\in G'}) = (\epsilon_{g^{-1}h} y_{|g^{-1}h|})_{h\in G'}.$$

Observe that if $g, k \in G$, then

$$T_k T_g((y_h)_h) = T_k((\epsilon_{g^{-1}h} y_{|g^{-1}h|})_h) = (\epsilon_{k^{-1}h}\epsilon_{g^{-1}|k^{-1}h|} y_{|g^{-1}|k^{-1}h||})_h.$$

Since $j$ commutes with any element of $G$, we have $|g^{-1}|k^{-1}h|| = |g^{-1}k^{-1}h|$ and it is easy to see that $\epsilon_{k^{-1}h}\epsilon_{g^{-1}|k^{-1}h|} = \epsilon_{g^{-1}k^{-1}h}$, therefore

$$T_k T_g((y_h)_h) = (\epsilon_{(kg)^{-1}h} y_{(kg)^{-1}h})_h = T_{kg}((y_h)_h).$$

From this we deduce that the map $g \mapsto T_g$ is a group homomorphism, and therefore we may assume that $G$ is a bounded group of isomorphisms on $X$ containing $-Id$ (here identified with $j$).

Let $x_0$ be a unit vector in the summand of the decomposition indexed by $1$. We observe that $\|x_0 - (-x_0)\| = 2$ and that for any $g \in G, g \notin \{-Id, Id\}$,

$$\|x_0 - gx_0\| \geq c,$$

where $c$ is the constant of the basis $(s_g)_{g\in G'}$ of $S$. Therefore $x_0$ separates $G$ and has discrete orbit. Finally, when $X$ is a dual space, Theorem 26 applies.

When $X$ is isomorphic to $c_0(Y)$ or $\ell_p(Y)$ for some $1 \leq p < +\infty$, we use the existence of a LUR norm on $X$ for which the $T_g$'s are isometries. The existence of the LUR norm may be found in the Appendix, Lemma 38 for $\ell_p(Y)$, Lemma 39 for $c_0(Y)$, modulo the result of Kadec that any separable space $Y$ has an equivalent LUR norm. Therefore $G$ is representable as a group of isometries containing $-Id$ for an LUR norm on $X$. Any unit vector $x_0$ in the first summand of the decomposition separates $G$ and has discrete orbit, therefore Theorem 25 applies.

Finally in the case when $G$ is finite, we may index a symmetric decomposition of $X$ on $\cup_{i\in\mathbb{N}} G'_i$ where the $G'_i$ are disjoint copies of $G'$. We may then use the previous method to represent $G$, up to renorming, as a group of isometries containing $-Id$ on each space spanned by the sum of the summands indexed on $G'_i$, and therefore globally as a group of isometries containing $-Id$ on X. The rest of the proof is as before. □

**Corollary 33** *A countable group is representable in the real space $c_0$, resp. $C([0,1])$, $l_p$ for $1 \leq p < +\infty$, $L_p$ for $1 \leq p < +\infty$, if and only if it admits a normal subgroup with two elements.*

From Theorem 32 we may also deduce the following theorem.

**Theorem 34** *Let $G$ be a countable group and $X$ be an infinite-dimensional separable real Banach space which contains a complemented subspace with a symmetric basis. Then $\{-1,1\} \times G$ is representable in $X$.*

*Proof :* By Theorem 30 we may assume that $G$ is infinite. Let $Y$ be a complemented subspace $Y$ of $X$ with a symmetric basis, and write $X = Y \oplus Z$. Since a symmetric basis is unconditional, $Y$ is either reflexive or contains a complemented subspace isomorphic to $c_0$ or $l_1$, therefore we may assume that $Y$ is isomorphic to a dual space or is isomorphic to $c_0$. By Theorem 32 we may assume that $\{-1,1\} \times G$ is a group of isometries on $Y$ containing $-Id$ (here identified with $(-1, 1_G)$).

When $Y$ is a dual space, we may by applying the result of Lancien, Theorem 15, also assume that the new norm is LUR. Since $Z$ is separable we may also assume it is equipped with an LUR norm, and we equip $X$ with the $l_2$-sum norm $|||.|||$, i.e. $X = Y \oplus_2 Z$. It is classical that the norm $|||.|||$ is LUR on $X$.

Furthermore, for any $(\epsilon, g)$ in $\{-1,1\} \times G$, the map $A_{\epsilon,g}$ defined on $X = Y \oplus_2 Z$ by

$$A_{\epsilon,g}(y,z) = ((\epsilon,g).y, \epsilon z)$$

is an isometry on $X$ for $|||.|||$. Therefore $\{-1,1\} \times G$ is isomorphic to a group of isometries on $(X, |||.|||)$ containing $-Id$. As in the proof of Theorem 32, the point $x_0 = e_1$ separates $G$ and has discrete orbit, where $e_1$ is the first vector of the symmetric basis of $Y$, so finally Theorem 25 applies.

When $Y$ is isomorphic to $c_0$, we may use Lemma 39 to see $\{-1,1\} \times G$ as a group of isometries containing $-Id$ for an LUR norm on $Y$. The rest of the proof is as in the first case. $\square$

Observe that Theorem 34 applies whenever $X$ is a subspace of $\ell_p$, $1 \leq p < +\infty$, or, by Sobczyk's Theorem, [9] Th. 2.f.5, whenever $X$ is separable and contains a copy of $c_0$.

Because of Theorem 32, it is natural to ask whether Theorem 30 and Theorem 34 extend to the case when one replaces groups of the form $\{-1,1\} \times G$ by groups which admit a normal subgroup with two elements. We provide examples to show that the answer is negative in general.

The space denoted $X_{GM}$ is the real HI space of W.T. Gowers and B. Maurey [5]. Every operator on $X_{GM}$ is of the form $\lambda Id + S$, $\lambda \in \mathbb{R}$, $S$ strictly singular, and therefore every isometry is of the form $\pm Id + S$. The complex version of $X_{GM}$ is such that every isometry is of the form $\lambda Id + S$, $\lambda \in C$, $S$ strictly singular. For the definition of the ideal of strictly singular operators we refer to [9].

**Proposition 35** *Any group which is representable in the real (resp. the complex) $X_{GM}$ is of the form $\{-1, 1\} \times G$ (resp. $C \times G$). In particular a finite group is representable in the real $X_{GM}$ if and only if it is of the form $\{-1, 1\} \times G$.*

*Proof :* The last part of the proposition is a consequence of the initial part and of Theorem 30. We prove the initial part. Let $H$ be the group of isometries on the real (resp. complex) $X_{GM}$ in some equivalent norm. Let $G$ be the subgroup of $H$ of isometries of the form $Id + S$, $S$ strictly singular. For $T \in H$, let $\lambda_T$ be the element of $\{-1, 1\}$ (resp. $C$) such that $T - \lambda_T Id$ is strictly singular. It is then easy to see, using the ideal properties of strictly singular operators, that by mapping $T$ to $(\lambda_T, T/\lambda_T)$ we provide an isomorphism of $H$ onto the group $\{-1, 1\} \times G$ (resp. $C \times G$). $\square$

**Proposition 36** *Let $S$ be a Banach space with a symmetric basis. Any group which is representable in $S \oplus X_{GM}$ is of the form $\{-1, 1\} \times G$ in the real case (resp. $C \times G$ in the complex case). In particular, in the real case, a countable group is representable in $S \oplus X_{GM}$ if and only if it is of the form $\{-1, 1\} \times G$.*

*Proof :* The last part of the proposition is a consequence of the initial part and of Theorem 34. We prove the initial part. Let $X = S \oplus X_{GM}$. We observe that, since $S$ and $X_{GM}$ are totally incomparable, any operator $T$ on $X$ may be written as a matrix of the form

$$\begin{pmatrix} A & s_1 \\ s_2 & \lambda_T Id + s \end{pmatrix},$$

where $A \in L(S)$, and $s_1 \in L(X_{GM}, S), s_2 \in L(S, X_{GM}), s \in L(X_{GM})$ are strictly singular; and $\lambda_T \neq 0$ if $T$ is an isomorphism. If $T$ is an isometry then since $T_{|X_{GM}}$ is a strictly singular perturbation of $\lambda_T i_{X_{GM},X}$, where $i_{X_{GM},X}$ denotes the canonical injection of $X_{GM}$ into $X$, $\lambda_T$ must belong to $\{-1, 1\}$ (resp. $C$).

Let $H$ be the group of isometries on $S \oplus X_{GM}$ for some equivalent norm. Let $G$ be the subgroup of $H$ defined by $G = \{T \in H : \lambda_T = 1\}$. Clearly mapping $T$ to $(\lambda_T, T/\lambda_T)$ we provide an isomorphism of $H$ onto the group $\{-1, 1\} \times G$ (resp. $C \times G$). $\square$

It remains open for a given separable infinite dimensional real space $X$ exactly which finite (resp. countable) groups are representable. We have the maximal case of $c_0$, $C([0,1])$, $\ell_p, 1 \leq p < +\infty$ or $L_p, 1 \leq p < +\infty$, in which all countable groups admitting a normal subgroup with two elements are representable, and the minimal case of $X_{GM}$, in which only groups of the form $\{-1,1\} \times G$ are representable. Apparently quite various situations may occur. Indeed we also show that a space constructed in [3] provide a third example which is "in between" the cases of $\ell_p$ and $X_{GM}$: in the following $X(\mathbb{C})$ denotes, seen as real, the separable complex space defined in [3] on which every $\mathbb{R}$-linear operator is of the form $\lambda Id + S$, where $\lambda \in \mathbb{C}$ and $S$ is strictly singular.

**Proposition 37** *The class of finite groups representable in $X(\mathbb{C})$ is neither equal to the class of finite groups which admit a normal subgroup with two elements, nor to the class of finite groups of the form $\{-1,1\} \times G$.*

*Proof:* For any $n \in \mathbb{N}, n \geq 1$, the group $\{e^{ik\pi/2n} Id, 0 \leq k \leq 4n-1\} \simeq \mathbb{Z}/4n\mathbb{Z}$ is a finite group of isomorphisms on $X(\mathbb{C})$ containing $-Id$. Therefore by Theorem 27 it is representable in $X(\mathbb{C})$; however it is not of the form $\{-1,1\} \times G$.

On the other hand, let $\{1, i, j, k\}$ be the generators of the algebra $\mathbb{H}$ of quaternions, and let $G$ be the group $\{\pm 1, \pm i, \pm j, \pm k\}$. The group $\{-1,1\}$ is a normal subgroup of $G$ with two elements, and we prove that $G$ is not representable in $X(\mathbb{C})$.

Assume on the contrary that $\alpha$ is an isomorphism from $G$ onto $H$, where $H$ is the group of isometries on $X(\mathbb{C})$ in some equivalent norm. Since $-Id \in H$, $(-Id)^2 = Id$ and $-1$ is the only element of square $1$ in $G \setminus \{1\}$, we have $\alpha(-1) = -Id$. Therefore from $ij = -ji$ we deduce $\alpha(i)\alpha(j) = -\alpha(j)\alpha(i)$. Let, for $T$ an operator on $X(\mathbb{C})$, $\lambda_T$ be the unique complex number such that $T - \lambda_T Id$ is strictly singular. The map $T \mapsto \lambda_T$ induces an homomorphism of $H$ into $C$. We deduce $\lambda_{\alpha(i)}\lambda_{\alpha(j)} = -\lambda_{\alpha(j)}\lambda_{\alpha(i)}$, which is impossible in $C$. $\square$

# 4 Appendix

We give the proof of two lemmas used in Section 4. They are inspired by [2] Theorem 7.4 page 72 and by the properties of Day's norm on $c_0$, as studied in [2] page 69.

**Lemma 38** *Let $Y$ be a Banach space with an LUR norm, let $1 \leq p < +\infty$, and let $X = l_p(Y)$. Then there exists an equivalent LUR norm on $X$ for which any* *map $T$ defined on $X$ by $T((y_n)_{n\in\mathbb{N}}) = (\epsilon_n y_{\sigma(n)})_{n\in\mathbb{N}}$, where $\epsilon_n = \pm 1$ for all $n \in \mathbb{N}$ and $\sigma$ is a permutation on $\mathbb{N}$, is an isometry.*

*Proof :* Fix an equivalent LUR norm $\|.\|$ on $Y$, and let $\|.\| = \|.\|_p$ be the corresponding $l_p$-norm on $X$, when $p > 1$. When $p = 1$, let $\|.\|_1$ denote the corresponding $l_1$-norm, $\|.\|_2$ denote the corresponding $l_2$-norm (via the canonical "identity" map from $l_1$ into $l_2$), and let $\|.\|$ be the equivalent norm defined on $X$ by

$$\|x\|^2 = \|x\|_1^2 + \|x\|_2^2 ,$$

as in Example 13. To prove that $\|.\|$ is LUR let $x = (y_k)_k \in X$ and $x_n = (y_{n,k})_k \in X$ with $\lim_n \|x_n\| = \|x\|$ and $\lim_n \|x + x_n\| = 2\|x\|$. We need to prove that $\lim_n x_n = x$.

We first assume that $p = 1$. We have that

$$\lim_n 2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 = 0. \tag{1}$$

Using [2] Fact 2.3 p 45, (1) implies

$$\lim_n 2\|x\|_1^2 + 2\|x_n\|_1^2 - \|x + x_n\|_1^2 = 0 \tag{2}$$

and

$$\lim_n 2\|x\|_2^2 + 2\|x_n\|_2^2 - \|x + x_n\|_2^2 = 0. \tag{3}$$

By [2] Fact 2.3 again, (3) implies, for all $k \in \mathbb{N}$,

$$\lim_n 2\|y_k\|^2 + 2\|y_{n,k}\|^2 - \|y_k + y_{n,k}\|^2 = 0,$$

whence, since the norm on $Y$ is LUR, by [2] Proposition 1.2. p 42,

$$\lim_n y_{n,k} = y_k, \forall k \in \mathbb{N}, \tag{4}$$

and from (2) we have, see [2] p 42,

$$\lim_n \|x_n\|_1 = \|x\|_1 . \tag{5}$$

Now assume $p > 1$. We have that

$$\lim_n \|x_n\|_p = \|x\|_p \tag{6}$$

which means that
$$\lim_n \sum_k \|y_{n,k}\|^p = \sum_k \|y_k\|^p . \tag{7}$$
Let $|.|_p$ also denote the norm on $\ell_p$. Since

$$\|x_n + x\| = |(\|y_{n,k} + y_k\|)_k|_p \leq |(\|y_{n,k}\| + \|y_k\|)_k|_p$$

$$\leq |(\|y_{n,k}\|)_k|_p + |(\|y_k\|)_k|_p = \|x_n\| + \|x\|$$

and both $\|x_n + x\|$ and $\|x_n\| + \|x\|$ converge to $2\|x\|$, we deduce that

$$\lim_n |(\|y_{n,k}\| + \|y_k\|)_k|_p = 2|(\|y_k\|)_k|_p. \tag{8}$$

Since $|.|_p$ is LUR on $\ell_p$, we deduce from (7) and (8) that $\lim_n |(\|y_{n,k}\| - \|y_k\|)_k|_p = 0$, in particular
$$\forall k \in \mathbb{N}, \lim_n \|y_{n,k}\| = \|y_k\| . \tag{9}$$
Since $\|x + x_n\|$ converges to $2\|x\|$ we also have

$$\lim_n \sum_k \|y_{n,k} + y_k\|^p = 2^p \sum_k \|y_k\|^p . \tag{10}$$

Fix $k_0 \in \mathbb{N}$ and $\epsilon > 0$. We may find some $k_1 > k_0$ such that

$$\sum_{k \geq k_1} \|y_k\|^p < \epsilon. \tag{11}$$

Therefore by (7), (9), and (11), for $n$ large enough,

$$\sum_{k \geq k_1} \|y_{n,k}\|^p < 2\epsilon. \tag{12}$$

Using (9), (11) and (12), we deduce that for $n$ large enough,

$$\sum_k \|y_{n,k} + y_k\|^p < 2^p \sum_{k \neq k_0, k < k_1} \|y_k\|^p + \epsilon + 2^p.3\epsilon + \|y_{n,k_0} + y_{k_0}\|^p , \tag{13}$$

while by (10) and (11), for $n$ large enough,

$$\sum_k \|y_{n,k} + y_k\|^p > 2^p \sum_{k \neq k_0, k < k_1} \|y_k\|^p + 2^p \|y_{k_0}\|^p - 2^p\epsilon - \epsilon. \tag{14}$$

From (13) and (14) we deduce that for $n$ large enough,

$$2^p \|y_{k_0}\|^p < (2 + 4.2^p)\epsilon + \|y_{n,k_0} + y_{k_0}\|^p ,$$

and we deduce, using also (9), that

$$\lim_n \|y_{n,k_0} + y_{k_0}\| = 2 \|y_{k_0}\| . \tag{15}$$

From (9) and (15), and from the fact that the norm on $Y$ is LUR, it follows that

$$\forall k \in \mathbb{N}, \lim_n y_{n,k} = y_k. \tag{16}$$

Going back to the general case, fix $\epsilon > 0$ and let $k_1 \in \mathbb{N}$ be such that $\sum_{k\geq k_1} \|y_k\|^p < \epsilon$, then

$$\|x - x_n\|_p^p = \sum_{k<k_1} \|y_k - y_{n,k}\|^p + \sum_{k\geq k_1} \|y_k - y_{n,k}\|^p$$

$$\leq \sum_{k<k_1} \|y_k - y_{n,k}\|^p + 2^p \sum_{k\geq k_1} \|y_k\|^p + 2^p \sum_{k\geq k_1} \|y_{n,k}\|^p$$

$$= \sum_{k<k_1} \|y_k - y_{n,k}\|^p + 2^p(2 \sum_{k\geq k_1} \|y_k\|^p + (\|x_n\|_p^p - \|x\|_p^p) + \sum_{k<k_1} (\|y_k\|^p - \|y_{n,k}\|^p)).$$

So by (4) and (5) when $p = 1$, or by (6) and (16) when $p > 1$, we obtain that $\|x - x_n\|_p^p < 3.2^p\epsilon$ for $n$ large enough. □

**Lemma 39** *Let $Y$ be a Banach space with an LUR norm and let $X = c_0(Y)$. Then there exists an equivalent LUR norm on $X$ for which any map $T$ defined on $X$ by $T((y_n)_{n\in\mathbb{N}}) = (\epsilon_n y_{\sigma(n)})_{n\in\mathbb{N}}$, where $\epsilon_n = \pm 1$ for all $n \in \mathbb{N}$ and $\sigma$ is a permutation on $\mathbb{N}$, is an isometry.*

Let $|.|_D$ denote the equivalent Day's norm on $c_0$ defined in Example 12, that is for $x = (x_n)_n \in c_0$,

$$|x|_D = \sup(\sum_{i=1}^{k} x_{n_i}^2 / 4^i)^{1/2},$$

where the sup is taken over $k \in \mathbb{N}$ and all $k$-tuples $(n_1, \ldots, n_k)$ of distincts elements of $\mathbb{N}$. let $\|.\|$ denote the corresponding norm on $X = c_0(Y)$, therefore for $x = (y_k)_k \in X$,

$$\|x\| = \sup(\sum_{i=1}^{k} \|y_{n_i}\|^2 / 4^i)^{1/2},$$

and let $\|.\|_\infty$ denote the sup norm on $X$, $\|x\|_\infty = \sup_k \|y_k\|$ . Note that isomorphisms associated to a permutation on $\mathbb{N}$ and a sequence of signs are isometries on $X$ for $\|.\|$. It remains to prove that $\|.\|$ is LUR. Let $x = (y_k)_k \in X$ and $x_n = (y_{n,k})_k \in X$ be such that

$$\lim_n \|x_n\| = \|x\| \tag{17}$$

and

$$\lim_n \|x + x_n\| = 2\|x\| . \tag{18}$$

We need to prove that $\lim_n \|x - x_n\| = 0$ or equivalently $\lim_n \|x - x_n\|_\infty = 0$. Since $(x_n)_n$ is arbitrary satisfying (17) and (18) it is enough to prove that some subsequence of $(x_n)_n$ satisfies $\lim_n \|x - x_n\|_\infty = 0$.

Since, by elementary properties of $|.|_D$,

$$\|x + x_n\| = |(\|y_k + y_{n,k}\|)_k|_D \leq |(\|y_k\| + \|y_{n,k}\|)_k|_D \leq \|x\| + \|x_n\| ,$$

we deduce from (17) and (18) that

$$\lim_n |(\|y_k\| + \|y_{n,k}\|)_k|_D = 2|(\|y_k\|)_k|_D. \tag{19}$$

Since $|.|_D$ is LUR on $c_0$, [2] Theorem 7.3 p 69, we deduce from (17) and (19) that

$$\lim_n |(\|y_k\| - \|y_{n,k}\|)_k|_D = 0,$$

therefore

$$\lim_n \max_k |\, \|y_{n,k}\| - \|y_k\| \,| = 0. \tag{20}$$

For any $n \in \mathbb{N}$, let $k_n \in \mathbb{N}$ be such that

$$\|x - x_n\|_\infty = \|y_{k_n} - y_{n,k_n}\| . \tag{21}$$

Note that if $\lim_n k_n = +\infty$, then $\|x - x_n\|_\infty \leq 2\|y_{k_n}\| + \max_k |\, \|y_{n,k}\| - \|y_k\| \,|$ converges to $0$. So passing to a subsequence we may assume that $(k_n)_n$ is constant equal to some $k_0 \in \mathbb{N}$. If $y_{k_0} = 0$ then by (20), $\lim_n y_{n,k_0} = 0$ and $\lim_n \|x - x_n\|_\infty = \lim_n \|y_{k_0} - y_{n,k_0}\| = 0$. Therefore we may assume that $y_{k_0} \neq 0$.

Let $m \in \mathbb{N}$ be such that $m \geq |\{i \in \mathbb{N} : \|y_i\| \geq \frac{1}{2}\|y_{k_0}\|\}|$. Let $\beta = \frac{1}{2}\frac{\|y_{k_0}\|}{2^m}$. We prove that for $n$ large enough,

$$\|y_{k_0} + y_{n,k_0}\| \geq \beta. \tag{22}$$

Indeed if (22) is contradicted then it is easy to see by the expression of $|.|_D$ that we may assume that for all $n$,

$$\|x+x_n\|^2 \leq \sum_{i=1}^{+\infty} \frac{\left\|y_{k_i^n}+y_{n,k_i^n}\right\|^2}{4^i}+\beta^2,$$

for some sequence $(k_i^n)_{i\geq 1}$ of distinct integers different from $k_0$. Let $\epsilon$ be positive. By (20) we deduce, for $n$ large enough,

$$\|x+x_n\|^2 \leq (1+\epsilon)4\sum_{i=1}^{+\infty} \frac{\left\|y_{k_i^n}\right\|^2}{4^i}+\beta^2,$$

So

$$\|x+x_n\|^2 \leq (1+\epsilon)4\sum_{i=1}^{+\infty} \frac{\|y_{j_i}\|^2}{4^i}+\beta^2,$$

where $(\|y_{j_i}\|)_{i\geq 1}$ is a non-increasing enumeration of $\{\|y_k\|, k\neq k_0\}$. Passing to the limit in $n$ and $\epsilon$, and using (18), we deduce

$$4\|x\|^2 \leq 4\sum_{i=1}^{+\infty} \frac{\|y_{j_i}\|^2}{4^i}+\beta^2 \leq 4\sum_{i=1}^{m} \frac{\|y_{j_i}\|^2}{4^i}+\|y_{k_0}\|^2\sum_{i=m+1}^{+\infty}\frac{1}{4^i}+\beta^2,$$

therefore

$$4\|x\|^2+\frac{\|y_{k_0}\|^2}{4^m} \leq 4(\sum_{i=1}^{m} \frac{\|y_{j_i}\|^2}{4^i}+\frac{\|y_{k_0}\|^2}{4^{m+1}})+\frac{\|y_{k_0}\|^2}{3.4^m}+\beta^2 \leq 4\|x\|^2+\frac{\|y_{k_0}\|^2}{3.4^m}+\beta^2.$$

We deduce that $\frac{2}{3.4^m}\|y_{k_0}\|^2 \leq \beta^2$, a contradiction. Therefore (22) is proved. Now

$$2\|x\|^2+2\|x_n\|^2-\|x+x_n\|^2 = 2\sum_{i=1}^{+\infty}\frac{\|y_{l_i}\|^2}{4^i}+2\sum_{i=1}^{+\infty}\frac{\left\|y_{n,l_i^n}\right\|^2}{4^i}-\sum_{i=1}^{+\infty}\frac{\left\|y_{n,m_i^n}+y_{m_i^n}\right\|^2}{4^i},$$

where $(\|y_{l_i}\|)_i$, $(\left\|y_{n,l_i^n}\right\|)_i$, and $(\left\|y_{n,m_i^n}+y_{m_i^n}\right\|)_i$ are non-increasing enumerations of $(\|y_k\|)_k$, $(\|y_{n,k}\|)_k$, and $(\|y_k+y_{n,k}\|)_k$, respectively. Therefore

$$2\|x\|^2+2\|x_n\|^2-\|x+x_n\|^2 \geq 2\sum_{i=1}^{+\infty}\frac{\left\|y_{m_i^n}\right\|^2}{4^i}+2\sum_{i=1}^{+\infty}\frac{\left\|y_{n,m_i^n}\right\|^2}{4^i}-\sum_{i=1}^{+\infty}\frac{\left\|y_{n,m_i^n}+y_{m_i^n}\right\|^2}{4^i}.$$

Since by (17) and (18),

$$\lim_n 2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 = 0,$$

we deduce by [2] Fact 2.3 p 45 that

$$\forall i \in \mathbb{N}, \lim_n 2\left\|y_{m_i^n}\right\|^2 + 2\left\|y_{n,m_i^m}\right\|^2 - \left\|y_{n,m_i^n} + y_{m_i^n}\right\|^2 = 0. \qquad (23)$$

Let $K \in \mathbb{N}$ be such that for $k > K$, $\|y_k\| \leq \frac{\beta}{4}$. By (20), we have for $n$ large enough and $k > K$,

$$\|y_k + y_{n,k}\| \leq 2\|y_k\| + \frac{\beta}{4} \leq \frac{\beta}{2}.$$

By (22) we deduce that for $n$ large enough, $k_0 \in \{m_1^n, \ldots, m_K^n\}$. There exists $i$ such that $k_0 = m_i^n$ for infinitely many $n$'s. Therefore from (23) we deduce, passing to a subsequence,

$$\lim_n 2\|y_{k_0}\|^2 + 2\|y_{n,k_0}\|^2 - \|y_{k_0} + y_{n,k_0}\|^2 = 0.$$

Since the norm $\|.\|$ on $Y$ is LUR, this implies by [2]Proposition 1.2 p 42 that $\lim_n y_{n,k_0} = y_{k_0}$. Finally

$$\lim_n \|x - x_n\|_\infty = \lim_n \|y_{k_0} - y_{n,k_0}\| = 0.$$

$\square$

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# References

[1] S.F. Bellenot, *Banach spaces with trivial isometries*, Israel Journal of Math. **56** (1986), no. 1, 89–96.

[2] R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in pure and applied mathematics, Longman Scientific and Technical Ed., (1993).

[3] V. Ferenczi, *Uniqueness of complex structure and real hereditarily indecomposable Banach spaces*, Advances in Math. **213**, 1 (2007), 462–488.

[4] Y. Gordon and R. Loewy, *Uniqueness of* ($\Delta$) *bases and isometries of Banach spaces*, Math. Ann. **241** (1979), 159–180.

[5] W.T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. **6** (1993), 4, 851–874.

[6] K. Jarosz, *Any Banach space has an equivalent norm with trivial isometries*, Israel Journal of Math. **64** (1988), no. 1, 49–55.

[7] A. S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, 156, Springer-Verlag, New York, 1995.

[8] G. Lancien, *Dentability indices and locally uniformly convex renormings*, Rocky Mountain J. Math. 23 (1993), no. 2, 635–647.

[9] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Springer-Verlag, New York, Heidelberg, Berlin (1979).

[10] J. Stern, *Le groupe des isométries d'un espace de Banach* (French), Studia Math. **64** (1979), no. 2, 139–149.



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